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Notes on chiral hydrodynamics within the effective theory approach

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We address the issue of evaluating chiral effects (such as the newly discovered chiral separation) in hydrodynamic approximation. The main tool we use is effective theory which defines interaction in terms of chemical potentials μ, μ_5. In the lowest order in μ, μ_5 we reproduce recent results based on thermodynamic considerations. In higher orders the results depend on details of infrared cutoff. Another point of our interest is an alternative way of the anomaly matching through introduction of effective scalar fields arising in the hydrodynamic approximation.

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I. INTRODUCTION

There is a clear renewed interest in the relativistic hydrodynamics, in view of its success in describing properties of the (strongly interacting) quark-gluon plasma (see, e.g., [1] and references therein). Recently some new hydrodynamic phenomena resulted from existence of novel transport coefficients have been found [2,3]. One considers a medium made of chiral fermions such that there exist two conserved charges Q and Q_5 and one can introduce two corresponding chemical potentials: μ and μ_5. Then in the standard hydrodynamic approximation one would use for the phenomenological vector and axial-vector currents the following textbook expressions:

$$J^\mu = n \cdot u^\mu, \qquad J_5^\mu = n_5 \cdot u^\mu, \tag{1}$$

where n and n_5 are densities of particles with the corresponding charges, while u^μ is the 4 velocity of an element of the liquid. As is argued in [2,3], one can add another term to the axial-vector current

$$\delta J_5^\mu = c_\omega \cdot \omega^\mu, \tag{2}$$

where $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$ and the new term describes chirality separation through rotation. Indeed, if the liquid is rotated with angular velocity $\mathbf{\Omega}$ then according to (2) there arises chiral current along the vector $\mathbf{\Omega}$. A remarkable feature of the new term is that, according to [3], the coefficient c_ω [in our approach see (10)] is uniquely fixed by the requirement of positivity of the divergence of the entropy current, $\partial_\mu s^\mu > 0$.

An intriguing point in the evaluation of the coefficient c_ω in Ref. [3] is that it is mostly given in thermodynamic terms. The only field-theoretic input is the chiral anomaly in the presence of both electric and magnetic fields. A puzzling observation is that the coefficient c_ω is related to the coefficient in front of the chiral anomaly although the chiral separation (2) persists in the absence of any electromagnetic fields. Some progress in understanding relation of Eq. (2) to the field theory was made in Ref. [4] where it was observed that the chirality separation, similar to (2) arises also in the flavor-related superfluidity (at zero temperature). The derivation [4] is based on the construction [5] of the conserved currents corresponding to Wess-Zumino effective action in the Goldstone phase.

We address the issue of the chiral effects in the hydrodynamic approximation within an effective theory. The interaction in this effective theory is determined by the chemical potentials, μ and μ_5, and the perturbation theory looks as an expansion in them. The first term in the expansion reproduces the leading contribution to c_ω. Further terms appear to depend on details of the infrared regularization.

II. EFFECTIVE FIELD THEORY

Our set up is as follows. There exists a fundamental Lagrangian constructed on massless left-handed, ψ_L and right-handed, ψ_R fermions. The fundamental interaction is assumed to conserve chirality and be nonanomalous, while in presence of external electromagnetic field A_μ there is chiral anomaly. As an example one could think of two flavors of massless quarks interacting with gluons in the standard flavor-blind way. This interaction is assumed to be responsible for formation of dense quark-gluon matter state, whose relevant effective long-distance description is given in terms of relativistic hydrodynamics. We choose microscopic currents as

$$j^{i,\mu} = \bar\psi\hat\tau^i\gamma^\mu\psi, \qquad j_5^{i,\mu} = \bar\psi\hat\tau^i\gamma^\mu\gamma_5\psi, \tag{3}$$

where the quark field $\psi = (u, d)$, $\hat\tau^0 = \hat 1$, and $\hat\tau^{1,2,3}$ are the standard Pauli matrices in flavor space. Classically these currents are conserved: $\partial_\mu j^\mu = \partial_\mu j_5^\mu = 0$.

To imitate the effects of the medium one introduces chemical potentials, conjugated to the corresponding conserved charges:

$$\delta H = \mu^i Q^i + \mu_5^i Q_5^i, \tag{4}$$

where $Q^i = \int d^3x\psi^\dagger\hat\tau^i\psi$ and $Q_5^i = \int d^3x\psi^\dagger\gamma_5\hat\tau^i\psi$. At Lagrangian level (4) corresponds to

$$\delta L = \bar\psi\,\hat\mu\,\gamma^0\psi + \bar\psi\hat\mu_5\gamma^0\gamma_5\psi + i\bar\psi\,\hat\epsilon(\mu, \mu_5, p_0)\psi, \tag{5}$$

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where we have introduced the notation $\hat{\mu} = \mu^i \hat{\tau}^i$, $\hat{\mu}_5 = \mu_5^i \hat{\tau}^i$. It is worth mentioning that the condition for the axial current to be nonanomalous with respect to the strong interactions dictates its isovector nature, i.e. one must have $\mu_5^0 = 0$. There is no such restriction for the vector current which may contain singlet component. The fact of currents conservation allows to introduce self-consistently the chemical potentials.

As is well known, the chemical potential μ conjugated to the charge Q in quantum field theory brings two aspects absent at zero density: first, it shifts energy levels of the system $p_0 \to p_0 - \mu$ (since all the levels below μ are occupied), and second, $i\epsilon$ prescription for propagator poles changes to μ and p-dependent one (see, e.g. [6]), and in this sense the theory becomes nonlocal in coordinate space. It is worth stressing that both effects correspond to a change of the vacuum state. However, as we will argue below, from effective-theory point of view they play rather different roles.

So far the discussion proceeded in terms of fundamental microscopic fields. We can now address the most important issue of transition to effective theory. First, one is to introduce "physically microscopic volume" of the medium, V_y, centered around the point y, in which infrared, long-distance fields of effective theory may be regarded as being uniform. Performing Lorentz boost to the rest frame one gets

$$S = \int_{V_y} d^4x \bar{\psi} \gamma_\mu (i\partial^\mu + (\hat{\mu} + \hat{\mu}_5 \gamma_5) u^\mu + \hat{q} A^\mu) \psi + i \int_{V_y} d^4x \bar{\psi}\, \hat{\epsilon}(\mu, \mu_5, p, u) \psi + S_{\rm int}, \tag{6}$$

where u^μ is the four-velocity of a given element of the fluid in the center-of-energy rest frame. As an infrared variable, it is supposed to be x independent. We have also introduced coupling to the external electromagnetic field A^μ with the charge matrix $\hat{q} = {\rm diag}(q_u, q_d)$. This field is also treated as long-distance nondynamical one. The term $S_{\rm int}$ stays for strong interaction part of the total action whose exact form is not relevant here.

To proceed one has to introduce summation over V_y (i.e. integration over effective-theory coordinates y) and integrate out microscopic fields: quarks and gluons. However for our purposes we need not to know the final result of this complex procedure. Instead, we notice that the effective currents anomalous (non)conservation cannot be affected by μ dependence of $i\epsilon$ term in (6). The latter is an infrared effect vanishing at $\mu = 0$ and corresponding to vacuum state change. At large momenta one always comes back to the standard $i\epsilon$ prescription with constant ϵ. This is completely analogous to the results on independence of chiral anomaly on temperature/density found in [7]. Speaking more technically, coupling of the effective scalar mass field $m - i\epsilon$ to fermions is nonanomalous regardless of its dynamics (encoded in μ, p and u dependence of its complex $i\epsilon$ part in our case), even if the real part of the mass vanishes. Thus the μ, μ_5 dependence of the anomaly can come only from the $\bar{\psi}\gamma_\mu(\hat{\mu} + \hat{\mu}_5\gamma_5)u^\mu\psi$ term in the effective action. This term, on the other hand, can be dealt with on exactly the same footing as the standard gauge field term $\bar{\psi}\gamma_\mu \hat{q} A^\mu \psi$. For finding the anomaly it is convenient to use the Fujikawa-Vergeles method [8]. According to this method, anomalies emerge due to noninvariance of the path-integral measure under field transformations. For the sake of simplicity we consider below only diagonal components of the currents (i.e. $i = 0$, 3 components), however we continue to use invariant isospin notation. Consider the following transformation

$$\psi \to e^{i\hat{\alpha}\gamma_5 + i\hat{\beta}} \psi. \tag{7}$$

One readily finds[1]

$$\partial_\mu j_5^{i,\mu} = -\frac{1}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} {\rm Tr}(\hat{\tau}^i (\partial^\mu (\hat{q} A^\nu + \hat{\mu} u^\nu) \partial^\alpha (\hat{q} A^\beta + \hat{\mu} u^\beta) + \partial^\mu (\hat{\mu}_5 u^\nu) \partial^\alpha (\hat{\mu}_5 u^\beta))) \tag{8}$$

$$\partial_\mu j^{i,\mu} = -\frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} {\rm Tr}(\hat{\tau}^i \partial^\mu (\hat{q} A^\nu + \hat{\mu} u^\nu) \partial^\alpha (\hat{\mu}_5 u^\beta)). \tag{9}$$

In hydrodynamic approximation one has correspondence between microscopic currents $j^{i,\mu} = \bar{\psi}\hat{\tau}^i\gamma^\mu\psi$, $j_5^{i,\mu} = \bar{\psi}\hat{\tau}^i\gamma^\mu\gamma_5\psi$ and macroscopic effective ones $J^{i,\mu}$, $J_5^{i,\mu}$ carrying the same conserved quantum numbers and given by (1). The collinear nature of the effective currents is an important feature of nonsuperfluid hydrodynamics, where all charges densities propagate with the same velocity u^μ. Having in mind that the right-hand sides of (8) and (9) contain only effective long-distance fields one can replace divergencies of microscopic currents ∂j in the left-hand sides by those of the effective currents ∂J. In particular, the anomaly Eq. (8) can be rewritten as

$$\partial_\mu \left(n_5^i u^\mu + {\rm Tr}\hat{\tau}^i \left(\frac{1}{2\pi^2} (\hat{\mu}^2 + \hat{\mu}_5^2) \omega^\mu + \frac{1}{2\pi^2} \hat{\mu}\, \hat{q}\, B^\mu \right) \right) = -\frac{1}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} {\rm Tr}(\hat{\tau}^i \hat{q}^2 \partial^\mu A^\nu \partial^\alpha A^\beta), \tag{10}$$

where $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$ is the magnetic field in the rest frame. We will come back to discuss the implications of this equation later. Now, let us emphasize that in the effective theory we have an expansion in μ, μ_5. The anomalous terms are exhausted by the triangle graph or by terms quadratic in μ, μ_5 kept explicit in (9) and (10). Another observation is that to keep the coefficient of the term ω^μ nonvanishing one must have nonzero singlet component in the vector current (otherwise ${\rm Tr}\hat{\tau}^i \hat{\mu}^2 = 0$).

[1]We could have defined anomaly in such a way that it does not contribute to $\partial_\mu j_5^\mu$. In presence of both chemical potentials μ and μ_5 there is no physical motivation for such a regularization.

The result for the coefficient c_ω implied by (10) coincides with that of the paper [3] in the lowest nontrivial order in the chemical potentials. The comparison is not absolutely straightforward, though. The reason is that we introduce two chemical potentials, μ, μ_5, and assume interactions to conserve parity (so that μ_5 is actually a pseudoscalar). On the other hand, the simplest version considered in most detail in [3] deals with a single chiral current, with no parity conservation.

It is worth emphasizing that there exist contributions of higher orders in μ, μ_5 to the hydrodynamic currents. The higher-order terms, however, do not contribute to the anomalous divergences evaluated within the effective theory and for this reason do not enter Eq. (8). Moreover, such terms depend, generally speaking, on details of the infrared cutoff, and we make no attempt to find them in closed form. For the sake of an estimate, consider the μ^3 contribution to the ω^μ term. It is of the order

$$\delta c_\omega \sim \frac{\mu^3}{2\pi^2}\frac{1}{\epsilon_{IR}}, \tag{11}$$

where we keep the $(2\pi^2)^{-1}$ factor just to indicate that it is a one-loop correction and ϵ_{IR} is an infrared cutoff in the energy/momentum integration. In the hydrodynamic limit, the following estimate for ϵ_{IR} seems reasonable:

$$\epsilon_{IR} \sim (\epsilon + p)/n,$$

where ϵ, p are the energy density and pressure, respectively, and n is the particle density. Moreover, the ratio $(\epsilon + p)/n$ is known to play the role of the mass in the relativistic hydrodynamic. The effective-theory estimate (11) reproduces then the structure of the explicit contribution to c_ω found in [3], $\delta c_\omega \sim \mu^3 n/(\epsilon + p)$.

Our final remark in this section is that the expression (9) for the vector current is closely related to another chiral effect in the hydrodynamic approximation, that is the chiral magnetic effect thoroughly discussed in the literature, for a review and references see [9]. In particular, in case $u^0 = 1$, $u^a = 0$ and with one flavor we have

$$J_a = -\frac{1}{2\pi^2}\mu_5 B_a,$$

where $a = 1, 2, 3$, and $B_a = \frac{1}{2}\epsilon_{abc}F^{bc}$ are components of the external magnetic field and reproduce well known results [9] for the chiral magnetic effect. In case of nontrivial u^μ Eq. (8) demonstrates existence of hydrodynamic corrections. For more details see [10].

III. COMPARISON WITH THERMODYNAMIC APPROACH

We can compare the results obtained within the effective theory (6) with the results of Ref [3], which are entirely based on thermodynamics (plus equations of state). The results within the thermodynamic approach can be summarized in the following way:

$$J^\mu_{\rm ch} = n_{\rm ch}u^\mu + \xi_\omega \omega^\mu + \xi_B B^\mu, \tag{12}$$

where $J^\mu_{\rm ch}$ is a chiral current associated with, say, left-handed fermions. Imposing the positivity of the flow of the entropy one fixes the coefficients uniquely. In particular,

$$\xi_\omega = \frac{\mu^2_{\rm ch}}{2\pi^2}\left(1 - \frac{2}{3}\frac{\mu_{\rm ch}n_{\rm ch}}{\epsilon + p}\right), \tag{13}$$

where $n_{\rm ch}$ and $\mu_{\rm ch}$ are the density of the chiral particles and the corresponding chemical potential, respectively.

One can readily see that the $\mu^2_{\rm ch}$ terms are the same as within the effective-theory approach above. However, for higher-order terms the predictions vary. According to the effective-theory approach the μ^3 terms are dependent on details of the infrared cut off and not fixed in this sense. According to (13), on the other hand, the $\mu^3_{\rm ch}$ terms are uniquely determined.

To appreciate the meaning of this discrepancy one should have in mind that some further assumptions were made in [3]. In particular, the ideal-liquid equations of motion are used. More generally, as is emphasized in [3] the vorticity term ω^μ appears within the standard hydrodynamic approximation in higher orders in derivatives than the leading term, which is $J^\mu_{\rm ch} \approx n_{\rm ch}u^\mu$. Keeping systematically all the terms of the next order in derivatives (not only ω^μ) would modify equations of motion and destroy (13). Moreover, as is noticed in [11] there is a kind of or gauge invariance in relativistic formulation of hydrodynamics which allows to redefine higher-derivatives terms in expansion of the energy-momentum tensor and of currents.

Thus, we believe that it is the extra assumptions made which allow to fix the μ^3 terms within the thermodynamic approach. Within the effective-theory approach the μ^3 terms are infrared dependent. It is worth mentioning that the infrared dependence of higher-order terms was found also within a superfluid version of hydrodynamics considered in [4].[2]

IV. CONSERVATION LAWS IN HYDRODYNAMIC APPROXIMATION

To summarize, within the model considered we have two types of anomalies. First, there is the common chiral anomaly in the divergence of the fundamental axial current, see Eq. (8). Second, there are anomalies presented in the effective theory alone.

Let us first consider the case when the product $(\mathbf{E}\cdot\mathbf{H}) = 0$ for external fields and the fundamental currents are conserved. The origin of the anomalies in the effective theory is that the symmetry of the effective theory

[2]Actually the Ref [4] does not state explicitly that the μ^3 terms are not uniquely fixed. Rather, a particular prescription for an infrared cut off is picked up.

is not the same as of the fundamental interaction. The point is clearly illustrated by the toy model (3) and (5). The strong interaction of quarks and gluons is flavor blind. We choose however asymmetric initial conditions by fixing the average number of u quarks but not of d quarks. Introduction of the effective interaction (5) transfers this flavor-asymmetry of the initial condition into the flavor-dependent effective interaction. Hence, anomalies emerge in the language of the effective theory.

The first example of this type was seemingly given in Ref. [5]. In that case, effective Lagrangian for the interaction of Goldstone bosons π^a is constructed in terms of the matrix

$$U = \exp\left(\frac{2i}{F_\pi}\lambda^a\pi^a\right), \qquad U^R_\mu \equiv U^{-1}\partial_\mu U.$$

The conservation condition is then a sum of two terms, naive and anomalylike ones:

$$-\partial_\mu F_\pi^2 U^R_\mu + (i/2)\epsilon_{\mu\nu\alpha\beta}U^R_\mu U^R_\nu U^R_\alpha U^R_\beta = 0. \qquad (14)$$

As is noted first in Ref. [4] the analogy between (14) and the hydrodynamic models we are considering might be much more direct than it appears at first sight. We come back to discuss this point later.

Note that the effective theory fixes all the anomalous terms in the left-hand side of the condition (10) in the absence of the genuine chiral anomaly which arises in case of $(\mathbf{E}\cdot\mathbf{H}) \neq 0$. This is again in analogy with the case considered in Ref. [5] and in contrast with the thermodynamic derivation of Ref. [3], where the genuine chiral anomaly is a necessary input to fix the vorticity coefficient c_ω. The simplicity of the Eq. (10) is somewhat deceptive, however, since the current itself, generally speaking, contains divergence-free terms dependent on the infrared cut off.

Turn now to the case of $(\mathbf{E}\cdot\mathbf{H}) \neq 0$. Then there is an immediate problem that the axial current $j_{\mu 5}$ is no longer conserved and the introduction of the chemical potential, see (6), conjugated to a nonconserved charge is not consistent, for recent discussion see [12] and references therein. As is known since long, see in particular [13] one can introduce a new conserved axial current also in the presence of external fields with $(\mathbf{E}\cdot\mathbf{H}) \neq 0$:

$$\partial_\mu \tilde{J}^\mu_5 = 0, \qquad (15)$$

where

$$\tilde{J}^\alpha_5 = n_5 u^\alpha + \frac{1}{2\pi^2}(\mu^2 + \mu_5^2)\omega^\alpha + \frac{\mu}{2\pi^2}B^\alpha - K^\alpha, \quad (16)$$

where $B^\mu \equiv \tilde{F}^{\mu\nu}u_\nu$. Although the current K_μ is not gauge invariant, the corresponding charge, $\int d^3xK_0$, is gauge invariant. However, we are considering the approximation of external electromagnetic fields, and the value of $(\mathbf{E}\cdot\mathbf{H})$ is not dynamical. In this sense the chiral charge of the medium is changed under influence of external fields, independent of possible redefinition (16). Therefore Eq. (10) is rather to be understood as an approximate, in the sense that we neglect the change of the axial charge due to the $(\mathbf{E}\cdot\mathbf{H}) \neq 0$ (the change of the axial charge of the medium is proportional to time for constant $\mathbf{E}$ and $\mathbf{H}$).

Turn now to a more delicate point, that is how the 't Hooft matching condition [14] is realized in the hydrodynamic approximation. Let us begin with commonly accepted points. At vanishing temperatures, the alternatives are that there exist either massless, not confined, quarks or massless pseudoscalars. Clearly, in QCD the Goldstone mode is realized at $T = 0$. At small finite temperatures there are well defined corrections to the pseudoscalar decays constants [15]. Finally, at the temperature of deconfining phase transition $T = T_c$ pions become massive and the matching involves massless, not confined quarks.

Now we are considering hydrodynamic approximation which assumes averaging over distances Δx larger than the free path of the quarks:

$$\Delta x \gg l_{\text{free path}}.$$

As a result of the interaction liquid is formed and one considers hydrodynamics, that is a classical approximation. The central point is then that there are no fermionic classical fields. In other words, in the hydrodynamic approximation there can be no fermionic (massless) excitations which would trivially saturate the 't Hooft matching condition. Thus, we are led to conclude that in the hydrodynamic approximation the 't Hooft consistency condition is to be satisfied on the bosonic massless modes.[3]

To reiterate, for fermions interacting only with external magnetic field the chiral anomaly is realized through chiral zero modes, which are nothing else but particular Landau levels, for details see [9,16]. However, now we consider the case when the fermions interact among themselves and form a liquid as a result of this interaction. This approximation corresponds to

$$l_{\text{free path}} < R_{\text{Landau}}, \qquad (17)$$

where $l_{\text{free path}}$ is the free-path length in the liquid and R_{Landau} is the radius of the would-be zero-energy Landau level. Note also that the hydrodynamic approximation assumes coarse graining at scale larger than $l_{\text{free path}}$. Then the fermions lose coherence and Landau levels do not exist as solutions any longer. Therefore in the hydrodynamic approximation the matching of the chiral anomaly to physical fermionic excitations is questionable.

Thus, matching of the quark anomaly to massless pseudoscalar degrees of freedom seems to be a viable

[3]In reality, even the light u and d quarks might be too heavy to be considered massless on the distances Δx relevant to the hydrodynamics.

alternative. In that case one would introduce an analog, $\tilde{U}^R_\mu$, to the matrix U^R_μ above and construct an effective current in terms of a new field $\pi^a_{\rm hydro}$, which might be called ''hydrodynamic shadow'' of the pion. As is argued in Ref. [4] the two terms in the current (14) can be interpreted in the hydrodynamic approximation as naive current $n_5 u_\mu$ and the vorticity term $c_\omega \omega_\mu$.

In our case, introduction of a specifically hydrodynamic massless excitations $\pi^a_{\rm hydro}$ would result, generally speaking, in a two-component liquid:

$$j_5^\mu \sim n_5 u^\mu + \tilde{n}_5 v^\mu + \ldots, \tag{18}$$

where u_μ, v_μ are two independent 4 velocities. Relativistic version of the hydrodynamics of two-component liquids has been intensely discussed recently, see in particular [17]. Generically, existence of massless bosonic excitations results in superfluidity.

V. CONCLUSION

In this note we have attempted to evaluate the newly discovered hydrodynamic chiral effects within the effective-theory approach. The perturbation theory within this approach is an expansion in chemical potentials, μ, μ_5. The effective theory is anomalous. The chiral anomaly fixes uniquely the μ^2 terms. Moreover, the anomaly is exhausted, as usual, by its lowest nontrivial order. The μ^2 terms generated by the anomaly coincide with the μ^2 terms found earlier within a pure thermodynamic approach [3]. Higher-order terms turn to be infrared dependent within the effective-theory approach.

Our interest in the problems considered stems not so much from phenomenological applications but rather from the fact that hydrodynamics appears to provide a novel example of realization of chiral symmetries. Namely, the symmetry which is not anomalous in the fundamental theory turns to be anomalous within the hydrodynamic approximation. One of the mechanisms of generating anomalous effective theories is the introduction of chemical-potential terms. Originally, chemical potentials reflect initial conditions, in terms of conserved charges. The symmetry of the initial conditions does not necessarily coincide with symmetry of the fundamental interactions. Also, we argued that the 't Hooft matching condition in the hydrodynamics favors massless boson excitations. Thus, even at temperatures above the deconfining phase transition there could exist specific hydrodynamic excitations $\pi^a_{\rm hydro}$ with quantum numbers of ordinary pions.

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