Electromagnetic Casimir piston in higher-dimensional spacetimes

L. P. Teo[*]

Department of Applied Mathematics, Faculty of Engineering, University of Nottingham Malaysia Campus,
Jalan Broga, 43500, Semenyih, Selangor Darul Ehsan, Malysia

We consider the Casimir effect of the electromagnetic field in a higher-dimensional spacetime of the form $M \times \mathcal{N}$, where M is the four-dimensional Minkowski spacetime and $\mathcal{N}$ is an n-dimensional compact manifold. The Casimir force acting on a planar piston that can move freely inside a closed cylinder is investigated. Different combinations of perfectly conducting boundary conditions and infinitely permeable boundary conditions are imposed on the cylinder and the piston. It is verified that if the piston and the cylinder have the same boundary conditions, the piston is always going to be pulled towards the closer end of the cylinder. However, if the piston and the cylinder have different boundary conditions, the piston is always going to be pushed to the middle of the cylinder. By taking the limit where one end of the cylinder tends to infinity, one obtains the Casimir force acting between two parallel plates inside an infinitely long cylinder. The asymptotic behavior of this Casimir force in the high temperature regime and the low temperature regime are investigated for the case where the cross section of the cylinder in M is large. It is found that if the separation between the plates is much smaller than the size of $\mathcal{N}$, the leading term of the Casimir force is the same as the Casimir force on a pair of large parallel plates in the $(4 + n)$-dimensional Minkowski spacetime. However, if the size of $\mathcal{N}$ is much smaller than the separation between the plates, the leading term of the Casimir force is $1 + h/2$ times the Casimir force on a pair of large parallel plates in the four-dimensional Minkowski spacetime, where h is the first Betti number of $\mathcal{N}$. In the limit the manifold $\mathcal{N}$ vanishes, one does not obtain the Casimir force in the four-dimensional Minkowski spacetime if h is nonzero. Therefore the data obtained from Casimir experiments suggest that the first Betti number of the extra dimensions should be zero.

I. INTRODUCTION

In 1948, Casimir proposed the existence of a force of magnitude

$$F = \frac{\pi^2 \hbar c A}{240 a^4}$$

between two parallel perfectly conducting (PC) plates of area A which are separated by a distance a due to the vacuum fluctuations of electromagnetic field [1]. Since 1970s, Casimir effect has aroused the interest of many researchers for its close relations with many other areas of physics such as quantum field theory, atomic physics, condensed matter physics, nanotechnology, astrophysics, and mathematical physics [2]. Although the original proposal of Casimir considered only the electromagnetic field, nowadays the scope of Casimir effect includes all other quantum fields. The Casimir effect or vacuum energy is an vital component of quantum field theory.

Higher-dimensional spacetimes have become ubiquitous in the theories of high energy physics especially in superstring and supergravity theories. Different spacetimes with extra dimensions have been proposed to solve some fundamental problems in physics such as the hierarchy problem and the dark energy problem. The Casimir energy plays an important role in the studies of these extra dimensional spacetimes. On the one hand, it has been investigated as a candidate for the dark energy that accounts for the accelerated expansion of the Universe [3–9]. On the other hand, it was studied for its role in the stabilization mechanism of extra dimensions [10–19].

In recent years, there is an increased interest in studying the Casimir effect on objects in spacetimes with extra dimensions [3,10,20–40]. It is desirable to investigate how the presence of extra dimensions would change the direction and the magnitude of the Casimir force. However, most of these works studied the scalar fields. Only very few works tried to address the problem on electromagnetic fields since it is more complicated due to the presence of gauge freedom. In some works, the simple relation between the Casimir force on a pair of large parallel plates due to a massless scalar field and the Casimir force on a pair of large parallel plates due to an electromagnetic field was wrongly extended to other scenarios.

By definition, the zero temperature Casimir energy is a divergent sum of the zero point energies of a quantum field. There are various methods such as cutoff method and zeta regularization method to remove the divergence and extract a physically meaningful Casimir energy. However, these divergence removal procedures can sometimes lead to ambiguities. In 2004, the piston configuration was introduced [41] and it quickly attracted a lot of attention

[*]LeePeng.Teo@nottingham.edu.my

because the divergence of the Casimir energy in this configuration can be unambiguously removed. The zero temperature Casimir force acting on a piston due to scalar fields or electromagnetic fields in the 4*D* Minkowski spacetime was soon investigated in [42–45]. This was then extended to rectangular piston in Minkowski spacetimes of arbitrary dimensions [46,47] and to the finite temperature effect [48]. There are also a number of works which considered the generalized piston configurations [49–57]. Lately, there is an interest in considering the piston configuration in spacetimes with extra dimensions such as the Kaluza-Klein spacetime and the Randall-Sundrum spacetime [26,28–33,36–38]. However, as mentioned above, the works in this direction were restricted to scalar fields. To the best of our knowledge, no work has considered the electromagnetic Casimir effect on a piston in higher-dimensional spacetimes. Although the results on scalar fields might give some general picture about the case of electromagnetic fields, however, it has been known that the direction and the magnitude of the Casimir force is very sensitive to the type of the fields, the boundary conditions, and the geometry of the objects [2]. A careful study of the electromagnetic Casimir effect in higher-dimensional spacetime is essential to avoid drawing unfounded conclusions from the results for scalar fields. In fact, we are going to see from this work that the electromagnetic Casimir effect imposes a topological condition on the extra dimensions, which is something that have never been discovered from the study of Casimir effect due to scalar fields.

The electromagnetic Casimir effect on a pair of large parallel perfectly conducting plates in the Kaluza-Klein spacetime with internal space S^1 and in the Randall-Sundrum spacetime have been considered in [27] and [40], respectively. As pointed out in [40], for an electromagnetic field in spacetimes with extra dimensions, one can either treat the field as a bulk field and impose the perfectly conducting boundary conditions introduced in [58] or one can use dimensional reduction to reduce the electromagnetic field to a tower of massive vector fields in the 4*D* Minkowski spacetime and impose the 4*D* perfectly conducting conditions on the massive vector fields. These two approaches lead to different Casimir effects. The first approach is a genuine higher-dimensional Casimir effect, whereas the second approach is essentially the Casimir effect of 4*D* massive vector fields.

In this work, we are going to consider the bulk electromagnetic Casimir effect in higher-dimensional spacetimes with arbitrary compact extra dimensions. The spacetime $\mathcal{M}$ is assumed to have the form $M \times \mathcal{N}$, where M is the 4*D* Minkowski spacetime and $\mathcal{N}$ is a manifold of dimension n, assumed to be compact and connected. We are interested in the Casimir effect in the spacetime $\mathcal{M}$ due to the vacuum fluctuations of the electromagnetic field in a piston system. The piston system consists of a cylinder of length L and a piston which can move freely inside the
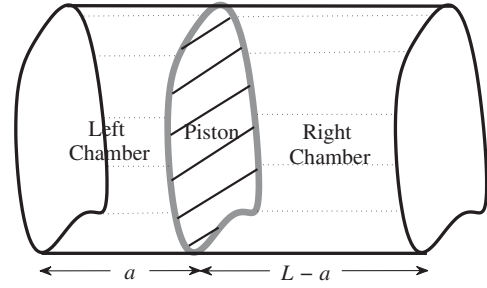

FIG. 1.   A piston system.

cylinder (See Fig. 1). The position of the piston is given by $x^1 = a$. The cross section of the cylinder and the piston are the same and assumes the general form $\Omega \times \mathcal{N}$, where Ω is a two-dimensional simply connected domain with boundary $\partial\Omega$ a smooth curve. For the boundary conditions on the walls of the cylinder and the piston, we impose either the perfectly conducting boundary conditions or the infinitely permeable boundary conditions proposed by Ambjørn and Wolfram [58]. Since the works by Marachevsky [44] and Kirsten and Fulling [28], there has been a trend in considering piston with arbitrary cross sections. Although studying a piston with rectangular cross section is enough for drawing conclusion about the Casimir effect on large parallel plates, it is desirable to know whether the properties of the Casimir force would be affected by the size and the geometry of the piston. In fact, a number of works have shown that the geometry of the piston has strong effect on the strength of the Casimir force [49–51,55]. In view of the interest in studying Casimir effect in generalized pistons [49–57], we think that it is necessary for us to consider piston with arbitrary cross section here.

The Casimir energy of the piston system is given by the sum of the Casimir energy inside the left chamber, the Casimir energy inside the right chamber and the Casimir energy outside the cylinder. The latter does not depend on a and therefore will not contribute to the Casimir force acting on the piston [41]. Omitting this term, we have

$$E_{\text{Cas}}^{\text{piston}} = E_{\text{Cas}}^{\text{cylinder}}(a) + E_{\text{Cas}}^{\text{cylinder}}(L - a), \qquad (1)$$

where $E_{\text{Cas}}^{\text{cylinder}}(a)$ is the Casimir energy inside a cylinder of length a. Using zeta regularization method, it is given by

$$E_{\text{Cas}}^{\text{cylinder}}(a) = -\frac{T}{2}(\zeta_T'(0; a) + \log[\lambda^2]\zeta_T(0; a)), \qquad (2)$$

where λ is a normalization constant and $\zeta_T(s)$ is the zeta function

$$\zeta_T(s; a) = \sum_{\omega} \sum_{p=-\infty}^{\infty} (\omega^2 + [2\pi pT]^2)^{-s},$$

which contains a sum over all ω that are the eigenfrequencies of the electromagnetic field inside the cylinder.

In Sec. II, we write down the eigenmodes of the electromagnetic field in a cylinder. We then proceed to compute the Casimir energy and the Casimir force in Sec. III. A discussion of the asymptotic behavior of the Casimir force in different limits is also given in Sec. III. In Sec. IV, we consider the specific examples where $\mathcal{N}$ is an n torus or an n sphere.

In the following, we use the units where $\hbar = c = k_B = 1$.

II. THE EIGENMODES OF THE FIELD INSIDE A CYLINDER

In this article, we consider spacetime $\mathcal{M}$ of the form $M \times \mathcal{N}$, where M is the four-dimensional Minkowski spacetime and $\mathcal{N}$ is an n-dimensional compact connected manifold. Let

$$ds^2 = g_{\mu\nu}dz^\mu dz^\nu = \eta_{\alpha\beta}dx^\alpha dx^\beta - G_{ab}dy^a dy^b$$

be the metric on $\mathcal{M}$, where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ is the usual four-dimensional metric on M, and $ds^2_{\mathcal{N}} = G_{ab}dy^a dy^b$ a Riemannian metric on $\mathcal{N}$. x denotes collectively the coordinates on M, y denotes collectively the coordinates on $\mathcal{N}$ and $z = (x, y)$. The action of the electromagnetic field is given by

$$S = -\frac{1}{4} \int \sqrt{|g|} F_{\mu\nu} F^{\mu\nu} d^N z, \quad (3)$$

where $N = 4 + n$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is an antisymmetric rank two tensor and $F^{\mu\nu} = g^{\mu\kappa} g^{\nu\eta} F_{\kappa\eta}$. The equation of motion is

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial z^\mu} (\sqrt{|g|} F^{\mu\nu}) = 0. \quad (4)$$

The perfectly conducting boundary condition is given by [58]

$$n^\mu (*F)_{\mu\nu_1...\nu_{N-3}} = 0, \quad (5)$$

and the infinitely permeable boundary condition is given by [58]

$$n^\mu F_{\mu\nu} = 0. \quad (6)$$

Here n^μ is a unit vector normal to the boundary surface, and $*F$ is the dual tensor of F.

As discussed in Appendix A, we can consider the eigenmodes of the electromagnetic field of the form

(I) $U_\alpha dx^\alpha$ with $U_0 = 0$ and $\delta_M U = 0$;

(II) $q_j(y)U_\alpha(x)dx^\alpha$ with $\delta_M U = 0$, $\Delta_M U + m_j^2 U = 0$, $j = 1, 2, \ldots$, where $q_j(y)$ is an eigenfunction with nonzero eigenvalue m_j^2 of the Laplace operator on functions on $\mathcal{N}$;

(III) $p(x)V_{j,a}(y)dy^a$ with $\Delta_M p + \mu_j^2 p = 0$, $j = 1, 2, \ldots$, where $V_{j,a}(y)dy^a$ is a coclosed eigen-one-form with eigenvalue μ_j^2 of the Laplace operator on $\mathcal{N}$.

In the following, we find the eigenmodes of the electromagnetic field in the cylinder $[0, a] \times \Omega \times \mathcal{N}$ with combinations of perfectly conducting or infinitely permeable boundary conditions on the sidewall $[0, a] \times \partial\Omega \times \mathcal{N}$, the bottom $x^1 = 0$, and the top $x^1 = a$.

The eigenmodes can be divided into TE modes which are modes with $F_{01} = 0$, and TM modes which are modes with $F_{\mu\nu} = 0$ for all $\mu, \nu \neq 0, 1$. Denote by $\bar{x} = (x^2, x^3)$. Let $\varphi_1(\bar{x}), \varphi_2(\bar{x}), \ldots$ be the eigenfunctions of the Laplace operator with Dirichlet boundary conditions on Ω, with eigenvalues $\varpi_1^2, \varpi_2^2, \ldots$, and let $\psi_0(\bar{x}), \psi_1(\bar{x}), \psi_2(\bar{x}), \ldots$ be the eigenfunctions of the Laplace operator with Neumann boundary conditions on Ω, with eigenvalues $\aleph_0^2, \aleph_1^2, \aleph_2^2, \ldots$. $\psi_0(\bar{x})$ is the constant function with eigenvalue $\aleph_0^2 = 0$.

A. Perfectly conducting condition on the whole cylinder

When the whole cylinder is perfectly conducting (5), one can show that the set of eigenmodes of the electromagnetic field is given by

Type A TE modes:

$$A_2 = -\sin\frac{\pi k x}{a} \frac{\partial \psi_l(\bar{x})}{\partial x^3} e^{-i\omega t} q_j(y),$$

$$A_3 = \sin\frac{\pi k x}{a} \frac{\partial \psi_l(\bar{x})}{\partial x^2} e^{-i\omega t} q_j(y), \quad \text{all other } A_\mu = 0,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \aleph_l^2 + m_j^2, \quad k, l = 1, 2, \ldots; j = 0, 1, 2, \ldots.$$

The $j = 0$ modes are type I modes, and the $j \geq 1$ modes are type II modes. By convention, $m_0 = 0$ and $q_0(y) = 1$.

Type B TE modes: These include all the type III modes where

$$A_\alpha = 0,$$

$$A_a = \sin\frac{\pi k x}{a} \varphi_l(\bar{x}) e^{-i\omega t} V_{j,a}(y),$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \varpi_l^2 + \mu_j^2, \quad k, l, j = 1, 2, \ldots.$$

Type A TM modes:

$$A_0 = A_a = 0,$$

$$A_1 = \varpi_l^2 \cos\frac{\pi k x}{a} \varphi_l(\bar{x}) e^{-i\omega t} q_j(y),$$

$$A_\gamma = -\frac{\pi k}{a} \sin\frac{\pi k x}{a} \frac{\partial \varphi_l(\bar{x})}{\partial x^\gamma} e^{-i\omega t} q_j(y), \quad \gamma = 2, 3,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \varpi_l^2 + m_j^2, \quad k, j = 0, 1, 2, \ldots; \quad l = 1, 2, \ldots.$$

The $j = 0$ modes are type I modes, and the $j \geq 1$ modes are type II modes.

Type B TM modes: These are type II modes with

$$A_1 = A_a = 0,$$

$$A_0 = \varpi_l^2 \sin\frac{\pi kx}{a} \varphi_l(\bar{x})e^{-i\omega t}q_j(y),$$

$$A_\gamma = i\omega \sin\frac{\pi kx}{a} \frac{\partial \varphi_l(\bar{x})}{\partial x^\gamma}e^{-i\omega t}q_j(y), \qquad \gamma = 2, 3,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \varpi_l^2 + m_j^2, \qquad k, l, j = 1, 2, \ldots.$$

B. Infinitely permeable condition on the whole cylinder

When the whole cylinder is infinitely permeable (6), one can show that the set of eigenmodes of the electromagnetic field is given by

Type A TE modes:

$$A_2 = -\cos\frac{\pi kx}{a} \frac{\partial \varphi_l(\bar{x})}{\partial x^3}e^{-i\omega t}q_j(y),$$

$$A_3 = \cos\frac{\pi kx}{a} \frac{\partial \varphi_l(\bar{x})}{\partial x^2}e^{-i\omega t}q_j(y), \quad \text{all other} \ \ A_\mu = 0,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \varpi_l^2 + m_j^2, \quad l = 1, 2, \ldots; \quad k, j = 0, 1, 2, \ldots.$$

Type B TE modes:

$$A_\alpha = 0,$$

$$A_a = \cos\frac{\pi kx}{a} \psi_l(\bar{x})e^{-i\omega t}V_{j,a}(y),$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \aleph_l^2 + \mu_j^2, \quad k, l = 0, 1, 2, \ldots; \quad j = 1, 2, \ldots.$$

Notice that the space of one-forms V on $\mathcal{N}$ with $\delta_{\mathcal{N}}V = 0$ contains harmonic one-forms where $\Delta_{\mathcal{N}}V = 0$. Let h denote the first Betti number of $\mathcal{N}$— the dimension of the vector space of harmonic one-forms on $\mathcal{N}$, which is a topological invariant. Then the set of μ_j^2 contains h zeros. Without loss of generality, let $\mu_1^2, \ldots, \mu_h^2$ be equal to zero. Then the modes with $l = 0$ and $j = 1, 2, \ldots, h$ are also TM modes. Therefore, they are TEM modes.

Type A TM modes:

$$A_0 = A_a = 0,$$

$$A_1 = \aleph_l^2 \sin\frac{\pi kx}{a} \psi_l(\bar{x})e^{-i\omega t}q_j(y),$$

$$A_\gamma = \frac{\pi k}{a} \cos\frac{\pi kx}{a} \frac{\partial \psi_l(\bar{x})}{\partial x^\gamma}e^{-i\omega t}q_j(y), \quad \gamma = 2, 3,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \aleph_l^2 + m_j^2, \quad j = 0, 1, 2, \ldots; \quad k, l = 1, 2, \ldots.$$

Type B TM modes: These are type II modes with

$$A_1 = A_a = 0,$$

$$A_0 = \aleph_l^2 \cos\frac{\pi kx}{a} \psi_l(\bar{x})e^{-i\omega t}q_j(y)$$

$$A_\gamma = i\omega \cos\frac{\pi kx}{a} \frac{\partial \psi_l(\bar{x})}{\partial x^\gamma}e^{-i\omega t}q_j(y), \quad \gamma = 2, 3,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + \aleph_l^2 + m_j^2, \quad k = 0, 1, 2, \ldots; \quad l, j = 1, 2, \ldots.$$

or

$$A_0 = \frac{\pi k}{a} \cos\frac{\pi kx}{a}e^{-i\omega t}q_j(y),$$

$$A_1 = -i\omega \sin\frac{\pi kx}{a}e^{-i\omega t}q_j(y), \quad \text{all other} \ A_\mu = 0,$$

$$\omega^2 = \left(\frac{\pi k}{a}\right)^2 + m_j^2, \qquad k, j = 1, 2, \ldots.$$

C. Perfectly conducting condition on the sidewall and the bottom, infinitely permeable condition on the top

When the side wall and the bottom of the cylinder are perfectly conducting (5), and the top is infinitely permeable (6), it is immediate to check that the eigenmodes of the electromagnetic field are obtained by replacing the k for the modes in Sec. II A by $k + \frac{1}{2}$, where k runs from zero to infinity.

D. Infinitely permeable condition on the sidewall and the bottom, perfectly conducting condition on the top

When the side wall and the bottom of the cylinder are infinitely permeable (6), and the top is perfectly conducting (5), it is immediate to check that the eigenmodes of the electromagnetic field are obtained by replacing the k for the modes in Sec. II B by $k + \frac{1}{2}$, where k runs from zero to infinity.

Note that in the absence of the space $\mathcal{N}$, we only have the type A TE modes with $j = 0$ and the type A TM modes with $j = 0$. The type B TE modes and type B TM modes only exist in the presence of the space $\mathcal{N}$.

III. THE CASIMIR ENERGY AND THE CASIMIR FORCE

A. The cylinder and the piston are imposed with the same boundary conditions

When the cylinder and the piston are both perfectly conducting, the piston divides the cylinder $[0, L] \times \Omega \times \mathcal{N}$ into two cylinders $[0, a] \times \Omega \times \mathcal{N}$ and $[0, L - a] \times \Omega \times \mathcal{N}$, both of them are perfectly conducting everywhere. From the results of Sec. II A, we find that the zeta function $\zeta_T(s; a)$ is given by

$$\zeta_T(s;a) = \sum_{p=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} \left(\left[\frac{\pi k}{a} \right]^2 + \aleph_l^2 + m_j^2 + [2\pi pT]^2 \right)^{-s} + \sum_{p=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \left(\left[\frac{\pi k}{a} \right]^2 + \varpi_l^2 + \mu_j^2 + [2\pi pT]^2 \right)^{-s}$$

$$+ \sum_{p=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} \left(\left[\frac{\pi k}{a} \right]^2 + \varpi_l^2 + m_j^2 + [2\pi pT]^2 \right)^{-s} + \sum_{p=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \left(\left[\frac{\pi k}{a} \right]^2 + \varpi_l^2 + m_j^2 + [2\pi pT]^2 \right)^{-s}.$$

For the third term, the $k = 0$ terms do not depend on a. Therefore, the zeta function can be written as

$$\zeta_T(s; a) = \sum_{p=-\infty}^{\infty} \sum_{\alpha} \zeta_{\alpha,p}(s) + C(s),$$

$$\zeta_{\alpha,p}(s; a) := \sum_{k=1}^{\infty} \left(\left[\frac{\pi k}{a} \right]^2 + \tau_{\alpha,p}^2 \right)^{-s}, \qquad (7)$$

where $C(s)$ denotes a term independent of a whose value can change from one expression to another, and

$$\tau_{\alpha,p}^2 = \tau_\alpha^2 + (2\pi pT)^2.$$

The set of τ_α^2 contains (PC)
  (i) $\varpi_l^2 + m_j^2$, $j \geq 0$, $l \geq 1$, with multiplicity two if $j \neq 0$ and multiplicity one if $j = 0$,
  (ii) $\varpi_l^2 + \mu_j^2$, $j \geq 1$, $l \geq 1$, each with multiplicity one,
  (iii) $\aleph_l^2 + m_j^2$, $j \geq 0$, $l \geq 1$, each with multiplicity one.

Notice that none of these τ_α^2 is zero.

When the cylinder and the piston are both infinitely permeable, we have two infinitely permeable cylinders. The results in Sec. II B show that the zeta function $\zeta_T(s; a)$ can also be written in the form (7), where the set of τ_α^2 contains (IP)
  (i) $\aleph_l^2 + m_j^2$, $j \geq 0$, $l \geq 1$, with multiplicity two if $j \neq 0$ and multiplicity one if $j = 0$,
  (ii) $\aleph_l^2 + \mu_j^2$, $j \geq 1$, $l \geq 0$, each with multiplicity one,
  (iii) $\varpi_l^2 + m_j^2$, $j \geq 0$, $l \geq 1$, each with multiplicity one,
  (iv) m_j^2, $j \geq 1$, each with multiplicity one.

In this case, we find that there are h of the τ_α^2 that are equal to zero, which are the τ_α^2 corresponding to the TEM modes, i.e., $\aleph_0^2 + \mu_1^2, \ldots, \aleph_0^2 + \mu_h^2$.

Using the fact that

$$\sum_{k=1}^{\infty} \exp\left(-t \left[\frac{\pi k}{a} \right]^2 \right)$$

$$= -\frac{1}{2} + \frac{a}{2\sqrt{\pi}} t^{-(1/2)} + \frac{a}{\sqrt{\pi}} t^{-(1/2)} \sum_{k=1}^{\infty} \exp\left(-\frac{k^2 a^2}{t} \right),$$

we find that if $\tau_{\alpha,p}^2 \neq 0$,

$$\zeta_{\alpha,p}(s; a) = \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} \int_0^\infty t^{s-1} \exp\left\{ -t \left(\left[\frac{\pi k}{a} \right]^2 + \tau_{\alpha,p}^2 \right) \right\} dt$$

$$= C_{\alpha,p}(s) + a\mathcal{D}_{\alpha,p}(s) + \frac{2a}{\sqrt{\pi}\Gamma(s)}$$

$$\times \sum_{k=1}^{\infty} \left(\frac{ka}{\tau_{\alpha,p}} \right)^{s-(1/2)} K_{s-(1/2)}(2ka\tau_{\alpha,p}).$$

Here $C_{\alpha,p}(s)$ and $\mathcal{D}_{\alpha,p}(s)$ are terms independent of a. From this, we obtain

$$\zeta_{\alpha,p}(0; a) = C_{\alpha,p}(0) + a\mathcal{D}_{\alpha,p}(0),$$

$$\zeta'_{\alpha,p}(0; a) = C'_{\alpha,p}(0) + a\mathcal{D}'_{\alpha,p}(0)$$

$$+ \frac{2a}{\sqrt{\pi}} \sum_{k=1}^{\infty} \left(\frac{\tau_{\alpha,p}}{ka} \right)^{1/2} K_{1/2}(2ka\tau_{\alpha,p})$$

$$= C'_{\alpha,p}(0) + a\mathcal{D}'_{\alpha,p}(0) + \sum_{k=1}^{\infty} \frac{1}{k} e^{-2ka\tau_{\alpha,p}}.$$

On the other hand, if $\tau_{\alpha,p}^2 = 0$,

$$\zeta_{\alpha,p}(s) = \left(\frac{\pi}{a} \right)^{-2s} \zeta_R(2s).$$

It follows that

$$\zeta_{\alpha,p}(0; a) = \zeta_R(0) = -\frac{1}{2}$$

$$\zeta'_{\alpha,p}(0; a) = 2\zeta'_R(0) - 2\zeta_R(0) \log\frac{\pi}{a} = -\log(2\pi) + \log\frac{\pi}{a}$$

$$= -\log(2a).$$

Here $\zeta_R(s)$ is the Riemann zeta function.

From (2), we find that the Casimir energy of the piston system (1) is given by

$$E_{\text{Cas}}^{\text{piston}} = \mathcal{E}_0 - \frac{T}{2} \sum_{k=1}^{\infty} \sum_{\tau_{\alpha,p} \neq 0} \frac{1}{k} e^{-2ka\tau_{\alpha,p}} + \frac{T}{2} \sum_{\tau_{\alpha,p}=0} \log(\lambda a)$$

$$- \frac{T}{2} \sum_{k=1}^{\infty} \sum_{\tau_{\alpha,p} \neq 0} \frac{1}{k} e^{-2k(L-a)\tau_{\alpha,p}} + \frac{T}{2} \sum_{\tau_{\alpha,p}=0} \log(\lambda(L-a)),$$

where $\mathcal{E}_0$ is independent of a. It follows that the Casimir force acting on the piston is given by

$$F_{\text{Cas}}^{\text{piston}} = -\frac{\partial E_{\text{Cas}}^{\text{piston}}}{\partial a} = F_{\text{Cas}}^{\parallel}(a) - F_{\text{Cas}}^{\parallel}(L-a), \qquad (8)$$

where

$$F_{\text{Cas}}^{\parallel}(a) = -\frac{\partial}{\partial a}\left(-\frac{T}{2}\sum_{k=1}^{\infty}\sum_{\tau_{\alpha,p}\neq 0}\frac{1}{k}e^{-2ka\tau_{\alpha,p}}\right.$$

$$\left.+\frac{T}{2}\sum_{\tau_{\alpha,p}=0}\log(\lambda a)\right)$$

$$= -T\sum_{k=1}^{\infty}\sum_{\tau_{\alpha,p}\neq 0}\tau_{\alpha,p}e^{-2ka\tau_{\alpha,p}} - \sum_{\tau_{\alpha,p}=0}\frac{T}{2a}$$

$$= -T\sum_{\tau_{\alpha,p}\neq 0}\frac{\tau_{\alpha,p}}{e^{2a\tau_{\alpha,p}} - 1} - \sum_{\tau_{\alpha,p}=0}\frac{T}{2a}. \tag{9}$$

In fact, $F_{\text{Cas}}^{\parallel}(a)$ is the limit of the Casimir force acting on the piston $F_{\text{Cas}}^{\text{piston}}$ when the cylinder becomes infinitely long, i.e., $L \to \infty$. Therefore, it is the Casimir force acting between two perfectly conducting plates or two infinitely permeable plates separated by a distance a moving inside an infinitely long cylinder with cross section $\Omega \times \mathcal{N}$.

When both the cylinder and the piston are infinitely permeable, the last term in (9) which comes from the TEM modes is nonzero if h, the dimension of the space of harmonic one-forms of $\mathcal{N}$, is nonzero. In contrast to the contribution to the force from the terms with $\tau_{\alpha,p} \neq 0$ which decays to zero exponentially fast when $a \to \infty$, the contribution from the terms with $\tau_{\alpha,p} = 0$ has power law decay which is much slower. This gives rise to a long range Casimir force [59]. It is interesting to note that the

long range term $-\frac{T}{2a}$ is a limit of $-T\frac{\tau_{\alpha,p}}{e^{2a\tau_{\alpha,p}} - 1}$ when $\tau_{\alpha,p} \to 0$, which exhibits some kind of smooth transition.

From the expression (9), it is obvious that the Casimir force acting between two perfectly conducting plates or two infinitely permeable plates is always attractive and is a monotonically decreasing function of the distance between the plates. Therefore, it can be inferred from (8) that in a closed cylinder of finite length, the Casimir force acting on the piston which has the same boundary condition with the cylinder always tends to pull the piston to the closer end.

In the following, we study the asymptotic behavior of the Casimir force (9) at different limits. Denote by $r = \sqrt[n]{V(N)}$ a measure of the size of the manifold $\mathcal{N}$ and $R = \sqrt{\mathcal{A}(\Omega)}$ a measure of the size of the domain Ω. We will investigate the behavior of the Casimir force when the length scales a, r, and R are such that $r \ll a \ll R$ or $a \ll r, R$, and when $aT \ll 1$ or $aT \gg 1$.

(9) is the high temperature expansion of the Casimir force. It shows that when $aT \gg 1$, the Casimir force is dominated by the term

$$F_{\text{Cas}}^{\parallel}(a) \sim -T\sum_{\tau_{\alpha}\neq 0}\frac{\tau_{\alpha}}{e^{2a\tau_{\alpha}} - 1} - \sum_{\tau_{\alpha}=0}\frac{T}{2a},$$

which is linear in T. This term is called the classical term. The sum of the remaining terms decay exponentially.

In Appendix A 1, we show that the Casimir force (9) can be rewritten as

$$F_{\text{Cas}}^{\parallel}(a) = -\sum_{\tau_{\alpha}=0}\left\{\frac{\pi}{24a^2} + \frac{\pi T^2}{6}\right\} - \sum_{\tau_{\alpha}\neq 0}\left\{\frac{1}{2\pi a}\sum_{k=1}^{\infty}\frac{\tau_{\alpha}}{k}K_1(2ka\tau_{\alpha}) + \frac{1}{\pi}\sum_{k=1}^{\infty}\tau_{\alpha}^2 K_0(2ka\tau_{\alpha}) + \frac{T}{\pi}\sum_{p=1}^{\infty}\frac{\tau_{\alpha}}{p}K_1\left(\frac{p\tau_{\alpha}}{T}\right)\right\}$$

$$+ \frac{\pi^2}{a^3}\sum_{\tau_{\alpha}}\sum_{k=1}^{\infty}\frac{k^2}{\sqrt{\left[\frac{\pi k}{a}\right]^2 + \tau_{\alpha}^2}}\left(\exp\left(\frac{1}{T}\sqrt{\left[\frac{\pi k}{a}\right]^2 + \tau_{\alpha}^2}\right) - 1\right)^{-1}.$$

This shows that at zero temperature, the Casimir force is given by

$$F_{\text{Cas}}^{\parallel, T=0}(a) = -\sum_{\tau_{\alpha}=0}\frac{\pi}{24a^2} - \sum_{\tau_{\alpha}\neq 0}\left\{\frac{1}{2\pi a}\sum_{k=1}^{\infty}\frac{\tau_{\alpha}}{k}K_1(2ka\tau_{\alpha}) + \frac{1}{\pi}\sum_{k=1}^{\infty}\tau_{\alpha}^2 K_0(2ka\tau_{\alpha})\right\}.$$

In the case the two infinitely permeable plates are placed inside an infinitely permeable cylinder and the first Betti number h of $\mathcal{N}$ is nonzero, the leading term of the thermal correction is

$$-\frac{\pi h}{6}T^2.$$

Otherwise, the thermal correction goes to zero exponentially fast when $aT \to 0$.

Next we consider the behavior of the Casimir force (9) when the separation between the plates a is much smaller than the size R of the domain Ω, i.e., $a \ll R$. In this case, the result of Appendix C shows that the first two leading terms of the Casimir force is given by

$$F_{\text{Cas}}^{\parallel}(a) \sim \mathcal{A}(\Omega)\left\{-(2+h)\frac{T}{8\pi a^3}\zeta_R(3) - \frac{T}{8\pi}\sum_{k=1}^{\infty}\sum_{\sigma_{\beta,p}\neq 0}\left(\frac{2\sigma_{\beta,p}^2}{ka} + \frac{2\sigma_{\beta,p}}{k^2 a^2} + \frac{1}{k^3 a^3}\right)e^{-2ka\sigma_{\beta,p}}\right\}$$

$$\pm l(\partial\Omega)\left\{h\frac{\pi T}{96a^2} + \frac{T}{8\pi}\sum_{k=1}^{\infty}\sum_{\xi_{\gamma,p}\neq 0}\frac{\xi_{\gamma,p}}{ka}K_1(2ka\xi_{\gamma,p}) + \frac{T}{4\pi}\sum_{k=1}^{\infty}\sum_{\xi_{\gamma,p}\neq 0}\xi_{\gamma,p}^2 K_0(2ka\xi_{\gamma,p})\right\}, \tag{10}$$

or

$$F_{\mathrm{Cas}}^{\parallel}(a) \sim \mathcal{A}(\Omega)\Bigg\{-(2+h)\left(\frac{\pi^2}{480a^4}+\frac{\pi^2 T^4}{90}\right)-\frac{3}{8\pi^2}\sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0}e^{2\pi i k\chi}\left(\frac{\sigma_\beta}{ka}\right)^2 K_2(2ka\sigma_\beta)-\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0}\frac{\sigma_\beta^3}{ka}K_1(2ka\sigma_\beta)$$

$$-\frac{T^2}{2\pi^2}\sum_{p=1}^{\infty}\sum_{\sigma_\beta \neq 0}\left(\frac{\sigma_\beta}{p}\right)^2 K_2\left(\frac{p\sigma_\beta}{T}\right)+\frac{\pi T}{2a^3}\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\sigma_\beta}\frac{k^2}{p}\exp\left(-\frac{p}{T}\sqrt{\left[\frac{\pi k}{a}\right]^2+\sigma_\beta^2}\right)\Bigg\} \pm l(\partial\Omega)\Bigg\{h\left(\frac{\zeta_R(3)}{32\pi a^3}+\frac{\zeta_R(3)T^3}{8\pi}\right)$$

$$+\frac{1}{32\pi}\sum_{k=1}^{\infty}\sum_{\xi_\gamma \neq 0}e^{-2ka\xi_\gamma}\left(\frac{2\xi_\gamma^2}{ka}+\frac{2\xi_\gamma}{k^2 a^2}+\frac{1}{k^3 a^3}\right)+\frac{T^2}{8\pi}\sum_{p=1}^{\infty}\sum_{\xi_\gamma \neq 0}e^{-(p\xi_\gamma/T)}\left(\frac{\xi_\gamma}{p^2}+\frac{T}{p^3}\right)$$

$$-\frac{\pi}{4a^3}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\sum_{\xi_\gamma}k^2 K_0\left(\frac{p}{T}\sqrt{\left[\frac{\pi k}{a}\right]^2+\xi_\gamma^2}\right)\Bigg\}, \tag{11}$$

where $\sigma_{\beta,p}^2 = \sigma_\beta^2 + [2\pi pT]^2$, $\xi_{\gamma,p}^2 = \xi_\gamma^2 + [2\pi pT]^2$; $A(\Omega)$ is the area of Ω which is of order R^2, and $l(\partial\Omega)$ is the length of the boundary $\partial\Omega$ of Ω which is of order R. The set of σ_β^2 contains m_0^2 with multiplicity two, m_j^2, $j \geq 1$, with multiplicity three, and μ_j^2, $j \geq 1$, with multiplicity one; the set of ξ_γ^2 contains m_j^2, $j \geq 1$, and μ_j^2, $j \geq 1$, each with multiplicity one. The plus signs on the second line in (10) and the third line in (11) are for the case where the

cylinder is perfectly conducting, and the minus signs are for the case where the cylinder is infinitely permeable.

In the high temperature regime, i.e., $aT \gg 1$, the leading terms of the Casimir force when $a \ll R$ is given by the sum of the $p = 0$ terms in (10). In the low temperature regime, i.e., $aT \ll 1$, (11) shows that the leading terms of the Casimir force is given by

$$F_{\mathrm{Cas}}^{\parallel}(a) \sim \mathcal{A}(\Omega)\Bigg\{-(2+h)\frac{\pi^2}{480a^4}-\frac{3}{8\pi^2}\sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0}e^{2\pi i k\chi}\left(\frac{\sigma_\beta}{ka}\right)^2 K_2(2ka\sigma_\beta)-\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0}\frac{\sigma_\beta^3}{ka}K_1(2ka\sigma_\beta)\Bigg\}$$

$$\pm l(\partial\Omega)\Bigg\{h\frac{\zeta_R(3)}{32\pi a^3}+\frac{1}{32\pi}\sum_{k=1}^{\infty}\sum_{\xi_\gamma \neq 0}e^{-2ka\xi_\gamma}\left(\frac{2\xi_\gamma^2}{ka}+\frac{2\xi_\gamma}{k^2 a^2}+\frac{1}{k^3 a^3}\right)\Bigg\}-(2+h)\mathcal{A}(\Omega)\frac{\pi^2 T^4}{90}\pm hl(\partial\Omega)\frac{\zeta_R(3)T^3}{8\pi}. \tag{12}$$

The first two terms give the zero temperature contribution, and the last two terms give the thermal correction terms which are of polynomial order in T. The remaining terms go to zero exponentially fast when $aT \to 0$.

(10) and (11) can also be used to study the leading behavior of the Casimir force when $r \ll a \ll R$. Since σ_β and ξ_γ are proportional to r^{-1}, we find that in the limit $r/a \to 0$, the leading term of the Casimir force is given by

$$F_{\mathrm{Cas}}^{\parallel}(a) \sim \frac{2+h}{2}\mathcal{A}(\Omega)\Bigg\{-\frac{T}{4\pi a^3}\zeta_R(3)-\frac{T}{\pi}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{4\pi^2 p^2 T^2}{ka}+\frac{2\pi pT}{k^2 a^2}+\frac{1}{2k^3 a^3}\right)e^{-4\pi kpaT}\Bigg\}, \tag{13}$$

or

$$F_{\mathrm{Cas}}^{\parallel}(a) \sim \frac{2+h}{2}\mathcal{A}(\Omega)\Bigg\{-\frac{\pi^2}{240a^4}-\frac{\pi^2 T^4}{45}+\frac{\pi T}{a^3}\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{k^2}{p}\exp\left(-\frac{\pi kp}{aT}\right)\Bigg\}. \tag{14}$$

Notice that the expressions in the brackets of (13) and (14) are the Casimir force per unit area acting on a pair of perfectly conducting or infinitely permeable plates in the 4D Minkowski spacetime [48]. They are also equal to twice the Casimir force per unit area acting on a pair of Dirichlet or Neumann plates [48]. Therefore in the limit the size of the manifold $\mathcal{N}$ goes to zero, one recovers the Casimir force between a pair of infinite parallel plates in the 4D Minkowski spacetime if and only if $h = 0$, i.e.,

the first Betti number of $\mathcal{N}$ is zero. For general h, one finds that when the size of $\mathcal{N}$ goes to zero, one has h extra copies of Casimir force acting on a pair of Dirichlet plates. Recall that h is the number of zero modes for the Laplace operator on one-forms on $\mathcal{N}$. Therefore, the presence of extra h copies of the Casimir force on a pair of Dirichlet plates when the size of the manifold $\mathcal{N}$ goes to zero can be considered as a kind of instantonic effect.

When $aT \gg 1$, (13) shows that the leading term of the Casimir force is

$$F_{\text{Cas}}^{\parallel}(a) \sim -\frac{2+h}{2}\mathcal{A}(\Omega)\frac{T}{4\pi a^3}\zeta_R(3). \qquad (15)$$

When $aT \ll 1$, (14) shows that the leading term of the Casimir force is

$$F_{\text{Cas}}^{\parallel}(a) \sim -\frac{2+h}{2}\mathcal{A}(\Omega)\frac{\pi^2}{240a^4}. \qquad (16)$$

Up to the factor $(1 + h/2)$, these are the familiar leading behavior of the Casimir force acting on a pair of perfectly conducting plates in the $4D$ Minkowski spacetime in the high temperature regime and in the low temperature regime, respectively.

Finally, we consider the case where separation between the plates is much smaller than the sizes of both Ω and $\mathcal{N}$, i.e., $a \ll r, R$. In this case, the results of Appendix C show that the first two leading terms of the Casimir force is given by

$$
\begin{aligned}
F_{\text{Cas}}^{\parallel}(a) \sim {}& (n+2)\mathcal{V}(\mathcal{S})\Bigg\{ -T\frac{(n+2)\Gamma(\frac{n+3}{2})}{(4\pi)^{(n+3)/2}a^{n+3}}\zeta_R(n+3) - (n+2)\frac{T^{(n+5)/2}}{2^{(n-1)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{p}{ka}\right)^{(n+3)/2}K_{(n+3)/2}(4\pi kpaT) \\
& - \frac{\pi T^{(n+7)/2}}{2^{(n-5)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{p^{(n+5)/2}}{(ka)^{(n+1)/2}}K_{(n+1)/2}(4\pi kpaT)\Bigg\} \mp \frac{n\mathcal{V}(\partial\mathcal{S})}{4}\Bigg\{ -T\frac{(n+1)\Gamma(\frac{n+2}{2})}{(4\pi)^{(n+2)/2}a^{n+2}}\zeta_R(n+2) \\
& - (n+1)\frac{T^{(n+4)/2}}{2^{(n-2)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{p}{ka}\right)^{(n+2)/2}K_{(n+2)/2}(4\pi kpaT) - \frac{\pi T^{(n+6)/2}}{2^{(n-6)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{p^{(n+4)/2}}{(ka)^{n/2}}K_{n/2}(4\pi kpaT)\Bigg\}, \qquad (17)
\end{aligned}
$$

or

$$
\begin{aligned}
F_{\text{Cas}}^{\parallel}(a) \sim {}& (n+2)\mathcal{V}(\mathcal{S})\Bigg\{ -\frac{(n+3)\Gamma(\frac{n+4}{2})}{(4\pi)^{(n+4)/2}a^{n+4}}\zeta_R(n+4) - \frac{\Gamma(\frac{n+4}{2})}{\pi^{(n+4)/2}}\zeta_R(n+4)T^{n+4} \\
& + \frac{\pi T^{(n+1)/2}}{2^{(n+1)/2}a^{(n+7)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{k^{(n+5)/2}}{p^{(n+1)/2}}K_{(n+1)/2}\left(\frac{\pi kp}{aT}\right)\Bigg\} \mp \frac{n\mathcal{V}(\partial\mathcal{S})}{4}\Bigg\{ -\frac{(n+2)\Gamma(\frac{n+3}{2})}{(4\pi)^{(n+3)/2}a^{n+3}}\zeta_R(n+3) \\
& - \frac{\Gamma(\frac{n+3}{2})}{\pi^{(n+3)/2}}\zeta_R(n+3)T^{n+3} + \frac{\pi T^{n/2}}{2^{n/2}a^{(n+6)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{k^{(n+4)/2}}{p^{n/2}}K_{n/2}\left(\frac{\pi pk}{aT}\right)\Bigg\}. \qquad (18)
\end{aligned}
$$

Here $\mathcal{S} = \Omega \times \mathcal{N}$ is the cross section of the cylinder, $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$; $\mathcal{V}(\mathcal{S})$ and $\mathcal{V}(\partial\mathcal{S})$ are, respectively, the volumes of $\mathcal{S}$ and $\partial\mathcal{S}$. The first terms in (17) and (18) are the Casimir force acting on a pair of perfectly conducting plates in the $N = 4 + n$-dimensional Minkowski spacetime [48]. When $aT \gg 1$, the leading term of the Casimir force is

$$F_{\text{Cas}}^{\parallel}(a) \sim -\mathcal{V}(\mathcal{S})T\frac{(n+2)^2\Gamma(\frac{n+3}{2})}{(4\pi)^{(n+3)/2}a^{n+3}}\zeta_R(n+3),$$

and when $aT \ll 1$, the leading term of the Casimir force is

$$F_{\text{Cas}}^{\parallel}(a) \sim -\mathcal{V}(\mathcal{S})\frac{(n+2)(n+3)\Gamma(\frac{n+4}{2})}{(4\pi)^{(n+4)/2}a^{n+4}}\zeta_R(n+4).$$

In the high temperature regime, we find that the leading term of the Casimir force is of order T/a^{n+3} if the separation between the plates is much smaller than the sizes of Ω and $\mathcal{N}$ but is of order T/a^3 if the size of $\mathcal{N}$ is much smaller than the separation between the plates and the separation between the plates is much smaller than the size of Ω. In the low temperature regime, we find that the leading term of the Casimir force is of order $1/a^{n+4}$ if the separation between the plates is much smaller than the sizes of Ω and $\mathcal{N}$ but is of order $1/a^4$ if the size of $\mathcal{N}$ is much smaller than the separation between the plates and the separation between the plates is much smaller than the size of Ω. Therefore, we find that the strength of the Casimir force depends strongly on the relative magnitude of a, r, and R.

B. The cylinder and the piston are imposed with different boundary conditions

If the cylinder is perfectly conducting and the piston is infinitely permeable, the piston divides the cylinder $[0, L] \times \Omega \times \mathcal{N}$ into two cylinders $[0, a] \times \Omega \times \mathcal{N}$ and $[0, L - a] \times \Omega \times \mathcal{N}$, both of them have perfectly conducting sidewall and bottom, and infinitely permeable top. If the cylinder is infinitely permeable and the piston is perfectly conducting, then the two cylinders have infinitely permeable sidewall and bottom, and perfectly conducting top. From the results of Sec. II C and II D we find that the zeta function $\zeta_T(s; a)$ can be written as

$$\zeta_T(s; a) = \sum_{p=-\infty}^{\infty} \sum_{\alpha} \zeta_{\alpha,p}(s; a) + \mathcal{C}(s),$$

$$\zeta_{\alpha,p}(s; a) = \sum_{k=0}^{\infty} \left(\left[\frac{\pi(k+\frac{1}{2})}{a} \right]^2 + \tau_{\alpha,p}^2 \right)^{-s},$$

where the set of τ_α^2 is given by (PC) if the cylinder is perfectly conducting and (IP) if the cylinder is infinitely permeable.

Using the fact that

$$\sum_{k=0}^{\infty} \exp\left(-t \left[\frac{\pi(k+\frac{1}{2})}{a} \right]^2 \right)$$

$$= \frac{a}{2\sqrt{\pi}} t^{-(1/2)} + \frac{a}{\sqrt{\pi}} t^{-(1/2)} \sum_{k=1}^{\infty} (-1)^k \exp\left(-\frac{k^2 a^2}{t} \right),$$

we find as in Sec. III A that if $\tau_{\alpha,p}^2 \neq 0$,

$$\zeta_{\alpha,p}(0; a) = \mathcal{C}_{\alpha,p}(0) + a \mathcal{D}_{\alpha,p}(0),$$

$$\zeta_{\alpha,p}'(0; a) = \mathcal{C}_{\alpha,p}'(0) + a \mathcal{D}_{\alpha,p}'(0) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-2ka\tau_{\alpha,p}}.$$

On the other hand, if $\tau_{\alpha,p}^2 = 0$,

$$\zeta_{\alpha,p}(s) = (2^{2s} - 1) \left(\frac{\pi}{a} \right)^{-2s} \zeta_R(2s).$$

Therefore,

$$\zeta_{\alpha,p}(0; a) = 0 \quad \text{and} \quad \zeta_{\alpha,p}'(0; a) = -\log 2.$$

Hence, the Casimir energy of the piston system is

$$E_{\text{Cas}}^{\text{piston}} = \mathcal{E}_0 - \frac{T}{2} \sum_{k=1}^{\infty} \sum_{\tau_{\alpha,p} \neq 0} \frac{(-1)^k}{k} e^{-2ka\tau_{\alpha,p}}$$

$$- \frac{T}{2} \sum_{k=1}^{\infty} \sum_{\tau_{\alpha,p} \neq 0} \frac{(-1)^k}{k} e^{-2k(L-a)\tau_{\alpha,p}},$$

where $\mathcal{E}_0$ is independent of a. It follows that the Casimir force acting on the piston is given by (8), where the Casimir force acting between a pair of parallel plates with different boundary conditions inside the infinitely long cylinder is

$$F_{\text{Cas}}^{\parallel}(a) = -\frac{\partial}{\partial a} \left(-\frac{T}{2} \sum_{k=1}^{\infty} \sum_{\tau_{\alpha,p} \neq 0} \frac{(-1)^k}{k} e^{-2ka\tau_{\alpha,p}} \right)$$

$$= -T \sum_{k=1}^{\infty} \sum_{\tau_{\alpha,p}} (-1)^k \tau_{\alpha,p} e^{-2ka\tau_{\alpha,p}}$$

$$= T \sum_{\tau_{\alpha,p}} \frac{\tau_{\alpha,p}}{e^{2a\tau_{\alpha,p}} + 1}. \tag{19}$$

Notice that contrary to the previous case where the two plates are both infinitely permeable, at finite temperature the force acting on a pair of plates, one perfectly conducting and one infinitely permeable, does not have a long range term even though there are TEM modes for which $\tau_{\alpha,p}^2 = 0$.

Equation (19) shows that the Casimir force acting between one perfectly conducting plate and one infinitely permeable plate is always repulsive and is a monotonically decreasing function of the distance between the plates. Therefore, in a closed cylinder with finite length, the Casimir force acting on the piston which has different boundary conditions always tends to push the piston to the middle of the cylinder, which is the equilibrium position.

In the following, we study the asymptotic behavior of the Casimir force (19) at different limits. In the high temperature limit $aT \gg 1$, the Casimir force (19) is dominated by the classical term given by

$$F_{\text{Cas}}^{\parallel}(a) \sim T \sum_{\tau_\alpha} \frac{\tau_{\alpha,p}}{e^{2a\tau_\alpha} + 1}.$$

The remaining terms decay exponentially. The result of Sec. II B shows that the Casimir force (19) can also be written as

$$F_{\text{Cas}}^{\parallel}(a) = \sum_{\tau_\alpha = 0} \left\{ \frac{\pi}{48a^2} - \frac{\pi T^2}{6} \right\} + \sum_{\tau_\alpha \neq 0} \left\{ \frac{1}{2\pi a} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\tau_\alpha}{k} K_1(2ka\tau_\alpha) + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \tau_\alpha^2 K_0(2ka\tau_\alpha) \right\}$$

$$- \frac{T}{\pi} \sum_{p=1}^{\infty} \frac{\tau_\alpha}{p} K_1\left(\frac{p\tau_\alpha}{T} \right) \right\} + \frac{\pi^2}{a^3} \sum_{\tau_\alpha} \sum_{k=0}^{\infty} \frac{(k+\frac{1}{2})^2}{\sqrt{[\frac{\pi(k+\frac{1}{2})}{a}]^2 + \tau_\alpha^2}} \left\{ \exp\left(\frac{1}{T} \sqrt{\left[\frac{\pi(k+\frac{1}{2})}{a} \right]^2 + \tau_\alpha^2} \right) - 1 \right\}^{-1}.$$

Therefore the zero temperature Casimir force is given by

$$F_{\text{Cas}}^{\parallel, T=0}(a) = \sum_{\tau_\alpha = 0} \frac{\pi}{48a^2} + \sum_{\tau_\alpha \neq 0} \left\{ \frac{1}{2\pi a} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\tau_\alpha}{k} K_1(2ka\tau_\alpha) + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \tau_\alpha^2 K_0(2ka\tau_\alpha) \right\}$$

Notice that the zero temperature Casimir force contains a long range term if the plates are placed inside an infinitely permeable cylinder and the first Betti number h of $\mathcal{N}$ is nonzero. In this case, the leading term of the thermal correction is also of order T^2.

Using the results of Appendix C, we can also derive the leading terms of the Casimir force when $r \ll a \ll R$ and $a \ll r, R$. In the high temperature regime, we find that if $r \ll a \ll R$, then

$$F_{\text{Cas}}^{\parallel}(a) \sim \frac{2 + h}{2} \mathcal{A}(\Omega) \frac{3T}{16\pi a^3} \zeta_R(3), \qquad (20)$$

and if $a \ll r, R$,

$$F_{\text{Cas}}^{\parallel}(a) \sim \mathcal{V}(\mathcal{S}) T \frac{(n + 2)^2 \Gamma(\frac{n+3}{2})}{(4\pi)^{(n+3)/2} a^{n+3}} \zeta_R(n + 3)(1 - 2^{-n-2}). \tag{21}$$

In the low temperature regime, if $r \ll a \ll R$,

$$F_{\text{Cas}}^{\parallel}(a) \sim \frac{2 + h}{2} \mathcal{A}(\Omega) \frac{7\pi^2}{1920 a^4}, \qquad (22)$$

and if $a \ll r, R$,

$$F_{\text{Cas}}^{\parallel}(a) \sim \mathcal{V}(\mathcal{S}) \frac{(n + 2)(n + 3)\Gamma(\frac{n+4}{2})}{(4\pi)^{(n+4)/2} a^{n+4}} \zeta_R(n + 4)(1 - 2^{-n-3}).$$

Again we find that in the high temperature regime, the leading term of the Casimir force is of order T/a^{n+3} when $a \ll r, R$ but is of order T/a^3 when $r \ll a \ll R$. In the low temperature regime, the leading term of the Casimir force is of order $1/a^{n+4}$ when $a \ll r, R$ but is of order $1/a^4$ when $r \ll a \ll R$. We also find that when the size of the manifold $\mathcal{N}$ goes to zero, then the Casimir force reduces to the Casimir force in the 4D Minkowski spacetime if and only if the first Betti number h of $\mathcal{N}$ is zero. Otherwise, there are some extra contributions.

Since recent Casimir experiments have shown excellent agreement between the data and the theory derived in 4D Minkowski spacetime [60–63], our asymptotic analysis above suggests that a physical extra dimensional spacetime should be subjected to the topological constraint that the first Betti number of the extra dimensions is zero.

IV. EXPLICIT EXAMPLES

In this section, we consider the specific examples where the manifold $\mathcal{N}$ is an n-dimensional torus T^n or an n-dimensional sphere S^n with volume r^n. Assume that T^n are n copies of S^1 with the same radius r_1. Then the volume of T^n is r^n implies that $r_1 = r/(2\pi)$. The radius of the sphere S^n with volume r^n is

$$r_2 = \left(\frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \right)^{1/n} r.$$

On T^n, the spectrum $\{m_j : j = 0, 1, 2, \ldots\}$ of the Laplace operator on functions is given by

$$\left\{ \frac{j_1^2 + \ldots + j_n^2}{r_1^2} : j_1, \ldots, j_n \in \mathbb{Z} \right\}, \qquad (23)$$

and the spectrum $\{\mu_j^2 : j = 1, 2, \ldots\}$ of the Laplace operator on coclosed one-forms is given by $(n - 1)$ copies of (23) plus one zero. $h = n$ for T^n.

On S^n, the spectrum $\{m_j : j = 0, 1, 2, \ldots\}$ of the Laplace operator on functions is given by [64]

$$b_l^0 = \frac{l(l + n - 1)}{r_1^2}, \qquad l = 0, 1, 2, \ldots,$$

with multiplicities

$$d_l^0 = \frac{(2l + n - 1)(l + n - 2)!}{l!(n - 1)!};$$

and the spectrum $\{\mu_j^2 : j = 1, 2, \ldots\}$ of the Laplace operator on coclosed one-forms is given by [64]

$$b_l^1 = l(l + n - 1) + n - 2, \qquad l = 1, 2, \ldots, \qquad (24)$$

with multiplicities

$$d_l^1 = \frac{l(l + n - 1)(2l + n - 1)(l + n - 3)!}{(n - 2)!(l + 1)!}.$$

$h = 0$ for S^n.

We only consider the case where the size of the domain Ω is much larger than a and r. Then the Casimir force is given by the first term in (10) or (11).

In Fig. 2, we show the graphs of the dimensionless Casimir force $\frac{F_{\text{Cas}}^{\parallel}}{A(\Omega)} r^4$ as a function of a/r when $rT = 1$ for $\mathcal{N} = T^2, T^3, S^2$, and S^3. From these graphs, one can see that as a/r gets smaller, the magnitude of the Casimir force for the same value of n agrees more, and it is larger for larger n, in agreement with the $(rT)/(a/r)^{n+3}$ or $1/(a/r)^{n+4}$ behavior for small a/r. It is not easy to read from the graphs in Fig. 2 the difference between the Casimir forces when a/r gets larger. Therefore, in Fig. 3 and Fig. 4, we show the ratio of the Casimir force when $\mathcal{N}$ is T^n to the Casimir force when $\mathcal{N}$ is S^n. We plot the graphs for $rT = 0.1, 0.5, 1$, and 2. The $n = 2$ case is shown in Fig. 3 and the $n = 3$ case is shown in Fig. 4. These graphs show that for small a/r, the ratios of the Casimir forces for fixed n indeed approaches unity. For larger a/r, the ratios of the Casimir forces approaches $1 + n/2$, in agreement with the fact that as $a \gg r$, the leading term of the Casimir force is $1 + h/2$ times the Casimir force in the 4D Minkowski spacetime. Since $h = n$ for T^n and $h = 0$
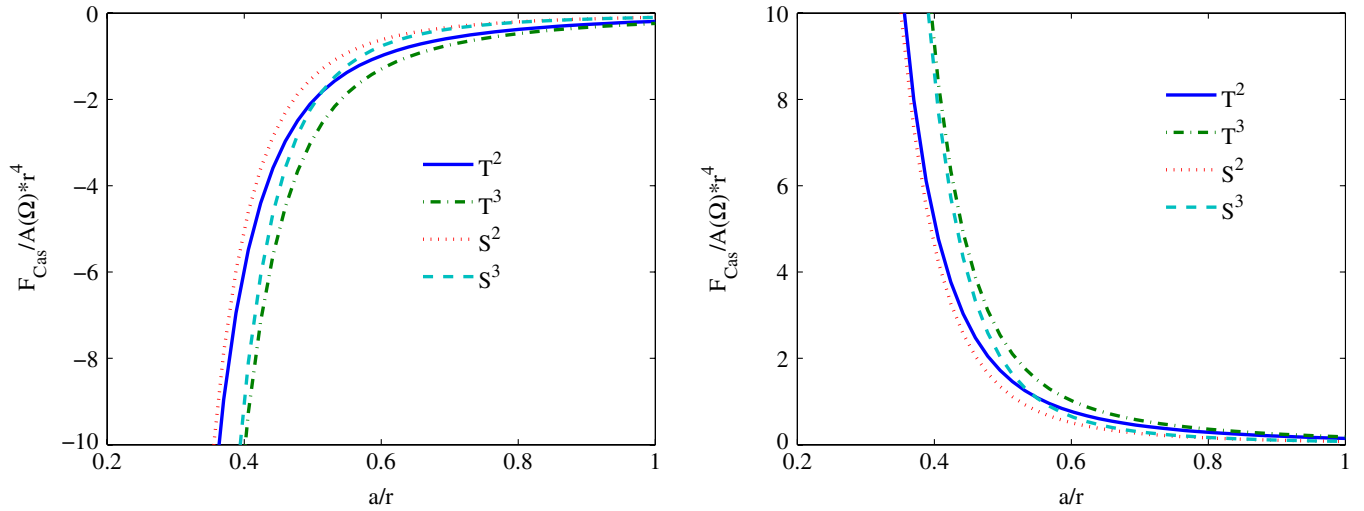
FIG. 2 (color online). $\frac{F^{\|}_{Cas}}{A(\Omega)} r^4$ as a function of a/r for $\mathcal{N} = T^2, T^3, S^2, S^3$. Here $rT = 1$. For the graph on the left, the two plates are both perfectly conducting or infinitely permeable. For the graph on the right, one plate is perfectly conducting and one is infinitely permeable.
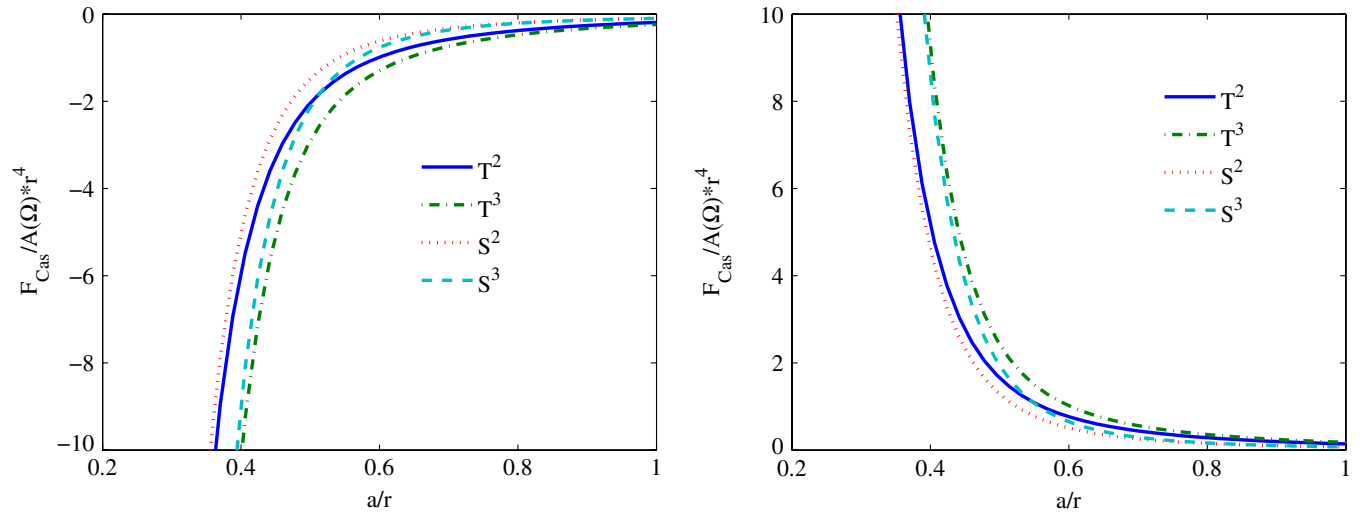

FIG. 3 (color online). The ratio of the Casimir force on T^2 to the Casimir force on S^2 as a function of a/r, for $rT = 0.1, 0.5, 1, 2$. For the graph on the left, the two plates are both perfectly conducting or infinitely permeable. For the graph on the right, one plate is perfectly conducting and one is infinitely permeable.

for S^n, therefore, the ratio of the Casimir force when $\mathcal{N}$ is T^n to the Casimir force when $\mathcal{N}$ is S^n should approach $1 + n/2$ when $a \gg r$.

V. CONCLUSION

In this article, we have investigated the electromagnetic Casimir effect on a piston in a higher-dimensional spacetime of the form $M \times \mathcal{N}$, where M is the 4D Minkowski spacetime and $\mathcal{N}$ is an arbitrary compact manifold. As we have stressed in the introduction, we consider the Casimir effect due to the electromagnetic field, which is a physically measurable Casimir effect. One of the reasons the Casimir effect of electromagnetic fields is a much more

difficult problem than the Casimir effect of scalar fields is that electromagnetic fields have gauge degree of freedom. It is crucial to choose gauges that eliminate all the gauge freedom and which facilitate the correct counting of eigenmodes of the electromagnetic field. This is also an important problem from the perspective of quantum field theory. We have discussed this issue in detail in Appendix A and Sec. B.

In Sec. III, we compute the Casimir force acting on the piston for different combinations of perfectly conducting boundary conditions and infinitely permeable boundary conditions on the cylinder and on the piston. It is shown that if the cylinder and the piston have the same boundary conditions, then the Casimir force always tends to pull the
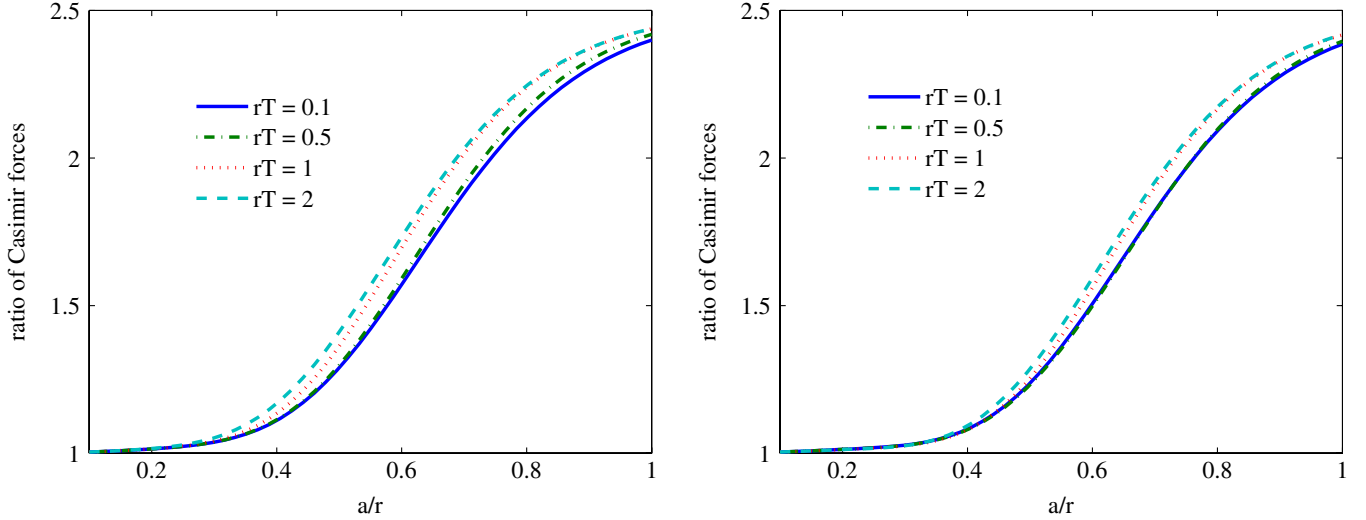
FIG. 4 (color online).   The ratio of the Casimir force on T^3 to the Casimir force on S^3 as a function of a/r, for $rT = 0.1, 0.5, 1, 2$. For the graph on the left, the two plates are both perfectly conducting or infinitely permeable. For the graph on the right, one plate is perfectly conducting and one is infinitely permeable.

piston to the closer end of the cylinder. However, if the cylinder and the piston have different boundary conditions, the Casimir force always tends to push the piston to the equilibrium position in the middle of the cylinder. We find that the geometry of the piston does not affect the direction of the Casimir force, although it does affect the magnitude. On the other hand, it is also discovered that if the cylinder is closed and infinitely permeable, then the Casimir force has a long range term if the first Betti number h of $\mathcal{N}$ which counts the dimension of harmonic one-forms on $\mathcal{N}$ is nonzero.

The asymptotic behavior of the Casimir force acting on a pair of parallel plates which are obtained by taking one end of the cylinder to infinity is discussed. We consider the high temperature asymptotic behavior and the low temperature asymptotic behavior for different relative magnitude of a—the separation between the plates, r—the size of $\mathcal{N}$, and R—the size of the cross-section of the cylinder in M. It is shown that if $a \ll r, R$, then the leading term of the Casimir force is the Casimir force between two large parallel plates in the $(4 + n)$-dimensional Minkowski spacetime, which is of order $1/a^{n+4}$ in the low temperature regime and of order T/a^{n+3} in the high temperature regime. On the other hand, if $r \ll a \ll R$, then the leading term of the Casimir force is $1 + h/2$ times the Casimir force between two large parallel plates in the four-dimensional Minkowski spacetime. In particular, this shows that if the size r of $\mathcal{N}$ reduces to zero, one will obtain the Casimir force in the four-dimensional Minkowski spacetime if and only if h, the number of zero modes of the Laplace operator on one-forms on $\mathcal{N}$, is zero. This poses a topological condition on the extra dimensions. Such a constraint has not been observed in the study of the Casimir effect due to scalar fields.

For future works, it would be interesting to investigate the electromagnetic Casimir effect in other higher-dimensional spacetime models such as brane models. Besides, it would also be interesting to consider generalized pistons such as in the works [49–51,55,57].

APPENDIX A: GAUGE FIXING OF THE ELECTROMAGNETIC FIELD

In this section, we will discuss the gauge fixing of the electromagnetic field in the spacetime $\mathcal{M} = M \times \mathcal{N}$. Using the language of differential geometry [65,66], the electromagnetic field $F_{\mu\nu}dz^\mu dz^\nu$ is an exterior two-form on $\mathcal{M}$ and $A_\mu dz^\mu$ is an exterior one-form on $\mathcal{M}$. Moreover, $F_{\mu\nu}dz^\mu dz^\nu = d(A_\mu dz^\mu)$.

Given an exterior k-form $\psi = \psi_{\mu_1\ldots\mu_k}dz^{\mu_1}\ldots dz^{\mu_k}$, the dual $(N-k)$-form $*\psi = (*\psi)_{\nu_1\ldots\nu_{N-k}}dz^{\nu_1}\ldots dz^{\nu_{N-k}}$ is given by

$$(*\psi)_{\nu_1\ldots\nu_{N-k}} = \frac{1}{k!}\sqrt{|g|}\varepsilon_{\nu_1\ldots\nu_{N-k}\mu_1\ldots\mu_k}\psi^{\mu_1\ldots\mu_k}, \qquad \text{(A1)}$$

where $\varepsilon_{\mu_1\ldots\mu_N}$ is a totally antisymmetric rank-N tensor, which is equal to one if and only if $\mu_1, \ldots, \mu_N$ is an even permutation of $(0, 1, 2, \ldots, N-1)$. Using the $*$ operator and the exterior differentiation d, one can define the co-differential operator δ by

$$\delta \psi = (-1)^{N(k-1)} * d * \psi,$$

which maps a k-form ψ to a $k - 1$-form. More explicitly,

$$(\delta \psi)^{\mu_1 \cdots \mu_{k-1}} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial z^\nu} (\sqrt{|g|} \psi^{\nu \mu_1 \cdots \mu_{k-1}}).$$

The Laplacian operator Δ mapping k-forms to k-forms is then defined as

$$\Delta = d\delta + \delta d.$$

Using these notations, the equation of motion (4) is equivalent to

$$\delta F = \delta dA = 0. \tag{A2}$$

The action (3) is invariant under the gauge transformation $A \mapsto A + d\phi$ for any function ϕ on $\mathcal{M}$. To fix the gauge, notice that for any one-form A, it is always possible to find a function ϕ satisfying

$$\Delta \phi = \delta d\phi = -\delta A. \tag{A3}$$

This implies that we can impose the Lorentz gauge

$$\delta A = \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} A^\mu = 0. \tag{A4}$$

The equation of motion (A2) is then equivalent to

$$\Delta A = 0, \tag{A5}$$

i.e., $A^\mu dz^\mu$ is a harmonic one-form on $\mathcal{M}$. However, (A3) only defines ϕ up to those solutions satisfying $\Delta \phi = 0$. In the following, we are going to show that we can use this remaining gauge freedom to impose stronger gauge conditions.

Notice that $\sqrt{|g|} = \sqrt{G}$. Therefore,

$$\delta A = \partial_\alpha A^\alpha + \frac{1}{\sqrt{G}} \partial_a (\sqrt{G} A^a).$$

We want to show that we can impose the gauges

$$\partial_\alpha A^\alpha = 0, \qquad \frac{1}{\sqrt{G}} \partial_a (\sqrt{G} A^a) = 0. \tag{A6}$$

Any one-form on $M \times \mathcal{N}$ can be written as linear combinations of one-forms of the form:

$$A_\mu(x, y) dz^\mu = q(y) U_\alpha(x) dx^\alpha + p(x) V_a(y) dy^a, \tag{A7}$$

where $p(x)$ and $q(y)$ are nonzero, but $U_\alpha(x) dx^\alpha$ or $V_a(y) dy^a$ can be zero. For such a one-form,

$$\partial_\alpha A^\alpha = q \delta_M U, \qquad \frac{1}{\sqrt{G}} \partial_a (\sqrt{G} A^a) = p \delta_{\mathcal{N}} V,$$
$$\delta_{\mathcal{M}} A = \partial_\alpha A^\alpha + \frac{1}{\sqrt{G}} \partial_a (\sqrt{G} A^a) = q \delta_M U + p \delta_{\mathcal{N}} V, \tag{A8}$$

where $\delta_{\mathcal{M}}$, δ_M, and $\delta_{\mathcal{N}}$ are the δ operators on $\mathcal{M}$, M, and $\mathcal{N}$, respectively. In the following, similar conventions will be used for other operators. Equation (A8) implies that $\delta_{\mathcal{M}} A = 0$ if and only if

$$\frac{\delta_M U(x)}{p(x)} = -\frac{\delta_{\mathcal{N}} V(y)}{q(y)}.$$

This happens if and only if there exists a constant c so that

$$\delta_M U = cp, \qquad \delta_{\mathcal{N}} V = -cq. \tag{A9}$$

If $c = 0$, then we are done. Otherwise, since

$$F = d_{\mathcal{M}} A = d_{\mathcal{N}} q U + q d_M U + d_M p V + p d_{\mathcal{N}} V,$$

the equation of motion gives

$$\begin{aligned}
0 &= \delta_{\mathcal{M}} F \\
&= \Delta_{\mathcal{N}} q U - \delta_M U d_{\mathcal{N}} q + q \delta_M d_M U + \Delta_M p V \\
&\quad - d_M p \delta_{\mathcal{N}} V + p \delta_{\mathcal{N}} d_{\mathcal{N}} V \\
&= \Delta_{\mathcal{N}} q U + q \delta_M d_M U + cq d_M p + \Delta_M p V \\
&\quad - cp d_{\mathcal{N}} q + p \delta_{\mathcal{N}} d_{\mathcal{N}} V \\
&= \Delta_{\mathcal{N}} q U + q \Delta_M U + \Delta_M p V + p \Delta_{\mathcal{N}} V. \tag{A10}
\end{aligned}$$

Comparing the components, we have

$$\Delta_{\mathcal{N}} q U + q \Delta_M U = 0, \qquad \Delta_M p V + p \Delta_{\mathcal{N}} V = 0.$$

Therefore, there must exist constants λ_1 and λ_2 such that

$$\begin{aligned}
\frac{\Delta_{\mathcal{N}} q}{q} &= -\frac{\Delta_M U}{U} = \lambda_1, \\
-\frac{\Delta_M p}{p} &= \frac{\Delta_{\mathcal{N}} V}{V} = \lambda_2.
\end{aligned} \tag{A11}$$

From (A9) and the fact that $\delta_M^2 = 0$, we find that

$$\begin{aligned}
-c\lambda_2 p = c\Delta_M p = c\delta_M d_M p &= \delta_M d_M \delta_M U \\
&= \delta_M (d_M \delta_M + \delta_M d_M) U = \delta_M \Delta_M U \\
&= -\lambda_1 \delta_M U = -c\lambda_1 p.
\end{aligned}$$

This implies that $\lambda_1 = \lambda_2$. Let $\lambda = \lambda_1 = \lambda_2$. If $\lambda = 0$, then $\Delta_{\mathcal{N}} V = 0$ implies that $\delta_{\mathcal{N}} V = 0$. Hence $cq = 0$. This is a contradiction since we assume that $c \neq 0$ and $q \neq 0$. Therefore $\lambda \neq 0$. Consider the function

$$\phi = \frac{c}{\lambda} pq.$$

It is easy to verify that $\Delta \phi = 0$. Therefore one can use the remaining gauge freedom to transform A to A', where

$$\begin{aligned}
A' &= A + d\phi = qU' + pV' \\
&= q \left(U + \frac{c}{\lambda} d_M p \right) + p \left(V + \frac{c}{\lambda} d_{\mathcal{N}} q \right).
\end{aligned}$$

It follows that

$$\delta_M U' = \delta_M\left(U + \frac{c}{\lambda}d_M p\right) = \delta_M U + \frac{c}{\lambda}\delta_M d_M p = 0,$$

$$\delta_N V' = \delta_N\left(V + \frac{c}{\lambda}d_N q\right) = \delta_N V + \frac{c}{\lambda}\delta_N d_N q = 0,$$

which show that A' has the desired property (A6). As a conclusion, it is possible to impose the gauges (A6) which are equivalent to $\delta_M U = 0$, $\delta_N V = 0$ if A has the form (A7). After fixing these gauges, one can show that since N is assumed to be compact and connected, one only has the gauge freedom of adding to A the differential of a function $\varphi(x)$ satisfying $\Delta_M \varphi = 0$. The gauge condition $\delta_N V = 0$ can be considered as a generalization of the almost axial gauge used in [67] when $N = S^1$.

Before ending this section, we would like to remark that for general $A = qU + pV$ satisfying $\delta_M U = 0$ and $\delta_N V = 0$, the equation of motion (A10) still implies (A11), but in general $\lambda_1 \neq \lambda_2$. Therefore, we can separately consider one-forms A of the form qU, with

$$\delta_M U = 0, \qquad \Delta_M U = -\lambda_1 U, \qquad \Delta_N q = \lambda_1 q,$$

and of the form pV with

$$\delta_N V = 0, \qquad \Delta_M p = -\lambda_2 p, \qquad \Delta_N V = \lambda_2 V.$$

When $\lambda_1 = 0$, q is a constant and therefore we can further impose the gauge condition $U_0 = 0$ on U.

APPENDIX B: ALTERNATIVE EXPRESSIONS FOR THE CASIMIR FORCE

1. The plates have the same boundary conditions

As shown in Sec. III A, the Casimir force acting between two plates with the same boundary conditions is the sum over all τ_α of the following expression:

$$-T\sum_{p=-\infty}^{\infty}\frac{\tau_{\alpha,p}}{e^{2a\tau_{\alpha,p}}-1}.$$

This expression can be written as

$$\frac{T}{2}\frac{\partial}{\partial a}\sum_{p=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-2ka\sqrt{\tau_\alpha^2+[2\pi pT]^2}} = \frac{T}{2\sqrt{\pi}}\frac{\partial}{\partial a}\left\{a\int_0^\infty t^{-(1/2)}\sum_{p=-\infty}^{\infty}\sum_{k=1}^{\infty}\exp\left(-tk^2a^2 - \frac{\tau_\alpha^2+[2\pi pT]^2}{t}\right)dt\right\}$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial a}\left\{a\int_0^\infty\sum_{p=-\infty}^{\infty}\sum_{k=1}^{\infty}\exp\left(-tk^2a^2 - \frac{tp^2}{4T^2} - \frac{\tau_\alpha^2}{t}\right)dt\right\}$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial a}\left\{a\int_0^\infty\sum_{k=1}^{\infty}\exp\left(-tk^2a^2 - \frac{\tau_\alpha^2}{t}\right)dt - a\int_0^\infty\sum_{p=1}^{\infty}\exp\left(-\frac{tp^2}{4T^2} - \frac{\tau_\alpha^2}{t}\right)dt\right.$$

$$\left. + 2\sqrt{\pi}\int_0^\infty t^{-(1/2)}\sum_{p=1}^{\infty}\sum_{k=0}^{\infty\prime}\exp\left(-\frac{tp^2}{4T^2} - \frac{1}{t}\left(\left[\frac{\pi k}{a}\right]^2 + \tau_\alpha^2\right)\right)dt\right\}.$$

If $\tau_\alpha^2 \neq 0$, this is equal to

$$\frac{1}{4\pi}\frac{\partial}{\partial a}\left\{2\sum_{k=1}^{\infty}\frac{\tau_\alpha}{k}K_1(2ka\tau_\alpha) - 4aT\sum_{p=1}^{\infty}\frac{\tau_\alpha}{p}K_1\left(\frac{p\tau_\alpha}{T}\right) + 4\pi T\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{p}\exp\left(-\frac{p\sqrt{[\frac{\pi k}{a}]^2 + \tau_\alpha^2}}{T}\right)\right\}$$

$$= -\frac{1}{2\pi a}\sum_{k=1}^{\infty}\frac{\tau_\alpha}{k}K_1(2ka\tau_\alpha) - \frac{1}{\pi}\sum_{k=1}^{\infty}\tau_\alpha^2 K_0(2ka\tau_\alpha) - \frac{T}{\pi}\sum_{p=1}^{\infty}\frac{\tau_\alpha}{p}K_1\left(\frac{p\tau_\alpha}{T}\right)$$

$$+ \frac{\pi^2}{a^3}\sum_{k=1}^{\infty}\frac{k^2}{\sqrt{[\frac{\pi k}{a}]^2 + \tau_\alpha^2}}\frac{1}{\exp(\frac{1}{T}\sqrt{[\frac{\pi k}{a}]^2 + \tau_\alpha^2}) - 1}.$$

But if $\tau_\alpha^2 = 0$, it is equal to

$$\frac{1}{4\pi}\frac{\partial}{\partial a}\left\{\frac{\pi^2}{6a} - \frac{2\pi^2}{3}aT^2 + 4\pi T\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{p}\exp\left(-\frac{\pi kp}{aT}\right)\right\} = -\frac{\pi}{24a^2} - \frac{\pi T^2}{6} + \frac{\pi}{a^2}\sum_{k=1}^{\infty}\frac{k}{\exp(\frac{\pi k}{aT}) - 1}.$$

2. The plates have different boundary conditions

As shown in Sec. III B, the Casimir force acting between two plates with different boundary conditions is the sum over all τ_α of the following expression:

$$T \sum_{p=-\infty}^{\infty} \frac{\tau_{\alpha,p}}{e^{2a\tau_{\alpha,p}} + 1} = \frac{T}{2} \frac{\partial}{\partial a} \sum_{p=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-2ka\sqrt{\tau_\alpha^2 + [2\pi pT]^2}}.$$

As in Appendix A 1, one can show that if $\tau_\alpha^2 \neq 0$, this expression is equal to

$$-\frac{1}{2\pi a} \sum_{k=1}^{\infty} (-1)^k \frac{\tau_\alpha}{k} K_1(2ka\tau_\alpha) - \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \tau_\alpha^2 K_0(2ka\tau_\alpha) - \frac{T}{\pi} \sum_{p=1}^{\infty} \frac{\tau_\alpha}{p} K_1\left(\frac{p\tau_\alpha}{T}\right)$$

$$+ \frac{\pi^2}{a^3} \sum_{k=0}^{\infty} \frac{(k + \frac{1}{2})^2}{\sqrt{[\frac{\pi(k+\frac{1}{2})}{a}]^2 + \tau_\alpha^2}} \left\{ \exp\left(\frac{1}{T} \sqrt{\left[\frac{\pi(k+\frac{1}{2})}{a}\right]^2 + \tau_\alpha^2}\right) - 1 \right\}^{-1},$$

and if $\tau_\alpha^2 = 0$, it is equal to

$$\frac{\pi}{48a^2} - \frac{\pi T^2}{6} + \frac{\pi}{a^2} \sum_{k=0}^{\infty} \frac{k + \frac{1}{2}}{\exp(\frac{\pi(k+\frac{1}{2})}{aT}) - 1}.$$

APPENDIX C: ASYMPTOTIC BEHAVIOR OF THE CASIMIR FORCE

To find the asymptotic behavior of the Casimir force when $r \ll a \ll R$ and $a \ll r, R$, let us define the global heat kernels

$$K_{\Omega,D}(t) = \sum_{l=1}^{\infty} e^{-t\varpi_l^2}, \quad K_{\Omega,N}(t) = \sum_{l=0}^{\infty} e^{-t\aleph_l^2},$$

$$K_{\mathcal{N},0}(t) = \sum_{j=0}^{\infty} e^{-tm_j^2}, \quad K_{\mathcal{N},1}(t) = \sum_{j=1}^{\infty} e^{-tm_j^2} + \sum_{j=1}^{\infty} e^{-t\mu_j^2}.$$

$K_{\Omega,D}(t)$ is the heat kernel of the Laplace operator with Dirichlet boundary conditions on functions on Ω. $K_{\Omega,N}(t)$ is the heat kernel of the Laplace operator with Neumann boundary conditions on functions on Ω. $K_{\mathcal{N},0}(t)$ is the heat kernel of the Laplace operator on functions on $\mathcal{N}$. For $K_{\mathcal{N},1}(t)$, notice that given a one-form V which is an eigenvector of the Laplace operator with eigenvalue $m^2 \neq 0$, if it is not coclosed, i.e., if $\delta_{\mathcal{N}} V \neq 0$, then $\delta_{\mathcal{N}} V$ is a nonzero function on $\mathcal{N}$. Moreover,

$$\Delta_{\mathcal{N}}(\delta_{\mathcal{N}} V) = (\delta_{\mathcal{N}} d_{\mathcal{N}} + d_{\mathcal{N}} \delta_{\mathcal{N}}) \delta_{\mathcal{N}} V = \delta_{\mathcal{N}} d_{\mathcal{N}} \delta_{\mathcal{N}} V$$
$$= \delta_{\mathcal{N}}(d_{\mathcal{N}} \delta_{\mathcal{N}} + \delta_{\mathcal{N}} d_{\mathcal{N}}) V = \delta_{\mathcal{N}} \Delta_{\mathcal{N}} V$$
$$= m^2 \delta_{\mathcal{N}} V.$$

Namely, $\delta_{\mathcal{N}} V$ is an eigenfunction of the Laplace operator with eigenvalue m^2. Conversely, if q is an eigenfunction of the Laplace operator with eigenvalue $m^2 \neq 0$, then dq is a one-form, and

$$\Delta_{\mathcal{N}}(d_{\mathcal{N}} q) = (d_{\mathcal{N}} \delta_{\mathcal{N}} + \delta_{\mathcal{N}} d_{\mathcal{N}}) d_{\mathcal{N}} q = d_{\mathcal{N}} \delta_{\mathcal{N}} d_{\mathcal{N}} q$$
$$= d_{\mathcal{N}}(\delta_{\mathcal{N}} d_{\mathcal{N}} + d_{\mathcal{N}} \delta_{\mathcal{N}}) q = d_{\mathcal{N}}(\Delta_{\mathcal{N}} q)$$
$$= m^2 d_{\mathcal{N}} q.$$

Namely, $d_{\mathcal{N}} q$ is an eigen-one-form of the Laplace operator with eigenvalue m^2. Therefore the union of m_j^2, $j \geq 1$, and μ_j^2, $j \geq 1$, is the set of all eigenvalues of the Laplace operator on one-forms on $\mathcal{N}$. Hence, $K_{\mathcal{N},1}(t)$ is the heat kernel of the Laplace operator on one-forms on $\mathcal{N}$.

It is well known [68–70] that as $t \to 0^+$, the heat kernels $K(t)$ have asymptotic expansions of the form

$$K(t) \sim \sum_{i=0}^{\infty} c_i t^{(i-d)/2} = c_0 t^{-(d/2)} + c_1 t^{-(d-1)/2} + \dots,$$

where d is the dimension of the manifold. More specifically,

$$K_{\Omega,D}(t) = \frac{\mathcal{A}(\Omega)}{4\pi} t^{-1} - \frac{l(\partial\Omega)}{8\sqrt{\pi}} t^{-(1/2)} + O(1),$$

$$K_{\Omega,N}(t) = \frac{\mathcal{A}(\Omega)}{4\pi} t^{-1} + \frac{l(\partial\Omega)}{8\sqrt{\pi}} t^{-(1/2)} + O(1),$$

$$K_{\mathcal{N},0}(t) = \frac{\mathcal{V}(\mathcal{N})}{(4\pi)^{n/2}} t^{-(n/2)} + O(t^{1-(n/2)}),$$

$$K_{\mathcal{N},1}(t) = n \frac{\mathcal{V}(\mathcal{N})}{(4\pi)^{n/2}} t^{-(n/2)} + O(t^{1-(n/2)}).$$

Here $\mathcal{A}(\Omega)$ is the area of Ω, $l(\partial\Omega)$ is the arc length of $\partial\Omega$, and $\mathcal{V}(\mathcal{N})$ is the volume of $\mathcal{N}$.

As is shown in Appendix A 1 and B 2 the Casimir force acting on a pair of parallel plates is given by

or

$$F_{\text{Cas}}^{\parallel}(a) = \frac{T}{2\sqrt{\pi}} \frac{\partial}{\partial a} \left\{ a \int_0^{\infty} t^{-(1/2)} \sum_{p=-\infty}^{\infty} \sum_{k=1}^{\infty} e^{2\pi ik\chi} \exp\left(-tk^2 a^2 - \frac{[2\pi pT]^2}{t} \right) \sum_{\tau_\alpha} \exp\left(-\frac{\tau_\alpha^2}{t} \right) dt \right\}, \tag{C1}$$

$$F_{\text{Cas}}^{\parallel}(a) = \frac{1}{4\pi}\frac{\partial}{\partial a}\Bigg\{ a\int_0^\infty \sum_{k=1}^\infty e^{2\pi i k\chi}\exp(-tk^2a^2)\sum_{\tau_\alpha}\exp\left(-\frac{\tau_\alpha^2}{t}\right)dt - a\int_0^\infty \sum_{p=1}^\infty \exp\left(-\frac{tp^2}{4T^2}\right)\sum_{\tau_\alpha}\exp\left(-\frac{\tau_\alpha^2}{t}\right)dt$$

$$+ 2\sqrt{\pi}\int_0^\infty t^{-(1/2)}\sum_{p=1}^\infty\sum_{k=0}^\infty \exp\left(-\frac{tp^2}{4T^2}-\frac{1}{t}\left(\frac{\pi(k+\chi)}{a}\right)^2\right)\sum_{\tau_\alpha}\exp\left(-\frac{\tau_\alpha^2}{t}\right)dt\Bigg\}, \tag{C2}$$

where $\chi = 1$ if the two plates have the same boundary condition, and $\chi = 1/2$ if the two plates have different boundary conditions.

When the cylinder is perfectly conducting, the set of τ_α^2 is given by (PC). Therefore,

$$\sum_{\tau_\alpha}\exp\left(-\frac{\tau_\alpha^2}{t}\right) = 2K_{\Omega,D}(t^{-1})\sum_{j=0}^\infty{}'\exp\left(-\frac{m_j^2}{t}\right) + K_{\Omega,D}(t^{-1})\sum_{j=1}^\infty\exp\left(-\frac{\mu_j^2}{t}\right) + (K_{\Omega,N}(t^{-1})-1)\sum_{j=0}^\infty\exp\left(-\frac{m_j^2}{t}\right)$$

$$\sim 2\left(\frac{\mathcal{A}(\Omega)}{4\pi}t - \frac{l(\partial\Omega)}{8\sqrt{\pi}}t^{1/2}\right)\sum_{j=0}^\infty{}'\exp\left(-\frac{m_j^2}{t}\right) + \left(\frac{\mathcal{A}(\Omega)}{4\pi}t - \frac{l(\partial\Omega)}{8\sqrt{\pi}}t^{1/2}\right)\sum_{j=1}^\infty\exp\left(-\frac{\mu_j^2}{t}\right)$$

$$+ \left(\frac{\mathcal{A}(\Omega)}{4\pi}t + \frac{l(\partial\Omega)}{8\sqrt{\pi}}t^{1/2}\right)\sum_{j=0}^\infty\exp\left(-\frac{m_j^2}{t}\right)$$

$$= \frac{\mathcal{A}(\Omega)}{4\pi}t\sum_{\beta}\exp\left(-\frac{\sigma_\beta^2}{t}\right) - \frac{l(\partial\Omega)}{8\sqrt{\pi}}t^{1/2}\sum_{\gamma}\exp\left(-\frac{\xi_\gamma^2}{t}\right). \tag{C3}$$

The set of σ_β^2 contains m_0^2 with multiplicity two, m_j^2, $j\geq 1$ with multiplicity three, and μ_j^2, $j\geq 1$ with multiplicity one. The set of ξ_γ^2 contains m_j^2, $j\geq 1$, and μ_j^2, $j\geq 1$, each with multiplicity one. On the other hand, when the cylinder is infinitely permeable, the set of τ_α^2 is given by (IP), which gives

$$\sum_{\tau_\alpha}\exp\left(-\frac{\tau_\alpha^2}{t}\right) = 2(K_{\Omega,N}(t^{-1})-1)\sum_{j=0}^\infty{}'\exp\left(-\frac{m_j^2}{t}\right) + K_{\Omega,N}(t^{-1})\sum_{j=1}^\infty\exp\left(-\frac{\mu_j^2}{t}\right)$$

$$+ K_{\Omega,D}(t^{-1})\sum_{j=0}^\infty\exp\left(-\frac{m_j^2}{t}\right) + \sum_{j=1}^\infty\exp\left(-\frac{m_j^2}{t}\right)$$

$$\sim \frac{\mathcal{A}(\Omega)}{4\pi}t\sum_{\beta}\exp\left(-\frac{\sigma_\beta^2}{t}\right) + \frac{l(\partial\Omega)}{8\sqrt{\pi}}t^{1/2}\sum_{\gamma}\exp\left(-\frac{\xi_\gamma^2}{t}\right). \tag{C4}$$

Notice that the leading terms of (C3) and (C4) are the same and is given by

$$\frac{\mathcal{A}(\Omega)}{4\pi}t\sum_{\beta}\exp\left(-\frac{\sigma_\beta^2}{t}\right). \tag{C5}$$

But the subleading terms

$$\mp\frac{l(\partial\Omega)}{8\sqrt{\pi}}t^{1/2}\sum_{\gamma}\exp\left(-\frac{\xi_\gamma^2}{t}\right) \tag{C6}$$

have opposite signs. Substituting (C5) into (C1), we find that when $a\ll R$, the leading term of the Casimir force is given by

$$\mathcal{F}_0(a) = \frac{T\mathcal{A}(\Omega)}{8\pi^{3/2}}\frac{\partial}{\partial a}\Bigg\{\frac{\sqrt{\pi}}{2}\frac{2+h}{a^2}\sum_{k=1}^\infty\frac{e^{2\pi i k\chi}}{k^3} + 2a\sum_{k=1}^\infty\sum_{\sigma_{\beta,p}\neq 0}e^{2\pi i k\chi}\left(\frac{\sigma_{\beta,p}}{ka}\right)^{3/2}K_{3/2}(2ka\sigma_{\beta,p})\Bigg\}$$

$$= -\frac{T\mathcal{A}(\Omega)}{8\pi}\frac{2+h}{a^3}\sum_{k=1}^\infty\frac{e^{2\pi i k\chi}}{k^3} - \frac{T\mathcal{A}(\Omega)}{8\pi}\sum_{k=1}^\infty\sum_{\sigma_{\beta,p}\neq 0}e^{2\pi i k\chi}\left(\frac{2\sigma_{\beta,p}^2}{ka}+\frac{2\sigma_{\beta,p}}{k^2a^2}+\frac{1}{k^3a^3}\right)e^{-2ka\sigma_{\beta,p}}, \tag{C7}$$

where $\sigma_{\beta,p}^2 = \sigma_\beta^2 + [2\pi pT]^2$. The first term in (C7) comes from those terms with $\sigma_\beta^2 = 0$ and $p = 0$. The sum

$$\sum_{k=1}^\infty\frac{e^{2\pi i k\chi}}{k^s}$$

is equal to $\zeta_R(s)$ if $\chi = 1$ and is equal to $(2^{1-s} - 1)\zeta_R(s)$ if $\chi = 1/2$. (C7) gives the leading behavior of the Casimir force when $a \ll R$ and $aT \gg 1$. If $aT \ll 1$, substituting (C3) into (C2) gives

$$
\begin{aligned}
\mathcal{F}_0(a) &= \frac{\mathcal{A}(\Omega)}{16\pi^2} \frac{\partial}{\partial a} \Bigg\{ (2+h)\bigg(\frac{1}{a^3}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^4} - \frac{8\pi^4 T^4 a}{45} \bigg) + 2a \sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0} e^{2\pi ik\chi}\Big(\frac{\sigma_\beta}{ka}\Big)^2 K_2(2ka\sigma_\beta) - 8aT^2 \sum_{p=1}^{\infty}\sum_{\sigma_\beta \neq 0}\Big(\frac{\sigma_\beta}{p}\Big)^2 \\
&\quad \times K_2\Big(\frac{p\sigma_\beta}{T}\Big) + 2^{7/2}\sqrt{\pi}T^{3/2} \sum_{p=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\sigma_\beta} p^{-(3/2)}\bigg(\Big[\frac{\pi(k+\chi)}{a}\Big]^2 + \sigma_\beta^2 \bigg)^{3/4} K_{3/2}\bigg(\frac{p}{T}\sqrt{\Big[\frac{\pi(k+\chi)}{a}\Big]^2 + \sigma_\beta^2}\bigg) \Bigg\} \\
&= -(2+h)\mathcal{A}(\Omega)\bigg(\frac{3}{16\pi^2 a^4}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^4} + \frac{\pi^2 T^4}{90} \bigg) - \frac{3\mathcal{A}(\Omega)}{8\pi^2}\sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0} e^{2\pi ik\chi}\Big(\frac{\sigma_\beta}{ka}\Big)^2 K_2(2ka\sigma_\beta) \\
&\quad - \frac{\mathcal{A}(\Omega)}{4\pi^2}\sum_{k=1}^{\infty}\sum_{\sigma_\beta \neq 0} e^{2\pi ik\chi}\frac{\sigma_\beta^3}{ka}K_1(2ka\sigma_\beta) - \frac{\mathcal{A}(\Omega)T^2}{2\pi^2}\sum_{p=1}^{\infty}\sum_{\sigma_\beta \neq 0}\Big(\frac{\sigma_\beta}{p}\Big)^2 K_2\Big(\frac{p\sigma_\beta}{T}\Big) \\
&\quad + \frac{\pi\mathcal{A}(\Omega)T}{2a^3}\sum_{p=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\sigma_\beta}\frac{(k+\chi)^2}{p}\exp\bigg(-\frac{p}{T}\sqrt{\Big[\frac{\pi(k+\chi)}{a}\Big]^2 + \sigma_\beta^2} \bigg).
\end{aligned}
\tag{C8}
$$

For the subleading term $\mathcal{F}_1(a)$, substitute (C6) into (C1) and (C2), respectively. Similar computations show that

$$
\mathcal{F}_1(a) = \pm\frac{Tl(\partial\Omega)}{16\pi}\Bigg\{ \frac{h}{a^2}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^2} + 2\sum_{k=1}^{\infty}\sum_{\xi_{\gamma,p}\neq 0} e^{2\pi ik\chi}\frac{\xi_{\gamma,p}}{ka}K_1(2ka\xi_{\gamma,p}) + 4\sum_{k=1}^{\infty}\sum_{\xi_{\gamma,p}\neq 0} e^{2\pi ik\chi}\xi_{\gamma,p}^2 K_0(2ka\xi_{\gamma,p}) \Bigg\},
$$

with $\xi_{\gamma,p}^2 = \xi_\gamma^2 + [2\pi pT]^2$, or

$$
\begin{aligned}
\mathcal{F}_1(a) &= \pm l(\partial\Omega)\Bigg\{ \frac{h}{32\pi}\bigg(\frac{1}{a^3}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^3} + 4T^3\zeta_R(3) \bigg) + \frac{1}{32\pi}\sum_{k=1}^{\infty}\sum_{\xi_\gamma\neq 0} e^{2\pi ik\chi}e^{-2ka\xi_\gamma}\bigg(\frac{2\xi_\gamma^2}{ka} + \frac{2\xi_\gamma}{k^2 a^2} + \frac{1}{k^3 a^3} \bigg) \\
&\quad + \frac{T^2}{8\pi}\sum_{p=1}^{\infty}\sum_{\xi_\gamma\neq 0} e^{-(p\xi_\gamma/T)}\bigg(\frac{\xi_\gamma}{p^2} + \frac{T}{p^3} \bigg) - \frac{\pi}{4a^3}\sum_{k=0}^{\infty}\sum_{p=1}^{\infty}\sum_{\xi_\gamma}(k+\chi)^2 K_0\bigg(\frac{p}{T}\sqrt{\Big[\frac{\pi(k+\chi)}{a}\Big]^2 + \xi_\gamma^2} \bigg) \Bigg\},
\end{aligned}
$$

where the plus sign is for perfectly conducting cylinder and the minus sign is for infinitely permeable cylinder.

To find the asymptotic behavior when $a \ll r, R$, we use the fact that

$$
\sum_\beta \exp\Big(-\frac{\sigma_\beta^2}{t}\Big) = 2 + 3\sum_{j=1}^{\infty}\exp\Big(-\frac{m_j^2}{t}\Big) + \sum_{j=1}^{\infty}\exp\Big(-\frac{\mu_j^2}{t}\Big) = 2K_{\mathcal{N},0}(t^{-1}) + K_{\mathcal{N},1}(t^{-1}) \sim (n+2)\frac{\mathcal{V}(\mathcal{N})}{(4\pi)^{n/2}}t^{n/2}
$$

$$
\sum_\gamma \exp\Big(-\frac{\xi_\gamma^2}{t}\Big) = \sum_{j=1}^{\infty}\exp\Big(-\frac{m_j^2}{t}\Big) + \sum_{j=1}^{\infty}\exp\Big(-\frac{\mu_j^2}{t}\Big) = K_{\mathcal{N},1}(t^{-1}) \sim n\frac{\mathcal{V}(\mathcal{N})}{(4\pi)^{n/2}}t^{n/2}.
$$

Therefore,

$$
\sum_\alpha \exp\Big(-\frac{\tau_\alpha^2}{t}\Big) \sim (n+2)\frac{\mathcal{V}(\mathcal{S})}{(4\pi)^{(n+2)/2}}t^{(n+2)/2} \mp n\frac{\mathcal{V}(\partial\mathcal{S})}{4(4\pi)^{(n+1)/2}}t^{(n+1)/2},
\tag{C9}
$$

where the minus sign is for perfectly conducting cylinder, the plus sign is for infinitely permeable cylinder, $\mathcal{S} = \Omega \times \mathcal{N}$ is the cross section of the cylinder, and $\mathcal{V}(\mathcal{S}) = \mathcal{A}(\Omega) \times \mathcal{V}(\mathcal{N})$ is its volume. Substituting (C9) into (C1) and (C2), we find that the first two leading terms of the Casimir force when $a \ll r, R$ are given by

$$\mathfrak{F}_0(a) = (n+2)\mathcal{V}(\mathcal{S})\Bigg\{-T\frac{(n+2)\Gamma(\frac{n+3}{2})}{(4\pi)^{(n+3)/2}a^{n+3}}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^{n+3}} - (n+2)\frac{T^{(n+5)/2}}{2^{(n-1)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}e^{2\pi ik\chi}\Big(\frac{p}{ka}\Big)^{(n+3)/2}K_{(n+3/2)}(4\pi kpaT)$$

$$-\frac{\pi T^{(n+7)/2}}{2^{(n-5)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}e^{2\pi ik\chi}\frac{p^{(n+5)/2}}{(ka)^{(n+1)/2}}K_{(n+1)/2}(4\pi kpaT)\Bigg\},$$

$$\mathfrak{F}_1(a) = \mp\frac{n\mathcal{V}(\partial\mathcal{S})}{4}\Bigg\{-T\frac{(n+1)\Gamma(\frac{n+2}{2})}{(4\pi)^{(n+2)/2}a^{n+2}}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^{n+2}} - (n+1)\frac{T^{(n+4)/2}}{2^{(n-2)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}e^{2\pi ik\chi}\Big(\frac{p}{ka}\Big)^{(n+2)/2}K_{(n+2)/2}(4\pi kpaT)$$

$$-\frac{\pi T^{(n+6)/2}}{2^{(n-6)/2}}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}e^{2\pi ik\chi}\frac{p^{(n+4)/2}}{(ka)^{n/2}}K_{n/2}(4\pi kpaT)\Bigg\},$$

or

$$\mathfrak{F}_0(a) = (n+2)\mathcal{V}(\mathcal{S})\Bigg\{-\frac{(n+3)\Gamma(\frac{n+4}{2})}{(4\pi)^{(n+4)/2}a^{n+4}}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^{n+4}} - \frac{\Gamma(\frac{n+4}{2})}{\pi^{(n+4)/2}}\zeta_R(n+4)T^{n+4}$$

$$+\frac{\pi T^{(n+1)/2}}{2^{(n+1)/2}a^{(n+7)/2}}\sum_{k=0}^{\infty}\sum_{p=1}^{\infty}\frac{(k+\chi)^{(n+5)/2}}{p^{(n+1)/2}}K_{(n+1)/2}\Big(\frac{\pi p(k+\chi)}{aT}\Big)\Bigg\},$$

$$\mathfrak{F}_1(a) = \mp\frac{n\mathcal{V}(\partial\mathcal{S})}{4}\Bigg\{-\frac{(n+2)\Gamma(\frac{n+3}{2})}{(4\pi)^{(n+3)/2}a^{n+3}}\sum_{k=1}^{\infty}\frac{e^{2\pi ik\chi}}{k^{n+3}} - \frac{\Gamma(\frac{n+3}{2})}{\pi^{(n+3)/2}}\zeta_R(n+3)T^{n+3}$$

$$+\frac{\pi T^{n/2}}{2^{n/2}a^{(n+6)/2}}\sum_{k=0}^{\infty}\sum_{p=1}^{\infty}\frac{(k+\chi)^{(n+4)/2}}{p^{n/2}}K_{n/2}\Big(\frac{\pi p(k+\chi)}{aT}\Big)\Bigg\}.$$

---

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **B51,** 793 (1948).

[2] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).

[3] L. Perivolaropoulos, Phys. Rev. D **77,** 107301 (2008).

[4] E. Elizalde, Phys. Lett. B **516,** 143 (2001).

[5] C. L. Gardner, Phys. Lett. B **524,** 21 (2002).

[6] K. A. Milton, Gravitation Cosmol. **9,** 66 (2003).

[7] M. R. Setare, Eur. Phys. J. C **38,** 373 (2004).

[8] G. Mahajan, S. Sarkar, and T. Padmanabhan, Phys. Lett. B **641,** 6 (2006).

[9] M. Doran and J. Jaeckel, J. Cosmol. Astropart. Phys. 08 (2006) 010.

[10] F. Pascoal, L. F. A. Oliveira, F. S. S. Rosa, and C. Farina, Braz. J. Phys. **38,** 581 (2008).

[11] H. Davoudiasl and E. Ponton, J. High Energy Phys. 10 (2010) 102.

[12] F. S. Accetta, Phys. Rev. D **34,** 1798 (1986).

[13] W. D. Goldberger and I. Z. Rothstein, Phys. Lett. B **491,** 339 (2000).

[14] S. Nam, J. High Energy Phys. 10 (2000) 044.

[15] W. H. Huang, J. High Energy Phys. 11 (2000) 041.

[16] E. Ponton and E. Poppitz, J. High Energy Phys. 06 (2001) 019.

[17] R. Hofmann, P. Kanti, and M. Pospelov, Phys. Rev. D **63,** 124020 (2001).

[18] W. H. Huang, Phys. Lett. B **497,** 317 (2001).

[19] N. Ahmed and I. G. Moss, Nucl. Phys. B **833,** 133 (2010).

[20] K. Poppenhaeger, S. Hossenfelder, S. Hofmann, and M. Bleicher, Phys. Lett. B **582,** 1 (2004).

[21] M. Frank, I. Turan, and L. Ziegler, Phys. Rev. D **76,** 015008 (2007).

[22] R. Linares, H. A. Morales-Técotl and O. Pedraza, Phys. Rev. D **77,** 066012 (2008).

[23] M. Frank, N. Saad, and I. Turan, Phys. Rev. D **78,** 055014 (2008).

[24] A. A. Saharian, Phys. Rev. D **73,** 064019 (2006).

[25] A. A. Saharian, Phys. Rev. D **73,** 044012 (2006).

[26] H. B. Cheng, Phys. Lett. B **668,** 72 (2008).

[27] A. Edery and V. N. Marachevsky, J. High Energy Phys. 12 (2008) 035.

[28] K. Kirsten and S. A. Fulling, Phys. Rev. D **79,** 065019 (2009).

[29] E. Elizalde, S. D. Odintsov, and A. A. Saharian, Phys. Rev. D **79,** 065023 (2009).

[30] L. P. Teo, Phys. Lett. B **672,** 190 (2009).

[31] L. P. Teo, Nucl. Phys. B **819,** 431 (2009).

[32] L. P. Teo, J. High Energy Phys. 06 (2009) 076.

[33] A. Flachi and T. Tanaka, Phys. Rev. D **80,** 124022 (2009).

[34] S. Bellucci and A. A. Saharian, Phys. Rev. D **80,** 105003 (2009).

[35] S. Bellucci and A. A. Saharian, Phys. Rev. D **79,** 085019 (2009).

[36] L. P. Teo, J. High Energy Phys. 11 (2009) 095.
[37] M. Rypestol and I. Brevik, New J. Phys. **12**, 013022 (2010).
[38] M. Rypestol and I. Brevik, Phys. Scr. **82**, 035101 (2010).
[39] A. A. Saharian and M. R. Setare, Classical Quantum Gravity **27**, 225009 (2010).
[40] L. P. Teo, J. High Energy Phys. 10 (2010) 019.
[41] R. M. Cavalcanti, Phys. Rev. D **69**, 065015 (2004).
[42] M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio, Phys. Rev. Lett. **95**, 250402 (2005).
[43] M. P. Hertzberg, R. L. Jaffe, M. Kardar, A. Scardicchio, Phys. Rev. D **76**, 045016 (2007).
[44] V. N. Marachevsky, Phys. Rev. D **75**, 085019 (2007).
[45] A. Edery, Phys. Rev. D **75**, 105012 (2007).
[46] A. Edery and I. Macdonald, J. High Energy Phys. 09 (2007) 005.
[47] A. Edery and V. N. Marachevsky, Phys. Rev. D **78**, 025021 (2008).
[48] S. C. Lim and L. P. Teo, Eur. Phys. J. C **60**, 323 (2009).
[49] M. Schaden, Phys. Rev. A **79**, 052105 (2009).
[50] M. Schaden, J. Phys. Conf. Ser. **161**, 012021 (2009).
[51] E. Álvarez, F. D. Mazzitelli, A. G. Monastra, and D. A. Wisniacki, Phys. Rev. A **82**, 052504 (2010).
[52] K. A. Milton, J. Wagner, and K. Kirsten, Phys. Rev. D **80**, 125028 (2009).
[53] A. Edery, N. Graham, and I. McDonald, Phys. Rev. D **79**, 125018 (2009).
[54] S. A. Fulling, L. Kaplan, K. Kirsten, Z. H. Liu, and K. A. Milton, J. Phys. A **42**, 155402 (2009).
[55] M. Schaden, Phys. Rev. Lett. **102**, 060402 (2009).
[56] G. Barton, Phys. Rev. D **73**, 065018 (2006).
[57] G. Fucci and K. Kirsten, J. High Energy Phys. 03 (2011) 016.
[58] Jan Ambjørn and S. Wolfram, Ann. Phys. (N.Y.) **147**, 1 (1983).
[59] E. Alvarez and F. D. Mazzitelli, Phys. Rev. D **79**, 045019 (2009).
[60] F. Chen, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Phys. Rev. A **69**, 022117 (2004).
[61] R. S. Decca, D. López, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Phys. Rev. D **75**, 077101 (2007).
[62] R. S. Decca, D. López, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Eur. Phys. J. C **51**, 963 (2007).
[63] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. **81**, 1827 (2009).
[64] M. A. Rubin and C. Ordonez, J. Math. Phys. Sci. **26**, 65 (1985).
[65] B. Felsager, *Geometry, Particles, and Fields* (Springer Verlag, New York, 1998).
[66] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Springer Verlag, New York, 1983).
[67] R. Sundrum, arXiv:hep-th/0508134.
[68] V. K. Patodi, J. Indian Math. Soc. **34**, 269 (1970).
[69] T. P. Branson and P. B. Gilkey, Commun. Partial Differ. Equ. **15**, 245 (1990).
[70] N. Blazic, N. Bokan, and P. B. Gilkey, Indian J. Pure Appl. Math. **23**, 103 (1992).