PHYSICAL REVIEW D **83,** 105009 (2011)

Can the renormalization group improved effective potential be used to estimate the Higgs mass in the conformal limit of the standard model?

F. A. Chishtie,[1] T. Hanif,[2,8] J. Jia,[1] R. B. Mann,[3] D. G. C. McKeon,[1,4] T. N. Sherry,[5,6,*] and T. G. Steele[7]

[1]*Department of Applied Mathematics, The University of Western Ontario, London, ON N6A 5B7, Canada*
[2]*Department of Physics and Astronomy, The University of Western Ontario, London, ON N6A 5B7, Canada*
[3]*Department of Physics, University of Waterloo, Waterloo, ON N2L 3G1, Canada*
[4]*Department of Mathematics and Computer Science, Algoma University, Sault St. Marie, ON N6A 2G4, Canada*
[5]*School of Mathematics, Statistics and Applied Mathematics, NUI Galway, University Road, Galway, Ireland*
[6]*School of Theoretical Physics, Dublin Institute for Advanced Studies, Burlington Road, Dublin 4, Ireland*
[7]*Department of Physics and Engineering Physics, University of Saskatchewan, Saskatoon, SK S7N 5E2, Canada*
[8]*Department of Theoretical Physics, University of Dhaka, Dhaka-1000, Bangladesh*

(Received 19 December 2010; published 12 May 2011)

We consider the effective potential V in the standard model with a single Higgs doublet in the limit that the only mass scale μ present is radiatively generated. Using a technique that has been shown to determine V completely in terms of the renormalization group (RG) functions when using the Coleman-Weinberg renormalization scheme, we first sum leading-log (LL) contributions to V using the one loop RG functions, associated with five couplings (the top quark Yukawa coupling x, the quartic coupling of the Higgs field y, the $SU(3)$ gauge coupling z, and the $SU(2) \times U(1)$ couplings r and s). We then employ the two loop RG functions with the three couplings x, y, z to sum the next-to-leading-log (NLL) contributions to V and then the three to five loop RG functions with one coupling y to sum all the $N^2LL \ldots N^4LL$ contributions to V. In order to compute these sums, it is necessary to convert those RG functions that have been originally computed explicitly in the minimal subtraction scheme to their form in the Coleman-Weinberg scheme. The Higgs mass can then be determined from the effective potential: the LL result is $m_H = 219\ \mathrm{GeV}/c^2$ and decreases to $m_H = 188\ \mathrm{GeV}/c^2$ at N^2LL order and $m_H = 163\ \mathrm{GeV}/c^2$ at N^4LL order. No reasonable estimate of m_H can be made at orders V_{NLL} or V_{N^3LL} since the method employed gives either negative or imaginary values for the quartic scalar coupling. The fact that we get reasonable values for m_H from the LL, N^2LL, and N^4LL approximations is taken to be an indication that this mechanism for spontaneous symmetry breaking is in fact viable, though one in which there is slow convergence towards the actual value of m_H. The mass 163 GeV/c^2 is argued to be an upper bound on m_H.

DOI: 10.1103/PhysRevD.83.105009 PACS numbers: 12.38.Cy

I. INTRODUCTION

The leading-logarithm (LL) contribution to the effective potential V in the standard model in which there is a single scalar field and no mass scale in the classical limit, has been used to estimate the Higgs mass to be $m_H = 224\ \mathrm{GeV}/c^2$ [1]. Subsequent investigations indicate that contributions beyond LL to V do not destabilize this result [2]. In this paper we propose to significantly improve the methods used in Refs. [1,2] and compute the resulting modification to the estimate of m_H. The value of m_H obtained using these improvements is much more realistic.

Since these results were obtained, it has been established that when the Coleman-Weinberg (CW) renormalization scheme is used to compute V, all N^pLL contributions to V can be computed using the $(p + 1)$ loop renormalization group (RG) functions when there is a single scalar field ϕ without a classical mass term for this scalar in the action [3].

*Corresponding author: tom.sherry@nuigalway.ie

We first show how these techniques can be used to refine the approach of [1,2]. In doing so, we overcome several shortcomings of the original calculation. First of all, the RG functions we use are those appropriate to the CW renormalization scheme, not the minimal subtraction (MS) scheme. This conversion from the MS scheme (in which the RG functions were originally computed) to the CW scheme was not carried out in [1,2]. Next, we show how the N^pLL contributions to V can be expressed exactly in terms of the $(p + 1)$ loop CW RG functions. This shows that once the $(p + 1)$ loop CW RG functions are known, we have an exact expression for the $(p + 1)$ loop contributions to V without having to compute any Feynman diagrams and, in addition, we can sum all the N^pLL contributions to V coming from all orders in the loop expansion. In [1,2] these contributions were only given as a power series in the couplings x, z, r and s. Finally, we compute the counterterm that takes into account all log-independent contributions to V beyond the N^pLL order in a more consistent way than was done in [1,2]; rather than fixing this counterterm by the LL calculation and then using this value at higher order, we determine the value of this counterterm at each order separately thereby taking into account how the value of the coupling y is adjusted. It is the methods of Ref. [3] that allow us to fix all log-independent contributions to V in terms of the RG functions when using the CW scheme. Our analytic approach supplements numerical techniques for investigating V using the RG equation (see e.g., Ref. [4]).

In the next section we review how N^pLL contributions to V can be computed in terms of the RG functions when the CW renormalization scheme is used, first considering the case in which there is a single $O(N)$ scalar field with only a quartic self coupling and no classical mass term in the Lagrangian. The only mass scale in such a theory is radiatively induced. This is then extended so that the scalar couples to other fields (both vectors and spinors). The details of the solution at NLL are presented in Appendix A along with an explanation of how the methodology can be extended to N^2LL and higher-order. Appendix B presents a method of computing terms in the derivative expansion of the one loop effective action.

We have employed the CW renormalization scheme, as in this scheme all logarithmic dependence on the external field comes through a single form of logarithm, $\ln(\phi^2/\mu^2)$. Having this single logarithm simplifies the ansatz we make for V when there are multiple couplings (see Eq. (19) below), making it possible to find V in terms of the CW RG functions. If there are multiple couplings (say x and y) then both $\ln(x\phi^2/\mu^2)$ and $\ln(y\phi^2/\mu^2)$ arise when using the MS renormalization scheme. This complicates the ansatz one has for V, making it no longer feasible to find V in terms of the MS RG functions. Furthermore, one must compute the radiative corrections dependent on ϕ to the kinetic term $(\partial_\mu\phi)^2$ in the effective Lagrangian when determining the radiatively generated Higgs mass m_H; this is unknown (and presumably nontrivial) in the MS scheme, whereas in the CW scheme it is defined to be equal to one at the value of ϕ that minimizes V. [See Eq. (18) below.] For these reasons we use the CW scheme in our analysis.

We also note that the inclusion of a quadratic mass term $m^2\phi^2$ for the $O(4)$ scalar field in the classical action results in multiple forms of the logarithm occurring in the ansatz for V (see Ref. [5]) and also necessitates consideration of a "cosmological term" (see Ref. [6]). These factors considerably complicate employing the RG equation to find the N^pLL contributions to V; we thus restrict ourselves to the classically conformal case $m = 0$ as originally suggested in Ref. [7].

We then discuss the conversion of the RG functions from the MS scheme, in which they have been originally computed, to the CW scheme, which is necessary to implement our procedure for computing the N^pLL contribution to V. We finally apply these results to the simplest version of the standard model in which there is a single scalar which is an $SU(2)$ doublet and which has no mass at the classical level. The resulting expression for the effective potential at N^4LL order leads to an estimate of 163 GeV$/c^2$ for the mass of the Higgs Boson. We regard this as an upper limit on the Higgs mass as lower order calculations lead to estimates that are considerably higher than this. In any case, the proposal [7] that the Higgs mechanism is a consequence of radiative corrections to the effective potential in the conformally invariant classical limit of the standard model is seen to be viable.

We note that the potential V being considered here is the sum of all one particle irreducible (1PI) diagrams with external scalar fields whose momentum vanishes. This 1PI potential has been argued to be distinct from the "effective potential," a quantity shown in Ref. [8] to be convex and real. The relationship between the 1PI potential and the effective potential is discussed in Refs. [9,10] and reviewed in Refs. [11–13]. However, resolution of the convexity problem continues to be debated in the literature (see Refs. [14–16]). The most recent examination of the convexity problem explores the distinctions between the Euclidean and Minkowskian formulations of the effective potential [17].

Although our work adopts the conventional approach of ascribing physical meaning to the 1PI potential [9–12], it is important to note that our Higgs mass predictions in the standard model rely upon only the *local* properties of the 1PI potential near the minimum as extracted from the RG equation. Since this minimum occurs at nonzero field values, the minimum corresponds to the qualitative nonperturbative form of a spontaneous symmetry breaking effective potential [18] and provides the lower bound on the region where the effective potential and 1PI potential coincide [9,10,12]. Therefore our analysis is not in conflict with Ref. [14], which argues that the 1PI and effective potential must agree near the minimum and advocates the use of RG methods.

Finally we note that nonperturbative approaches are not isolated from the convexity problem. For example, the constraint effective potential [19] in lattice approaches is nonconvex at finite volumes [20], and lattice results are found to agree with the perturbative 1PI potential in appropriate regions of parameter space [21]. Functional flows of the exact renormalization group can be used to calculate an effective *average* action [22,23] and convexity constrains the regulators used in various truncation schemes used in these methods [24].[1] Other alternatives to the effective potential include the Gaussian effective potential [26] which is well suited to variational techniques.

[1]The average effective action is calculated for scalar QED in Ref. [25].

II. SUMMING LOGARITHMS IN THE EFFECTIVE POTENTIAL

We begin by considering an $O(N)$ scalar field ϕ with a classical potential $V_{\rm cl}$

$$V_{\rm cl} = \lambda\phi^4 = \pi^2 y\phi^4, \tag{1}$$

where λ is the usual scalar coupling constant but y is more useful as it removes explicit factors of π^2 in RG functions. The coupling y is renormalized so that the effective potential V satisfies the CW renormalization condition [7]

$$\frac{d^4V(\phi)}{d\phi^4}\bigg|_{\phi=\mu} = 24\pi^2 y \tag{2}$$

is satisfied. Radiative corrections to the effective potential [7,11,27,28] with this renormalization condition take the form

$$V(y,\phi,\mu) = \pi^2\sum_{n=0}^{\infty}\sum_{m=0}^{n} y^{n+1}T_{nm}L^m\phi^4, \tag{3}$$

where $L = \ln(\frac{\phi^2}{\mu^2})$. In order that there be no net dependence on the renormalization scale parameter μ, V must satisfy

$$\mu\frac{dV}{d\mu} = 0 = \left(\mu\frac{\partial}{\partial\mu} + \beta(y)\frac{\partial}{\partial y} - \phi\gamma(y)\frac{\partial}{\partial\phi}\right)V, \tag{4}$$

where

$$\mu\frac{dy}{d\mu} = \beta(y) = \sum_{n=2}^{\infty} b_n y^n \tag{5}$$

and

$$\frac{\mu}{\phi}\frac{d\phi}{d\mu} = -\gamma(y) = -\sum_{n=1}^{\infty} g_n y^n. \tag{6}$$

The RG Eq. (4) and its solution for V in Eq. (3) corresponds to the situation where there is no quadratic term for the scalar field, consistently maintaining the massless nature of the theory. In particular, extension to massive theories is achieved by including a mass term and anomalous mass dimension into the RG equation (4) [see Ref. [6] for an analysis of a single-component massive scalar theory]. It is therefore not necessary for us to impose the $V''(\phi = 0) = 0$ renormalization condition used by Coleman & Weinberg [7] to eliminate quadratic divergences. Furthermore, vacuum graphs do not generate divergences that are eliminated by renormalization of the cosmological term [6]. As outlined below, we also do not introduce quadratic counterterms into the phenomenological analysis of V.

If now the N^pLL contribution to V in Eq. (3) is defined to be $V_{N^pLL} = \pi^2 y^{p+1}S_p(yL)\phi^4$, where

$$S_n(yL) = \sum_{m=0}^{\infty} T_{n+m,m}(yL)^m \tag{7}$$

so that

$$V = \pi^2\sum_{n=0}^{\infty} y^{n+1}S_n(yL)\phi^4, \tag{8}$$

then Eq. (4) is satisfied at order y^{n+2} provided $S_n(\xi)$ satisfies

$$\left[(-2+b_2\xi)\frac{d}{d\xi} + b_2 - 4g_1\right]S_0 = 0 \tag{9}$$

and

$$\begin{aligned}&\left[(-2+b_2\xi)\frac{d}{d\xi} + (n+1)b_2 - 4g_1\right]S_n \\ &\quad + \sum_{m=0}^{n-1}\left[-2g_{n-m} + b_{n-m+2}\xi\frac{d}{d\xi}\right. \\ &\quad \left. + (m+1)b_{n-m+2} - 4g_{n-m+1}\right]S_m = 0\end{aligned} \tag{10}$$

with the boundary condition

$$S_n(0) = T_{n0}. \tag{11}$$

Thus V can be determined by solving the coupled Eqs. (9) and (10) provided the boundary values T_{n0} are known. These are fixed by the CW condition of Eq. (2); since $L = 0$ when $\phi = \mu$ Eqs. (2) and (8) together imply that

$$\begin{aligned}24y = \sum_{k=0}^{\infty} y^{k+1}[&16y^4S_k''''(0) + 80y^3S_k'''(0) \\ &+ 140y^2S_k''(0) + 100yS_k'(0) + S_k(0)].\end{aligned} \tag{12}$$

Since $g_1 = 0$, together (9) and (12) lead to

$$T_{00} = 1, \tag{13}$$

$$S_0(\xi) = \frac{1}{w}, \tag{14}$$

where $w = 1 - \frac{1}{2}b_2\xi$. Equation (12) then gives

$$T_{10} = -\tfrac{25}{12}b_2 \tag{15}$$

so that Eq. (10) can be solved when $n = 1$

$$\begin{aligned}S_1(\xi) &= \frac{4g_2}{b_2 w} - \frac{4g_2 + \frac{25}{12}b_2^2}{b_2 w^2} - \frac{b_3}{b_2 w^2}\ln|w| \\ &= \frac{1}{4w} + \left(\frac{1}{4}\ln|w| - \frac{51}{4}\right)\frac{1}{w^2}\end{aligned} \tag{16}$$

(for $N = 4$).

This process can be continued indefinitely; $S_p(\xi)$ can be determined in terms of $b_2 \ldots b_{p+2}$, $g_2 \ldots g_{p+1}$ where these RG function coefficients are those appropriate to the CW scheme.

If in addition to y there are other couplings $g_i (i = 1 \ldots N)$ (Yukawa, gauge, etc.) in the theory then the CW renormalization condition (2) must be supplemented by additional conditions. For example, in massless scalar electrodynamics in which a complex scalar ϕ is coupled to a $U(1)$ gauge field A_μ with coupling e, then the effective action takes the form [7]

$$\Gamma = \int d^4x\Big[-V(\phi) + \frac{1}{2}Z(\phi)|(\partial_\mu - ieA_\mu)\phi|^2 - \frac{1}{4}H(\phi)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \ldots\Big]. \tag{17}$$

Infinities arise when computing V, Z and H and so in addition to (2) one requires renormalization conditions which we take to be

$$H(\phi = \mu) = 1 = Z(\phi = \mu). \tag{18}$$

Application of the RG equation to determine higher-order corrections to $Z(\phi)$ is discussed in Ref. [29].

Suppose that x and y are the only two couplings. (It is easy to extend our considerations to include more than two.) The expansion of Eq. (3) now generalizes to

$$V = \pi^2 \sum_{n=1}^{\infty}\sum_{r=0}^{n+k}\sum_{k=0}^{\infty} T_{n+k-r,r,k}y^{n+k-r}x^r L^k \tag{19}$$

and V satisfies the RG equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta^x\frac{\partial}{\partial x} + \beta^y\frac{\partial}{\partial y} - \phi\gamma\frac{\partial}{\partial\phi}\Big)V = 0. \tag{20}$$

The RG functions are

$$\beta^x = \mu\frac{dx}{d\mu} = \sum_{n=2}^{\infty}\beta_n^x = \sum_{n=2}^{\infty}\sum_{r=0}^{n} b^x_{n-r,r}x^r y^{n-r}, \tag{21}$$

$$\beta^y = \mu\frac{dy}{d\mu} = \sum_{n=2}^{\infty}\beta_n^y = \sum_{n=2}^{\infty}\sum_{r=0}^{n} b^y_{n-r,r}x^r y^{n-r}, \tag{22}$$

$$\gamma = -\frac{\mu}{\phi}\frac{d\phi}{d\mu} = \sum_{n=1}^{\infty}\gamma_n = \sum_{n=1}^{\infty}\sum_{r=0}^{n} g_{n-r,r}x^r y^{n-r}. \tag{23}$$

The N^pLL contribution to V is now given by

$$V_{N^pLL} = \pi^2\sum_{k=0}^{\infty} p^k_{k+p+1}L^k\phi^4, \tag{24}$$

where

$$p_n^k(x,y) = \sum_{r=0}^{n} T_{n-r,r,k}y^{n-r}x^r \qquad (n \geq k+1) \tag{25}$$

so that

$$V = \sum_{p=0}^{\infty} V_{N^pLL}. \tag{26}$$

The CW condition of Eq. (2) now shows that for all n

$$24y\delta_{n0} = 24p_n^0 + 100p_n^1 + 280p_n^2 + 480p_n^3 + 384p_n^4. \tag{27}$$

Furthermore, the RG Eq. (20) leads to

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\Big[-2kp_n^kL^{k-1} + \sum_{m=2}^{\infty}\Big(\beta_m^x\frac{\partial}{\partial x} + \beta_m^y\frac{\partial}{\partial y}\Big)p_n^kL^k - \sum_{m=1}^{\infty}(4\gamma_m p_n^kL^k + 2k\gamma_m p_n^kL^{k-1})\Big] = 0. \tag{28}$$

Together, (27) and (28) fix V in terms of the CW RG functions.

We employ a novel way of treating the sums in Eq. (24), which involves using the method of characteristics [3]. Beginning with the definition

$$w_n^k(\bar{x}(t),\bar{y}(t),t) = p_n^k(\bar{x}(t),\bar{y}(t))\exp\Big[-4\int_0^t \gamma_1(\bar{x}(\tau),\bar{y}(\tau))d\tau\Big], \tag{29}$$

where

$$\frac{d\bar{x}(t)}{dt} = \beta_2^x(\bar{x}(t),\bar{y}(t)), \tag{30}$$

$$\frac{d\bar{y}(t)}{dt} = \beta_2^y(\bar{x}(t),\bar{y}(t)) \tag{31}$$

with $\bar{x}(0) = x$, $\bar{y}(0) = y$ we find that

$$\frac{d}{dt}w_n^k(\bar{x},\bar{y},t) = \Big(\beta_2^x(\bar{x},\bar{y})\frac{\partial}{\partial\bar{x}} + \beta_2^y(\bar{x},\bar{y})\frac{\partial}{\partial\bar{y}} - 4\gamma_1(\bar{x},\bar{y})\Big)w_n^k(\bar{x},\bar{y},t). \tag{32}$$

Equation (28) is satisfied to order $n-1$ in L and $n+1$ in the couplings x and y provided

$$p_{n+1}^n = \frac{1}{2n}\Big(\beta_2^x\frac{\partial}{\partial x} + \beta_2^y\frac{\partial}{\partial y} - 4\gamma_1\Big)p_n^{n-1} \tag{33}$$

so that by Eqs. (29), (32), and (33)

$$w_{n+1}^n(\bar{x},\bar{y},t) = \frac{1}{2n}\frac{d}{dt}w_n^{n-1}(\bar{x},\bar{y},t). \tag{34}$$

If now

$$\bar{V}_{N^pLL}(\bar{x},\bar{y},t) = \pi^2\sum_{k=0}^{\infty} w^k_{k+p+1}(\bar{x},\bar{y},t)L^k\phi^4 \tag{35}$$

so that if $t = 0$

$$\bar{V}_{N^pLL}(x,y,0) = V_{N^pLL}, \tag{36}$$

then by (29)

$$\bar{V}_{LL}(\bar{x}(t), \bar{y}(t), t) = \pi^2 \sum_{n=0}^{\infty} \frac{L^n}{2^n n!} \frac{d^n}{dt^n} w_1^0(\bar{x}(t), \bar{y}(t), t)\phi^4 = \pi^2 w_1^0\left(\bar{x}\left(t+\frac{L}{2}\right), \bar{y}\left(t+\frac{L}{2}\right), \frac{L}{2}\right)\phi^4 \tag{37}$$

and hence by (36) we finally have a closed form expression for V_{LL}.

$$V_{LL} = \pi^2 w_1^0\left(\bar{x}\left(\frac{L}{2}\right), \bar{y}\left(\frac{L}{2}\right), \frac{L}{2}\right)\phi^4. \tag{38}$$

The detailed computation of V_{NLL} presented in Appendix A gives Eq. (A27)

$$V_{NLL} = \pi^2\phi^4 \exp\left[-4\int_0^{L/2} d\tau \gamma_1(\bar{x}^i(\tau))\right]\left\{p_2^0\left(\bar{x}^i\left(\frac{L}{2}\right)\right) + \int_0^{L/2} d\tau[(-\gamma_1(\bar{x}^i(\tau))\beta_2^{x^i}(\bar{x}^i(\tau)) + \beta_3^{x^i}(\bar{x}^i(\tau)))\mathbf{U}_{ij}(0,\tau)] \times\left[\mathbf{U}_{jk}\left(\frac{L}{2},0\right)\frac{\partial}{\partial \bar{x}^k(\frac{L}{2})}p_1^0\left(\bar{x}^i\left(\frac{L}{2}\right)\right)\right] + 4\int_0^{L/2} d\tau[\gamma_1^2(\bar{x}^i(\tau)) - \gamma_2(\bar{x}^i(\tau))]p_1^0\left(\bar{x}^i\left(\frac{L}{2}\right)\right)\right\}, \tag{39}$$

where by Eqs. (27), (33), and (A1)

$$p_1^0 = y, \qquad p_2^1 = \frac{1}{2}\beta_2^y - 2\gamma_1 y, \qquad p_2^0 = -\frac{25}{6}p_2^1, \tag{40}$$

and by Eqs. (A14)–(A18)

$$\frac{d}{dt}\mathbf{U}(t,0) = \mathbf{U}(t,0)\mathbf{M}, \tag{41}$$

$$\mathbf{U}^{-1}(t,0) = \mathbf{U}(0,t) = 1 + \sum_{n=1}^{\infty}(-1)^n\int_0^t d\tau_1 \ldots \times \int_0^{\tau_{n-1}} d\tau_n[\mathbf{M}(\tau_1)\ldots\mathbf{M}(\tau_n)] \tag{42}$$

and

$$\mathbf{M}_{ij} = \frac{\partial \beta_2^{x^j}}{\partial \bar{x}^i}. \tag{43}$$

The techniques used to find V_{NLL} in Eq. (39) can be extended to obtain V_{N^2LL}. However, since the three loop RG functions needed for this extension have not been computed for the standard model, we will not pursue this calculation further.

We now will discuss how the CW RG functions can be found if the MS RG functions are known.

III. FINDING THE COLEMAN-WEINBERG RENORMALIZATION GROUP FUNCTIONS

The RG functions have been computed using dimensional regularization and minimal subtraction to five loop order in an $O(N)$ scalar theory [30] and to two loop order in the standard model [31]. We will now examine how from these known results one can find the RG functions in the CW renormalization scheme.

First, we quote the MS values of the $O(N)$ scalar model of Eq. (1) to five loop order [30]

$$\begin{aligned}\tilde{\beta}(y) = {} & \frac{N+8}{2}y^2 - \frac{3}{4}(3N+14)y^3 + \frac{1}{64}[33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)]y^4 \\ & - \frac{4}{3}\left(\frac{3}{2}\right)^5\frac{y^5}{7776}[-5N^3 + 6320N^2 + 80456N + 196648 + 96(63N^2 + 764N + 2332)\zeta(3). \\ & - 288(5N+22)(N+8)\zeta(4) + 1920(2N^2 + 55N + 186)\zeta(5)]] + \frac{4}{3}\left(\frac{3}{2}\right)^6\frac{y^6}{124416}[13N^4 + 12578N^3 \\ & + 808496N^2 + 6646336N + 13177344. + 16(-9N^4 + 1248N^3 + 67640N^2 + 552280N + 1314336)\zeta(3) \\ & + 768(-6N^3 - 59N^2 + 446N + 3264)\zeta^2(3) - 288(63N^3 + 1388N^2 + 9532N + 21120)\zeta(4) \\ & + 256(305N^3 + 7466N^2 + 66986N + 165084)\zeta(5). - 9600(N+8)(2N^2 + 55N + 186)\zeta(6) \\ & + 112896(14N^2 + 189N + 526)\zeta(7)] + \mathcal{O}(y^7)\end{aligned} \tag{44}$$

and

$$\begin{aligned}\tilde{\gamma}(y) = {} & \frac{N+2}{16}y^2 - \frac{(N+2)(N+8)}{128}y^3 + \left(\frac{3}{2}\right)^4\frac{y^4}{5184}(N+2)[5(-N^2 + 18N + 100)] \\ & - \left(\frac{3}{2}\right)^5\frac{y^5}{186624}(N+2)[39N^3 + 296N^2 + 22752N + 77056 \\ & - 48(N^3 - 6N^2 + 64N + 184)\zeta(3) + 1152(5N+22)\zeta(4)] + \mathcal{O}(y^6).\end{aligned} \tag{45}$$

We next provide the two loop RG functions in the standard model in which there is a single scalar doublet with no mass term for this field in the classical action. The quartic scalar coupling y appears in Eq. (1); the other couplings are the top quark Yukawa coupling

$$x = \frac{g_t^2}{4\pi^2} \tag{46}$$

the $SU(3)$ coupling

$$z = \frac{g_3^2}{4\pi^2} \tag{47}$$

and the $SU(2) \times U(1)$ couplings

$$r = \frac{g_2^2}{4\pi^2}, \tag{48}$$

$$s = \frac{g_1^2}{4\pi^2}. \tag{49}$$

To two loop order the RG functions in this simplest version of the standard model [31] in the MS renormalization scheme are

$$\begin{aligned} \tilde{\beta}^x &= \tilde{\mu}\frac{dx}{d\tilde{\mu}} \\ &= \left[\frac{9}{4}x^2 - 4xz - \frac{9}{8}xr - \frac{17}{24}xs\right] + \left[-\frac{3}{2}x^3 + \frac{131}{128}x^2 s + \frac{225}{128}x^2 r + \frac{9}{2}x^2 z - \frac{3}{2}x^2 y + \frac{1187}{1728}xs^2 \right. \\ &\quad \left. - \frac{3}{32}xrs + \frac{19}{72}xsz - \frac{23}{32}xr^2 + \frac{9}{8}xrz - \frac{27}{2}xz^2 + \frac{3}{4}xy^2\right] + \ldots, \end{aligned} \tag{50}$$

$$\begin{aligned} \tilde{\beta}^y &= \tilde{\mu}\frac{dy}{d\tilde{\mu}} \\ &= \left[6y^2 + 3xy - \frac{3}{2}x^2 - \frac{9}{4}yr - \frac{3}{4}ys + \frac{3}{32}s^2 + \frac{3}{16}rs + \frac{9}{32}r^2\right] + \left[-\frac{39}{2}y^3 - 9xy^2 + \frac{27}{4}y^2 r + \frac{9}{4}y^2 s - \frac{3}{16}x^2 y \right. \\ &\quad + 5xyz + \frac{45}{32}xyr + \frac{85}{96}xys - \frac{73}{128}yr^2 + \frac{39}{64}yrs + \frac{629}{384}ys^2 + \frac{15}{8}x^3 - 2x^2 z - \frac{1}{6}x^2 s - \frac{9}{64}xr^2 + \frac{21}{32}xrs \\ &\quad \left. - \frac{19}{64}xs^2 + \frac{305}{256}r^3 - \frac{289}{768}r^2 s - \frac{559}{768}rs^2 - \frac{379}{768}s^3\right] + \ldots, \end{aligned} \tag{51}$$

$$\tilde{\beta}^z = \tilde{\mu}\frac{dz}{d\tilde{\mu}} = \left[-\frac{7}{2}z^2\right] + \left[\frac{11}{48}sz^2 + \frac{9}{16}rz^2 - \frac{13}{4}z^3 - \frac{xz^2}{4}\right] + \ldots, \tag{52}$$

$$\tilde{\beta}^r = \tilde{\mu}\frac{dr}{d\tilde{\mu}} = \left[-\frac{19}{12}r^2\right] + \left[\frac{3}{16}r^2 s + \frac{35}{48}r^3 + \frac{3}{2}r^3 z - \frac{3}{16}xr^2\right] + \ldots, \tag{53}$$

$$\tilde{\beta}^s = \tilde{\mu}\frac{ds}{d\tilde{\mu}} = \left[\frac{41}{12}s^2\right] + \left[\frac{199}{144}s^3 + \frac{9}{16}rs^2 + \frac{11}{6}zs^2 - \frac{17}{48}xs^2\right] + \ldots, \tag{54}$$

and

$$\tilde{\gamma} = -\frac{\tilde{\mu}}{\phi}\frac{d\phi}{d\tilde{\mu}} = \left[\frac{3}{4}x - \frac{9}{16}r - \frac{3}{16}s\right] + \left[\frac{3}{8}y^2 - \frac{27}{64}x^2 + \frac{5}{4}xz + \frac{45}{128}xr + \frac{85}{384}xs - \frac{271}{512}r^2 + \frac{9}{256}rs + \frac{41}{1536}s^2\right] + \ldots \tag{55}$$

In the case of there being only an $O(N)$ scalar field ϕ, we follow the procedure outlined in Refs. [3,32] to convert from the RG functions of Eqs. (44) and (45) to those appropriate to the CW scheme. In the MS scheme, the computation results in an expansion of V that is similar to that of Eq. (3),

$$V = \pi^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n} y^{n+1}\tilde{T}_{nm}\tilde{L}^m \phi^4, \tag{56}$$

where now $\tilde{L} = \ln(\frac{y\phi^2}{\tilde{\mu}^2})$. If the RG scale $\tilde{\mu}$ in the MS scheme is rescaled

$$\tilde{\mu} = y^{1/2}\mu, \tag{57}$$

where μ is the RG scale in the CW scheme, then the form of the expansion of Eq. (56) becomes that of Eq. (3). Finite renormalizations of the form

$$y \to y(1 + a_1 y + a_2 y^2 + \ldots), \tag{58}$$

$$\phi \to \phi(1 + b_1 y + b_2 y^2 + \ldots) \tag{59}$$

may then be required to adjust the coefficients $\tilde{T}_{n0}$ in Eq. (56) so that the CW RG condition of Eq. (2) is satisfied, but this can be done without altering $\tilde{T}_{nm}(m > 0)$ and hence the terms in V that fix the RG functions are not changed [33].

With the rescaling of Eq. (57)

$$\beta(y) = \mu \frac{\partial y}{\partial \mu} = (\tilde{\mu} y^{-1/2}) \left(\frac{\partial (y^{1/2} \mu)}{\partial \mu} \right) \frac{\partial y}{\partial \tilde{\mu}} = \tilde{\beta}(y)/(1 - \tilde{\beta}(y)/(2y)) \tag{60}$$

and similarly

$$\gamma(y) = \tilde{\gamma}(y)/(1 - \tilde{\beta}(y)/(2y)). \tag{61}$$

Equations (60) and (61) allow one to pass from the MS RG functions of Eqs. (44) and (45) to the CW RG functions.

It is somewhat more complicated to convert the RG functions of Eqs. (45)–(50) to the CW scheme since more than one type of logarithm arises when V is computed using the MS renormalization scheme. A computation of V in the CW scheme would allow one to infer the CW RG functions, but to obtain in this way the RG functions to order n, one must compute V to order $(n + 1)$ [33]. Since V in the standard model has only been computed to second order [34] one cannot determine the CW RG functions to two loop order from V directly; other contributions to the effective action must be considered.

Suppose the couplings in a theory are g_i (with $g_i = (x, y, z, r, s)$ in the standard model) and that there is one scalar field ϕ. When computing V using MS, logarithms of the form $\tilde{L}_i = \ln(g_i \phi^2/\tilde{\mu}^2)$ arise. At one loop order in MS, only these types of logarithms occur; beyond one loop order other more complicated logarithms arise [34] but do not affect our discussion of how the MS and CW RG functions are related at two loop order. As in Refs. [3,5] we associate a separate renormalization scale κ_i with each of these logarithms so that now

$$\tilde{L}_i = \ln\left(\frac{g_i \phi^2}{\kappa_i^2}\right). \tag{62}$$

A rescaling similar to that of Eq. (57)

$$\kappa_i = g_i^{1/2} \mu \tag{63}$$

leads to

$$\beta^{g_i} = \mu \frac{\partial g_i}{\partial \mu} = \sum_j \tilde{\beta}_j^{g_i} \left(1 + \frac{\beta^{g_j}}{2g^j} \right), \tag{64}$$

$$\gamma = -\frac{\phi}{\mu} \frac{\partial \phi}{\partial \mu} = \sum_j \tilde{\gamma}_j \left(1 + \frac{\beta^{g_j}}{2g^j} \right), \tag{65}$$

where

$$\tilde{\beta}_j^{g_i} = \kappa_j \frac{\partial g_i}{\partial \kappa_j}, \tag{66}$$

$$\tilde{\gamma}_j = -\frac{\kappa_j}{\phi} \frac{\partial \phi}{\partial \kappa_j}. \tag{67}$$

Again, μ is the CW mass parameter. We also see that

$$\tilde{\beta}^{g_i} = \sum_j \tilde{\beta}_j^{g_i}, \tag{68}$$

$$\tilde{\gamma} = \sum_j \tilde{\gamma}_j, \tag{69}$$

where $\tilde{\beta}^{g_i}$ and $\tilde{\gamma}$ are the MS RG functions.

We now will use Eqs. (64) and (65) to find the CW RG functions to two loop order in the standard model, restricting ourselves to the limiting case in which only the three dominant couplings $g_1 = x$, $g_2 = y$ and $g_3 = z$ are considered. If we use Roman numeral subscripts with the RG functions to denote the number of coupling constants present in a perturbative expansion (e.g., $\tilde{\beta}^x_{1II}$ is the term in the expansion of the β function for x in the MS scheme associated with the mass scale κ_1 that has two powers of the coupling), then by Eqs. (64)–(67) we see that

$$\beta^{g_i}_{II} = \tilde{\beta}^{g_i}_{II}, \tag{70}$$

$$\gamma_I = \tilde{\gamma}_I; \tag{71}$$

that is at lowest order the RG functions in the CW and MS schemes are the same. It also follows that

$$\beta^{g_i}_{III} = \tilde{\beta}^{g_i}_{III} + \sum_j \frac{\tilde{\beta}^{g_i}_{jII} \beta^{g_j}_{II}}{2g_j}, \tag{72}$$

$$\gamma_{II} = \tilde{\gamma}_{II} + \sum_j \frac{\tilde{\gamma}_{jI} \beta^{g_j}_{II}}{2g_j}. \tag{73}$$

Equations (72) and (73) show that apart from standard RG functions, only the one loop multiscale RG quantities $\tilde{\beta}^{g_i}_{jII}$ and $\tilde{\gamma}_{jI}$ are needed to obtain the two loop CW RG functions $\beta^{g_i}_{III}$ and γ_{II}.

To find $\tilde{\gamma}_{jI}$ we note that the one loop scalar self energy in the standard model (with no classical mass term for the scalar and just the couplings x, y and z) only has a contribution coming from the top quark loop. Consequently the term $Z(\phi)(\partial_\mu \phi)^2$ in the effective action only receives a logarithmic contribution of the form $\ln(x\phi^2/\tilde{\mu}^2)$ and so we see that

$$\tilde{\gamma}_{1I} = \tilde{\gamma}_I, \tag{74}$$

$$\tilde{\gamma}_{2I} = \tilde{\gamma}_{3I} = 0. \tag{75}$$

To obtain $\tilde{\beta}^y_{1I}$, $\tilde{\beta}^y_{2II}$ and $\tilde{\beta}^y_{3II}$, we note that at leading-log one loop order in the model we are considering [31], V is given in the MS scheme by

$$V = \pi^2\left[y + \left(3y^2 \ln\frac{y\phi^2}{\tilde{\mu}^2} - \frac{3}{4}x^2 \ln\frac{x\phi^2}{\tilde{\mu}^2}\right)\right]\phi^4. \tag{76}$$

If the RG equation of Eq. (20) is to be satisfied for each of the three mass scales κ_j introduced in Eq. (62), we find that consistency with Eqs. (74) and (75) occurs if

$$\tilde{\beta}^{\,y}_{1II} = -\tfrac{3}{2}x^2 + 3xy, \tag{77}$$

$$\tilde{\beta}^{\,y}_{2II} = 6y^2, \tag{78}$$

and

$$\tilde{\beta}^{\,y}_{3II} = 0. \tag{79}$$

Determining $\tilde{\beta}^z_{1II}$, $\tilde{\beta}^z_{2II}$ and $\tilde{\beta}^z_{3II}$ is most easily done by considering the one loop contribution to the term $-\frac{1}{4}H(\phi)F^2$ in the effective action where $F^a_{\mu\nu}$ is the $SU(3)$ field strength. As only a quark loop can contribute at one loop order to $H(\phi)$, then the only logarithmic contribution to $H(\phi)$ at one loop order is $\ln(x\phi^2/\tilde{\mu}^2)$ in the MS scheme. However, $H(\phi)$ dictates the function $\tilde{\beta}^z$ on account of gauge invariance [35] and so

$$\tilde{\beta}^{\,z}_{1II} = \tilde{\beta}^z_{II} \tag{80}$$

and

$$\tilde{\beta}^{\,z}_{2II} = \tilde{\beta}^z_{3II} = 0. \tag{81}$$

For $\tilde{\beta}^x_{1II}$, $\tilde{\beta}^x_{2II}$ and $\tilde{\beta}^x_{3II}$ we note that the scalar-quark-quark vertex only receives a logarithmic contribution at one loop order of the form $\ln(x\phi^2/\tilde{\mu}^2)$ and hence

$$\tilde{\beta}^{\,x}_{1II} = \tilde{\beta}^x_{II}, \tag{82}$$

$$\tilde{\beta}^{\,x}_{2II} = \tilde{\beta}^x_{3II} = 0. \tag{83}$$

Together, Eqs. (74)–(83) result in Eqs. (72) and (73) yielding to two loop order in the CW scheme

$$\begin{aligned}\beta^x &= \left[\frac{9}{4}x^2 - 4xz\right] + \left[-\frac{3}{2}x^3 + \frac{9}{2}x^2z - \frac{3}{2}x^2y - \frac{27}{2}xz^2 + \frac{3}{4}xy^2\right] + \frac{1}{2x}\left[\frac{9}{4}x^2 - 4xz\right]^2 + \ldots \\ &= \left[\frac{9}{4}x^2 - 4xz\right] + \left[\frac{33}{32}x^3 - \frac{9}{2}x^2z - \frac{3}{2}x^2y + \frac{3}{4}xy^2 - \frac{11}{2}xz^2\right] + \ldots,\end{aligned} \tag{84}$$

$$\begin{aligned}\beta^y &= \left[6y^2 + 3xy - \frac{3}{2}x^2\right] + \left[-\frac{39}{2}y^3 - 9xy^2 - \frac{3}{16}x^2y + 5xyz + \frac{15}{8}x^3 - 2x^2z\right] + \frac{1}{2x}\left[-\frac{3}{2}x^2 + 3xy\right]\left[\frac{9}{4}x^2 - 4xz\right] \\ &\quad + \frac{1}{2y}[6y^2]\left[6y^2 + 3xy - \frac{3}{2}x^2\right] + \ldots \\ &= \left[6y^2 + 3xy - \frac{3}{2}x^2\right] + \left[-\frac{3}{2}y^3 + \frac{3}{16}x^3 + x^2z - xyz - \frac{21}{16}x^2y\right] + \ldots\end{aligned} \tag{85}$$

(which is the same result as is obtained from Eq. (60) if $x = z = 0$)

$$\beta^z = \left[-\frac{7}{2}z^2\right] + \left[-\frac{13}{4}z^3 - \frac{1}{4}xz^2\right] + \frac{1}{2x}\left[-\frac{7}{2}z^2\right]\left[\frac{9}{4}x^2 - 4xz\right] + \ldots = \left[-\frac{7}{2}z^2\right] + \left[\frac{15}{4}z^3 - \frac{67}{16}xz^2\right] + \ldots \tag{86}$$

and

$$\gamma = \left[\frac{3}{4}x\right] + \left[-\frac{27}{64}x^2 + \frac{3}{8}y^2 + \frac{5}{4}xz\right] + \frac{1}{2x}\left[\frac{3}{4}x\right]\left[\frac{9}{4}x^2 - 4xz\right] + \ldots = \left[\frac{3}{4}x\right] + \left[\frac{27}{64}x^2 + \frac{3}{8}y^2 - \frac{xz}{4}\right] + \ldots \tag{87}$$

(Exact solutions for the one loop characteristic functions $\bar{x}(t)$, $\bar{y}(t)$, $\bar{z}(t)$ appear in [36].)

With these CW RG functions we can compute V_{NLL} using Eq. (39) in the model we are considering.

IV. APPLICATION TO THE STANDARD MODEL

We now show how the results of the previous two sections can be applied to the standard model in order to estimate the mass of the Higgs Boson. We only consider the case in which there is a single Higgs doublet with no classical mass term.

As was pointed out in [1,2], there are three things to consider. First of all, we have the CW renormalization conditions of Eqs. (2) and (18). Next there is the stability condition

$$\frac{d}{d\phi}V(\phi = \mu) = 0. \tag{88}$$

This means that we identify μ with the vacuum expectation value of ϕ, that is $\mu = 2^{-1/4}G_F^{-1/2}$. Once these two requirements are satisfied, we can compute the Higgs mass by the formula

$$m_H^2 = \frac{d^2V(\phi = \mu)}{d\phi^2} / Z(\phi = \mu). \tag{89}$$

With the renormalization condition of Eq. (18) this just reduces to

$$m_H^2 = \frac{d^2V(\phi = \mu)}{d\phi^2}. \tag{90}$$

If V is expanded in the form

$$V = \sum_{p=0}^{\infty} V_{N^pLL}, \tag{91}$$

where V_{N^pLL} is the N^pLL contribution to V, then we begin by estimating V by

$$V_m = \sum_{p=0}^{m} V_{N^pLL} + \pi^2 K_m \phi^4. \tag{92}$$

The term $\pi^2 K_m \phi^4$ in Eq. (92) represents the parts of V coming from those terms in Eq. (91) beyond N^mLL which can be determined by imposing Eq. (2)—the renormalization condition. As is discussed in Sec. II above, V_{N^pLL} can be determined in terms of the CW RG functions if they are known to $p+1$ loop order. From Sec. II then, V_{LL} can be found using all five couplings (x, y, z, r, s), V_{NLL} can be found using the three couplings (x, y, z) and finally V_{N^2LL}, V_{N^3LL} and V_{N^4LL} can be found using the single coupling y.

The role of K_m in Eq. (92) is to ensure that the CW renormalization condition of Eq. (2) is satisfied. It is a "counterterm"; more explicitly in terms of the quantities p_n^k introduced in Eq. (25) [or the generalization of this expression to accommodate more than two couplings]

$$K_m = \sum_{n=m+2}^{\infty} p_n^0. \tag{93}$$

Equations (12) and (27) on their own only ensure that Eq. (2) is satisfied up to a finite order m in the coupling constant expansion; the inclusion of the counterterm ensures that Eq. (2) is satisfied to all orders. Once expressions for $V_{LL} \ldots V_{N^mLL}$ have been given in terms of the appropriate CW RG functions, there are still two unknowns: the counterterm K_m and the quartic scalar coupling y. These two are fixed by conditions (2) and (88), then V_m is used in conjunction with Eq. (90) to estimate m_H^2.

More explicitly, V_{LL} is given by Eq. (38) with Eq. (29) leading to

$$V_{LL} = \pi^2 p_1^0\left(\bar{x}\left(\frac{L}{2}\right), \bar{y}\left(\frac{L}{2}\right), \bar{z}\left(\frac{L}{2}\right), \bar{r}\left(\frac{L}{2}\right), \bar{s}\left(\frac{L}{2}\right)\right) \times \exp\left[-4\int_0^{L/2} d\tau \gamma_1(\bar{x}(\tau), \ldots, \bar{s}(\tau))\right]\phi^4. \tag{94}$$

We see by Eq. (27), $p_1^0 = y$ and by Eqs. (55) and (71), $\gamma_1 = \frac{3}{4}x - \frac{9}{16}r - \frac{3}{16}s$.

When one computes derivatives of the characteristic functions $\bar{x}(t) \ldots \bar{s}(t)$ when evaluating V'_{LL}, V''_{LL} and V''''_{LL} as required by Eqs. (2), (8), and (89) and Eqs. (50)–(54), the one loop contributions to $\tilde{\beta}^x \ldots \tilde{\beta}^s$ in Eqs. (84)–(86) are to be used since at one loop order the CW and MS RG functions are the same.

For V_{NLL} we need RG functions in the CW renormalization scheme to two loop order. These are given by Eqs. (84)–(86) for the limiting case in which the standard model with only the three couplings (x, y, z) is being considered. These are used in conjunction with V_{NLL} in Eq. (39). In this equation, we have

$$w_1^0\left(\bar{x}\left(\frac{L}{2}\right), \bar{y}\left(\frac{L}{2}\right), \bar{z}\left(\frac{L}{2}\right)\right) = \bar{y}\left(\frac{L}{2}\right)\exp\left[-4\int_0^{L/2} d\tau\left(\frac{3}{4}\bar{x}(\tau)\right)\right] \tag{95}$$

and since by Eqs. (40) and (84)–(87)

$$p_1^0 = y, \qquad p_2^1 = 3y^2 - \tfrac{3}{4}x^2, \qquad p_2^0 = -\tfrac{25}{6}p_2^1, \tag{96}$$

we also have

$$w_2^0 = \left[-\frac{25}{2}\bar{y}^2\left(\frac{L}{2}\right) + \frac{25}{8}\bar{x}^2\left(\frac{L}{2}\right)\right]\exp\left[-4\int_0^{L/2} d\tau\left(\frac{3}{4}\bar{x}(\tau)\right)\right]. \tag{97}$$

For consistency, the derivatives of $\bar{x}(t), \bar{y}(t), \bar{z}(t)$ that arise when computing V'_{NLL}, V''_{NLL} and V''''_{NLL} are given by the one loop contributions to β^x, β^y, β^z occurring in Eqs. (84)–(86).

Finally, for V_{N^2LL}, V_{N^3LL} and V_{N^4LL} we have at our disposal only the CW RG functions associated with the single scalar coupling y. These RG functions are found by combining Eqs. (44), (45), (60), and (61). Using them, the functions $S_2 \ldots S_4$ appearing in Eq. (8) are given by

$$S_2(\xi) = \frac{1}{4w} + \left(-\frac{175}{16} + \frac{1}{16}\ln|w| - \frac{21}{2}\zeta(3)\right)\frac{1}{w^2} + \left(\frac{1}{16}\ln^2|w| - \frac{103}{16}\ln|w| + \frac{3591}{16} + \frac{21}{2}\zeta(3)\right)\frac{1}{w^3}, \tag{98}$$

$$S_3(\xi) = \left(-\frac{7}{8}\zeta(3) - \frac{1}{96}\right)\frac{1}{w} + \left(-\frac{7\pi^4}{40} + \frac{1}{16}\ln|w| + \frac{365}{4}\zeta(5) + \frac{1205}{64} + \frac{239}{8}\zeta(3)\right)\frac{1}{w^2} + \left(\frac{16\,363}{64} + \frac{\ln^2|w|}{64} - \frac{21}{4}\zeta(3)\ln|w| + 273\zeta(3) - \frac{351}{64}\ln|w|\right)\frac{1}{w^3} + \left(\frac{7\pi^4}{40} + \frac{1}{64}\ln^3|w| - \frac{239\,263}{48} - \frac{1733}{4}\zeta(3) + \frac{2719}{16}\ln|w| - \frac{311}{128}\ln^2|w| + \frac{63}{8}\zeta(3)\ln|w| - \frac{365}{4}\zeta(5)\right)\frac{1}{w^4}, \tag{99}$$

and

$$S_4(\xi) = \left(-\frac{7\pi^4}{160} + \frac{45}{8}\zeta(3) - \frac{713}{768} + \frac{365}{32}\zeta(5)\right)\frac{1}{w} + \left[\frac{365\pi^6}{1008} - \frac{\ln|w|}{384} - \frac{3449}{6}\zeta(5) - \frac{4421}{24}\zeta(3) + \frac{139}{8}\zeta^2(3) - \frac{36\,897}{32}\zeta(7) - \frac{7}{32}\zeta(3)\ln|w| - \frac{5347}{48} + \frac{337\pi^4}{320}\right]\frac{1}{w^2} + \left[-\frac{19\,325}{32}\zeta(3) - \frac{37\,595}{16}\zeta(5) - \frac{115\,387}{256} + \frac{365}{8}\ln|w|\zeta(5) + \frac{441}{4}\zeta^2(3) + \frac{1203}{128}\ln|w| + \frac{1}{64}\ln^2|w| + \frac{239}{16}\ln|w|\zeta(3) + \frac{721\pi^4}{160} - \frac{7\pi^4}{80}\ln|w|\right]\frac{1}{w^3} + \left[-\frac{63}{32}\ln^2|w|\zeta(3) - \frac{1\,250\,731}{192} + \frac{1545}{8}\ln|w| + \frac{1}{256}\ln^3|w| - \frac{1323}{4}\zeta^2(3) + \frac{3297}{16}\ln|w|\zeta(3) - \frac{1055}{512}\ln^2|w| - \frac{365}{16}\zeta(5) - \frac{119\,837}{16}\zeta(3) + \frac{7\pi^4}{160}\right]\frac{1}{w^4} + \left[-\frac{365\pi^6}{1008} - \frac{3179\pi^4}{320} + \frac{7\pi^4}{40}\ln|w| + \frac{51\,712\,991}{384} + \frac{1625}{8}\zeta^2(3) + \frac{1}{256}\ln^4|w| - \frac{13\,927}{32}\ln|w|\zeta(3) + \frac{36\,897}{32}\zeta(7) + \frac{1\,505\,921}{96}\zeta(3) - \frac{965\,209}{192}\ln|w| + \frac{500\,849}{96}\zeta(5) - \frac{625}{768}\ln^3|w| + \frac{63}{16}\ln^2|w|\zeta(3) + \frac{43\,815}{512}\ln^2|w| - \frac{365}{4}\ln|w|\zeta(5)\right]\frac{1}{w^5}. \tag{100}$$

With one coupling, we have $V_{N^pLL} = \pi^2 y^{p+1} S_p(yL)\phi^4$ for $p = 2, 3, 4$.

It is now possible to implement our program for determining the mass of the Higgs. This requires knowledge of x, z, r and s at the mass scale v. The couplings x, z, r and s are defined in terms of the Yukawa and gauge couplings g_t, g_3, g_2 and g_1 by Eqs. (46)–(49). These in turn are related to the measured quantities m_t (the top quark mass), θ_w (the weak angle), M_W (the W- Boson mass), α_s (the strong structure constant) and α (the fine structure constant), all of which are known at the mass scale set by the Z Boson. These relations are

$$x_0 = \frac{\alpha}{2\pi}\left(\frac{m_t}{M_W \sin\theta_w}\right)^2, \tag{101}$$

$$z_0 = \frac{\alpha_s}{\pi}, \tag{102}$$

$$r_0 = \frac{\alpha}{\pi \sin^2\theta_w}, \tag{103}$$

$$s_0 = \frac{\alpha}{\pi \cos^2\theta_w}, \tag{104}$$

where the subscript 0 means that these are evaluated at the mass of the Z Boson. From the Particle Data Group [37], at the mass of the Z Boson (91.1876 GeV/c^2), $\alpha = 1/128.91$, $\alpha_s = .1176$, $\sin^2\theta_w = .23\,119$, $M_w = 80.398$ GeV/c^2 and $m_t = 171.3$ GeV/c^2. It is now necessary to evaluate these couplings at the vacuum expectation value $v = 2^{-1/4}G_F^{-1/2}$ (taking G_F to be 1.16637×10^{-5} (GeV/c^2)$^{-2}$). To do this, we use the one loop limit of the RG equations that follow from Eqs. (50)–(54) as a suitable approximation

$$\mu\frac{dx}{d\mu} = \frac{9}{4}x^2 - 4xz, \tag{105}$$

$$\mu\frac{dz}{d\mu} = -\frac{7}{2}z^2, \tag{106}$$

$$\mu\frac{dr}{d\mu} = -\frac{19}{12}r^2, \tag{107}$$

$$\mu\frac{ds}{d\mu} = \frac{41}{12}s^2. \tag{108}$$

Equations (106)–(108) have solutions [36]

$$z = \frac{z_0}{1 + \frac{7}{2}z_0\ln(\frac{\mu}{\mu_0})}, \tag{109}$$

$$r = \frac{r_0}{1 + \frac{19}{12}r_0\ln(\frac{\mu}{\mu_0})}, \tag{110}$$

$$s = \frac{s_0}{1 - \frac{41}{12} s_0 \ln(\frac{\mu}{\mu_0})}. \tag{111}$$

Dividing Eq. (105) by Eq. (106) leads to the homogeneous equation

$$\frac{dx}{dz} = -\frac{9}{14}\left(\frac{x}{z}\right)^2 + \frac{8}{4}\left(\frac{x}{z}\right), \tag{112}$$

whose solution is

$$x = \frac{(2/9)z}{1 - [(1 - 2/9(z_0/x_0)](z/z_0)^{-1/7}}. \tag{113}$$

Using $(x_0, z_0, r_0, s_0,)$ given by Eqs. (101)–(104) at the mass scale $\mu_0 = 91.1876\ \mathrm{GeV}/c^2$ then Eqs. (109)–(111) and (113) yield (x, z, r, s) at the mass scale $\mu = v = 2^{-1/4} G_F^{-1/2}$.

We can now proceed to compute the Higgs mass at each order of the expansion of V in the $N^p LL$ expanion. With Eq. (92) for V_m, we use Eq. (2) to fix K_m in terms of y and then use Eq. (88) to solve for y itself. In this paper the only acceptable values for y are positive in order to ensure physical stability of the theory for reasonable values of ϕ^2, as will be discussed below. With these values of y (and K_m) Eq. (92) can be used give an explicit expression for V_m. Equation (90) can then be used to evaluate m_H^2. Only real and positive values of m_H^2 are acceptable. We note that it is not necessary to find explicit results for the integrals and running couplings appearing in Eqs. (94), (95), and (97). The derivatives of these expressions at $\phi = v$ that are needed to evaluate the Higgs mass are determined completely in terms of the RG functions and boundary values at $\phi = v$. Thus our methodology can be applied to very complicated models and is an important tool in its own right.

We present, in Table I, the values of K_m, $\lambda = \pi^2 y$, m_H for $m = 0, 1, 2, 3, 4$ when (x, y, z, r, s) contribute at LL order, (x, y, z) contribute at NLL order and only y contributes beyond that. (The units for m_H are GeV/c^2.) It is important to emphasize that the values for K_m listed in Table I arise because of the functional dependence of K_m on the coupling y; first K_m is expressed in terms of y by using Eq. (2) and then y is fixed by Eq. (88).

No entry occurs for $m = 1$ or $m = 3$ as the values of y that follow from V_1 and V_3 are negative and unacceptable. This appears to be due to the large negative contribution to S_1 and S_3 coming from terms of order $\frac{1}{w^2}$ and $\frac{1}{w^4}$ respectively.

The second derivative of the order m estimate for the effective potential, normalized to the scale v^2,

$$M_m = \frac{1}{v^2} \frac{d^2}{d\phi^2} V_m|_{\phi=v} \tag{114}$$

can be viewed as a function of the scalar field coupling λ once the counterterm K_m has been expressed in terms of λ. In Fig. 1 we present curves for the dimensionless quantity $M_m (m = 0, 2, 4)$ for positive values of λ while M_m is positive. The crosses on the curves correspond to the values of λ and m_H found by our approach and listed in Table 1 for $m = (0, 2, 4)$. Table I and Fig. 1 suggest a tendency for both λ and m_H to decrease with increasing order m. We can gain further insight on this trend in the $O(4)$ scalar theory by extracting the counterterm from the second derivative, normalized to the scale v^2,

TABLE I. Calculated results for the standard model to three significant digits.

m	K_m	λ	m_H
0	−.0586	.536	219
1			
2	−.0431	.439	188
3			
4	−.0346	.363	163

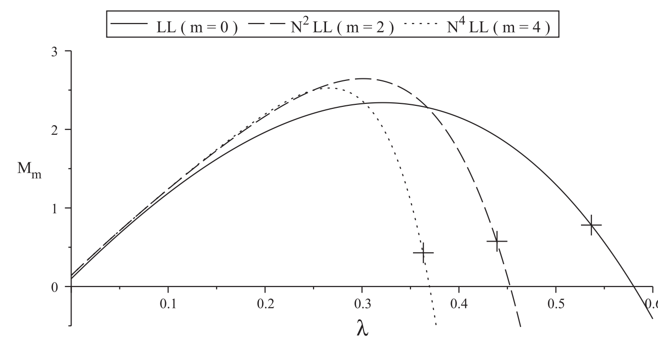

FIG. 1. The dimensionless ratio $M_m = \frac{1}{v^2} \frac{d^2}{d\phi^2} V_m|_{\phi=v}$ plotted as a function of λ.

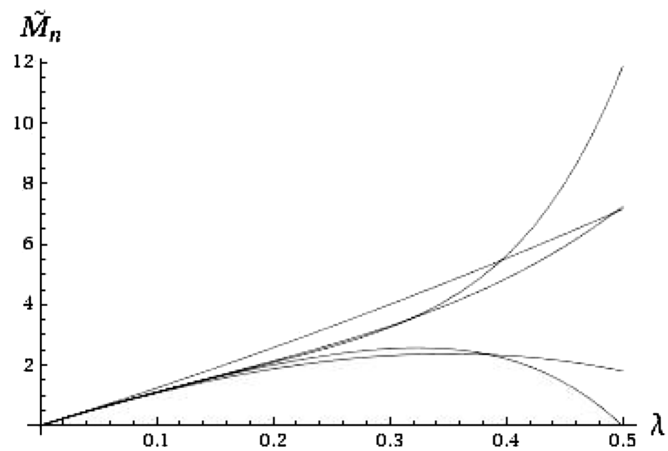

FIG. 2. The dimensionless quantity $\tilde{M}_n = \frac{1}{v^2} \frac{d^2(V_n - \pi^2 K_n \phi^4)}{d\phi^2}\Big|_{\phi=v}$ is plotted as a function of λ for the $O(4)$ scalar theory. The upper curves represent the even orders (n = 0, 2, 4) and the lower curves represent the odd orders (n = 1, 3).

$$\tilde{M}_n = \frac{1}{v^2} \frac{d^2(V_n - \pi^2 K_n \phi^4)}{d\phi^2}\bigg|_{\phi=v}. \qquad (115)$$

For the pure scalar field theory case the resulting dimensionless expressions are shown as a function of λ in Fig. 2. One can see the distinction between even and odd orders in the Figure, and one can also see evidence of slow convergence towards a result which would lie between the even and odd envelopes of the curves. Because $\tilde{M}_n$ represents the field-theoretical (i.e., counterterm-independent) contributions to the Higgs mass, it is evident that even orders provide an upper bound on m_H and odd orders provide a lower bound on m_H. Although the lower bound is trivial (i.e., $m_H = 0$), this does not obviate the interpretation of m_H at odd orders as an upper bound.

V. DISCUSSION

In this paper we have presented a systematic way of using the RG equation to sum all of the logarithms contributing to V at order N^pLL in terms of the $(p+1)$ order RG functions, provided we use the CW renormalization scheme and have only one form of logarithm (here $L = \log[\phi^2/\mu^2]$) contributing to V. We have applied our method of analysis to the conformal limit of the standard model with a single scalar field, as was originally envisaged by Coleman and Weinberg [7]. This has led to a surprisingly interesting sequence of estimates for the Higgs mass and the quartic scalar couplings.

It was not anticipated that the improvements to the approach, originally used in [1,2], introduced in this paper and [3] would lead to a sequence of decreasing estimates for the Higgs mass as listed in Table I above. The values of these estimates suggest that increasing the order m to 6 and beyond (if that were feasible) would lead to Higgs mass estimates closer to the generally expected range of possible values. A compilation of predictions of the Higgs mass in different scenarios is given in Ref. [38], and a discussion on its limits is given in Ref. [39]. In our approach we have made use of all known RG functions relevant to any part of the standard model. To make further progress using this approach will require knowledge of RG functions at a higher loop order than is currently available.

Even though we have not come up with a definitive prediction of the Higgs mass within the standard model, we feel that our results establish the viability of the Coleman-Weinberg mechanism to generate spontaneous symmetry breaking and to provide a mass for the Higgs scalar particle. We have done this by the use of the RG-improved effective potential. We propose that the masses generated in Sec. IV above be viewed as a decreasing sequence of upper bounds on the actual Higgs mass in the standard model.

A significant insight into the standard model effective potential can be gained by applying our method of analysis to a simplified pure $O(4)$ scalar field theory obtained

TABLE II. Calculated results for the $O(4)$ scalar theory to three significant digits.

m	K_m	λ	m_H
0	$-.0585$	.534	221
1	0	0	0
2	$-.0390$	.417	186
3	0	0	0
4	$-.0321$	.354	165

from the standard model by setting all couplings except $\lambda = \pi^2 y$ to zero. We present in Table II the results for K_m, λ and m_H in this simplified model using exactly the same steps as were used to derive the results in Table I for the standard model.

The similarity between the results of Tables I and II indicates that y is the dominant coupling in these considerations, much more than x, z, r or s. We note the vanishing values for K_m, λ and m_H in Table II for $m = 1, 3$. For this simplified model our method yields the acceptable but trivial solution $\lambda = 0$ for all values of m. In Table II we only include the nontrivial solutions for $m = 0, 2, 4$. For these nontrivial solutions we can plot V_m as a function of ϕ for values of ϕ near the VeV scale v, something which cannot be easily done in the standard model. This plot is provided in Fig. 3.

Remarkably, the plots of V_0, V_2 and V_4 have the well known shape of a spontaneous symmetry breaking potential when restricted to ϕ values near the location of the minimum. These potentials also have a singularity at $\phi = \pm v \exp(\pi^2/6\lambda)$ (i.e. when $w = 0$). This is significantly far from the region near the minimum.

In addition to the positive and zero λ-solutions in the pure $O(4)$ scalar field model referred to above, there are negative λ solutions. We have heretofore rejected negative λ solutions as unacceptable. In contrast to the standard model, in the $O(4)$ model we can plot V_m as a function of ϕ with these negative values of λ. We show the shape of $V_m(\phi)$ for the appropriate negative λ values for $m = 0, 2, 4$ in Fig. 4 and for $m = 1, 3$ in Fig. 5.

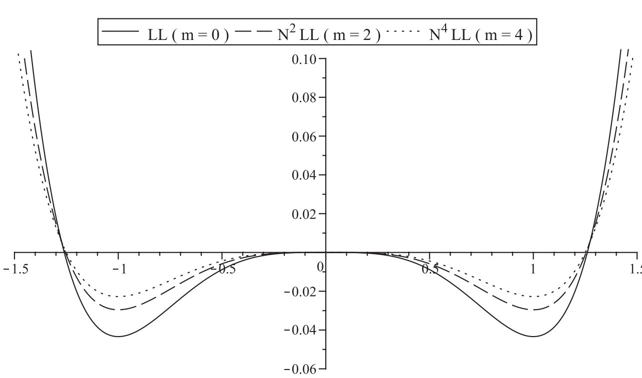

FIG. 3. V_m is plotted as a function of ϕ/v with λ as in Table II.

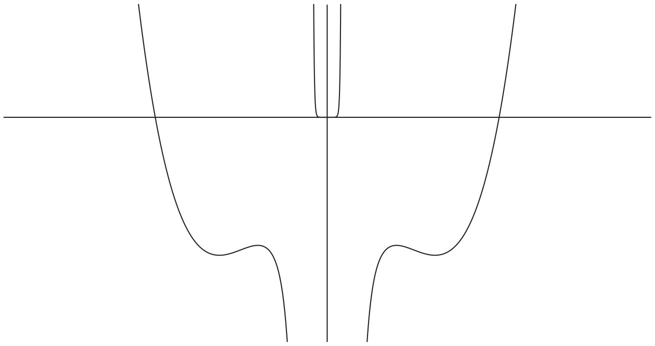

FIG. 4. Shape of V_m as a function of ϕ for $m = 0, 2, 4$.

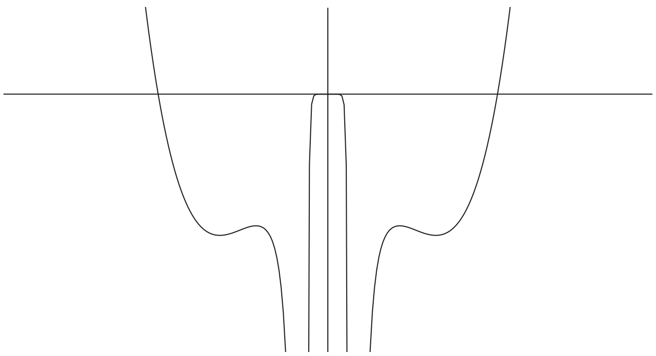

FIG. 5. Shape of V_m as a function of ϕ for $m = 1, 3$.

For the even m cases ($m = 0, 2, 4$) we note the existence of a tightly bound minimum at $\phi = 0$, singularities at $|\phi| < v$ (since $\lambda < 0$) and local minima at $\phi = \pm v$. On the other hand, for the odd m cases ($m = 1, 3$) we note the existence of a highly unstable maximum at $\phi = 0$, singularities at $|\phi| < v$ (since $\lambda < 0$) and local minima at $\phi = \pm v$. The occurrence of a singularity at $w = 0$ in V_m may be considered pathological but away from the singular points the form of V_m is interesting. Whether this feature has a role to play in the standard model is an open question which may be worth pursuing. It has been shown [3] in the scalar model that summing portions of the contributions to V_m beyond order $m = 4$ may shift such singularities.

We have attempted setting $K_m = 0$ in Eq. (92), and then determining the single remaining unknown y by using either Eq. (2) or Eq. (88). Neither of these attempts leads to acceptable values of y or m_H^2; one must employ the counterterm K_m in Eq. (92) to get reasonable values for these parameters at any value of m. In fact, by having introduced the counterterm, we are availing ourselves of information about terms, independent of $L = \log\frac{\phi^2}{\mu^2}$, beyond the N^pLL contribution to V. We have been unable to establish any other viable alternative to the counterterm approach.

Whereas in this paper we have used the CW renormalization scheme, preliminary investigations indicate that it may be possible to adapt our approach to incorporate the MS renormalization scheme, at least in the single coupling $O(4)$ scalar model. Using the MS renormalization scheme to compute the LL and NLL contributions to V when there is only the coupling y, realistic values of m_H^2 and y follow from Eqs. (88) and (90) only if the counterterm K_m of Eq. (92) is included and the condition of Eq. (2) is applied. Strictly speaking, Eq. (2) is not part of the MS renormalization scheme, though it might possibly be used to fix the physical value of y in the MS scheme in a way analogous to using the gap equation to fix a physical mass.

We hope to develop this formalism in several other ways. First, inclusion of a mass term $-m^2\phi^2$ into the classical action should be considered [40]. Next, the inclusion of more scalars beyond an $SU(2)$ doublet should be dealt with, as additional scalars are necessary [41] in any supersymmetric extension of the standard model. A further problem to be addressed concerns working with summing logarithmic contributions to V in the standard model using MS RG functions rather than converting them to the CW scheme, even though this would entail having a separate logarithm for each coupling [see Eq. (62)] and not being able to fix the terms p^0_{p+1} in Eq. (24) by using some analogue of Eq. (27). We would also like to see if the RG methods that have been developed could be employed in the consideration of other physical processes [42], or the contributions to the effective action arising due to an external magnetic field [43].

ACKNOWLEDGMENTS

This work was largely inspired by the late V. Elias. R. Macleod had a useful suggestion. There was helpful correspondence with C. Ford and S. Martin. NSERC (Natural Science & Engineering Research Council of Canada) provided funding for R. B. M. and T. G. S.

APPENDIX A: METHOD OF CHARACTERISTICS SOLUTION AT NLL AND N^2LL ORDER

The computation of V_{NLL} begins by noting that by Eq. (28)

$$p^n_{n+2} + \gamma_1 p^n_{n+1} = \frac{1}{2n}\left[\left(\beta^x_2\frac{\partial}{\partial x} + \beta^y_2\frac{\partial}{\partial y} - 4\gamma_1\right)p^{n-1}_{n+1} + \left(\beta^x_3\frac{\partial}{\partial x} + \beta^y_3\frac{\partial}{\partial y} - 4\gamma_2\right)p^{n-1}_n\right], \quad \text{(A1)}$$

so that together Eqs. (29), (32), (34), and (A1) imply that

$$w^n_{n+2} = \frac{1}{2n}\left[\frac{d}{dt}w^{n-1}_{n+1} + D(t)w^{n-1}_n\right], \quad \text{(A2)}$$

where

$$D(t) = -\gamma_1\left(\beta^x_2\frac{\partial}{\partial\bar{x}} + \beta^y_2\frac{\partial}{\partial\bar{y}} - 4\gamma_1\right) + \left(\beta^x_3\frac{\partial}{\partial\bar{x}} + \beta^y_3\frac{\partial}{\partial\bar{y}} - 4\gamma_2\right). \quad \text{(A3)}$$

Iterating Eq. (A2) shows that

$$w_{n+2}^n = \frac{1}{2^n n!}\left[\frac{d^n}{dt^n}w_2^0 + \left(\frac{d^{n-1}}{dt^{n-1}}D(t) + \frac{d^{n-2}}{dt^{n-2}}D(t)\frac{d}{dt} + \ldots + D(t)\frac{d^{n-1}}{dt^{n-1}}\right)w_1^0\right]. \tag{A4}$$

One can inductively prove the identity

$$\left(\frac{d^{n-1}}{dt^{n-1}}f + \frac{d^{n-2}}{dt^{n-2}}f\frac{d}{dt} + \ldots + \frac{d}{dt}f\frac{d^{n-2}}{dt^{n-2}} + f\frac{d^{n-1}}{dt^{n-1}}\right)g = \frac{d^n}{dt^n}(\phi g) - \phi\frac{d^n}{dt^n}g \qquad \left(\frac{d\phi}{dt} \equiv f\right). \tag{A5}$$

To employ Eq. (A5) to simplify Eq. (A4) we need to commute the functional derivatives appearing in $D(t)$ [see Eq. (A3)] through $\frac{d}{dt}$ so that they act on g before $\frac{d}{dt}$ does. (This step was not considered properly in Eq. (B22) of Ref. [3].) In order to do this, we first write $D(t)$ in Eq. (A3) in the form

$$D(t) = A^i\frac{\partial}{\partial \bar{x}^i(t)} + B, \tag{A6}$$

where

$$\bar{x}^1(t) \equiv \bar{x}(t), \tag{A7}$$

$$\bar{x}^2(t) \equiv \bar{y}(t), \tag{A8}$$

$$A^1(\bar{x}^i(t)) \equiv -\gamma_1\beta_2^x + \beta_3^x, \tag{A9}$$

$$A^2(\bar{x}^i(t)) \equiv -\gamma_1\beta_2^y + \beta_3^y, \tag{A10}$$

and

$$B(\bar{x}^i(t)) \equiv 4(\gamma_1^2 - \gamma_2). \tag{A11}$$

Furthermore, using Eqs. (30) and (31),

$$\frac{d}{dt} = \beta_2^x(\bar{x}(t),\bar{y}(t))\frac{\partial}{\partial\bar{x}(t)} + \beta_2^y(\bar{x}(t),\bar{y}(t))\frac{\partial}{\partial\bar{y}(t)} + \frac{\partial}{\partial t} \equiv \Lambda^i\frac{\partial}{\partial\bar{x}^i} + \frac{\partial}{\partial t}. \tag{A12}$$

We now note that

$$A^i\frac{\partial}{\partial\bar{x}^i}\frac{df}{dt} = A^i\frac{\partial}{\partial\bar{x}^i}\left(\Lambda^j\frac{\partial}{\partial\bar{x}^j} + \frac{\partial}{\partial t}\right)f = A^i\left[\left(\Lambda^j\frac{\partial}{\partial\bar{x}^j} + \frac{\partial}{\partial t}\right)\frac{\partial f}{\partial\bar{x}^i} + \frac{\partial\Lambda^j}{\partial\bar{x}^i}\frac{\partial f}{\partial\bar{x}^j}\right] = A^i\left[\frac{d}{dt}\delta_{ij} + (\mathbf{M})_{ij}\right]\frac{\partial f}{\partial\bar{x}^j}, \tag{A13}$$

where

$$(\mathbf{M})_{ij} = \frac{\partial\Lambda_j}{\partial\bar{x}^i}, \tag{A14}$$

and so by iterating we obtain

$$A^i\frac{\partial}{\partial\bar{x}^i}\left(\frac{d}{dt}\right)^p f = A^i\left[\left(\frac{d}{dt} + \mathbf{M}\right)^p\right]_{ij}\frac{\partial f}{\partial\bar{x}^j}. \tag{A15}$$

If we now define

$$(\mathbf{U}(t,0))_{ij} = \delta_{ij} + \sum_{n=1}^{\infty}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{n-1}}d\tau_n[\mathbf{M}(\tau_n)\mathbf{M}(\tau_{n-1})\ldots\mathbf{M}(\tau_2)\mathbf{M}(\tau_1)]_{ij}, \tag{A16}$$

then it is evident that

$$\frac{d}{dt}(\mathbf{U}(t,0)f) = \mathbf{U}(t,0)\left(\frac{d}{dt} + \mathbf{M}\right)f \tag{A17}$$

and that

$$\mathbf{U}^{-1}(t,0) = \mathbf{U}(0,t) = 1 + \sum_{n=1}^{\infty}(-1)^n\int_0^t d\tau_1\ldots\times\int_0^{\tau_{n-1}}d\tau_n[\mathbf{M}(\tau_1)\ldots\mathbf{M}(\tau_n)]. \tag{A18}$$

(An operator analogous to $\mathbf{U}$ arises in standard perturbation theory.) Together, Eqs. (A14)–(A18) show that

$$A^i\frac{\partial}{\partial\bar{x}^i}\left(\frac{d}{dt}\right)^p f = A^i\left[\mathbf{U}(0,t)\left(\frac{d}{dt}\right)^p\mathbf{U}(t,0)\right]_{ij}\frac{\partial}{\partial\bar{x}_j}f. \tag{A19}$$

We now find that by Eqs. (A5), (A6), and (A19),

$$\left(\frac{d^{n-1}}{dt^{n-1}}D(t) + \frac{d^{n-2}}{dt^{n-2}}D(t)\frac{d}{dt} + \ldots + \frac{d}{dt}D(t)\frac{d^{n-2}}{dt^{n-2}} + D(t)\frac{d^{n-1}}{dt^{n-1}}\right)w_1^0(\bar{x}^i(t),t) = \frac{d^n}{dt^n}(\tilde{Z}_j(t)\zeta_{1j}^0(\bar{x}^i(t),t)) - \tilde{Z}_j(t)\frac{d^n}{dt^n}\zeta_{1j}^0(\bar{x}^i(z),t) + \frac{d^n}{dt^n}(\tilde{B}(t)w_1^0(\bar{x}^i(t),t)) - \tilde{B}(t)\frac{d^n}{dt^n}w_1^0(\bar{x}^i(t),t), \tag{A20}$$

where

$$\tilde{Z}_j(t) \equiv \left(\int_0^t d\tau A^i(\bar{x}^i(\tau)) \mathbf{U}_{ij}(0, \tau) \right), \tag{A21}$$

$$\tilde{\zeta}^{\,0}_{1j}(\bar{x}^i(t), t) \equiv \mathbf{U}_{jk}(t, 0) \frac{\partial}{\partial \bar{x}^k(t)} w_1^0(\bar{x}^i(t), t), \tag{A22}$$

and

$$\tilde{B}(t) = \int_0^t d\tau B(\bar{x}^i(\tau)). \tag{A23}$$

Upon combining Eqs. (35), (A4), and (A20) we obtain

$$\bar{V}_{NLL}(\bar{x}^i(t), t) = \pi^2 \phi^4 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{L}{2}\right)^k \left[\left(\frac{d}{dt}\right)^k w_2^0(\bar{x}^i(t), t) + \left(\frac{d}{dt}\right)^k (\tilde{Z}_j(t) \zeta_{1j}^0(\bar{x}^i(t), t)) - \tilde{Z}_j(t) \left(\frac{d}{dt}\right)^k \zeta_{1j}^0(\bar{x}^i(t), t) + \left(\frac{d}{dt}\right)^k (\tilde{B}(t) w_1^0(\bar{x}^i(t), t)) - \tilde{B}(t) \left(\frac{d}{dt}\right)^k w_1^0(\bar{x}^i(t), t) \right]. \tag{A24}$$

If we now employ Taylor's theorem with Eq. (A24), it follows that

$$\bar{V}_{NLL} = \pi^2 \phi^4 \left[w_2^0\left(\bar{x}^i\left(t + \frac{L}{2}\right), t + \frac{L}{2}\right) + \left(\tilde{Z}_j\left(t + \frac{L}{2}\right) - \tilde{Z}_j(t)\right) \zeta_{1j}^0\left(\bar{x}^i\left(t + \frac{L}{2}\right), t + \frac{L}{2}\right) + \left(\tilde{B}\left(t + \frac{L}{2}\right) - \tilde{B}(t)\right) w_1^0\left(\bar{x}^i\left(t + \frac{L}{2}\right), t + \frac{L}{2}\right) \right] \tag{A25}$$

and so by Eq. (36)

$$V_{NLL} = \pi^2 \phi^4 \left[w_2^0\left(\bar{x}^i\left(\frac{L}{2}\right), \frac{L}{2}\right) + \tilde{Z}_j\left(\frac{L}{2}\right) \zeta_{1j}^0\left(\bar{x}^i\left(\frac{L}{2}\right), \frac{L}{2}\right) + \tilde{B}\left(\frac{L}{2}\right) w_1^0\left(\bar{x}^i\left(\frac{L}{2}\right), \frac{L}{2}\right) \right] \tag{A26}$$

or, more explicitly

$$V_{NLL} = \pi^2 \phi^4 \exp\left[-4 \int_0^{L/2} d\tau \gamma_1(\bar{x}^i(\tau)) \right] \left\{ p_2^0\left(\bar{x}^i\left(\frac{L}{2}\right)\right) + \int_0^{L/2} d\tau [(-\gamma_1(\bar{x}^i(\tau)) \beta_2^{x^i}(\bar{x}^i(\tau)) + \beta_3^{x^i}(\bar{x}^i(\tau))) \mathbf{U}_{ij}(0, \tau)] \cdot \left[\mathbf{U}_{jk}\left(\frac{L}{2}, 0\right) \frac{\partial}{\partial \bar{x}^k(\frac{L}{2})} p_1^0\left(\bar{x}^i\left(\frac{L}{2}\right)\right) \right] + 4 \int_0^{L/2} d\tau [\gamma_1^2(\bar{x}^i(\tau)) - \gamma_2(\bar{x}^i(\tau))] p_1^0\left(\bar{x}^i\left(\frac{L}{2}\right)\right) \right\}. \tag{A27}$$

We have used the fact that $\tilde{B}(0) = 0 = \tilde{Z}_i(0)$. V_{N^2LL} can be computed using the approach used to obtain V_{NLL}. To begin, just as Eq. (A1) follows from Eq. (28), we find that

$$p_{n+3}^n + \gamma_1 p_{n+2}^n + \gamma_2 p_{n+1}^n = \frac{1}{2n} \left[\left(\beta_2^x \frac{\partial}{\partial x} + \beta_2^y \frac{\partial}{\partial y} - 4\gamma_1 \right) p_{n+2}^{n-1} + \left(\beta_3^x \frac{\partial}{\partial x} + \beta_3^y \frac{\partial}{\partial y} - 4\gamma_2 \right) p_{n+1}^{n-1} + \left(\beta_4^x \frac{\partial}{\partial x} + \beta_4^y \frac{\partial}{\partial y} - 4\gamma_3 \right) p_n^{n-1} \right]. \tag{A28}$$

With the definitions of Eqs. (29)–(31), we see that Eqs. (34), (A2), and (A28) together lead to

$$\begin{aligned} w_{n+3}^n(\bar{x}(t), \bar{y}(t), t) &= \frac{1}{2n} \left[-\gamma_2 \frac{d}{dt} w_n^{n-1} - \gamma_1 \left(\frac{d}{dt} w_{n+1}^{n-1} + D(t) w_n^{n-1} \right) + \frac{d}{dt} w_{n+2}^{n-1} + \left(\beta_3^x \frac{\partial}{\partial \bar{x}} + \beta_3^y \frac{\partial}{\partial \bar{y}} - 4\gamma_2 \right) w_{n+1}^{n-1} + \left(\beta_4^x \frac{\partial}{\partial \bar{x}} + \beta_4^y \frac{\partial}{\partial \bar{y}} - 4\gamma_3 \right) w_n^{n-1} \right] \\ &= \frac{1}{2n} \left[\frac{d}{dt} w_{n+2}^{n-1} + D(t) w_{n+1}^{n-1} + \Delta(t) w_n^{n-1} \right], \end{aligned} \tag{A29}$$

where

$$\Delta(t) = [\gamma_1^2 - \gamma_2]\left[\beta_2^x \frac{\partial}{\partial \bar{x}} + \beta_2^y \frac{\partial}{\partial \bar{y}} - 4\gamma_1\right] - \gamma_1\left[\beta_3^x \frac{\partial}{\partial \bar{x}} + \beta_3^y \frac{\partial}{\partial \bar{y}} - 4\gamma_2\right] + \left[\beta_4^x \frac{\partial}{\partial \bar{x}} + \beta_4^y \frac{\partial}{\partial \bar{y}} - 4\gamma_3\right]. \tag{A30}$$

Again one can iterate Eq. (A29) to obtain w_{n+3}^n in terms of w_1^0, w_2^0 and w_3^0 as well as the two and three loop RG functions in the CW scheme. The summations needed to compute V_{N^2LL} can then be performed using the same techniques as were used to find V_{NLL} in Eq. (A27). However, since the three loop RG functions have not been computed for the standard model, we will not pursue this calculation further.

APPENDIX B: THE DERIVATIVE EXPANSION OF THE EFFECTIVE ACTION

This paper has been concerned with contributions to the effective action coming from the first few terms in the derivative expansion when the background field is either a scalar or vector field [44]. In this appendix we show how terms in this derivative expansion can be computed. Operator regularization [45] will be used in calculation. This technique has the advantages of not explicitly breaking any classical symmetries of the theory (since no regulating parameter is inserted into the initial action) and of avoiding all explicit divergences at every stage of the calculation.

To illustrate this technique, we first consider a simple scalar model with a classical action

$$S^{(0)} = \int d^4x\left(-\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{6}\mu\phi^3 - \frac{1}{24}\lambda\phi^4\right). \tag{B1}$$

If we split ϕ into the sum of a background part f and a quantum fluctuation h then performing the path integral over the quantum fluctuation leads to the one loop contribution to the effective action

$$iS^{(1)} = -\tfrac{1}{2}\mathrm{tr}\ln(p^2 + m^2 + \mu f + \tfrac{1}{2}\lambda f^2), \qquad (p \equiv -i\partial). \tag{B2}$$

Regulating the logarithm in Eq. (B2) using the zeta function [45]

$$\ln H = -\frac{d}{ds}\bigg|_0 H^{-s} = -\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\int_0^\infty dit(it)^{s-1}e^{-iHt}, \tag{B3}$$

we see that Eq. (B2) can be written

$$iS^{(1)} = \frac{1}{2}\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty dit(it)^{s-1} \times \mathrm{tr}\left\{\exp - i\left(p^2 + m^2 + \mu f + \frac{1}{2}\lambda f^2\right)t\right\}\bigg|_0. \tag{B4}$$

If now $f \to v + f$ where v is a constant, and if $H = H_0 + H_1$ where

$$H_0 = p^2 + m^2 + \mu v + \tfrac{1}{2}\lambda v^2, \tag{B5}$$

$$H_1 = (\mu + \lambda v)f + \frac{\lambda f^2}{2}, \tag{B6}$$

then upon applying the Schwinger expansion [45]

$$\mathrm{tr} e^{-i(H_0+H_1)t} = \mathrm{tr}\left[e^{-iH_0t} + (-it)H_1e^{-iH_0t} + \frac{1}{2}(-it)^2 \times \int_0^1 duH_1e^{-i(1-u)H_0t}H_1e^{-iuH_0t} + \dots\right] \tag{B7}$$

and keeping terms at most quadratic in f we obtain

$$iS_2^{(1)} = \frac{1}{2}\frac{d}{ds}\frac{\kappa^{2s}}{\Gamma(s)}\int_0^\infty dit(it)^{s-1}\mathrm{tr}\left\{(-it)e^{-iH_0t}\left[(\mu + \lambda v)f + \frac{\lambda f^2}{2}\right] + \frac{(-it)^2}{2}\int_0^1 due^{-i(1-u)H_0t}(\mu + \lambda v)fe^{-iuH_0t}(\mu + \lambda v)f\right\}\bigg|_0, \tag{B8}$$

where κ^{2s} is a dimensionful parameter inserted to ensure that $S^{(1)}$ is dimensionless. [One could have introduced κ^2 in Eq. (B2) to keep the argument of the logarithm dimensionless in that equation.]

The functional trace in Eq. (A8) can most easily be computed using momentum eigenstates $|p>$, $|q>$ and configuration eigenstates $|x>$, $|y>$ wherein n dimensions $(2\pi)^{n/2}\langle x|p\rangle = e^{ip\cdot x}$ so that

$$iS_2^{(1)} = \frac{1}{2}\frac{d}{ds}\frac{\kappa^{2s}}{\Gamma(s)}\int_0^\infty dit(it)^{s-1}e^{-i(m^2+\mu v+(1/2)\lambda v^2)t}\bigg\{(-it)\int dpdx\langle p|e^{-ip^2t}|x\rangle\langle x|(\mu+\lambda v)f+\frac{\lambda f^2}{2}|p\rangle + \frac{1}{2}(-it)^2\int dpdqdxdy\int_0^1 du\langle p|e^{-i(1-u)p^2t}|x\rangle\langle x|(\mu+\lambda v)f|q\rangle\langle q|e^{-iuq^2}|y\rangle\langle y|(\mu+\lambda v)f|p>\bigg\}\bigg|_0, \quad \text{(B9)}$$

$$iS_2^{(1)} = \frac{1}{2}\frac{d}{ds}\frac{\kappa^2 s}{\Gamma(s)}\int_0^\infty dite^{-i(m^2+\mu v+(1/2)\lambda v^2)t}\bigg\{-(it)^s\int\frac{dpdx}{(2\pi)^4}e^{-ip^2t}\bigg[(\mu+\lambda v)f(x)+\frac{\lambda}{2}f^2(x)\bigg] + \frac{1}{2}(it)^{s+1}(\mu+\lambda v)^2\int\frac{dpdqdxdy}{(2\pi)^8}e^{-i[(1-u)p^2+uq^2]t}e^{-i(p-q)\cdot(x-y)}f(x)f(y)\bigg\}\bigg|_0. \quad \text{(B10)}$$

To obtain those terms which contribute to the effective action at one loop order which are second order in derivatives of the background field, we expand $f(y)$ about x up to second order so that

$$\int\frac{dpdqdxdy}{(2\pi)^8}e^{-i[(1-u)p^2+uq^2]t}e^{-i(p-q)\cdot(x-y)}f(x)f(y) \approx \int\frac{dpdqdxdy}{(2\pi)^8}e^{-i[(1-u)p^2+uq^2]t}e^{-i(p-q)\cdot(x-y)}f(x)\bigg[f(x)+(x-y)^\alpha f_{,\alpha}(x)+\frac{1}{2}(x-y)^\alpha(x-y)^\beta f_{,\alpha\beta}(x)\bigg]. \quad \text{(B11)}$$

If now we write in Eq. (B11)

$$(x-y)^\alpha e^{-i(p-q)\cdot(x-y)} = -i\frac{\partial}{\partial q^\alpha}e^{-i(p-q)\cdot(x-y)}, \quad \text{(B12)}$$

$$(x-y)^\alpha(x-y)^\beta e^{-i(p-q)\cdot(x-y)} = (-i)^2\frac{\partial}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}e^{-i(p-q)\cdot(x-y)} \quad \text{(B13)}$$

and then perform an integration by parts with respect to q we find that

$$iS_2^{(2)} = \frac{1}{2}\frac{d}{ds}\frac{\kappa^2 s}{\Gamma(s)}\int_0^\infty dite^{-i(m^2+\mu v+\frac{1}{2}\lambda v^2)t}\bigg\{-(it)^s\frac{i}{(4\pi it)^s}\int dx\bigg[(\mu+\lambda v)f(x)+\frac{1}{2}\lambda f^2(x)\bigg] + \frac{1}{2}(it)^{s+1}\int_0^1 du(\mu+\lambda v)^2[f^2(x)+(it)u(1-u)f(x)\partial^2 f(x)]\bigg\}\bigg|_0, \quad \text{(B14)}$$

where we have used the integral

$$\int\frac{d^n p}{(2\pi)^n}e^{-ip^2t} = \frac{i}{(4\pi it)^{n/2}}. \quad \text{(B15)}$$

The integrals over t and u are now standard and we end up with

$$iS_2^{(1)} = \frac{i}{32\pi^2}\int dx\bigg\{\bigg[(\mu+\lambda v)f(x)+\frac{1}{2}\lambda f^2(x)\bigg]\bigg[m^2+\lambda v+\frac{1}{2}\lambda v^2\bigg]\bigg[1-\ln\bigg(\frac{m^2+\mu v+\frac{1}{2}\lambda v^2}{\kappa^2}\bigg)\bigg] - \frac{1}{2}(\mu+\lambda v)^2 f^2(x)\ln\bigg(\frac{m^2+\mu v+\frac{1}{2}\lambda v^2}{\kappa^2}\bigg)+\frac{1}{2}\frac{(\mu+\lambda v)^2 f(x)\partial^2 f(x)}{(m^2+\mu v+\frac{1}{2}\lambda v^2)}\bigg\}. \quad \text{(B16)}$$

Equation (B16) agrees with what was obtained using different techniques in Ref. [44].

The approach outlined for the simple scalar model of Eq. (B1) can easily be applied to compute terms in the derivative expansion of the effective action in more complicated models. For scalar electrodynamics with the classical action

$$S_\phi = \int d^4x\bigg[-(\partial_\mu+ieV_\mu)\phi^*(\partial^\mu-ieV^\mu)\phi - \lambda(\phi^*\phi)^2-\frac{1}{4}(\partial_\mu V_\nu-\partial_\nu V_\mu)^2\bigg], \quad \text{(B17)}$$

we again let $\phi = f + h$ where f is the background field. Using the gauge fixing term

$$S_{gf} = -\frac{1}{2\alpha}\int d^4x\bigg[\partial\cdot V+\frac{ie\alpha}{2}(f^*h-fh^*)\bigg]^2 \quad \text{(B18)}$$

and the attendant ghost action

$$S_{gh} = \int d^4x\bar{c}\bigg[\partial^2-\frac{1}{2}e^2\alpha(2f^*f+f^*h+fh^*)\bigg], \quad \text{(B19)}$$

we find that the one loop effective action is given by

$$iS^{(1)} = \ln\det[p^2 + e^2\alpha(f_1^2 + f_2^2)] - \frac{1}{2}\ln\det\begin{bmatrix} p^2 + 3\lambda f_1^2 + (\lambda + \alpha e^2)f_2^2 & (2\lambda - \alpha e^2)f_1 f_2 & -2ef_{2,\nu} \\ (2\lambda - \alpha e^2)f_1 f_2 & p^2 + (\lambda + \alpha e^2)f_1^2 + 3\lambda f_2^2 & 2ef_{1,\nu} \\ -2ef_{2,\mu} & 2ef_{1,\mu} & p^2\left(T + \frac{1}{\alpha}L\right)_{\mu\nu} + e^2(f_1^2 + f_2^2)g_{\mu\nu} \end{bmatrix}, \tag{B20}$$

where f_1 and f_2 are the real and imaginary parts of f and $T_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu/p^2$, $L_{\mu\nu} = p_\mu p_\nu/p^2$ are a complete set of orthogonal projection operators.

Operator regularization can now be applied to this expression in the same way as it was applied to Eq. (B2); after the replacement $f_1 \to v + f_1$ the Schwinger expansion is used to obtain all terms second order in f_1 and f_2 and these fields can then be expanded out to second order in a Taylor expansion about some point x. One could also expand in powers of the external field strength and its derivatives.

[1] V. Elias, R. B. Mann, D. G. C. McKeon, and T. G. Steele, Phys. Rev. Lett. **91**, 251601 (2003); Nucl. Phys. **B678**, 147 (2004); **B703**, 413(E) (2004).
[2] F. A. Chishtie, V. Elias, R. B. Mann, D. G. C. McKeon, and T. G. Steele, Nucl. Phys. **B743**, 104 (2006).
[3] F. A. Chishtie, T. Hanif, D. G. C. McKeon, and T. G. Steele, Phys. Rev. D **77**, 065007 (2008).
[4] K. Meissner and H. Nicolai, Phys. Lett. B **648**, 312 (2007).
[5] C. Ford and C. Wiesendanger, Phys. Rev. D **55**, 2202 (1997).
[6] B. Kastening, Phys. Rev. D **54**, 3965 (1996); Phys. Rev. D **57**, 3567 (1998).
[7] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[8] K. Symanzik, Commun. Math. Phys. **16**, 48 (1970); J. Iliopoulos, C. Itzykson, and Andre Martin, Rev. Mod. Phys. **47**, 165 (1975).
[9] E. Weinberg and A. Wu, Phys. Rev. D **36**, 2474 (1987).
[10] A. Dannenberg, Phys. Lett. B **20**, 110 (1989).
[11] M. Sher, Phys. Rep. **179**, 273 (1989).
[12] V. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).
[13] E. N. Argyres, M. T. M. van Kessel, and R. H. P. Kleiss, Eur. Phys. J. C **64**, 319 (2009).
[14] V. Branchina and H. Faivre, Phys. Rev. D **72**, 065017 (2005).
[15] M. B. Einhorn and D. R. Timothy Jones, J. High Energy Phys. 04 (2007) 051.
[16] V. Branchina, H. Faivre, and V. Pangon, J. Phys. G **36**, 015006 (2009).
[17] E. N. Argyres, M. T. M. van Kessel, and R. H. P. Kleiss, Eur. Phys. J. C **65**, 303 (2009).
[18] D. J. E. Callaway, Phys. Rev. D **27**, 2974 (1983).
[19] R. Fukuda and E. Kyriakopoulos, Nucl. Phys. **B85**, 354 (1975).
[20] L. O'Raifeartaigh, A. Wipf, and H. Yoneyama, Nucl. Phys. **B271**, 653 (1986).
[21] K. Holland, Nucl. Phys. B, Proc. Suppl. **140**, 155 (2005); K. Holland and J. Kuti, Nucl. Phys. B, Proc. Suppl. **129–130**, 765 (2004).
[22] C. Wetterich, Nucl. Phys. B **352**, 529 (1991).
[23] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
[24] D. F. Litim, J. M. Pawlowski, and L. Vergara, arXiv:hep-th/0602140 .
[25] D. Litim and C. Wetterich, Mod. Phys. Lett. A **12**, 2287 (1997).
[26] P. M. Stephenson, Phys. Rev. D **30**, 1712 (1984).
[27] S. Weinberg, Phys. Rev. D **7**, 2887 (1973).
[28] R. Jackiw, Phys. Rev. D **9**, 1686 (1974).
[29] F. A. Chishtie, J. Jia, and D. G. C. McKeon, Phys. Rev. D **76**, 105006 (2007).
[30] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin, and S. A. Larin, Phys. Lett. B **272**, 39 (1991); **319**, 545(E) (1993).
[31] C. Ford, D. R. T. Jones, P. W. Stephenson, and M. B. Einhorn, Nucl. Phys. **B395**, 17 (1993).
[32] C. Ford and D. R. T. Jones, Phys. Lett. B **274**, 409 (1992); **285**, 398(E) (1992).
[33] A. Kotikov and D. G. C. McKeon, Can. J. Phys. **72**, 250 (1994).
[34] C. Ford, I. Jack, and D. R. T. Jones, Nucl. Phys. **B387**, 373 (1992).
[35] L. Abbott, Nucl. Phys. **B185**, 189 (1981).
[36] F. A. Chishtie, D. G. C. McKeon, T. G. Steele, and I. Zakout, Can. J. Phys. **86**, 1067 (2008).
[37] C. Amsler *et al.* (Particle Data Group) Particle Data Guide, Phys. Lett. B **667**, 1 (2008) and 2009 update for the 2010 edition.
[38] T. Schucker, arXiv:0708.3344 .
[39] A. Hoecker *et al.* (Gfitter Collaboration) Proc. Sci. EPS-HEP2009 (2009) 366 [arXiv:0909.0961].
[40] B. Kastening, Phys. Lett. B **283**, 287 (1992).
[41] T. N. Sherry, ICTP Report No. IC/79/105.
[42] M. R. Ahmady, F. A. Chishtie, V. Elias, A. H. Fariborz, N. Fattahi, D. G. C. McKeon, T. N. Sherry, and T. G. Steele, Phys. Rev. D **66**, 014010 (2002).
[43] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[44] J. Iliopoulos, C. Itzykson, and A. Martin, Rev. Mod. Phys. **47**, 165 (1975); L.H. Chan, Phys. Rev. D **38**, 3739 (1988).I. J.R. Aitcheson and C.M. Fraser, Phys. Rev. D **31**, 2605 (1985); **32**, 2190 (1985); O. Cheyette, Phys. Rev. Lett. **55**, 2394 (1985); J. Hauknes, Ann. Phys. (N.Y.) **156**, 303 (1984); H.W. Lee, P.Y. Pac, and H.K. Shin, Phys. Rev. D **40**, 4202 (1989); V. Gusynin and I. A. Skovkovy, Can. J. Phys. **74**, 282 (1996); D. Cangemi, E. D'Hoker, and G. Dunne, Phys. Rev. D **51**, R2513 (1995); D. G. C. McKeon, Phys. Rev. D **55**, 7989 (1997).

[45] D. G. C. McKeon and T. N. Sherry, Phys. Rev. D **35**, 3854 (1987); Ann. Phys. (N.Y.) **218**, 325 (1992).