Generalizations of teleparallel gravity and local Lorentz symmetry

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We analyze the relation between teleparallelism and local Lorentz invariance. We show that generic modifications of the teleparallel equivalent to general relativity will not respect local Lorentz symmetry. We clarify the reasons for this and explain why the situation is different in general relativity. We give a prescription for constructing teleparallel equivalents for known theories. We also explicitly consider a recently proposed class of generalized teleparallel theories, called $f(T)$ theories of gravity, and show why restoring local Lorentz symmetry in such theories cannot lead to sensible dynamics, even if one gives up teleparallelism.

I. INTRODUCTION

A key feature of general relativity is observer independence, expressed through general covariance. This makes it possible to formulate the theory in terms of spacetime tensors, without making any reference to the tangent space of each point of the spacetime manifold (even though this is where vectors and tensors are naturally defined). Also, in general relativity, the spacetime geometry can be fully described by the metric alone, because the connection that defines parallel transport is assumed to be the Levi-Civita connection of the metric. The effects of the gravitational interaction are described in terms of the curvature of spacetime, which responds to the distribution and motion of mass-energy.

The avoidance of any reference to the tangent space in general relativity is not mandatory. It is straightforward to introduce an orthonormal basis for the tangent space at each point, the vierbein or tetrad fields, $\mathbf{h}_a(x^\mu)$, project along this basis and formulate the theory in terms of the projected quantities. This is generally referred to as the tetrad or vierbein formalism. Such an approach has advantages, especially when working with fermions. In the tangent-space picture of general relativity, distances are measured with the flat metric, but there still exists a nonvanishing connection with nonvanishing curvature.

Alternatively, we could consider constructing a theory where, at least in a suitable class of frames, the connection in the tangent space would have zero curvature, without vanishing altogether. This can be achieved if torsion is not zero and the corresponding connection is called the Weitzenbock connection [1]. Such a theory is called teleparallel gravity [2,3]. The simplest form of this theory is actually equivalent to general relativity [3]. This appears surprising, given that the role of curvature is so central in the latter. It is entirely consistent though, since the zero-curvature Weitzenbock connection does not coincide with the Levi-Civita connection of the metric. This will be explained in more detail below.

The teleparallel formulation of general relativity, which we will refer to simply as teleparallel gravity below, allows a different physical interpretation of the gravitational interaction in terms of torsion instead of curvature. It has attracted interest in the past because it allows us to interpret general relativity as a gauge theory.

Very recently, there have been proposals for constructing generalizations of teleparallel gravity in Refs. [4–20] which followed the spirit of $f(R)$ gravity (see Ref. [21] for a review) as a generalization of general relativity. That is, the Lagrangians of the theories were generalised to the form $f(T)$, where f is some suitably differentiable function and T is the Lagrangian of teleparallel gravity. The interest in these theories was aroused by the claim that their dynamics differ from those of general relativity but their equations are still second order in derivatives and, therefore, they might be able to account for the accelerated expansion of the universe and remain free of pathologies. We showed in Ref. [22], however, that this last expectation was unfounded: these theories are not locally Lorentz invariant and appear to harbour extra degrees of freedom.

Our aim here is to elaborate on the findings of Ref. [22]. More specifically, we clarify below the role of violations of local Lorentz invariance in generalized teleparallel theories. We provide an illustrative example of an analogous situation with general covariance in ordinary field theory. We will also explain why general relativity, which does respect local Lorentz invariance, can nevertheless admit a teleparallel formulation. It is argued that this is not a sign of the uniqueness for general relativity, and a prescription is given for constructing teleparallel equivalents of other known gravity theories. Finally, we focus on $f(T)$ theories and argue that, even if we decide to give up teleparallelism, such actions would not make sense as descriptions of the dynamics of gravity if local Lorentz symmetry was restored.

II. SPACETIME AND TANGENT-SPACE DESCRIPTIONS

If we want to describe a spacetime in a coordinate basis, we need a metric $g_{\mu\nu}$ and a connection $\Gamma^\lambda{}_{\mu\nu}$. The connection does not have to be related to the metric. Instead of working with a coordinate basis we could choose to associate a tangent space to each spacetime point and work in terms of that tangent space. The vierbein or tetrad fields, $\mathbf{h}_a(x^\mu)$, would then form an orthonormal basis for the tangent space at each point of the manifold with spacetime coordinates x^μ. Latin indices label tangent-space coordinates while Greek indices label spacetime coordinates. All indices run from 0 to 3. Clearly, $\mathbf{h}_a(x^\mu)$ is a vector in the tangent space, and can be described in a coordinate basis by its components h_a^μ. So, h_a^μ also transforms as a vector in spacetime.

The spacetime metric, $g_{\mu\nu}$, is given by

$$g_{\mu\nu} = \eta_{ab} h_\mu^a h_\nu^b, \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric for the tangent space. It follows that

$$h_a^\mu h_\nu^a = \delta_\nu^\mu, \qquad h_a^\mu h_\mu^b = \delta_a^b, \tag{2}$$

where Einstein's summation convention has been used.

A general connection cannot be described just in terms of the tetrad (in the same way that the Christoffel symbols are not generically related to the metric components). The following relations hold

$$\Gamma^\lambda{}_{\mu\nu} \equiv h_b^\lambda \partial_\nu h_\mu^b + h_a^\lambda A^a{}_{b\nu} h_\mu^b \equiv h_b^\lambda D_\nu h_\mu^b, \tag{3}$$

which also implicitly define the Lorentz covariant derivative D_ν. Here, $A^a{}_{b\nu}$ is the spin connection and solving for it we find

$$A^a{}_{b\nu} = h_\lambda^a \partial_\nu h_b^\lambda + h_\lambda^a \Gamma^\lambda{}_{\mu\nu} h_b^\mu \equiv h_\lambda^a \nabla_\nu h_b^\lambda, \tag{4}$$

where ∇_μ denotes the covariant derivative associated with $\Gamma^\lambda{}_{\mu\nu}$. Note that $\Gamma^\lambda{}_{\mu\nu}$ is a Lorentz scalar (as long as $A^a{}_{b\nu}$ is left unrestricted). The torsion tensor is defined by

$$T^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}. \tag{5}$$

If we denote the Levi-Civita connection by

$$\bar{\Gamma}^\lambda{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}), \tag{6}$$

then

$$K^\rho{}_{\mu\nu} \equiv \Gamma^\rho{}_{\mu\nu} - \bar{\Gamma}^\rho_{\mu\nu} \tag{7}$$

is defined to be the contorsion tensor. Note that the Levi-Civita connection does not have vanishing $A^a{}_{b\nu}$; instead

$$\bar{A}^a{}_{b\nu} = h_\lambda^a \bar{\nabla}_\nu h_b^\lambda, \tag{8}$$

where a bar is used to denote all quantities associated with the Levi-Civita connection. Finally, it is possible to show

(after expressing the equation in terms of the connection explicitly) that *if*

$$\nabla_\lambda g_{\mu\nu} = 0, \tag{9}$$

then

$$K^\rho{}_{\mu\nu} = \frac{1}{2}(T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}). \tag{10}$$

That is, if the connection is metric compatible (i.e. it has vanishing nonmetricity), then the contorsion tensor can be expressed in terms of the torsion. From now on we will only consider metric compatible connections, but not necessarily symmetric ones. Equation (10) can be solved for the torsion to give

$$T_{\rho\mu\nu} = K_{\rho\nu\mu} - K_{\rho\mu\nu}. \tag{11}$$

Next, we define the tensor $S^{\rho\mu\nu}$ as

$$S^{\rho\mu\nu} \equiv K^{\mu\nu\rho} - g^{\rho\nu}T^{\sigma\mu}{}_\sigma + g^{\rho\mu}T^{\sigma\nu}{}_\sigma, \tag{12}$$

and the associated invariant is

$$T \equiv \frac{1}{2} S^{\rho\mu\nu} T_{\rho\mu\nu} = -S^{\rho\mu\nu} K_{\rho\mu\nu}$$

$$= \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu. \tag{13}$$

Using the definitions given above, and without any restrictions on $A^a{}_{b\nu}$ apart from those implied by metric compatibility for $\Gamma^\lambda{}_{\mu\nu}$, we see that $T^\lambda{}_{\mu\nu}$ is a spacetime tensor and a Lorentz scalar, which makes T both a spacetime scalar and a Lorentz scalar. On the other hand,

$$R^\rho{}_{\mu\lambda\nu} \equiv \partial_\lambda \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\lambda} + \Gamma^\rho{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}, \tag{14}$$

and

$$R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu} \tag{15}$$

are spacetime tensors and Lorentz scalars, and

$$R \equiv g^{\mu\nu} R_{\mu\nu}, \tag{16}$$

is both a spacetime scalar and a Lorentz scalar, just like T. The same properties hold for $\bar{R}^\rho{}_{\mu\lambda\nu}$, $\bar{R}_{\mu\nu}$ and $\bar{R}$. Using the definitions listed above, it is a straightforward exercise to show that

$$R^\rho{}_{\mu\lambda\nu} = \bar{R}^\rho{}_{\mu\lambda\nu} + \bar{\nabla}_\lambda K^\rho{}_{\mu\nu} - \bar{\nabla}_\nu K^\rho{}_{\mu\lambda} + K^\rho{}_{\sigma\lambda} K^\sigma{}_{\mu\nu}$$

$$- K^\rho{}_{\sigma\nu} K^\sigma{}_{\mu\lambda}, \tag{17}$$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\rho K^\rho{}_{\mu\nu} - \bar{\nabla}_\nu K^\rho{}_{\mu\rho} + K^\rho{}_{\sigma\rho} K^\sigma{}_{\mu\nu}$$

$$- K^\rho{}_{\sigma\nu} K^\sigma{}_{\mu\rho}, \tag{18}$$

$$R = \bar{R} + T + 2\bar{\nabla}^\mu(T^\nu{}_{\mu\nu}). \tag{19}$$

Our aim in this section was just to give some basic definitions in two *equivalent* descriptions of spacetime.

So, the important message is that the spacetime descriptions with a metric $g_{\mu\nu}$ and a (metric compatible, but only due to our assumption) connection $\Gamma^\lambda{}_{\mu\nu}$, is dual to a description which refers to a tangent space and uses a tetrad h_a^μ, and a spin connection $A^a{}_{b\nu}$. We intend to exploit the equivalence of the two descriptions in what comes next.

III. TELEPARALLELISM AND LOCAL LORENTZ INVARIANCE

The main requirement of teleparallelism is that there exist a class of frames where the spin connection vanishes, i.e. where

$$A^a{}_{b\nu} = 0. \tag{20}$$

In these frames

$$\Gamma^\lambda{}_{\mu\nu} \equiv h_b^\lambda \partial_\nu h_\mu^b = -h_\mu^b \partial_\nu h_b^\lambda \tag{21}$$

which implies

$$R^\rho{}_{\mu\lambda\nu} = 0, \tag{22}$$

but nonzero torsion. A very important observation is that, since $R^\rho{}_{\mu\lambda\nu}$ is a Lorentz scalar, if there exists some class of frames where it is zero, then it will actually be zero in all frames. But this requirement *cannot* be imposed without introducing some prior geometry.

Let us examine this in more detail. Suppose we are working in the tangent-space picture, where the fundamental fields are considered to be the tetrad h_a^μ, and the spin connection $A^a{}_{b\nu}$. Clearly, if it is a characteristic of the theory that there exists a class of frames in which $A^a{}_{b\nu} = 0$ then we can always choose to work in one of these frames (that is, define the theory as a preferred-frame theory *a priori*). Then, we have $R = 0$, and $T^\lambda{}_{\mu\nu}$ is manifestly not a Lorentz scalar anymore. The same holds for T. This was the approach followed in Ref. [22] and many other papers in the literature.

On the other hand, since $A^a{}_{b\nu}$ is a connection, it can be nonzero in other frames. Therefore, another option is to define all quantities in a manifestly Lorentz covariant way, as done above, and enforce the teleparallelism condition, i.e. that $A^a{}_{b\nu} = 0$ in some class of frames, as a constraint on the form of $A^a{}_{b\nu}$. Such constraints are usually imposed either by the explicit use of a Lagrange multiplier, or implicitly by allowing only variations that respect them when extremizing the action. Even though in this formulation the action can be made manifestly covariant, it is not really a way to restore local Lorentz invariance at the level of the solutions due to the existence of the constraint. This is best seen in the dual picture where the tangent space is abandoned and the theory is described by the metric $g_{\mu\nu}$ and the connection $\Gamma^\lambda{}_{\mu\nu}$ (clearly, the Levi-Civita connection of the metric exists as well but it is not an independent field). As mentioned earlier, in this picture, the requirement of teleparallelism, that there be a class of frames where

$A^a{}_{b\nu} = 0$, translates to the requirement that $R^\rho{}_{\mu\lambda\nu} = 0$ in *all* frames because $R^\rho{}_{\mu\lambda\nu}$ is a Lorentz scalar. Suppose that the action of such a theory is written in a manifestly (spacetime and of course Lorentz since we have abandoned the use of tangent space) covariant way. Enforcing the constraint $R^\rho{}_{\mu\lambda\nu} = 0$ at the level of the field equation (e.g. through a suitable covariant Lagrange multiplier) implies the existence of a second metric which is always forced to be flat. Obviously, such a theory cannot generically respect local Lorentz covariance.

An illuminating example of an analogous situation in a much simpler theory is that of a relativistic massless scalar field in flat spacetime, which is usually described by the action

$$S_\phi = \int d^4x \, \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{23}$$

Variation with respect to ϕ yields

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0. \tag{24}$$

Now consider the action

$$S_\phi^c = \int d^4x \sqrt{-g} (g^{\mu\nu} \bar\nabla_\mu \phi \bar\nabla_\nu \phi + M_\rho{}^{\mu\lambda\nu} \bar R^\rho{}_{\mu\lambda\nu}). \tag{25}$$

Variation with respect to ϕ yields

$$g^{\mu\nu} \bar\nabla_\mu \bar\nabla_\nu \phi = 0, \tag{26}$$

whereas variation with respect to $A_\rho{}^{\mu\lambda\nu}$ yields

$$\bar R^\rho{}_{\mu\lambda\nu} = 0. \tag{27}$$

The last equation has the unique solution $g_{\mu\nu} = \eta_{\mu\nu}$. Then, Eq. (26) becomes identical to Eq. (24). There will also be a third equation coming from the variation with respect to $g^{\mu\nu}$, which will determine $M_\rho{}^{\mu\lambda\nu}$. However, the dynamics of $M_\rho{}^{\mu\lambda\nu}$ become irrelevant as there is no coupling to ϕ, not even an indirect one since Eq. (27) forces the metric to be flat.

The point of this example is to illustrate that we can write the action or the field equations of a scalar field in flat space in a manifestly covariant way. If action (25) is taken at face value the theory appears to be invariant under diffeomorphisms. However, at the level of the solutions this theory is only invariant under global Lorentz transformations, exactly like the theory described by action (23).

IV. TELEPARALLEL FORMULATION OF GENERAL RELATIVITY AND LOCAL LORENTZ INVARIANCE

Let us now return to teleparallel gravity. Superficially, there appears to be a contradiction in the statements made above: it was claimed in Sec. I that, on the one hand, teleparallel gravity can provide an alternative formulation of general relativity, which is intrinsically a locally

Lorentz-invariant theory. On the other hand, it was argued in the last section that the requirement that there is a class of frames where the spin connection vanishes, which is the cornerstone of teleparallelism, cannot generically be enforced without violating local Lorentz invariance. However, there is no real contradiction, and the resolution lies on the exact form of the Lagrangian (28).

The action for (ordinary) teleparallel gravity is given by

$$S_T \equiv \frac{1}{16\pi G} \int d^4x h T, \qquad (28)$$

in which $h = \sqrt{-g}$ is the determinant of h_a^λ and g is the determinant of the metric $g_{\mu\nu}$, G is the gravitational constant. Equation (22) implies that $R = 0$ in teleparallel theories, and Eq. (19) yields

$$T = -\bar{R} - 2\bar{\nabla}^\mu(T^\nu{}_{\mu\nu}). \qquad (29)$$

Consequently, the action differs from the Einstein–Hilbert action only by a boundary term, and will therefore lead to the same field equations. This is why teleparallel gravity can be considered as an alternative formulation of general relativity.

The presence of the boundary term is crucial. Without it, the action is the Einstein–Hilbert action with the usual symmetries. Adding it and enforcing the teleparallelism condition, the action becomes that of teleparallel gravity, with Lagrangian (28), and is no longer locally Lorentz covariant. In conclusion, the Einstein–Hilbert action, which of course leads to a fully diffeomorphism invariant and Lorentz invariant theory, can be written as the sum of two pieces: the teleparallel action and a boundary term. Neither of these two pieces is locally Lorentz invariant once teleparallelism is imposed (though they sum up to a locally Lorentz-invariant quantity). Imposing the last condition however, and formally subtracting the boundary term, is crucial for the interpretation of the theory as a teleparallel theory of gravity. Of course, this does not alter at all the dynamical context of the theory, or its real symmetries in the spacetime picture.

In conclusion, the reason that the teleparallel theory described by the action (28) does not really violate local Lorentz symmetry, even though it respects teleparallelism, is because its action only differs from a locally Lorentz covariant action by a boundary term. In fact, this action turns out to be that of general relativity. Of course this cannot be a property of a general teleparallel action. If a Lagrangian is constructed with the tetrad and the torsion tensor, then even if this Lagrangian is a spacetime and Lorentz scalar initially, it will not be a local Lorentz scalar once teleparallelism has been imposed.

V. TELEPARALLEL FORMULATION OF GRAVITY THEORIES

The previous discussion does not imply that general relativity is the only gravity theory that can be cast into

a teleparallel formulation and take on a teleparallel interpretation. In fact, given Eqs. (17)–(19) it should be clear that any action constructed with curvature invariants of the metric can be cast into a teleparallel formulation. Additionally, all teleparallel theories (i.e. theories whose Lagrangians are constructed with the curvature-free Weitzenbock connection and the tetrad) whose action differs from a diffeomorphism invariant and locally Lorentz-invariant action only by a boundary term, will lead to locally Lorentz-invariant theories. This follows from straightforwardly generalizing the example of general relativity considered above.

To understand this better, let us consider some simple examples. Let us start from Brans-Dicke theory [23], which is the best known alternative theory of gravity. The action of the theory is

$$S_{\rm BD} = \int d^4x\sqrt{-g}\left[\phi\bar{R} - \frac{\omega_0}{\phi}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi \right] + S_M(g_{\mu\nu}, \psi), \qquad (30)$$

where ω_0 is the Brans-Dicke parameter, S_M is the matter action and ψ collectively denotes the matter fields. Simply using Eq. (19) and the fact that teleparallelism requires $R = 0$, this action can take the form of a teleparallel theory

$$S_{\rm BD} = \int d^4x h\left[-\phi T - \frac{\omega_0}{\phi}\nabla_\mu\phi\nabla^\mu\phi + 2T^\nu{}_{\mu\nu}\nabla^\mu\phi \right] + S_M(g_{\mu\nu}, \psi), \qquad (31)$$

and can acquire a teleparallel interpretation. Note that we have discarded a boundary term.

Another example is $f(\bar{R})$ gravity. The action for $f(\bar{R})$ gravity is

$$S_f = \int d^4x\sqrt{-g}f(\bar{R}) + S_M(g_{\mu\nu}, \psi). \qquad (32)$$

As is well known [24–27], as long as $f''(\bar{R}) \neq 0$ this action can be brought into the form

$$S_f = \int d^4x\sqrt{-g}[\phi\bar{R} - V(\phi)] + S_M(g_{\mu\nu}, \psi), \qquad (33)$$

where

$$V(\phi) = f(\chi) - \chi\phi, \qquad (34)$$

and χ implicitly defined through $\phi = f'(\chi)$. The prime denotes differentiation with respect to the argument. Again, using Eq. (19) to replace $\bar{R}$, imposing teleparallelism and discarding a boundary term, we get

$$S_f = \int d^4x\sqrt{-g}[-\phi T + 2T^\nu{}_{\mu\nu}\nabla^\mu\phi - V(\phi)] + S_M(g_{\mu\nu}, \psi). \qquad (35)$$

These simple examples demonstrate how we can construct teleparallel versions of known gravity theories.

On the other hand, we could also use $f(T)$ theories of gravity as a characteristic example of why generic *ad hoc* teleparallel actions will not respect local Lorentz invariance. Let us consider the action

$$S_{f(T)} \equiv \int d^4x h f(T). \tag{36}$$

In an analogous manner to the procedure followed above for $f(\bar{R})$ gravity, this action can be brought into the form

$$S_{f(T)} = \int d^4x h[\phi T - V(\phi)] + S_M(g_{\mu\nu}, \psi). \tag{37}$$

To demonstrate this, first consider the action

$$S_1 = \int d^4x h[f(\chi) - \phi(\chi - T)] + S_M(g_{\mu\nu}, \psi). \tag{38}$$

Variation with respect to ϕ yields the algebraic constraint $\chi = T$. Replacing this constraint in Eq. (38) gives Eq. (36), implying the dynamical equivalence of these two actions. On the other hand, variation with respect to χ yields another algebraic constraint: $\phi = f'(\chi)$. Placing this constraint back in Eq. (38), suitably defining $V(\phi)$ and writing the action in terms of ϕ instead of χ, yields the equivalent action Eq. (37). Suppose now that we want to use Eq. (19) together with the teleparallelism constraint $R = 0$ in order to eliminate T in favor of $\bar{R}$. We would then get

$$S_{f(T)} = \int d^4x \sqrt{-g}[-\phi\bar{R} + 2T^\nu{}_{\mu\nu}\bar{\nabla}^\mu\phi - V(\phi)]$$
$$+ S_M(g_{\mu\nu}, \psi). \tag{39}$$

It is the presence of the $T^\nu{}_{\mu\nu}\bar{\nabla}^\mu\phi$ term that leads to the violations of local Lorentz invariance as, under the constraint of teleparallelism, $T^\nu{}_{\mu\nu}$ is not a Lorentz scalar anymore. Indeed, this term makes the difference between actions Eq. (39) and (33) or Eq. (37) and (35) (the sign differences are not important here as they can be absorbed by a redefinition of ϕ). Note that, action Eq. (28) differs from the Einstein-Hilbert action only by a boundary term, whereas the difference between Eq. (36) and (32) is not simply a boundary term.

VI. A COVARIANT VERSION OF $f(T)$ THEORIES?

We have established that general teleparallel theories will not respect local Lorentz invariance and we have argued that this stems from the fact that teleparallelism cannot generally be imposed without prior geometry. We also saw that teleparallel $f(T)$ theories of gravity are typical examples of theories that suffer from this problem. Suppose now that, for some reason, we wish to restore local Lorentz invariance in these theories without changing the form of the action. This can clearly be achieved only by giving up teleparallelism. This is because, as explained in Sec. II, if no restrictions related to teleparallelism are

imposed on $A^a{}_{b\nu}$, then T and consequently $f(T)$, will be local Lorentz scalars.[1]

What would giving up the teleparallelism restriction on $A^a{}_{b\nu}$ mean for the dynamics for $f(T)$ theories? The best way to understand the answer is to consider the duality between the $(h^\mu_a, A^a{}_{b\nu})$ and the $(g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu})$ descriptions. The action is defined in the former as in Eq. (36), but since $h = \sqrt{-g}$ it can take the form

$$S_{f(T)} \equiv \int d^4x \sqrt{-g} f(T). \tag{40}$$

What remains is to determine T in terms of $g_{\mu\nu}$ and $\Gamma^\lambda{}_{\mu\nu}$. Recall that $\Gamma^\lambda{}_{\mu\nu}$ is an independent connection which satisfies Eq. (9). This implies that the part of this connection which is independent of the metric is just the contorsion tensor $K^\rho{}_{\mu\nu}$. Therefore, the independent fields with respect to which we must vary the action are $g_{\mu\nu}$ and $K^\rho{}_{\mu\nu}$. Equation (11) expresses the torsion in terms of the contorsion. Taking a trace of the same equation yields

$$T^\rho{}_{\mu\rho} = -K^\rho{}_{\mu\rho}. \tag{41}$$

Replacing the last expression in Eq. (13), we can obtain an expression for T in terms of $g_{\mu\nu}$ and $K^\rho{}_{\mu\nu}$ only:

$$T = -K^{\mu\nu\rho}K_{\rho\mu\nu} - K^{\sigma\mu}{}_\sigma K^\rho{}_{\mu\rho}. \tag{42}$$

But with T given in terms of $K^\rho{}_{\mu\nu}$ by Eq. (42), it becomes evident that the action (40) is dynamically trivial, as it contains no derivatives of either of the two fundamental fields: the metric and the contorsion. In fact, variations with respect to the metric and the contorsion yield respectively

$$- f'^\rho{}_{\sigma(\mu}K^\sigma{}_{\nu)\rho} - f'^\sigma{}_{\mu\sigma}K^\rho{}_{\nu\rho} - \frac{f}{2}g_{\mu\nu} = S_{\mu\nu}, \tag{43}$$

$$(K^{\mu\nu}{}_\rho + K^\nu{}_\rho{}^\mu + K^{\sigma\mu}{}_\sigma\delta^\nu_\rho - K^\sigma{}_{\rho\sigma}g^{\mu\nu})f' = 0, \tag{44}$$

where we have taken into account the fact that the metric is symmetric, the contorsion tensor is antisymmetric in the first two indices and

$$S_{\mu\nu} \equiv -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{45}$$

After some mathematical manipulations, and provided that $f'(T) \neq 0$ generally, Eq. (44) yields

$$K^\rho{}_{\mu\nu} = 0. \tag{46}$$

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[1]One might wonder why imposing the teleparallelism constraint on $A^a{}_{b\nu}$, which implies a constraint on $\Gamma^\lambda{}_{\mu\nu}$, does not allow T to be a Lorentz scalar, whereas imposing that $\Gamma^\lambda{}_{\mu\nu}$ does not contain any part leading to nonmetricity does not cause such a problem. The reason is that the latter restriction just expresses part of the connection in terms of another dynamical field (the metric), whereas the former requires the introduction of prior geometry as discussed in Sec. III.

This implies that Eq. (43) is trivially satisfied *in vacuo* and inconsistent with the presence of matter. In addition, $g_{\mu\nu}$ remains indeterminate. The choice $f'(T) = 0$ clearly does not improve things.

These results should not come as a surprise and are not in conflict with the fact that, after teleparallelism is enforced, the same action leads to dynamical equations for the tetrad h_a^μ. The reason for this is that if $\Gamma^\lambda{}_{\mu\nu}$ is to have zero curvature then we must be able to express $K^\rho{}_{\mu\nu}$ in terms of the tetrad (much as the requirements of zero nonmetricity and torsion yield the expression for the Levi-Civita connection). This introduces derivatives of the tetrad in the action and leads to a dynamical (yet locally Lorentz-violating) theory.

Note that all of the arguments in this section could have be made in terms of $T^\rho{}_{\mu\nu}$ instead of $K^\rho{}_{\mu\nu}$, given the algebraic relation between the two quantities in Eq. (10).

VII. CONCLUSIONS

We have investigated the relation between teleparallelism and local Lorentz symmetry violations, and have concluded that generically imposing teleparallelism requires the introduction of prior geometry in the theory, which leads to violations of local Lorentz symmetry. A special class of teleparallel actions create an exception: those which differ from diffeomorphism-invariant and locally Lorentz-invariant actions (without constraints) only by a boundary term. The Einstein-Hilbert action belongs to this exceptional class. A prescription for constructing teleparallel equivalents of known theories was also given. Finally, we focussed on $f(T)$ theories, which have attracted a lot of recent attention as possible dark energy equivalents, and demonstrated that giving up teleparallelism in order to restore local Lorentz invariance leads to a dynamically trivial theory, which also becomes inconsistent if matter is added. Therefore, there seems to be no way to get sensible dynamics from such an action, while simultaneously satisfying local Lorentz invariance.

It would be interesting to explore further the generation of teleparallel equivalents of known gravity theories and also to study their properties. It could provide some insight into the interpretation of other gravity theories as gauge theories.

ACKNOWLEDGMENTS

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