Some generalizations of the Raychaudhuri equation

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The Raychaudhuri equation has seen extensive use in general relativity, most notably in the development of various singularity theorems. In this rather technical article we shall generalize the Raychaudhuri equation in several ways. First an improved version of the standard timelike Raychaudhuri equation is developed, where several key terms are lumped together as a divergence. This already has a number of interesting applications, both within the Arnowitt-Deser-Misner formalism and elsewhere. Second, a spacelike version of the Raychaudhuri equation is briefly discussed. Third, a version of the Raychaudhuri equation is developed that does not depend on the use of normalized congruences. This leads to useful formulae for the ''diagonal'' part of the Ricci tensor. Fourth, a ''two vector'' version of the Raychaudhuri equation is developed that uses two congruences to effectively extract ''off-diagonal'' information concerning the Ricci tensor.

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I. INTRODUCTION

The Raychaudhuri equation [1] has become one of the standard workhorses of general relativity, particularly as applied to the singularity theorems. For textbook presentations see for instance [2–5]. Some measure of the level of interest the Raychaudhuri equation can be inferred from the dedicated conference on this topic just five years ago [6–16], as well as the fact that new research articles continue to appear to this day [17]. Nevertheless, we feel that there are still some interesting ways in which the general formalism can be extended. There are four specific extended versions of the Raychaudhuri equation we wish to explore in this article:

(i) *Single timelike unit vector field.*
By collecting several terms in the usual formulation into a divergence, we obtain a particularly useful version that finds many applications in the derivation and use of the Arnowitt-Deser-Misner (ADM) formalism and other situations.

(ii) *Single spacelike unit vector field.*
This situation is most typically ignored. We will make a few hopefully clarifying comments.

(iii) *Single non-normalized vector field.*
This somewhat simplifies the Raychaudhuri equation, at the cost of no longer having nice positivity properties.

(iv) *Two non-normalized vector fields.*
This allows us to probe the off-diagonal components of the Ricci tensor.

These four extensions of the Raychaudhuri equation will soon be seen to each be useful in their own way, and to yield quite different information. We shall provide numerous examples below.

II. SINGLE UNIT TIMELIKE VECTOR FIELD

This is the standard case. Let u^a be a field of unit timelike vectors (a congruence). This does not have to be the 4-velocity of a physical fluid (though it might be), it applies just as well to the 4-velocities of an imaginary collection of ''fiducial observers'' (FIDOs). Then it is a purely geometrical result (see, for example, Hawking and Ellis [2], pp. 82–84, or Wald [3], or Carroll [4], or Poisson [5], or even Wikipedia, (note that there are sometimes minor disagreements of notation—typically just a factor of 2 in odd places) that

$$\frac{d\theta}{ds} = -R_{ab}u^a u^b + \omega^2 - \sigma^2 - \frac{1}{3}\theta^2 + \nabla_a\left(\frac{du^a}{ds}\right). \quad (1)$$

This is the standard form of the Raychaudhuri equation. This first form of the standard Raychaudhuri equation is written as a propagation equation for the expansion scalar θ. To set up the formalism, first consider the spatial projection tensor

$$h_{ab} = g_{ab} + u_a u_b. \quad (2)$$

This projection tensor has signature $\{0, +1, +1, +1\}$. Various shear and expansion related quantities are

$$\theta_{ab} = h_{ac}\nabla^{(c}u^{d)}h_{db}; \quad (3)$$

$$\theta = g^{ab}\theta_{ab} = h^{ab}\theta_{ab} = \nabla_a u^a; \quad (4)$$

$$\sigma_{ab} = \theta_{ab} - \tfrac{1}{3}h_{ab}\theta; \qquad \sigma^2 = \sigma_{ab}\sigma^{ab} \geq 0. \quad (5)$$

Vorticity related quantities are

$$\omega_{ab} = h_{ac}\nabla^{[c}u^{d]}h_{db}; \qquad \omega^2 = \omega_{ab}\omega^{ab} \geq 0. \quad (6)$$

With these definitions we have the usual decomposition:

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$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - \frac{\mathrm{d}u_a}{\mathrm{d}s} u_b. \quad (7)$$

See (for example) pp. 82–84 of Hawking and Ellis [2]. Equation (1) is Wald's equation (9.2.11) [3], supplemented with the $\nabla_a(\mathrm{d}u^a/\mathrm{d}s)$ term due to allowing a nongeodesic congruence; you can deduce the presence of this term from the second line in his (9.2.10) by not assuming geodesic motion.

Though less common in the literature, it is sometimes useful to rewrite the $\nabla_a(\mathrm{d}u^a/\mathrm{d}s)$ divergence term by writing $\mathsf{a}^a = \mathrm{d}u^a/\mathrm{d}s$ and noting [17]

$$\begin{aligned}\nabla_a \mathsf{a}^a &= h_{ab}\nabla^a \mathsf{a}^b - u_a u_b \nabla^a \mathsf{a}^b \\ &= h_{ab}\nabla^a \mathsf{a}^b + u_a(\nabla^a u_b) a^b \\ &= h^{ab}\nabla_a \mathsf{a}_b + \mathsf{a}_a \mathsf{a}^a \\ &= h^{ab}\nabla_a \mathsf{a}_b + \mathsf{a}^2, \end{aligned} \quad (8)$$

where we are now guaranteed that $\mathsf{a}^2 \geq 0$ because a^a is spacelike. This now permits us to write

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -R_{ab}u^a u^b + \omega^2 - \sigma^2 - \frac{1}{3}\theta^2 + h^{ab}\nabla_a \mathsf{a}_b + \mathsf{a}^2. \quad (9)$$

This second form of the standard Raychaudhuri equation now focusses on the square of the 4-acceleration a.

Now consider the geometrical identity

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = u \cdot \nabla\theta = \nabla \cdot (\theta u) - \theta \nabla \cdot u = \nabla \cdot (\theta u) - \theta^2. \quad (10)$$

Using this identity we can also write the Raychaudhuri equation in the slightly unusual "divergence" form

$$\nabla_a\left(\theta u^a - \frac{\mathrm{d}u^a}{\mathrm{d}s}\right) = -R_{ab}u^a u^b + \omega^2 - \sigma^2 + \frac{2}{3}\theta^2. \quad (11)$$

This is our third form of the standard Raychaudhuri equation, now focussing on the spacetime divergence of a suitable vector field.

Alternatively, rearranging the equation to rephrase it as a statement about Ricci tensor components, we deduce

$$R_{ab}u^a u^b = \omega^2 - \sigma^2 + \frac{2}{3}\theta^2 - \nabla_a\left(\theta u^a - \frac{\mathrm{d}u^a}{\mathrm{d}s}\right). \quad (12)$$

This is our fourth form of the standard Raychaudhuri equation, now focussing on the constraints one can place on the Ricci tensor components. This minor extension/modification/rephrasing of the usual Raychaudhuri equation is "close" to, but significantly more general than, a key technical result used by Padmanabhan and Patel in Refs. [18–20]. Much of the discussion below will focus on this form of the Raychaudhuri equation.

Note that we have not yet said anything about any possible implications either by or for the Einstein equations or the stress-energy tensor. This is deliberate. The Raychaudhuri equation is in essence a purely geometrical statement, at this stage a statement regarding the behavior of an arbitrary timelike congruence. Only after we have extracted as much purely geometric information as possible will we turn to the implications regarding the stress-energy tensor.

III. APPLICATIONS: TIMELIKE CONGRUENCES

We now consider several applications of the above purely geometrical formalism—these applications basically amount to strategically choosing an appropriate timelike congruence.

A. Vorticity-free congruence

Let $\Psi(x)$ be an arbitrary scalar field and define a set of FIDOs by

$$u_a \propto \nabla_a \Psi. \quad (13)$$

Then normalizing we have

$$u_a = -\frac{\nabla_a \Psi}{\|\nabla\Psi\|}, \quad (14)$$

and furthermore

$$\omega_{ab} = 0. \quad (15)$$

The minus sign here is purely conventional, it guarantees that the u^a is "future pointing" in the direction of increasing Ψ. Conversely,

$$\omega_{ab} = 0 \Rightarrow u_a \propto \nabla_a \Psi. \quad (16)$$

This is guaranteed by the Frobenius theorem.

Then in this vorticity-free situation the extended nongeodesic Raychaudhuri equation reduces to

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -R_{ab}u^a u^b - \sigma^2 - \frac{1}{3}\theta^2 + \nabla_a\left(\frac{\mathrm{d}u^a}{\mathrm{d}s}\right), \quad (17)$$

or equivalently

$$R_{ab}u^a u^b = -\sigma^2 + \frac{2}{3}\theta^2 + \nabla_a\left(-\theta u^a + \frac{\mathrm{d}u^a}{\mathrm{d}s}\right), \quad (18)$$

or even

$$R_{ab}u^a u^b = -\theta_{ab}\theta^{ab} + \theta^2 + \nabla_a\left(-\theta u^a + \frac{\mathrm{d}u^a}{\mathrm{d}s}\right). \quad (19)$$

But since u^a is now hypersurface orthogonal we can use the slices of constant Ψ to define a spacelike foliation—the scalar Ψ serves (at least locally) as a "cosmic time" function. Then in terms of the extrinsic curvature K_{ab} of the constant Ψ hypersurfaces we have, (using Misner, Thorne, and Wheeler [21] sign conventions for the extrinsic curvature), the results:

$$\theta_{ab} = -K_{ab}; \qquad \theta = -K; \quad (20)$$

$$\sigma_{ab} = -(K_{ab} - \tfrac{1}{3}K h_{ab}); \tag{21}$$

and

$$\sigma^2 = \tfrac{1}{2}[K_{ab}K^{ab} - \tfrac{1}{3}K^2]. \tag{22}$$

But then

$$R_{ab}u^a u^b = -K_{ab}K^{ab} + K^2 + \nabla_a\left(K u^a + \frac{\mathrm{d}u^a}{\mathrm{d}s}\right). \tag{23}$$

This is effectively one of the key technical results used by Padmanabhan and Patel in [18–20], but now we see that this result is actually a special case of a considerably more general result, Eq. (30). Indeed Eq. (23) can be viewed as a relatively straightforward extension and then specialization of the Raychaudhuri equation.

A further step one can take when dealing with such congruences is to note that the vorticity-free condition $u^a = -\nabla^a\Psi/\|\nabla\Psi\|$ implies the purely geometrical result

$$\mathsf{a}^a = h^{ab}\nabla_b \ln\|\nabla\Psi\|. \tag{24}$$

(See for instance [22], where essentially the same result was derived in a fluid dynamics context. The result is however much more general in scope, applying to any vorticity-free congruence regardless of whether or not it arises from fluid motion.) In this situation one now has

$$\nabla_a \mathsf{a}^a = h^{ab}\nabla_a\nabla_b \ln\|\nabla\Psi\| + \theta\nabla_u \ln\|\nabla\Psi\| + \mathsf{a}^2. \tag{25}$$

While the last term is guaranteed non-negative, not much can be said about the first two terms, at least not at this stage.

B. ADM formalism

By definition, in any stably causal spacetime there is a globally defined "cosmic time" function $t(x)$ such that $\mathrm{d}t$ is always timelike. Then on the one hand the constant-t slices are always spacelike and can be used to set up an ADM decomposition of the metric, while on the other hand $u = -(\mathrm{d}t)^\sharp/\|\mathrm{d}t\|$ is a vorticity-free unit timelike congruence, so that the results of the previous subsection apply. (As usual, $\mathrm{d}t^\sharp$ denotes the vector obtained form the one-form $\mathrm{d}t$ by "raising the index," similarly $u^\flat$ will denote the one-form obtained from the vector u by "lowering the index.")

Consequently the extended Raychaudhuri equation can now be cast in the form

$$R_{\hat{t}\hat{t}} = -K_{ab}K^{ab} + K^2 + \nabla_a\left(K u^a + \frac{\mathrm{d}u^a}{\mathrm{d}s}\right). \tag{26}$$

This result complements and reinforces the information one obtains from the Gauss equations—see, for example, Misner, Thorne, and Wheeler [21] pp. 505–520, or Rendall [23] pp. 23–24. The Gauss equations (for a spacelike hypersurface) are

$${}^{(4)}R_{abcd} = {}^{(3)}R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}. \tag{27}$$

Contracting once

$${}^{(4)}R_{ab} = {}^{(3)}R_{ab} - {}^{(4)}R_{acbd}u^c u^d + \mathrm{tr}(K)K_{ab} - (K^2)_{ab}. \tag{28}$$

Contracting a second time

$${}^{(4)}R = {}^{(3)}R - 2\,{}^{(4)}R_{ab}u^a u^b + K^2 - \mathrm{tr}(K^2). \tag{29}$$

But now, since ${}^{(4)}R_{ab}u^a u^b$ has been given to us via the extended Raychaudhuri equation, we easily see that for a spacelike hypersurface

$${}^{(4)}R = {}^{(3)}R + \mathrm{tr}(K^2) - K^2 - 2\nabla_a\left(K u^a + \frac{\mathrm{d}u^a}{\mathrm{d}s}\right). \tag{30}$$

Traditional derivations of this result are sometimes somewhat less than transparent, and viewing Eq. (30) as an extension of the timelike Raychaudhuri equation is the cleanest derivation we have been able to develop. To see some of the deeper connections with the ADM formalism read (for example) Sec. 21.6 on pp. 519–520 of Misner, Thorne, and Wheeler [21]; note especially Eq. (21.88). See also exercise. (21.10) on p. 519. Also note the discussion by Padmanabhan and Patel in Refs. [18–20]. Also, we should warn the reader that Wald uses an opposite sign convention for the extrinsic curvature. See specifically Wald [3] Eq. (10.2.13) on p. 256.

C. Static spacetimes

Let us now take the discussion in a rather different direction, and assume that the spacetime is *static*. That is, there exists a hypersurface-orthogonal Killing vector k^a that is timelike at spatial infinity. Because it is hypersurface orthogonal then $k_a \propto \nabla_a\Psi$, and so $u^a = k^a/\|k\|$ is a set of FIDOs of the type considered in the previous section. But since k^a is also a Killing vector we have $k_{(a;b)} = 0$ and so (temporarily setting $\|k\| = e^\varphi$ for calculational simplicity) obtain the quite standard result that in this situation

$$\begin{aligned} u_{(a;b)} &= \nabla_{(a}\{k e^{-\varphi}\}_{b)} = e^{-\varphi}(k_{(a;b)} - k_{(b}\nabla_{a)}\varphi) \\ &= -e^{-\varphi}k_{(b}\nabla_{a)}\varphi = -u_{(b}\nabla_{a)}\varphi = -u_{(b}\varphi_{,a)} \\ &= -u_{(a}\varphi_{,b)}. \end{aligned} \tag{31}$$

Hence

$$\theta_{ab} = 0 \Rightarrow K_{ab} = 0 \Rightarrow K = 0. \tag{32}$$

That is, in static spacetimes the extrinsic curvature of the time-slices is zero (in addition to the congruence being vorticity free). The Raychaudhuri equation then specializes to the particularly simple geometrical result

$$R_{ab}u^a u^b = \nabla_a\left(\frac{\mathrm{d}u^a}{\mathrm{d}s}\right). \tag{33}$$

This is essentially the technical result we used in our derivation of an entropy bound for static spacetimes

[24,25], though in those articles we had derived it from an old result due to Landau and Lifshitz [26]. (The original Landau–Lifshitz result is obtained via a straightforward but tedious series of index manipulations, with little geometrical insight.)

D. Stationary spacetime—Killing congruence

What can we now do for *stationary*, as opposed to *static* spacetimes? (This distinction is relevant to "rotating spacetimes," for example, Kerr spacetimes versus Schwarzschild spacetimes. See for instance [27–30].) The (asymptotically) timelike Killing vector $k = \partial_t$ [that is, $k^a = (1; 0, 0, 0)^a$] is no longer hypersurface orthogonal. Nevertheless we can still define the timelike Killing congruence

$$u^a = \frac{k^a}{\|k\|}. \tag{34}$$

This timelike congruence corresponds to a class of FIDOs [not ZAMOs, not zero angular momentum observers] that sit at fixed spatial coordinate position [31,32]. This timelike congruence, even though it is *not* hypersurface orthogonal, still satisfies Eq. (31). So even though there is no longer any interpretation of the shear in terms of an extrinsic curvature, we still have

$$\theta_{ab} = 0, \tag{35}$$

whence both

$$\sigma_{ab} = 0; \quad \text{and} \quad \theta = 0. \tag{36}$$

Therefore,

$$R_{ab}u^a u^b = \omega^2 + \nabla_a\left(\frac{du^a}{ds}\right). \tag{37}$$

However, unless further assumptions are made, we cannot do much with the ω^2 term. Generically (again temporarily setting $\|k\| = e^{\varphi}$ for calculational simplicity) we have

$$u_{[a;b]} = -\nabla_{[a}\{ke^{-\varphi}\}_{b]} = e^{-\varphi}(k_{[a;b]} - k_{[a}\nabla_{b]}\varphi) = \frac{k_{[a;b]}}{\|k\|} - u_{[a}\nabla_{b]}\varphi. \tag{38}$$

This implies

$$\omega^{ab} = h^{ac}h^{bd}\frac{k_{[c;d]}}{\|k\|}, \tag{39}$$

whence

$$R_{ab}u^a u^b = +\frac{h^{ac}h^{bd}k_{[a;b]}k_{[c;d]}}{\|k\|^2} + \nabla_a\left(\frac{du^a}{ds}\right). \tag{40}$$

Unfortunately this does not simplify any further, and without further assumptions for the timelike Killing congruence on a stationary spacetime we should just be satisfied by the *inequality*

$$R_{ab}u^a u^b \geq \nabla_a\left(\frac{du^a}{ds}\right). \tag{41}$$

E. Stationary axisymmetric spacetimes

In a stationary axisymmetric spacetime one could also consider the vorticity-free congruence of Sec. III A. (Not the Killing congruence of Sec. III D.) Because of the axisymmetry the congruence $u = -(dt)^\sharp/\|dt\|$ must then be a linear combination of the two Killing vectors, $k_t = \partial_t$ and $k_\phi = \partial_\phi$, in which case the expansion scalar is zero: $\theta = \nabla \cdot u = 0$. In this case Eq. (18) reduces to

$$R_{ab}u^a u^b = -\sigma^2 + \nabla_a\left(\frac{du^a}{ds}\right), \tag{42}$$

which implies, for the natural vorticity-free congruence on an stationary axisymmetric spacetime

$$R_{ab}u^a u^b \leq \nabla_a\left(\frac{du^a}{ds}\right). \tag{43}$$

It is this particular inequality that we used in Ref. [32] to place an entropy bound on rotating fluid blobs. (Note that the direction of the inequality has changed between Eqs. (41) and (43) but that is merely due to the fact that we are using *different* timelike congruences.)

IV. SINGLE UNIT SPACELIKE VECTOR FIELD

In counterpoint, in this section we now let v^a be a field of unit *spacelike* vectors. The projection tensor becomes

$$h_{ab} = g_{ab} - v_a v_b. \tag{44}$$

In contrast to the timelike situation the projection tensor is now of indefinite signature $\{-1, +1, +1, 0\}$. One can still formally define the quantities θ_{ab}, θ, σ_{ab}, and ω_{ab}, (now constructed using the spacelike vector field v_a), but they no longer have the same physical interpretation in terms of shear and vorticity. Furthermore since the projection tensor has indefinite signature we now *cannot* guarantee either $\sigma^2 \geq 0$ or $\omega^2 \geq 0$. On the other hand, the Raychaudhuri equation itself is formally unaffected. That is, the fundamental Eqs. (1), (9), (11), and (30), continue to hold as they stand.

If we now consider a vorticity-free spacelike congruence, it will be hypersurface orthogonal to a timelike hypersurface. (That is, the normal to the hypersurface is spacelike, while the tangent space to the hypersurface can be chosen to have a basis of one timelike and two spacelike tangent vectors.)

In this situation we can without loss of generality set $v = (d\Psi)^\sharp/\|d\Psi\|$. Then $\omega_{ab} \to 0$, while in terms of the extrinsic curvature $\sigma_{ab} \to -K_{ab}$ as for vorticity-free timelike congruencies. Thus, Eq. (23) is formally unaffected and can now be cast in the form

$$R_{\hat{n}\hat{n}} = -K_{ab}K^{ab} + K^2 + \nabla_a\left(Kv^a + \frac{dv^a}{ds}\right). \tag{45}$$

Note however that we can no longer guarantee the nonnegativity of $K^{ab}K_{ab}$. Furthermore, because v is now a spacelike normal to a timelike hypersurface there is a key sign flip in the Gauss equations, which now read

$$^{(4)}R_{abcd} = {}^{(3)}R_{abcd} - K_{ac}K_{bd} + K_{ad}K_{bc}. \tag{46}$$

Contracting twice

$$^{(4)}R = {}^{(3)}R + 2{}^{(4)}R_{ab}u^a u^b + \mathrm{tr}(K^2) - K^2. \tag{47}$$

Therefore for a timelike hypersurface we have

$$^{(4)}R = {}^{(3)}R - \mathrm{tr}(K^2) + K^2 + 2\nabla_a\left(K v^a + \frac{\mathrm{d}v^a}{\mathrm{d}s}\right). \tag{48}$$

In summary, for spacelike congruences the Raychaudhuri equation itself is formally unaffected (though the projection tensor is slightly different and we can no longer rely on the non-negativity of σ^2 and ω^2). However applications of the Raychaudhuri equation, specifically anything involving the Gauss equations for embedded hypersurfaces, typically exhibit a limited number of sign flips.

V. SINGLE NON-NORMALIZED VECTOR FIELD

Let us now consider an *non-normalized* vector field u^a, either spacelike, timelike, or null. What if anything can we say about the quantity

$$R_{ab}u^a u^b = ??? \tag{49}$$

Following and modifying the discussion of Wald [3], see. (E.2.28) on p. 464:

$$\begin{aligned} R_{ab}u^a u^b &= R^c{}_{acb}u^a u^b = -u^a[\nabla_a\nabla_b - \nabla_b\nabla_a]u^b \\ &= -\nabla_a(u^a\nabla_b u^b) + (\nabla_a u^a)(\nabla_b u^b) \\ &\quad + \nabla_b(u^a\nabla_a u^b) - (\nabla_b u^a)(\nabla_a u^b) \\ &= \nabla_a(-u^a\nabla_b u^b + u^b\nabla_b u^a) + (\nabla\cdot u)^2 \\ &\quad - (\nabla_b u_a)(\nabla^a u^b) \\ &= \nabla\cdot\{(u\cdot\nabla)u - (\nabla\cdot u)u\} + (\nabla\cdot u)^2 \\ &\quad - (\nabla_b u_a)(\nabla^a u^b). \end{aligned} \tag{50}$$

In obvious notation, using $\theta = \nabla\cdot u$, this can be cast as

$$\begin{aligned} R_{ab}u^a u^b &= \nabla\cdot\{\nabla_u u - \theta u\} + \theta^2 - \nabla_{(a}u_{b)}\nabla^{(a}u^{b)} \\ &\quad + \nabla_{[a}u_{b]}\nabla^{[a}u^{b]}. \end{aligned} \tag{51}$$

This result can be viewed as yet another generalization of the standard Raychaudhuri equation. The advantage of this particular formula is that we have not carried out any projections, and have not even committed ourselves to the nature of the congruence, be it spacelike, timelike, or null. One disadvantage is that because of the Lorentzian signature of spacetime we *cannot* (at least not without further assumptions) guarantee either

$$\nabla_{(a}u_{b)}\nabla^{(a}u^{b)} \geq 0??? \tag{52}$$

or

$$\nabla_{[a}u_{b]}\nabla^{[a}u^{b]} \geq 0??? \tag{53}$$

Two specific applications come readily to mind:

(i) For any Killing vector $u^a = k^a$ we have $\nabla_{(a}u_{b)} = 0$, and consequently $\theta = 0$. Therefore for any Killing vector whatsoever we have the particularly pleasant result

$$R_{ab}k^a k^b = \nabla\cdot\{\nabla_k k\} + \nabla_{[a}k_{b]}\nabla^{[a}k^{b]}. \tag{54}$$

(ii) For any one arbitrary exact one-form $u = d\Psi$, even a locally exact one-form, we have $\nabla_{[a}u_{b]} = 0$, while $\theta = \nabla^2\Psi$ and $\nabla_{(a}u_{b)}\nabla^{(a}u^{b)} = \Psi_{;a;b}\Psi^{;a;b}$. Therefore, for any locally exact one-form whatsoever we have

$$\begin{aligned} R^{ab}(\mathrm{d}\Psi)_a(\mathrm{d}\Psi)_b &= \nabla\cdot\{\nabla_{\mathrm{d}\Psi}\mathrm{d}\Psi - (\nabla^2\Psi)\mathrm{d}\Psi\} \\ &\quad + (\nabla^2\Psi)^2 - \Psi_{;a;b}\Psi^{;a;b}. \end{aligned} \tag{55}$$

In fact, Ψ could simply be one of the spacetime coordinates (defined on some suitable local coordinate patch) in which case this version of the Raychaudhuri equation turns into a statement about the diagonal components of the Ricci tensor in a coordinate basis

$$\begin{aligned} R^{\Psi\Psi} &= \nabla\cdot\{\nabla_{\mathrm{d}\Psi}\mathrm{d}\Psi - (\nabla^2\Psi)\mathrm{d}\Psi\} + (\nabla^2\Psi)^2 \\ &\quad - \Psi_{;a;b}\Psi^{;a;b}. \end{aligned} \tag{56}$$

More boldly, if one chooses Ψ to be a harmonic coordinate, ($\nabla^2\Psi = 0$), and this can always be done locally, then we have

$$R^{\Psi\Psi} = \nabla\cdot\{\nabla_{\mathrm{d}\Psi}\mathrm{d}\Psi\} - \Psi_{;a;b}\Psi^{;a;b}. \tag{57}$$

In summary, this extension of the Raychaudhuri equation to non-normalized vector fields has given us some useful computational formulae for the diagonal part of the Ricci tensor.

VI. TWO NON-NORMALIZED VECTOR FIELDS

We shall now ask if it is possible to extract any useful information by considering two different (non-normalized) congruences simultaneously.

A. Motivation

To motivate this particular extension of the Raychaudhuri equation, recall that many decades ago Landau and Lifshitz had shown that in any stationary spacetime [26] (see. Sec. 105, Eq. (105.22); for a recent application of this result see [24,25]):

$$R_0{}^0 = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{0a}\Gamma^i{}_{a0}). \tag{58}$$

(Here $a \in \{0, 1, 2, 3\}$; $i \in \{1, 2, 3\}$.) But because the metric is stationary (t independent) we can also write this as

$$R_0{}^0 = \frac{1}{\sqrt{-g_4}}\partial_b(\sqrt{-g_4}g^{0a}\Gamma^b{}_{a0}). \tag{59}$$

To begin converting this into a coordinate-free statement, note that

$$R_0{}^0 = R^a{}_b(\mathrm{d}t)_a(\partial_t)^b = R^a{}_b(\mathrm{d}t)_a k^b. \tag{60}$$

Here we have had to use *both* the timelike Killing vector k, for which $k^a = (\partial_t)^a = (1, 0, 0, 0)^a$, *and* the one-form $\mathrm{d}t$, for which $(\mathrm{d}t)_a = (1, 0, 0, 0)_a$. Then by direct computation we see

$$\begin{aligned} g^{0a}\Gamma^b{}_{a0} &= g^{ca}\Gamma^b{}_{ad}(\mathrm{d}t)_c k^d \\ &= \Gamma^b{}_{cd}(\mathrm{d}t)^c k^d \\ &= \Gamma^b{}_{cd}k^c(\mathrm{d}t)^d \\ &= \{\partial_d k^b + \Gamma^b{}_{cd}k^c\}(\mathrm{d}t)^d \\ &= (\nabla_d k^b)(\mathrm{d}t)^d \\ &= (\mathrm{d}t)^d(\nabla_d k^b). \end{aligned} \tag{61}$$

But then

$$\begin{aligned} R_0{}^0 &= \frac{1}{\sqrt{-g_4}}\partial_b(\sqrt{-g_4}g^{0a}\Gamma^b{}_{a0}) \\ &= \frac{1}{\sqrt{-g_4}}\partial_b(\sqrt{-g_4}(\mathrm{d}t)^d(\nabla_d k^b)) = \nabla_b\{(\mathrm{d}t)^d(\nabla_d k^b)\}. \end{aligned} \tag{62}$$

So the Landau–Lifshitz result is equivalent to the statement that in any stationary spacetime

$$R^a{}_b(\mathrm{d}t)_a k^b = \nabla_b\{(\mathrm{d}t)^d(\nabla_d k^b)\} = \nabla\cdot(\nabla_{\mathrm{d}t^\sharp}k). \tag{63}$$

So some linear combination of Ricci tensor components is given by a pure divergence. Note that two *different* vector fields are involved. This observation naturally leads to the question: Is it possible to come up with a variant of the Raychaudhuri equation that depends on *two* congruences u^a and v^a? Something of the form

$$R_{ab}u^a v^b = ??? \tag{64}$$

We shall see how this is done below.

For now, let us mention that

$$\begin{aligned} (\nabla_d k^b)(\mathrm{d}t)^d &= (\nabla^d k^b)(\mathrm{d}t)_d = -(\nabla^b k^d)(\mathrm{d}t)_d \\ &= -\nabla^b\{k^d(\mathrm{d}t)_d\} + k^d\nabla^b(\mathrm{d}t)_d \\ &= -\nabla^b\{1\} + k^d\nabla^b\nabla_d t = k^d\nabla^b\nabla_d t \\ &= k^d\nabla_d\nabla^b t. \end{aligned} \tag{65}$$

So the Landau-Lifshitz result can also be written in the alternative form

$$R^a{}_b(\mathrm{d}t)_a k^b = \nabla_b\{k^d\nabla_d\nabla^b t\} = \nabla\cdot(\nabla_k \mathrm{d}t^\sharp). \tag{66}$$

Finally, note that

$$(\nabla_d k^b)(\mathrm{d}t)^d(\mathrm{d}t)_b = (\nabla_d k_b)(\mathrm{d}t)^d(\mathrm{d}t)^b = 0, \tag{67}$$

so the vector $\nabla_{\mathrm{d}t^\sharp}k = \nabla_k \mathrm{d}t^\sharp$ is perpendicular to $\mathrm{d}t^\sharp$.

B. Construction

Following and modifying the discussion of Wald [3], see Eq. (E.2.28) on p. 464:

$$\begin{aligned} R_{ab}u^a v^b &= R^c{}_{acb}u^a v^b = -u^a[\nabla_a\nabla_b - \nabla_b\nabla_a]v^b \\ &= -\nabla_a(u^a\nabla_b v^b) + (\nabla_a u^a)(\nabla_b v^b) \\ &\quad + \nabla_b(u^a\nabla_a v^b) - (\nabla_b u^a)(\nabla_a v^b) \\ &= \nabla_a(-u^a\nabla_b v^b + u^b\nabla_b v^a) + (\nabla\cdot u)(\nabla\cdot v) \\ &\quad - (\nabla_b u_a)(\nabla^a v^b). \end{aligned} \tag{68}$$

With minor notational changes and given the symmetry of the Ricci tensor this can also be written as

$$\begin{aligned} R_{ab}u^a v^b &= \nabla\cdot\{(u\cdot\nabla)v - (\nabla\cdot v)u\} + (\nabla\cdot u)(\nabla\cdot v) \\ &\quad - (\nabla_b u_a)(\nabla^a v^b), \end{aligned} \tag{69}$$

and

$$\begin{aligned} R_{ab}u^a v^b &= \nabla\cdot\{(v\cdot\nabla)u - (\nabla\cdot u)v\} + (\nabla\cdot u)(\nabla\cdot v) \\ &\quad - (\nabla_b u_a)(\nabla^a v^b). \end{aligned} \tag{70}$$

Furthermore (in obvious notation) this can again be rewritten as

$$\begin{aligned} R_{ab}u^a v^b &= \nabla\cdot\{\nabla_u v - \theta_v u\} + \theta_u\theta_v - \nabla_{(a}u_{b)}\nabla^{(a}v^{b)} \\ &\quad + \nabla_{[a}u_{b]}\nabla^{[a}v^{b]}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} R_{ab}u^a v^b &= \nabla\cdot\{\nabla_v u - \theta_u v\} + \theta_u\theta_v - \nabla_{(a}u_{b)}\nabla^{(a}v^{b)} \\ &\quad + \nabla_{[a}u_{b]}\nabla^{[a}v^{b]}. \end{aligned} \tag{72}$$

Note the similarities to the single-congruence case, and note particularly the presence of a divergence term generalizing the standard Raychaudhuri equation. To check the equivalence of these two formulae note

$$\begin{aligned} &(\nabla_u v - \theta_v u) - (\nabla_v u - \theta_u v) \\ &\quad = [\nabla_u v + \theta_u v] - [\nabla_v u + \theta_v u] \\ &\quad = \nabla\cdot[u\otimes v - v\otimes u] \\ &\quad = \nabla\cdot[u\wedge v]. \end{aligned} \tag{73}$$

That is, the difference of these two currents is the divergence of a two-form, which makes it automatically closed.

C. Generalizing the Landau–Lifshitz result

Let us now take $u = k$ to be any Killing vector, and furthermore let $v^\flat$ be an arbitrary (locally) exact one-form. Then $v = (\mathrm{d}\Psi)^\sharp$ where $\Psi(x)$ is an arbitrary scalar.

Furthermore $\nabla_{(a}u_{b)} = \nabla_{(a}k_{b)} = 0$, and so we see $\theta_u = 0$. Finally, $\nabla_{[a}v_{b]} = \nabla_{[a}\nabla_{b]}\Psi = 0$, so from Eq. (71) we have

$$R_{ab}k^a\nabla^b\Psi = \nabla\cdot\{\nabla_k \mathrm{d}\Psi - (\nabla^2\Psi)k\}, \tag{74}$$

while from Eq. (72) we have

$$R_{ab}k^a\nabla^b\Psi = \nabla\cdot\{\nabla_{\mathrm{d}\Psi^\sharp}k\}. \tag{75}$$

These two equations nicely generalize the Landau–Lifshitz result to any arbitrary Killing vector and any arbitrary (locally) exact one form $\mathrm{d}\Psi$, not just $\mathrm{d}t$. (That these two formulae are equivalent follows from the discussion in the previous section above.) Note the (standard) Landau–Lifshitz result corresponds to the special case $k^a \to (\partial_t)^a$ and $\Psi \to t$.

Now choose a coordinate system adapted to the Killing vector k. Let $k = \partial_{\mathrm{K}}$ define a Killing coordinate K, so that all geometrical objects are independent of the coordinate K. Let Ψ also be viewed as a coordinate, relabel it as x^a, possibly distinct from K, and with no claim that x^a necessarily corresponds to a Killing vector. Then

$$R_{\mathrm{K}}{}^a = \nabla\cdot\{\nabla_{(\mathrm{d}x^a)^\sharp}\partial_{\mathrm{K}}\}. \tag{76}$$

Unwrapping the covariant derivatives we see

$$R_{\mathrm{K}}{}^a = \frac{1}{\sqrt{-g_4}}\partial_b(\sqrt{-g_4}g^{ac}\Gamma^b{}_{c\mathrm{K}}). \tag{77}$$

If we now let the index i range over every coordinate except the Killing coordinate K then, because all geometrical objects are independent of the coordinate K, we have

$$R_{\mathrm{K}}{}^a = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{ac}\Gamma^i{}_{c\mathrm{K}}). \tag{78}$$

This last equation, while ultimately based on our two-congruence extension of the Raychaudhuri Eq. (71), is now very much in Landau–Lifshitz form, but is definitely considerably more powerful than the original Landau-Lifshitz result.

D. Landau-Lifshitz in axial symmetry

In stationary axisymmetric spacetimes considerably more can be said: Since in a stationary spacetime with axial symmetry we have a second azimuthal Killing vector $k^a \to (\partial_\phi)^a$, and could also consider $\Psi \to \phi$, then there are three additional Landau-Lifshitz like results:

$$R_\phi{}^t = R_{ab}(\partial_\phi)^a\nabla^b t = \nabla\cdot\{\nabla_{\mathrm{d}t^\sharp}\partial_\phi\}; \tag{79}$$

$$R_t{}^\phi = R_{ab}(\partial_t)^a\nabla^b \phi = \nabla\cdot\{\nabla_{\mathrm{d}\phi^\sharp}\partial_t\}; \tag{80}$$

and

$$R_\phi{}^\phi = R_{ab}(\partial_\phi)^a\nabla^b \phi = \nabla\cdot\{\nabla_{\mathrm{d}\phi^\sharp}\partial_\phi\}. \tag{81}$$

Let the indices $A, B \in \{t, \phi\}$ then we can collect these results (four of them altogether) in a single equation

$$R_A{}^B = \nabla\cdot\{\nabla_{(\mathrm{d}x^B)^\sharp}\partial_A\}. \tag{82}$$

Unwrapping the covariant derivatives

$$R_A{}^B = \frac{1}{\sqrt{-g_4}}\partial_b(\sqrt{-g_4}g^{Ba}\Gamma^b{}_{aA}). \tag{83}$$

If we now let the index i range over every coordinate except the two Killing coordinates t and ϕ, then

$$R_A{}^B = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{Ba}\Gamma^i{}_{aA}). \tag{84}$$

Making this all very explicit, there are now four Landau-Lifshitz like results in total. They are

$$R_t{}^t = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{ta}\Gamma^i{}_{at}); \tag{85}$$

$$R_t{}^\phi = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{\phi a}\Gamma^i{}_{at}); \tag{86}$$

$$R_\phi{}^t = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{ta}\Gamma^i{}_{a\phi}); \tag{87}$$

$$R_\phi{}^\phi = \frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{\phi a}\Gamma^i{}_{a\phi}). \tag{88}$$

Furthermore, recall that in stationary axisymmetric spacetimes we can always choose coordinates to block diagonalize the metric: $g_{ab} = g_{AB}\oplus g_{ij}$. But then

$$g^{Ba}\Gamma^i{}_{aA} = g^{BC}\Gamma^i{}_{CA} = g^{BC}g^{ij}\Gamma_{jCA} = -\frac{1}{2}g^{BC}g^{ij}\partial_j g_{CA}. \tag{89}$$

So finally we have the relatively compact result

$$R_A{}^B = -\frac{1}{2}\frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{BC}g^{ij}\partial_j g_{CA}). \tag{90}$$

This can be rearranged in a number of different ways. As an illustration we point out

$$R_{AB} = -\frac{1}{2}\frac{1}{\sqrt{-g_4}}\partial_i(\sqrt{-g_4}g^{ij}\partial_j g_{AB}) + \frac{1}{2}g^{ij}\partial_i g_{AC}g^{CD}\partial_j g_{DB}. \tag{91}$$

We again see that our two-congruence extension of the Raychaudhuri equation has given us additional useful information regarding the Ricci tensor which might be difficult to extract by other means.

VII. EINSTEIN EQUATIONS AND STRESS ENERGY

We have very carefully avoided use of the Einstein equations up to this stage, to emphasize that the Raychaudhuri equations are in essence purely geometrical, relating components of the Ricci tensor to information about vector congruences. If we now wish to see the connections to the stress-energy tensor, let us first write the Einstein equations in the form

$$R_{ab} = 8\pi[T_{ab} - \tfrac{1}{2}T g_{ab}]. \tag{92}$$

Now, assuming u^a is a normalized unit timelike vector, it is always possible to decompose the stress energy in the form

$$T^{ab} = \rho u^a u^b + p h^{ab} + u^a q^b + q^a u^b + \pi^{ab}, \tag{93}$$

where ρ is the comoving energy density, p is the isotropic pressure, q^a is the energy flux, and π^{ab} is the anisotropic stress. We have

$$q^a u_a = 0; \qquad \pi^{ab} u_b = 0; \qquad \pi^{ab} h_{ab} = 0; \tag{94}$$

whence

$$T = -\rho + 3p, \tag{95}$$

and so

$$R_{ab} u^a u^b = 4\pi(\rho + 3p). \tag{96}$$

So whenever one is dealing with a timelike congruence u^a, the Raychaudhuri equation either gives information about or uses information about the combination $\rho + 3p$, and this is the combination of stress-energy components involved in the weak energy condition.

Now suppose one instead has a normalized spacelike unit vector v^a. One can always choose a unit timelike vector u^a that is orthogonal to this spacelike vector and again decompose the stress energy as above. Then

$$R_{ab} v^a v^b = 4\pi(\rho - p + 2\pi_{ab} v^a v^b). \tag{97}$$

That is, whenever one is dealing with a spacelike unit congruence v^a, the Raychaudhuri equation either gives information about or uses information about the combination $\rho - p$ in conjunction with the specific v-$v$component $\pi_{ab} v^a v^b$ of the anisotropic stress.

The cross term when one combines orthonormal spacelike and timelike unit vectors is

$$R_{ab} u^a v^b = -8\pi(q^a v_a). \tag{98}$$

That is, in this situation the Raychaudhuri equation either gives information about or uses information about the quantity of energy flux q^a in the direction v^a.

In all of these three cases you can either use information about the stress-energy tensor to provide information about the congruences, (typically leading to singularity theorems of some sort), or can use information about the congruences to provide information about the Ricci tensor, and hence the stress energy.

VIII. PERFECT FLUID MOTION

In this penultimate section, let us make even stronger assumptions about the matter content. Assume we are dealing with a perfect fluid so that

$$T^{ab} = \rho u^a u^b + p h^{ab}, \tag{99}$$

Assume further that the fluid is barotropic, so that we have $\rho = \rho(p)$. Then the Euler equation is

$$\mathsf{a}^a = -\frac{h^{ab}\nabla_b p}{\rho + p} \tag{100}$$

If we now define

$$H(p) = \int_0^p \frac{\mathrm{d}p}{\rho(p) + p}, \tag{101}$$

then

$$\mathsf{a}^a = -h^{ab}\nabla_b H. \tag{102}$$

Under these conditions

$$\nabla_a \mathsf{a}^a = -h^{ab}\nabla_a\nabla_b H - \theta\nabla_u H + \mathsf{a}^2. \tag{103}$$

So once one has a specific model for the physics underlying nongeodesic motion, one can sometimes decompose the terms appearing in the Raychauhuri equation a little further. In fact in the current context, if vorticity-free motion is additionally assumed, this yields the relativistic Bernoulli equation in the form [22]

$$H = -\ln\|\nabla\Psi\|. \tag{104}$$

IX. DISCUSSION AND CONCLUSIONS

In this somewhat technical article we have developed several useful and novel extensions of the usual Raychaudhuri equation. The main theme has been to relate various linear combinations of components of the Ricci tensor to divergences of suitably defined fluxes. [See for instance Eq. (30).] Sometimes this allows us to derive old results in a simpler and cleaner manner, sometimes one obtains new results, such as the non-normalized Raychauhuri-like equation (51) and the generalized Landau-Lifshitz type results in Eqs. (84). One potentially far-reaching result is the "two-congruence" extension of the Raychauduri equation presented in Eqs. (71) and (72)

We have worked with timelike congruences, spacelike congruences, and non-normalized congruences, in all cases being able to say considerably more (and sometimes much more) than the standard Raychaudhuri equation would imply. If we furthermore invoke the Einstein equations we can then relate various linear combinations of components of the stress-energy tensor to divergences of suitably defined fluxes.

Despite the long (55 year) history of the Raychaudhuri equation, it still has the potential to lead to new surprises.

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[1] A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955).
[2] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, England, 1973).
[3] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).
[4] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison–Wesley, San Francisco, 2004).
[5] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black Hole Mechanics* (Cambridge University Press, England, 2004).
[6] A. K. Raychaudhuri, Pramana J. Phys. **69**, 3 (2007).
[7] J. Ehlers, Pramana J. Phys. **69**, 7 (2007); See also Int. J. Mod. Phys. D **15**, 1573 (2006).
[8] N. Dadhich, Pramana J. Phys. **69**, 23 (2007).
[9] J. M. M. Senovilla, Pramana J. Phys. **69**, 31 (2007).
[10] S. Kar and S. SenGupta, Pramana J. Phys. **69**, 49 (2007).
[11] A. Ashtekar, Pramana J. Phys. **69**, 77 (2007).
[12] S. R. Das, Pramana J. Phys. **69**, 93 (2007).
[13] D. Brill, P. Khetarpal, and V. Kaul, Pramana J. Phys. **69**, 109 (2007).
[14] P. S. Joshi, Pramana J. Phys. **69**, 119 (2007).
[15] C. Mukku, S. M. Mahajan, and B. A. Bambah, Pramana J. Phys. **69**, 137 (2007).
[16] N. Ahmadi and M. Nouri-Zonoz, Pramana J. Phys. **69**, 147 (2007).
[17] A. P. Kouretsis and C. G. Tsagas, Phys. Rev. D **82**, 124053 (2010).
[18] T. Padmanabhan and A. Patel, arXiv:hep-th/0305165.
[19] T. Padmanabhan, Classical Quantum Gravity **21**, 4485 (2004).
[20] T. Padmanabhan and A. Patel, arXiv:gr-qc/0309053.
[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[22] M. Visser and C. Molina-Paris, New J. Phys. **12**, 095014 (2010).
[23] A. D. Rendall, *Partial Differential Equations in General Relativity* (Oxford University Press, England, 2008).
[24] G. Abreu and M. Visser, Phys. Rev. Lett. **105**, 041302 (2010).
[25] G. Abreu and M. Visser, arXiv:1011.4538.
[26] L. D. Landau and E. M. Lifschitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 2000), 4th ed. Chap. 11, 12, See esp Eq. (105.22), p. 358, Sec. 105.
[27] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[28] R. P. Kerr, published in: *Quasi-Stellar Sources and Gravitational Collapse: Including The Proceedings of the First Texas Symposium on Relativistic Astrophysics, Austin, Texas, on 16–18 December 1963*, edited by Ivor Robinson, Alfred Schild, and E. L. Schücking (University of Chicago Press, Chicago, 1965), p. 99.
[29] D. L. Wiltshire, M. Visser, and S. M. Scott, *The Kerr Spacetime: Rotating Black Holes in General Relativity* (Cambridge University Press, England, 2009).
[30] M. Visser, arXiv:0706.0622.
[31] K. S. Thorne, R. H. Price, and D. A. MacDonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, 1986).
[32] G. Abreu and M. Visser, J. High Energy Phys. 03 (2011) 056.