Nonspinning black holes in alternative theories of gravity

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We study two large classes of alternative theories, modifying the action through algebraic, quadratic curvature invariants coupled to scalar fields. We find one class that admits solutions that solve the vacuum Einstein equations and another that does not. In the latter, we find a deformation to the Schwarzschild metric that solves the modified field equations in the small-coupling approximation. We calculate the event horizon shift, the innermost stable circular orbit shift, and corrections to gravitational waves, mapping them to the parametrized post-Einsteinian framework.

I. INTRODUCTION

Although black holes (BHs) are one of the most striking predictions of general relativity (GR), they remain one of its least tested concepts. Electromagnetic observations have allowed us to infer their existence, but direct evidence of their nonlinear gravitational structure remains elusive. In the next decade, data from very long-baseline interferometry [1,2] and gravitational wave (GW) detectors [3–20] should allow us to image and study BHs in detail. Such observations will test GR in the dynamical, nonlinear, or strong-field regime, precisely where tests are currently lacking.

Testing strong-field gravity features of GR is of utmost importance to physics and astrophysics as a whole. This is because the particular form of BH solutions, such as the Schwarzschild and Kerr metrics, enters many calculations, including accretion disk structure, gravitational lensing, cosmology, and GW theory. The discovery that these metric solutions do not accurately represent real BHs could indicate a strong-field departure from GR with deep implications to fundamental theory.

Such tests require parametrizing deviations from Schwarzschild or Kerr. One such parametrization at the level of the metric is that of *bumpy BHs* [21–23], while another at the level of the GW observable is the *parametrized post-Einsteinian* (ppE) framework [24,25]. In both cases, such parametrizations are greatly benefited from knowledge of specific non-GR solutions, but few, 4D, analytic ones are known that represent regular BHs (except, perhaps, in dynamical Chern-Simons (CS) gravity [26,27] and Einstein-Dilaton-Gauss-Bonnet (EDGB) gravity [28–32]).

Most non-GR BH solutions are known through numerical studies. In this approach, one chooses a particular alternative theory, constructs the modified field equations, and then postulates a metric ansatz with arbitrary functions. One then derives differential equations for such arbitrary functions that are then solved and studied numerically. Such an approach was used, for example, to study BHs in EDGB gravity [28–32]

Another approach is to find non-GR BH solutions analytically through approximation methods. In this scheme, one follows the same route as in the numerical approach, except that the differential equations for the arbitrary functions are solved analytically through the aid of approximation methods, for example, by expanding in (a dimensionless function of) the coupling constants of the theory. Such a *small-coupling approximation* [26,33,34] treats the alternative theory as an *effective and approximate* model that allows for small GR deformations. This approach has been used to find an analytic, slowly rotating BH solution in dynamical CS modified gravity [26,27].

But not all BH solutions outside of GR must necessarily be different from standard GR ones. In fact, there exist many modified gravity theories where the Kerr metric remains a solution. This was the topic studied in [35], where it was explicitly shown that the Kerr metric is also a solution of certain $f(R)$ theories, nondynamical quadratic gravity theories, and certain vector-tensor gravity theories. Based on these fairly generic examples, it was then inferred that the astrophysical observational verification of the Kerr metric could not distinguish between GR and alternative theories of gravity.

Such an inference, however, is not valid, as it was later explicitly shown in [26]. Indeed, there are alternative gravity theories, such as dynamical CS modified gravity, where the Kerr metric is not a solution. This prompted us to study what class of modified gravity theories admit Kerr and which do not. We begin by considering the most general quadratic gravity theory with dynamical couplings, as this is strongly motivated by low-energy effective string actions [36–40]. When the couplings are static, we recover the results of [35], while, when they are dynamic, we find that the Kerr metric is not a solution. In the latter case, we find how the Schwarzschild metric must be modified to satisfy the corrected field equations. We explicitly compute the shift in the location of the event horizon and innermost stable circular orbit.

Such modifications to the BH nature of the spacetime induce corrections to the waveforms generated by binary

inspirals. We compute such modifications and show that they are of so-called second post-Newtonian (PN) order; i.e., they correct the GR result at $\mathcal{O}(v^4)$ relative to the leading-order Newtonian term, where v is the orbital velocity. We further show that one can map such corrections to the ppE framework [12], which proposes a model-independent, waveform family that interpolates between GR and non-GR waveform predictions. This result supports the suggestion that the ppE scheme can handle a large class of modified gravity models.

The remainder of this paper is organized as follows. Section II defines the set of theories we will investigate and computes the modified field equations. Section III solves for BH solutions in this class of theories. Section IV discusses properties of the solution, and Sec. V studies the impact that such BH modifications will have on the GW observable. Section VI concludes by pointing to future possible research directions. For the remainder of this paper, we use the following conventions: Latin letters in index lists stand for spacetime indices; parentheses and brackets in index lists stand for symmetrization and antisymmetrization, respectively, i.e., $A_{(ab)} = (A_{ab} + A_{ba})/2$ and $A_{[ab]} = (A_{ab} - A_{ba})/2$; and we use geometric units with $G = c = 1$.

II. QUADRATIC GRAVITY

Consider the wide class of alternative theories of gravity in four dimensions defined by modifying the Einstein-Hilbert action through all possible quadratic, algebraic curvature scalars, multiplied by constant or nonconstant couplings:

$$
\begin{aligned}
S \equiv \int d^4x \sqrt{-g} \{ &\kappa R + \alpha_1 f_1(\vartheta) R^2 + \alpha_2 f_2(\vartheta) R_{ab} R^{ab} \\
&+ \alpha_3 f_3(\vartheta) R_{abcd} R^{abcd} + \alpha_4 f_4(\vartheta) R_{abcd}{}^* R^{abcd} \\
&- \frac{\beta}{2} [\nabla_a \vartheta \nabla^a \vartheta + 2V(\vartheta)] + \mathcal{L}_{\text{mat}} \},
\end{aligned} \tag{1}
$$

where g is the determinant of the metric g_{ab}; $(R, R_{ab}, R_{abcd}{}^*, R_{abcd})$ are the Ricci scalar and tensor, the Riemann tensor, and its dual [27], respectively; $\mathcal{L}_{\text{mat}}$ is the Lagrangian density for other matter; ϑ is a scalar field; (α_i, β) are coupling constants; and $\kappa = (16\pi G)^{-1}$. All other quadratic curvature terms are linearly dependent, e.g., the Weyl tensor squared. Theories of this type are motivated from fundamental physics, such as in low-energy expansions of string theory [37–40].

Let us distinguish between two different types of theories: nondynamical and dynamical. In the former, all the couplings are constant ($f_i'(\vartheta) = 0$), and there is no scalar field ($\beta = 0$). Varying Eq. (1) with respect to the metric and setting $f_i(\vartheta) = 1$, we find the modified field equations

$$
G_{ab} + \frac{\alpha_1}{\kappa} \mathcal{H}_{ab} + \frac{\alpha_2}{\kappa} \mathcal{I}_{ab} + \frac{\alpha_3}{\kappa} \mathcal{J}_{ab} = \frac{1}{2\kappa} T_{ab}^{\text{mat}}, \tag{2}
$$

where T_{ab}^{mat} is the stress energy of matter, and

$$
\mathcal{H}_{ab} \equiv 2R_{ab}R - \tfrac{1}{2}g_{ab}R^2 - 2\nabla_{ab}R + 2g_{ab}\Box R, \tag{3a}
$$

$$
\begin{aligned}
\mathcal{I}_{ab} \equiv \Box R_{ab} &+ 2R_{acbd}R^{cd} - \tfrac{1}{2}g_{ab}R_{cd}R^{cd} \\
&+ \tfrac{1}{2}g_{ab}\Box R - \nabla_{ab}R,
\end{aligned} \tag{3b}
$$

$$
\begin{aligned}
\mathcal{J}_{ab} \equiv 8R^{cd}R_{acbd} &- 2g_{ab}R^{cd}R_{cd} + 4\Box R_{ab} \\
&- 2RR_{ab} + \tfrac{1}{2}g_{ab}R^2 - 2\nabla_{ab}R,
\end{aligned} \tag{3c}
$$

with $\nabla_a, \nabla_{ab} = \nabla_a \nabla_b$, and $\Box = \nabla_a \nabla^a$ the first and second covariant derivatives and the d'Alembertian, and using the Weyl identity $4C_a{}^{cde}C_{bcde} = g_{ab}C_{cdef}C^{cdef}$, with C_{abcd} the Weyl tensor.

The dynamical theory is specified through the action in Eq. (1), with $f_i(\vartheta)$ some function of the dynamical scalar field ϑ, with potential $V(\vartheta)$. For simplicity, we restrict attention here to functions that admit the Taylor expansion $f_i(\vartheta) = f_i(0) + f_i'(0)\vartheta + \mathcal{O}(\vartheta^2)$ about small ϑ, where $f_i(0)$ and $f_i'(0)$ are constants. The ϑ-independent terms, proportional to $f_i(0)$, lead to the nondynamical theory, and we thus ignore them henceforth. Let us then concentrate on $f_i(\vartheta) = c_i \vartheta$, where we reabsorb the constants $c_i = f_i'$ into α_i, such that $\alpha_i f_i(\vartheta) \rightarrow \alpha_i \vartheta$. The field equations are then

$$
\begin{aligned}
G_{ab} &+ \frac{\alpha_1}{\kappa} \mathcal{H}_{ab}^{(\vartheta)} + \frac{\alpha_2}{\kappa} \mathcal{I}_{ab}^{(\vartheta)} + \frac{\alpha_3}{\kappa} \mathcal{J}_{ab}^{(\vartheta)} + \frac{\alpha_4}{\kappa} \mathcal{K}_{ab}^{(\vartheta)} \\
&= \frac{1}{2\kappa} (T_{ab}^{\text{mat}} + T_{ab}^{(\vartheta)}),
\end{aligned} \tag{4}
$$

where $T_{ab}^{(\vartheta)} = \frac{\beta}{2} [\nabla_a \vartheta \nabla_b \vartheta - \frac{1}{2} g_{ab} (\nabla_c \vartheta \nabla^c \vartheta - 2V(\vartheta))]$ is the scalar field stress-energy tensor, and

$$
\begin{aligned}
\mathcal{H}_{ab}^{(\vartheta)} \equiv -4v_{(a}\nabla_{b)}R &- 2R\nabla_{(a}v_{b)} + g_{ab}(2R\nabla^c v_c + 4v^c \nabla_c R) \\
&+ \vartheta \left[2R_{ab}R - 2\nabla_{ab}R - \tfrac{1}{2}g_{ab}(R^2 - 4\Box R) \right],
\end{aligned} \tag{5a}
$$

$$
\begin{aligned}
\mathcal{I}_{ab}^{(\vartheta)} \equiv -v_{(a}\nabla_{b)}R &- 2v^c(\nabla_{(a}R_{b)c} - \nabla_c R_{ab}) + R_{ab}\nabla_c v^c \\
&- 2R_{c(a}\nabla^c v_{b)} + g_{ab}(v^c \nabla_c R + R^{cd}\nabla_c v_d) \\
&+ \vartheta \left[2R^{cd}R_{acbd} - \nabla_{ab}R + \Box R_{ab} \right. \\
&\left. + \tfrac{1}{2}g_{ab}(\Box R - R_{cd}R^{cd}) \right],
\end{aligned} \tag{5b}
$$

$$
\begin{aligned}
\mathcal{J}_{ab}^{(\vartheta)} \equiv -8v^c(\nabla_{(a}R_{b)c} - \nabla_c R_{ab}) &+ 4R_{acbd}\nabla^c v^d \\
- \vartheta [2(R_{ab}R - 4R^{cd}R_{acbd} &+ \nabla_{ab}R - 2\Box R_{ab}) \\
- \tfrac{1}{2}g_{ab}(R^2 - 4R_{cd}R^{cd})],&
\end{aligned} \tag{5c}
$$

$$
\mathcal{K}_{ab}^{(\vartheta)} \equiv 4v^c \epsilon_c{}^d{}_{e(a}\nabla^e R_{b)d} + 4\nabla_d v_c{}^* R_{(a}{}^c{}_{b)}{}^d,
$$

with $v_a \equiv \nabla_a \vartheta$, and ϵ^{abcd} the Levi-Civita tensor. Notice that $\alpha_4 \mathcal{K}_{ab} = \alpha_{\text{CS}} C_{ab}$, where α_{CS} and C_{ab} are the CS coupling constant and the CS C tensor [27]. The dynamical quadratic theory includes dynamical CS gravity as a special case. Variation of the action with respect to ϑ yields the scalar field equation of motion

$$\beta\Box\vartheta - \beta\frac{dV}{d\vartheta} = -\alpha_1 R^2 - \alpha_2 R_{ab}R^{ab} - \alpha_3 R_{abcd}R^{abcd}$$
$$- \alpha_4 R_{abcd}{}^*R^{abcd}. \quad (6)$$

Both the nondynamical and dynamical theories arise from a diffeomorphism invariant action, and thus, they lead to field equations that are covariantly conserved; i.e., the covariant divergence of Eq. (2) identically vanishes, while that of Eq. (5) vanishes upon the imposition of Eq. (6), unlike in nondynamical CS gravity [27].

III. NONSPINNING BLACK HOLE SOLUTION

A. Nondynamical theories

The modified field equations of the nondynamical theory have the interesting property that metrics for which the Ricci tensor vanishes are automatically solutions. One can see that if $R_{ab} = 0$, then Eqs. (3a)–(3c) vanish exactly, thus satisfying the modified field equations in Eq. (2). This generalizes the result in [35], as we here considered a more general action.

The reason for this simplification is the Gauss-Bonnet and Pontryagin identities. The integral of the Gauss-Bonnet term $\mathcal{G} \equiv R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ is proportional to the Euler characteristic $\mathcal{E}$, while that of the Pontryagin density $R_{abcd}{}^*R^{abcd}$ is proportional to the Chern number $\mathcal{C}$. Thus, the $R_{abcd}R^{abcd}$ and the $R_{abcd}{}^*R^{abcd}$ terms can be removed from the action in Eq. (1) in favor of $\mathcal{E}$ and $\mathcal{C}$. Since the variation of these constants vanishes identically, the field equations can be rewritten to depend only on the Ricci tensor and its trace.

This feature has a natural generalization for a wider class of alternative theories of gravity. If an action for an alternative theory contains the Riemann tensor or its dual *only* in a form that can be rewritten in terms of topological invariants (with no dynamical couplings), then the field equations will be free of Riemann, and thus, all vacuum GR solutions will also be solutions of such modified theories. Therefore, any action built from powers of the Ricci scalar or products of the Ricci tensor, possibly coupled to dynamical fields, and with Riemann tensors entering only as above, admits all vacuum GR solutions.

These results have important consequences for attempts to test GR in the strong field. Electromagnetic GR tests that aim at probing the Kerr nature of BHs would be insensitive to such modified theories. On the other hand, observations that probe the dynamics of the background, such as GW observations [4–20], would be able to constrain them.

B. Dynamical theories

The modified field equations in the dynamical theory, however, are not as simple, as clearly they are not satisfied when $R_{ab} = 0$. This is because $\mathcal{J}_{ab}^{(\vartheta)}$ depends on $\nabla^c v^d R_{abcd}$, and $\mathcal{K}_{ab}^{(\vartheta)}$ depends on $\nabla_d v_c{}^*R_{(a}{}^c{}_{b)}{}^d$. Let us search for small deformations away from the GR Schwarzschild metric that

preserve stationarity and spherical symmetry. The only relevant term here then is $\mathcal{J}_{ab}^{(\vartheta)}$, as $\mathcal{K}_{ab}^{(\vartheta)}$ vanishes in spherical symmetry, as already analyzed in [26].

We thus pose the ansatz

$$ds^2 = -f_0[1 + \epsilon h_0(r)]dt^2 + f_0^{-1}[1 + \epsilon k_0(r)]dr^2 + r^2 d\Omega^2, \quad (7)$$

and $\vartheta = \bar{\vartheta} + \epsilon\tilde{\vartheta}$, where $f_0 \equiv 1 - 2M_0/r$, with M_0 the "bare" or GR BH mass and (t, r, θ, ϕ) are Schwarzschild coordinates, while $d\Omega^2$ is the line element on the two-sphere. The free functions (h_0, k_0) are small deformations from the Schwarzschild metric, controlled by a function of the coupling constants (α_i, β) that we define below; ϵ is a bookkeeping parameter.

Before we solve the field equations, let us discuss the scalar field potential $V(\vartheta)$. There are two distinct choices for this potential: a flat ($V'(\vartheta) = 0$) or nonflat ($V'(\vartheta) \neq 0$) potential. For the nonflat case, the potential must be bounded from below for the theory to be globally stable, and thus it will contain one or more minima. The scalar field would tend towards the minimum of the potential, where the latter could be expanded as a quadratic function about the minimum (assumed here to be at zero): $V \approx \frac{1}{2}m_\vartheta^2\vartheta^2$. One might treat the flat potential as the limit $m_\vartheta \to 0$ of the above nonflat potential, but this limit is not continuous at the point $m_\vartheta = 0$. The massive case must thus be treated generically, and it turns out to be sufficiently complicated that we restrict our attention only to the massless (flat) case [41].

With this ansatz, we can solve the modified field equations and the scalar field's equation of motion order-by-order in ϵ. Through the small-coupling approximation, we treat $\alpha = \mathcal{O}(\epsilon)$ and $\beta = \mathcal{O}(\epsilon)$. To zeroth order in ϵ, the field equations are automatically satisfied because the Schwarzschild metric has a vanishing Ricci tensor. To this order, the scalar field equation can be solved to find

$$\bar{\vartheta} = \frac{\alpha_3}{\beta}\frac{2}{M_0 r}\left(1 + \frac{M_0}{r} + \frac{4}{3}\frac{M_0^2}{r^2}\right). \quad (8)$$

This is the same solution found in [36] for dilaton hair sourced in EDGB gravity. The scalar field depends only on α_3, since the term proportional to α_4 vanishes identically in a spherically symmetric background.

We can use this scalar field solution to solve the modified field equations to $\mathcal{O}(\epsilon)$. Requiring that the metric be asymptotically flat and regular at $r = 2M_0$, we find the unique solution $h_0 \equiv \mathcal{F}(1 + \tilde{h}_0)$ and $k_0 \equiv -\mathcal{F}(1 + \tilde{k}_0)$, where $\mathcal{F} \equiv -(49/40)\zeta(M_0/r)$ and

$$\tilde{h}_0 = \frac{2M_0}{r} + \frac{548}{147}\frac{M_0^2}{r^2} + \frac{8}{21}\frac{M_0^3}{r^3} - \frac{416}{147}\frac{M_0^4}{r^4} - \frac{1600}{147}\frac{M_0^5}{r^5},$$
$$\tilde{k}_0 = \frac{58}{49}\frac{M_0}{r} + \frac{76}{49}\frac{M_0^2}{r^2} - \frac{232}{21}\frac{M_0^3}{r^3} - \frac{3488}{147}\frac{M_0^4}{r^4} - \frac{7360}{147}\frac{M_0^5}{r^5},$$
$$(9)$$

and where we have defined the dimensionless coupling function $\zeta \equiv \alpha_3^2/(\beta \kappa M_0^4) = \mathcal{O}(\epsilon)$. This solution is the same as that found in EDGB gravity [42]. Our analysis shows that such a solution is the most general for all dynamical, algebraic, quadratic gravity theories, in spherical symmetry.

The demand that the metric deformation be regular everywhere outside the horizon has led to a term that changes the Schwarzschild BH mass; i.e., there is a correction to g_{tt} and g_{rr} that decays as $1/r$ at spatial infinity. We can then define the physical mass $M \equiv M_0[1 + (49/80)\zeta]$, such that the only modified metric components become $g_{tt} = -f(1 + h)$ and $g_{rr} = f^{-1}(1 + k)$ where $h = \zeta/(3f)(M/r)^3\tilde{h}$ and $k = -(\zeta/f)(M/r)^2\tilde{k}$, and

$$\tilde{h} = 1 + \frac{26M}{r} + \frac{66}{5}\frac{M^2}{r^2} + \frac{96}{5}\frac{M^3}{r^3} - \frac{80M^4}{r^4}, \quad (10)$$

$$\tilde{k} = 1 + \frac{M}{r} + \frac{52}{3}\frac{M^2}{r^2} + \frac{2M^3}{r^3} + \frac{16}{5}\frac{M^4}{r^4} - \frac{368}{3}\frac{M^5}{r^5}, \quad (11)$$

and where $f \equiv 1 - 2M/r$. Physical observables are related on the renormalized mass, not the bare mass. This renormalization was not performed by [42].

In fact, one need not fix the single constant of integration which appears in finding this solution. Any value of the integration constant, after renormalization, is absorbed into the renormalized mass. Rather than a family of spacetimes, there is a unique spacetime after renormalization.

The sign of the coupling constant can be determined by computing the energy carried by the scalar field in Eq. (8). The energy is $E_{(\vartheta)} \equiv \int_\Sigma T_{ab}^{(\vartheta)} t^a t^b \gamma^{1/2} d^3x$, where Σ is a $t = $ const. hypersurface outside of the horizon (so that it is spacelike everywhere), $t^a = (\partial/\partial t)^a$, and γ is the determinant of the metric intrinsic to Σ. We find that $E_{(\vartheta)} = (9/7)\zeta\kappa\pi M$. For stability reasons, we require that $E_{(\vartheta)} \geq 0$, which then implies $\zeta \geq 0$ and $\alpha_3^2/\beta \geq 0$.

Although we here considered nonspinning BHs, our analysis can be generalized to spinning ones, by separating the theory and its solutions into parity-even and parity-odd sectors. A parity transformation consists of the reflection $x^i \to -x^i$, which, for a spinning BH metric, implies $a \to -a$, where $|S^i| = M|a|$ is the magnitude of the spin angular momentum. Expanding the spinning BH solution as a power series in a/M, we see that the Kretschmann scalar $R_{abcd}R^{abcd}$ has only even powers of a/M (even-parity sector), while the Pontryagin density $^*RR$ has only odd powers of a/M (odd-parity sector). These quantities source the ϑ equation of motion, therefore, driving even and odd metric perturbations, respectively. The solution found here is of even parity and corresponds to the $\mathcal{O}(a^0)$ part of the metric expansion for a slowly spinning BH in dynamical quadratic gravity. The next order, $\mathcal{O}(a^1)$, is parity-odd and is sourced only by the Pontryagin density, since R^2, $R_{ab}R^{ab}$, and $R_{abcd}R^{abcd}$ are all even under parity. The solution sourced by just the Pontryagin density is

identical to that in dynamical Chern-Simons gravity (all $\alpha_i = 0$, except for α_4) and was found in [26]. From the parity arguments presented here, we see that the exact same modification arises at $\mathcal{O}(a^1)$ in the more general dynamical quadratic gravity considered here. Therefore, to $\mathcal{O}(a^1)$, the modification in dynamical quadratic gravity is simply the linear combination of the $\mathcal{O}(a^0)$ solution found here and the $\mathcal{O}(a^1)$ solution found in [26].

IV. PROPERTIES OF THE SOLUTION

The solution found is spherically symmetric, stationary, asymptotically flat, and regular everywhere except at $r = 0$. It represents a nonspinning BH with a real singularity at the origin, as evidenced by calculating the Kretschmann scalar expanded to $\mathcal{O}(\zeta)$: $K \equiv R_{abcd}R^{abcd} = \bar{K} - 32\zeta M^3/r^7\tilde{K}$, where $\bar{K} = 48M^2/r^6$, and

$$\tilde{K} = 1 + \frac{M}{2r} + \frac{72M^2}{r^2} + \frac{7M^3}{r^3} + \frac{64}{5}\frac{M^4}{r^4} - \frac{840M^5}{r^5}. \quad (12)$$

The location of the event horizon, i.e., the surface of infinite redshift, can be computed by solving $g_{tt} = 0$ to find $r_{\rm EH}/M = 2 - (49/40)\zeta$. The metric remains Lorentzian (i.e., $\text{sgn}(g) < 0$) everywhere outside $r_{\rm EH}$, provided ζ is sufficiently small (specifically, $0 < \zeta < (120/361)$).

One can also study point-particle motion in this background. Neglecting internal structure and spins, test-particle motion remains geodesic [43], and the equation of motion reduces to $\dot{r}^2/2 = V_{\rm eff}^{\rm GR} + \delta V_{\rm eff}$, where the overhead dot stands for differentiation with respect to proper time, and

$$V_{\rm eff}^{\rm GR} = \frac{E^2}{2} - \frac{L^2}{2r^2}f - \frac{f}{2}, \qquad \delta V_{\rm eff} = -\frac{1}{2}E^2 h - \frac{1}{2}V_{\rm eff}^{\rm GR}k, \quad (13)$$

where (E, L) are the conserved quantities induced by the timelike and azimuthal Killing vectors, i.e., the particle's energy and angular momentum per unit mass.

One can solve for the energy and angular momentum for circular orbits [44] through the conditions $\dot{r} = 0$ and $V_{\rm eff}' = 0$ to find $E = E_{\rm GR} + \delta E$ and $L = L_{\rm GR} + \delta L$, where $E_{\rm GR} = f(1 - 3M/r)^{-1/2}$, $L_{\rm GR} = (Mr)^{1/2}E_{\rm GR}/f$, and

$$\delta E = -\frac{\zeta}{12}\frac{M^3}{r^3}\left(1 - \frac{3M}{r}\right)^{-3/2}\left(1 + \frac{54M}{r} + \frac{198}{5}\frac{M^2}{r^2}\right. \\ \left. + \frac{252}{5}\frac{M^3}{r^3} - \frac{2384}{5}\frac{M^4}{r^4} + \frac{480M^5}{r^5}\right), \quad (14)$$

$$\delta L = -\frac{\zeta M}{4}\frac{M^{3/2}}{r^{3/2}}\left(1 - \frac{3M}{r}\right)^{-3/2}\left(1 + \frac{100}{3}\frac{M}{r} - \frac{30M^2}{r^2}\right. \\ \left. + \frac{16}{5}\frac{M^3}{r^3} - \frac{752}{3}\frac{M^4}{r^4} + \frac{320M^5}{r^5}\right). \quad (15)$$

From this expression, we can find the modified Kepler law by expanding $\omega \equiv L/r^2$ in the far field limit:

$$\omega^2 = \omega_{\mathrm{GR}}^2 \left[1 - \frac{\zeta}{2} \left(\frac{M}{r} \right)^2 \right], \qquad (16)$$

where $\omega_{\mathrm{GR}}^2 = M/r^3[1 + \mathcal{O}(M/r)]$. If, in addition to the above circular orbit conditions, one evaluates the marginal stability condition $V_{\mathrm{eff}}'' = 0$, one finds that the shift in the innermost stable circular orbit (ISCO) location is

$$\frac{r_{\mathrm{ISCO}}}{M} = 6 - \frac{16297}{9720} \zeta. \qquad (17)$$

V. IMPACT ON BINARY INSPIRAL GWS

As evidenced above, such a modified theory will introduce corrections to the binding energy of binary systems. Consider a binary with component masses $m_{1,2}$ and total mass $m = m_1 + m_2$. The binding energy, to leading $\mathcal{O}(m/r, \zeta)$, can be obtained from E_{GR} and δE in Eq. (14) by the transformation $m_1 m_2 \rightarrow m^2 \eta$ and expanding in $M/r \ll 1$. This trick works to leading order in ζ and in m/r only and it leads to

$$E_b(r) = -\frac{m^2 \eta}{2r} \left[1 + \frac{\zeta}{6} \left(\frac{m}{r} \right)^2 \right]. \qquad (18)$$

Using the modified Keplerian relation of the previous section, this becomes

$$E_b(F) = -\frac{1}{2}(2\pi m F)^{2/3} - \frac{1}{6} m \eta \zeta (2\pi m F)^2, \qquad (19)$$

to leading $\mathcal{O}(mF, \zeta)$, where F is the orbital frequency, and $\eta = m_1 m_2/m^2$ is the symmetric mass ratio. Such a modification to the binding energy will introduce corrections to the binary's orbital phase evolution at leading Newtonian order.

A calculation of the phase and amplitude waveform correction that accounts only for the leading-order binding energy modification is incomplete. First, higher $\mathcal{O}(m/r)$ terms in E_b are necessary for detailed GW tests. These terms, however, are not necessary to find the leading-order, functional form of the waveform correction; this is all one needs to map these modifications to the ppE scheme.

To be consistent, we must also consider the energy flux carried by the scalar field. This program involves solving for the perturbation on top of the background solution given in Eq. (8). The solution can be found using post-Newtonian integration techniques and is in preparation [45]. The modification to radiation reaction due to the scalar field is subdominant (of much higher post-Newtonian order) compared to the modification to the binding energy calculated here, as will be shown in a forthcoming paper [45].

Let us now compute the orbital phase correction due to modifications to the binding energy. The orbital phase for a binary in a circular orbit is simply

$$\phi(F) = \int^F (E')(\dot{E})^{-1} \omega \, d\omega, \qquad (20)$$

where $\omega = 2\pi F$ is the orbital angular frequency, $E' \equiv dE/d\omega$, and $\dot{E} = -(32/5)\eta^2 m^2 r^4 \omega^6$ is the loss of binding energy due to radiation. This expression for $\dot{E}$ is the GR quadrupole form, which was shown [17] to be valid in the small-coupling limit in asymptotically flat spacetimes when the action is of the form we use. Neglecting $\dot{E}^{(\vartheta)}$ and to leading $\mathcal{O}(m\omega, \zeta)$, the orbital phase

$$\phi = \phi_{\mathrm{GR}}[1 + \tfrac{25}{3}\zeta(2\pi m F)^{4/3}], \qquad (21)$$

where the GR phase is $\phi_{\mathrm{GR}} = -1/(32\eta)(2\pi m F)^{-5/3}$. The leading-order correction is of so-called second PN order, as it scales with $(mF)^{4/3}$ (down by $1/c^4$) relative to the leading-order GR result.

Similarly, we can compute the correction to the frequency-domain GW phase in the stationary phase approximation, by assuming that its rate of change is much more rapid than the GW amplitude's. This phase is (see, e.g., [46])

$$\Psi_{\mathrm{GW}} = 2\phi(t_0) - 2\pi f t_0, \qquad (22)$$

where t_0 satisfies the stationary phase condition $F(t_0) = f/2$, with f the GW frequency. Neglecting $\dot{E}^{(\vartheta)}$ and to leading $\mathcal{O}(m\omega, \zeta)$, we find that

$$\Psi_{\mathrm{GW}} = \Psi_{\mathrm{GW}}^{\mathrm{GR}}[1 + \tfrac{50}{3}\zeta \eta^{-4/5} u^{4/3}], \qquad (23)$$

where $u \equiv \pi \mathcal{M} f$ is the reduced frequency, and $\mathcal{M} = \eta^{3/5} m$ is the chirp mass. Similarly, the Fourier-domain amplitude scales as $|\tilde{h}| \propto \dot{F}(t_0)^{-1/2}$, which then leads to

$$|\tilde{h}| = |\tilde{h}|_{\mathrm{GR}} \left[1 + \tfrac{5}{6}\zeta u^{4/3} \eta^{-4/5} \right], \qquad (24)$$

where $|\tilde{h}_{\mathrm{GR}}|$ is the GW amplitude in GR. In principle, there could be additional corrections to $|h|$ from modifications to the first-order equations of motion, but [17] has shown that these vanish in the small-coupling approximation.

The modifications introduced to the inspiral waveforms can be mapped to parametrized waveform models that facilitate GR tests. In the ppE framework [24], the simplest parametrization is

$$\tilde{h} = |\tilde{h}|_{\mathrm{GR}}(1 + \alpha \eta^c u^a) \exp[i\Psi_{\mathrm{GW}}^{\mathrm{GR}}(1 + \beta \eta^d u^b)], \quad (25)$$

where $(\alpha, a, \beta, b, c, d)$ are ppE parameters. Our results clearly map to this parametrization, with $\alpha = (5/6)\zeta$, $\beta = (50/3)\zeta$, $a = 4/3 = b$, and $c = -4/5 = d$. Since the radiation carried by the scalar field is of higher post-Newtonian order, including it will not change these ppE parameters. Future GW constraint on these parameters could be translated into a bound on the class of alternative theories considered here.

Preliminary studies suggest that GW detectors, such as LIGO, could place interesting constraints on the parameter β. Given a signal-to-noise ratio of 20 for a comparable

mass binary inspiral signal, one might be able to constrain $\beta \lesssim 10^{-1}$ when $b = 4/3$ [47]. This bound would translate to a ζ constraint of $\zeta \lesssim 10^{-2}$, which should be compared to the current double binary pulsar constraint $\zeta \lesssim 10^7$ [25]. Since the effect calculated here occurs at second PN order, systems with strong gravity are required to probe it. Second PN order effects are unimportant in describing the spacetime of the solar system and known binary pulsars. GWs sourced in the strong field could place much stronger constraints on nonlinear strong-field deviations from GR relative to current solar system and binary pulsar bounds.

VI. FUTURE WORK

The study presented here shows that there is a wide class of modified gravity theories where Schwarzschild and Kerr are not solutions, yet their waveform modifications can be mapped to the ppE scheme. This study could be extended by investigating higher orders in ν and PN corrections to the waveform modifications. Such a calculation would require one to solve for the two-body metric in this specific class of theories. Although this can, in principle, be done within the PN scheme, in practice the calculation will be analytically quite difficult, due to the nonlinear terms introduced by the modified theory.

Another possible extension is to investigate the effect of different potential terms to the results presented here. For example, one could postulate a cosine potential and see how this modifies the solutions found. Such cosine potentials arise naturally due to nonlinear interactions in effective string actions. The inclusion of such a potential will probably render the problem nonanalytic, forcing us to solve the equations of motion for the scalar field numerically.

One other avenue of future research is to find analytic, closed-form solutions for BHs rotating arbitrarily fast in dynamical quadratic gravity. The analysis presented here applies only to nonrotating BHs, and we have discussed how it would be modified when considering slowly rotating BHs. Exact, closed-form solutions for rapidly rotating BHs, however, remain elusive. One might have to integrate the equations numerically to find such solutions. One possible line of attack is to evolve the field equations in a $3 + 1$ decomposition, starting with a dense and rotating scalar field configuration. Upon evolution, this scalar field will collapse into a rapidly rotating BH, yielding a numerical representation of the solution one seeks.

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