Using the Noether symmetry approach to probe the nature of dark energy

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We propose to use a model-independent criterion based on first integrals of motion, due to Noether symmetries of the equations of motion, in order to classify the dark energy models in the context of scalar field (quintessence or phantom) Friedmann-Lemaître-Robertson-Walker cosmologies. In general, the Noether symmetries play an important role in physics because they can be used to simplify a given system of differential equations as well as to determine the integrability of the system. The Noether symmetries are computed for nine distinct accelerating cosmological scenarios that contain a homogeneous scalar field associated with different types of potentials. We verify that all the scalar field potentials, presented here, admit the trivial first integral, namely, energy conservation, as they should. We also find that the exponential potential inspired from scalar field cosmology, as well as some types of hyperbolic potentials, include extra Noether symmetries. This feature suggests that these potentials should be preferred along the hierarchy of scalar field potentials. Finally, using the latter potentials, in the framework of either quintessence or phantom scalar field cosmologies that contain also a nonrelativistic matter (dark matter) component, we find that the main cosmological functions, such as the scale factor of the Universe, the scalar field, the Hubble expansion rate, and the metric of the Friedmann-Lemaître-Robertson-Walker space-time, are computed analytically. Interestingly, under specific circumstances the predictions of the exponential and hyperbolic scalar field models are equivalent to those of the ΛCDM model, as far as the global dynamics and the evolution of the scalar field are concerned. The present analysis suggests that our technique appears to be very competitive to other independent tests used to probe the functional form of a given potential and thus the associated nature of dark energy.

I. INTRODUCTION

The analysis of the available high quality cosmological data (supernovae type Ia, cosmic microwave background (CMB), galaxy clustering, power spectrum, etc.) have converged during the last decade towards a cosmic expansion history that involves a spatially flat geometry and a recently initiated accelerated expansion of the Universe (see [1–6] and references therein). From a theoretical point of view, an easy way to explain this expansion is to consider an additional energy component, usually called dark energy (DE) with negative pressure, that dominates the Universe at late times. The simplest DE candidate corresponds to the cosmological constant (see [7–9] for reviews). Indeed the so-called spatially flat concordance ΛCDM model, which includes cold dark matter (DM) and a cosmological constant, (Λ) fits accurately the current observational data and thus it is an excellent candidate to be the model which describes the observed Universe.

However, the concordance model suffers, among other [10], from two fundamental problems: (a) *The "old" cosmological constant problem* (or *fine-tuning problem*) i.e., the fact that the observed value of the vacuum energy density ($\rho_\Lambda = \Lambda c^2/8\pi G \simeq 10^{-47}$ GeV^4) is many orders of magnitude below the value found using quantum field theory [7], and (b) *the coincidence problem* [11] i.e., the fact that the matter energy density and the vacuum energy density are of the same order (just prior to the present epoch), despite the fact that the former is a rapidly decreasing function of time while the latter is stationary.

Attempts to solve the coincidence problem have been presented in the literature (see [8,9,12] and references therein), in which an easy way to overpass the coincidence problem is to replace the constant vacuum energy with a DE that evolves with time. Nowadays, the physics of DE is considered one of the most fundamental and challenging problems on the interface uniting astronomy, cosmology, and particle physics. In the last decade there have been theoretical debates among the cosmologists regarding the nature of this exotic component. Many candidates have been proposed in the literature, such as a cosmological constant Λ (vacuum), time-varying Λ(*t*) cosmologies, quintessence, *k*-essence, quartessence, vector fields, phantom, gravitational matter creation, tachyons, modifications of gravity, Chaplygin gas and the list goes on (see [7,13–29] and references therein).

In the original scalar field models [30] and later in the quintessence context, one can *ad hoc* introduce an adjusting or tracker scalar field φ [18] (different from the usual standard model Higgs field), rolling down the potential energy *V*(φ), which could resemble the DE [8,9,31–34]. However, it was realized that the idea of a scalar field

rolling down some suitable potential does not really solve the problem because ϕ has to be some high energy field of a grand unified theory, and this leads to an unnaturally small value of its mass, which is beyond all conceivable standards in Particle Physics. As an example, utilizing the simplest form for the potential of the scalar field, $V(\phi) = m_\phi^2 \phi^2/2$, the present value of the associated vacuum energy density is $\rho_\Lambda = \langle V(\phi) \rangle \sim 10^{-11}$ eV^4, so for $\langle \phi \rangle$ of order of a typical grand unified theory scale near the Planck mass, $M_P \sim 10^{19}$ GeV, the corresponding mass of ϕ is expected in the ballpark of $m_\phi \sim H_0 \sim 10^{-33}$ eV.

Notice that the presence of such a tiny mass scale in scalar field models of DE is generally expected also on the basis of structure formation arguments [35–37];, namely, from the fact that the DE perturbations seem to play an insignificant role in structure formation for scales well below the sound horizon. The main reason for this homogeneity of the DE is the flatness of the potential, which is necessary to produce a cosmic acceleration. Being the mass associated to the scalar field fluctuation proportional to the second derivative of the potential itself, it follows that m_ϕ will be very small and one expects that the magnitude of DE fluctuations induced by ϕ should be appreciable only on length scales of the order of the horizon. Thus, equating the spatial scale of these fluctuations to the Compton wavelength of ϕ (hence to the inverse of its mass) it follows once more that $m_\phi \lesssim H_0 \sim 10^{-33}$ eV.

Despite the above difficulties there is a class of viable models of quintessence based on supersymmetry, supergravity and string-theory which can protect, under of specific potentials, the light mass of quintessence (for a review see [38] and references therein). In spite of that, this class of DE models have been widely used in the literature due to their simplicity. Notice that DE models with a canonical kinetic term have a dark energy equation of state (EoS) parameter $-1 \le w_\phi < -1/3$. Models with ($w_\phi < -1$), sometimes called phantom DE [39], are endowed with a very exotic nature, like a scalar field with negative kinetic energy. In any case, in order to investigate the overall dynamics of the Universe we need to define the functional form of the potential energy. As we have already mentioned the issue of the potential energy has a long history in scalar field cosmology and indeed several parametrizations have been proposed (exponential, power law, hyperbolic, etc.).

The aim of the present work is to investigate which of the available scalar field potentials can accommodate basic geometrical symmetries (connected to the space-time), namely, Lie point and Noether. In fact the idea to use Noether symmetries as a cosmological tool is not new. In particular, it has been proposed that the existence of such symmetries are related with conserved quantities and thus they can be used as a selection criterion in order to discriminate the dark energy models, including those of $f(R)$ gravity (see [40–47]). From a mathematical point of view,

the Lie point/Noether symmetries play a vital role in physical problems because they provide Noether (first) integrals, which can be used in order to simplify a given system of differential equations and to determine the integrability of the system. A fundamental approach to derive the Lie point and Noether symmetries for a given dynamical problem moving in a Riemannian space has been proposed recently by Tsamparlis and Paliathanasis [48] (a similar analysis can be found in [49–55]).

The structure of the article is as follows. The basic theoretical elements of the problem are presented in Sec. II, where we also introduce the basic FLRW cosmological equations for various potentials of the scalar field. The geometrical symmetries of the scalar fields and their connections to the potential energy are discussed in Sec. III. In Sec. IV we provide for a first time (to our knowledge) analytical solutions in the light of either quintessence or phantom scalar field cosmologies that include nonrelativistic matter (dark matter). Finally, the main conclusions are summarized in Sec. V.

II. COSMOLOGY WITH A SCALAR FIELD

The scalar field contribution to the curvature of spacetime can be absorbed in Einstein's field equations as follows:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = k\tilde{T}_{\mu\nu}, \qquad k = 8\pi G, \qquad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor and $\tilde{T}_{\mu\nu}$ is the total energy-momentum tensor $\tilde{T}_{\mu\nu}$ given by $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} + T_{\mu\nu}(\phi)$. Here $T_{\mu\nu}(\phi)$ is the energy-momentum tensor associated with the scalar field ϕ, and $T_{\mu\nu}$ is the ordinary energy-momentum tensor of matter and radiation. Modeling the expanding Universe as a perfect fluid that includes radiation, matter and DE with 4-velocity U_μ, we have $\tilde{T}_{\mu\nu} = -Pg_{\mu\nu} + (\rho + P)U_\mu U_\nu$, where $\rho = \rho_m + \rho_\phi$ and $P = P_m + P_\phi$ are the total energy density and pressure of the cosmic fluid, respectively. Note that ρ_m is the proper isotropic density of matter-radiation, ρ_ϕ denotes the density of the scalar field and P_m, P_ϕ are the corresponding pressures. In the context of a FLRW metric with Cartesian coordinates

$$ds^2 = -dt^2 + a^2(t)\frac{1}{(1 + \frac{K_3}{4}\mathbf{x}^2)^2}(dx^2 + dy^2 + dz^2) \quad (2)$$

the Einstein's field Eqs. (1), for comoving observers ($U^\mu = \delta_0^\mu$), provide

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad (3)$$

$$R_{\mu\nu} = \left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + K_3}{a^2}\right]g_{\mu\nu}, \qquad (4)$$

where the overdot denotes derivative with respect to the cosmic time t, $a(t)$ is the scale factor of the Universe, and

$K_3 = 0, \pm 1$ is the spatial curvature parameter. Also, the contraction of the Ricci tensor provides the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + K_3}{a^2}\right]. \qquad (5)$$

Finally, the gravitational field equations boil down to Friedmann's equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{3}(\rho_m + \rho_\phi) - \frac{K_3}{a^2} \qquad (6)$$

and

$$3H^2 + 2\dot{H} = -k(P_m + P_\phi) - \frac{K_3}{a^2}, \qquad (7)$$

where $H(t) \equiv \dot{a}/a$ is the Hubble function. The Bianchi identity (which ensure the covariance of the theory) $\nabla^\mu \tilde{T}_{\mu\nu} = 0$ amounts to the following generalized local conservation law:

$$\dot{\rho}_m + \dot{\rho}_\phi + 3H(\rho_m + P_m + \rho_\phi + P_\phi) = 0. \qquad (8)$$

Note that the latter quantities obey the following relations:

$$(\rho_m, P_m) \equiv (-T_0^0, T_i^i), \quad (\rho_\phi, P_\phi) \equiv (-T_0^0(\phi), T_i^i(\phi)). \qquad (9)$$

Combining Eqs. (6)–(8), we obtain

$$\frac{\ddot{a}}{a} = -\frac{k}{6}[\rho_m + \rho_\phi + 3(P_m + P_\phi)]. \qquad (10)$$

Assuming negligible interaction between matter and scalar field, Eq. (8) leads to the following independent differential equations:

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0, \qquad (11)$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0. \qquad (12)$$

In this work, we will present the global dynamics of the Universe in the presence of a barotropic cosmic fluid whose the corresponding EoS parameters are given by $w_m = P_m/\rho_m$ and $w_\phi = P_\phi/\rho_\phi$. In what follows we assume a constant w_m which implies that $\rho_m = \rho_{m0}a^{-3(1+w_m)}$ (cold $w_m = 0$ and relativistic $w_m = 1/3$ matter), where ρ_{m0} is the matter density at the present time. Generically, some high energy field theories suggest that the dark energy EoS parameter is a function of cosmic time (see, for instance, [56]) and thus

$$\rho_\phi(a) = \rho_{\phi 0} \exp\left(\int_a^1 \frac{3[1 + w_\phi(\sigma)]}{\sigma} d\sigma\right), \qquad (13)$$

where $\rho_{\phi 0}$ is the DE density at the current epoch.

A. The scalar field

We consider a scalar field in the Friedmann-Lemaître-Robertson-Walker (FRLW) cosmology which is minimally coupled to gravity, such that the field satisfies the cosmological principle that is, ϕ inherits the symmetries of the metric. This means that the scalar field depends only on the cosmic time t, and consequently $\phi_{,\nu} = \dot{\phi}\delta_\nu^0$ where $\dot{\phi} = \frac{d\phi}{dt}$. A scalar field $\phi(t)$ with a potential $V(\phi)$ is defined by the energy-momentum tensor of the form (for review see [38] and references therein)

$$T_{\mu\nu}(\phi) = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_\phi)}{\delta g^{\mu\nu}}, \qquad (14)$$

where L_ϕ is the Lagrangian of the scalar field. Although in the current analysis we study generically, as much as possible, the problem we will focus on a scalar field with

$$L_\phi = -\tfrac{1}{2}\epsilon g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) \qquad (15)$$

or

$$L_\phi = \tfrac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \qquad (16)$$

where

$$\epsilon = \begin{cases} 1 & \text{Quintessence} \\ -1 & \text{Phantom.} \end{cases} \qquad (17)$$

Therefore, using the second equality of Eq. (9), (14), and (16) the energy density ρ_ϕ and the pressure P_ϕ of the scalar field are given by

$$\rho_\phi \equiv -T_0^0(\phi) = \tfrac{1}{2}\epsilon\dot{\phi}^2 + V(\phi) \qquad (18)$$

and

$$P_\phi \equiv T_i^i(\phi) = L_\phi = \tfrac{1}{2}\epsilon\dot{\phi}^2 - V(\phi). \qquad (19)$$

Inserting Eq. (18) and (19) into Eq. (12) it is routine to derive the Klein-Gordon equation which describes the time evolution of the scalar field. This is

$$\ddot{\phi} + \frac{3}{a}\dot{a}\,\dot{\phi} + \epsilon V_{,\phi} = 0, \qquad (20)$$

where $V_{,\phi} = dV/d\phi$. Obviously, if we use the current functional form of L_ϕ, then Eq. (7) takes the form

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}^2}{a^2} + \frac{K_3}{a^2}\right) + \frac{k}{2}\left(P_m + \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi)\right) = 0. \qquad (21)$$

The corresponding dark energy EoS parameter (defined before) is

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\epsilon(\dot{\phi}^2/2) - V(\phi)}{\epsilon(\dot{\phi}^2/2) + V(\phi)}. \qquad (22)$$

The quintessence ($\epsilon = 1$) cosmological model accommodates a late time cosmic acceleration in the case of $w_\phi < -1/3$ which implies that $\dot{\phi}^2 < V(\phi)$. On the other hand, if the kinetic term of the scalar field is negligible with respect to the potential energy $[\frac{\dot{\phi}^2}{2} \ll V(\phi)]$, then the equation of state parameter is $w_\phi \simeq -1$. In the case of a

phantom DE ($\epsilon = -1$), due to the negative kinetic term, one has $w_\phi < -1$ for ($\dot{\phi}^2/2) < V(\phi)$.

The unknown quantities of the problem are $a(t)$, $\phi(t)$ and $V(\phi)$ but we have only two independent differential equations available, namely, Eqs. (20) and (21). Thus, in order to solve this system of differential equations we need to assume a functional form of the scalar field potential energy, $V(\phi)$. In the literature, due to the unknown nature of the DE, there are many forms of potentials (for a review see [38]) which describe differently the physical features of the scalar field. Let us now briefly present various potentials whose free parameters can be constrained by using the current cosmological data.

(i) The power law potential [13,18]:

$$V(\phi) = \frac{M^{4+n}}{\phi^n} \qquad (23)$$

(ii) The exponential potential [57]:

$$V(\phi) = V_0 \exp(-\sqrt{k}\lambda\phi) \qquad (24)$$

(iii) The unified dark matter potential (hereafter UDM) [58]:

$$V(\phi) = V_0[1 + \cosh^2(\lambda\phi)] \qquad (25)$$

(iv) The pseudo-Nambu Goldstone boson potential [59]:

$$V(\phi) = \mu^4[1 + \cos(\phi/f)] \qquad (26)$$

(v) The exponential with inverse power [60]:

$$V(\phi) = M[\exp(\gamma/\phi) - 1]. \qquad (27)$$

(vi) The supergravity motivated potential [19]:

$$V(\phi) = M \exp(\phi^2)/\phi^\gamma. \qquad (28)$$

(vii) The early dark energy potential [61]:

$$V(\phi) = V_1 \exp(-\sqrt{k}\lambda_1\phi) + V_2 \exp(-\sqrt{k}\lambda_2\phi). \qquad (29)$$

Interestingly the potential

$$V(\phi) = V_0[\cosh(\lambda\phi) - 1]^p$$

provides predictions which are similar to those of the early DE model, as far as the global dynamics of the Universe is concerned [62].

(viii) The Albrecht-Skordis model [63]:

$$V(\phi) = M[A + (\phi - B)^2]\exp(-\gamma\phi). \qquad (30)$$

(ix) The Chaplygin gas from the ordinary scalar field viewpoint [38]:

$$V(\phi) = \frac{\sqrt{A}}{2}\left(\cosh\sqrt{3}k\phi + \frac{1}{\cosh\sqrt{3}k\phi}\right). \qquad (31)$$

Detailed analysis of these potentials exist in the literature, including their confrontation with the data (see [38] for extensive reviews). It is worth pointing out that for some special cases attempts to find analytical solutions can be found in [60,64–70] (and references therein).

III. DARK ENERGY VERSUS SPACE-TIME SYMMETRIES

In the last decade, a large number of experiments have been proposed in order to constrain DE and study its evolution. Naturally, in order to establish the evolution of the DE equation of state a realistic form of $H(a)$ is required while the included free parameters must be constrained through a combination of independent DE probes (for example SNIa, BAOs, CMB, etc.). However, there is always a range of parameters of the considered scalar field cosmological models for which a good fit with the observational data is provided. This implies that such DE models can not be distinguished observationally, since they provide similarly evolving Hubble functions.

Practically, the goal here is to define a method (selection criterion) that can distinguish the DE models on a more fundamental (e.g., geometrical) level. According to the theory of general relativity, the space-time symmetries (Killing and homothetic vectors) via the Einstein's field equations [see Eq. (1)], are also symmetries of the energy-momentum tensor (the matter generates the gravitational field). Owing to the fact that the scalar field is minimally coupled to gravity and it evolves in space-time one would expect that the scalar field must inherit the symmetries of the space-time as gravity does.

It is interesting to mention that besides the geometric symmetries one has to consider the dynamical symmetries, which are the symmetries of the field equations. These latter symmetries are known as Lie symmetries. In case the field equations are derived from a Lagrangian there is a special class of Lie symmetries, the Noether symmetries, which have the characteristic that lead to conserved currents or, equivalently, to first integrals of the equations of motion. The Noether integrals are used to reduce the order of the field equations or even to solve them. Therefore a sound requirement, which is possible to be made in Lagrangian theories is that they admit extra Noether symmetries. This assumption is model independent, because it

is imposed after the field equations have been derived, therefore it does not lead to conflict whereas with the geometric symmetries while, at the same time, serves the original purpose of a selection rule. Since the basic equations in the scalar field cosmologies follow from a Lagrangian we can apply the above ideas by looking for scalar field cosmologies which admit extra Noether symmetries.

In particular, let us consider a cosmological model which accommodates a late time accelerated expansion and it contains a scalar field $\phi(t)$, described by a potential $V(\phi)$. We pose the following question: *For the scalar field that lives into a two-dimensional Riemannian space $\{a, \phi\}$ and which is embedded in the space-time, how many (if any) of the previously presented potentials (see Sec.* II A) *can provide nontrivial Noether symmetries (or first integrals of motion)*? As an example, if we find a cosmological model (or a family of models) for which its scalar field produces nontrivial number of first integrals of motion with respect to the other DE cosmological models, then obviously this model contains an extra geometrical feature. Thus, we can use this geometrical characteristic in order to classify the explored DE cosmological model into a distinct category (see also [40–45]). Below, we present the geometrical method used in order to define the basic symmetries, namely, Lie point and Noether which lead to first integrals of motion.

A. Lie and Noether symmetries

We briefly present the main points of the method used to classify all two-dimensional Newtonian dynamical systems, which admit Lie point/Noether symmetries.[1] A first important ingredient is the use of two theorems which relate the Lie point and the Noether symmetries of a dynamical system moving in a Riemannian space with the special projective group and the homothetic group generators of the space, respectively. These theorems have been given by some of us in [48]. In particular in a recent paper Tsamparlis and Paliathanasis [48] have provided an alternative way to solve the system of Lie point/Noether symmetry conditions (see their Tables 1–15), for second order equations of the form

$$\ddot{x}^{\,i} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = F^i. \qquad (32)$$

Here $\Gamma^i_{jk}(x^r)$ are general functions, along the solution curves and $F^i(x^j)$ is a C^∞ vector field. Basically, Eqs. (32) are the equations of motion of a dynamical system in a Riemannian space in which the functions $\Gamma^i_{jk}(x^r)$ are the connection coefficients (Christofell sym-

bols) of the metric $\hat{g}_{ij}$ of the space (in our case $\{a, \phi\}$ see below). The key point, (see [48]), is to express the system of Lie point/Noether symmetry conditions of Eq. (32) in terms of collineation (usually referred as symmetries) conditions of the metric. If this is achieved, then the Lie point/ Noether symmetries of Eq. (32) will be related to the collineations of the metric. Therefore the determination of the Lie point/Noether symmetries of Eq. (32) will be transferred to the geometric problem of determining the generators of a specific type of collineations of the metric. Then it will be possible to use the plethora of results of Differential Geometry on collineations to produce the solution of the Lie point/Noether symmetry problem.

The natural question to ask is: *How one will select the Lie point/Noether symmetries of two different dynamical systems, which move in the same Riemannian space?* The answer is simple. The left-hand side of Eq. (32) contains the metric and its derivatives and it is *common to all* dynamical systems moving in the same Riemannian space. Therefore geometry enters in the left-hand side of Eq. (32) only. Each dynamical system is defined by the force field F^i, which enters into the right-hand side of Eq. (32) only. Therefore, there must exist constraints, which involve the components of the Lie point/Noether symmetry vectors and the force field, and which will have to be satisfied for a collineation of the metric in order for it to be a Lie point/Noether symmetry of the specific dynamical system. A similar approach can be found in [51–55].

In a subsequent work [48] Tsamparlis and Paliathanasis determined (among others) all two-dimensional potentials which admit at least one Lie point and/or Noether symmetry. The results of the calculations have been collected for convenience in Tables 1–15. From these tables one can read directly the aforementioned potentials and, furthermore, for each potential the admitted Lie point and Noether symmetries together with the corresponding Noether integrals. It is emphasized that no extra calculations are required. In the following we shall make use of these results.

B. Using the geometry of the space $\{a, \phi\}$ to constrain dark energy

In this section we apply the previous theory into the scalar field cosmology. Interestingly, we can easily prove that the main field Eqs. (20) and (21), described in Sec. II, can be produced by the following general Lagrangian:

$$L = -3a\dot{a}^2 + ka^3 L_\phi + ka^3 P_m + 3K_3 a \qquad (33)$$

in the space of the variables $\{a, \phi\}$. Also the action is

$$S = \int L d^3 x dt. \qquad (34)$$

Therefore, utilizing Eq. (16) and (33) we can also obtain the Hamiltonian of this system

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[1]Note that the Noether symmetries are a subalgebra of the algebra defined by the Lie symmetries [48]. A dot over a symbol in Eq. (32) indicates derivation with respect to the parameter s (the affine parameter along the trajectory—cosmic time in our case)

$$E = \tfrac{1}{2}(-6a\dot{a}^2 + ka^3\epsilon\dot{\phi}^2) + ka^3[V(\phi) - P_m] - 3K_3 a. \tag{35}$$

In order to compute the Lie point/Noether symmetries of equations of motion (20) and (21), we consider the Lagrangian as the sum of a kinetic energy which defines the metric in the space of $\{a, \phi\}$ and an external force field. In particular, this two-dimensional metric takes the form

$$d\hat{s}^2 = -6ada^2 + ka^3\epsilon d\phi^2, \tag{36}$$

which implies that

$$\Gamma^a_{aa} = \frac{1}{2a}, \qquad \Gamma^\phi_{a\phi} = \frac{3}{2a}, \qquad \Gamma^a_{\phi\phi} = \frac{k}{4}\epsilon a. \tag{37}$$

Obviously, in the case of phantom cosmology $\epsilon = -1$, Eq. (36) points that we have to replace ϕ with $i\phi$ for mathematical convenience. Using now Eq. (37) and inserting the variables $x^i = a(t), \phi(t)$ variables into Eq. (32) we find that

$$\ddot{a} + \frac{1}{2a}\dot{a}^2 + \frac{k}{4}a\epsilon\dot{\phi}^2 = F^a, \qquad \ddot{\phi} + \frac{3}{a}\dot{a}\,\dot{\phi} = F^\phi.$$

Comparing with the equations of motion (20) and (21), we can define the external "forces" in terms of the scalar field potential $V(\phi)$

$$F^a = -\frac{1}{2a}K_3 - \frac{ka}{2}[P_m - V(\phi)], \qquad F^\phi = -\epsilon V_{,\phi}.$$

On the other hand, using the above Γ^i_{jk} functions we find after some simple algebra that the curvature of the $\{a, \phi\}$ space is $\hat{R} = 0$ implying flatness (all two-dimensional spaces are Einstein spaces hence $\hat{R} = 0$ implies the space is flat). Also, the signature of the metric Eq. (36) is -1, hence the space is the two-dimensional Minkowski space. Therefore, according to theorems 1 and 2 of [48] the Lie point/Noether symmetries of the Eqs. (20) and (21) follow from the special projective group of the two-dimensional Minkowski space. To find explicitly these vectors and thus the corresponding Noether symmetries (or first integrals), we have to bring the two-dimensional metric of Eq. (36) into its canonical form (i.e., $d\hat{s}^2 = -dx^2 + dy^2$). Changing now the variables from (a, ϕ) to (r, θ) via the relations

$$r = \sqrt{\frac{8}{3}}a^{3/2}, \qquad \theta = \sqrt{\frac{3k\epsilon}{8}}\phi \tag{38}$$

the two-dimensional metric (36) is given by

$$d\hat{s}^2 = -dr^2 + r^2 d\theta^2, \tag{39}$$

that is, (r, θ) are hyperbolic spherical coordinates in the two-dimensional Minkowski space $\{a, \phi\}$. Next we introduce the new coordinates (x, y) with the transformation

$$x = r\cosh(\theta), \qquad y = r\sinh(\theta), \tag{40}$$

which implies that Eq. (39) becomes $d\hat{s}^2 = -dx^2 + dy^2$. We also point here that

$$r^2 = x^2 - y^2, \qquad \theta = \text{arctanh}(y/x). \tag{41}$$

The scale factor $(a(t) > 0)$ is now given by

$$a = \left[\frac{3(x^2 - y^2)}{8}\right]^{1/3}, \tag{42}$$

which means that the new variables have to satisfy the following inequality: $x \geq |y|$.

In the new coordinate system (x, y) the Lagrangian (33) and the Hamiltonian (35) are written

$$L = \tfrac{1}{2}(\dot{y}^2 - \dot{x}^2) - V_{\text{eff}}(x, y), \tag{43}$$

$$E = \tfrac{1}{2}(\dot{y}^2 - \dot{x}^2) + V_{\text{eff}}(x, y), \tag{44}$$

where

$$V_{\text{eff}}(x, y) = (x^2 - y^2)\left[\tilde{V}\!\left(\frac{y}{x}\right) - \tilde{P}_m\right] - 3\tilde{K}_3(x^2 - y^2)^{1/3}. \tag{45}$$

Note that we have used

$$\tilde{K}_3 = K_3\left(\frac{3}{8}\right)^{1/3} \qquad \tilde{P}_m = \frac{3k}{8}P_m \tag{46}$$

and

$$\tilde{V}(\theta) = \frac{3k}{8}V(\theta). \tag{47}$$

If the matter pressure P_m is constant, then the Lagrangian (or Hamiltonian) is time independent, thus the system is autonomous implying that theorems 1 and 2 of [48] apply. In this case we also have (for more details see Appendix A) that $\rho_m = \frac{|E|}{ka^3} - P_m$ which obeys Eq. (11).

We now proceed in an attempt to provide the Lie point and Noether symmetries of the current dynamical problem. Note, that for simplicity in the analytical treatment below, unless explicitly stated, we consider spatially flat FRLW ($K_3 = 0$) scalar field models that include nonrelativistic matter ($P_m = 0$ with $\rho_m = \frac{|E|}{ka^3}$). Taking the latter into account the effective potential of Eq. (45) is given by

$$V_{\text{eff}}(x, y) = (x^2 - y^2)\tilde{V}\!\left(\frac{y}{x}\right) = r^2\tilde{V}(\theta). \tag{48}$$

For this effective potential and with the aid of [48,49] we find the following cases that admit Lie point and Noether symmetries:

(i) *Trivial Noether symmetries:* First of all, it is well known (see for example [48] and references therein) that for a general effective potential $V_{\text{eff}}(x, y)$ we have only the Lie symmetry ∂_t. Additionally, for effective potentials given by Eq. (48) we have that

$$V_{\text{eff}}(x, y) = x^2\left(1 - \frac{y^2}{x^2}\right)\tilde{V}\left(\frac{y}{x}\right);$$

hence, the case of Table 8, line 5 (with $d = 0$) of [48] applies and we have the additional Lie symmetry $x\partial_x + y\partial_y$. A linear combination between ∂_t and $x\partial_x + y\partial_y$ provides the following Lie symmetry:

$$X_L = c_1\partial_t + c_2(x\partial_x + y\partial_y). \tag{49}$$

From the corresponding tables of theorem 2 of [48], we see that in this case there is only the trivial Noether symmetry[2] ∂_t, whose Noether integral is the Hamiltonian $E =$ constant ($\partial_t E = 0$). As it is expected, this result implies that all the scalar field potentials described in Sec. II A admit the trivial Noether symmetry, namely, energy conservation, as they should.

(ii) *Extra Noether symmetries:* Now we are looking for first integrals beyond the standard one. In particular, we are interested to check whether the scalar field potentials mentioned in this paper (see Sec. II A) can admit nontrivial Lie point/Noether symmetries. The argument is simple: if for a given potential we find extra Noether symmetries which are related with conserved quantities, then this particular model has an enhanced physical meaning (see also [42–45]). From the mathematical point of view the existence of extra integrals of motion points the existence of an analytical solution (see the next section). The novelty in the current work is that we find that the exponential potential [see Eq. (24)] and the UDM potential (see Eq. (25)) can be clearly distinguished from the other dark energy potentials because these are the only potentials from the list presented in Sec. II A, that accommodate extra Noether symmetries.

Hyperbolic-UDM Potential: Generically, we use the following potential:

$$\tilde{V}(\theta) = \frac{\omega_1\cosh^2(\theta) - \omega_2\sinh^2(\theta)}{2} \tag{50}$$

or

$$V_{\text{eff}}(x, y) = r^2\tilde{V}(\theta) = \frac{\omega_1 x^2 - \omega_2 y^2}{2}, \tag{51}$$

where we have used Eqs. (40) and (48). The corresponding Noether symmetries, X_n, are known (see for example [49]). These are (for $\omega_1\omega_2 \neq 0$ otherwise see Appendix B):

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[2]The operator ∂_z denotes $\partial/\partial z$. Also, there is a difference of the results presented in [48] due to the Lorentzian character of the metric. This affects only the rotational part of the metric (non gradient Killing vectors) and gives $y\partial_x + x\partial_y$ instead of the Euclidean $y\partial_x - x\partial_y$.

$$X_{n_1} = \partial_t, \qquad X_{n_2} = \sinh(\sqrt{\omega_1}t)\partial_x,$$
$$X_{n_3} = \cosh(\sqrt{\omega_1}t)\partial_x, \qquad X_{n_4} = \sinh(\sqrt{\omega_2}t)\partial_y,$$
$$X_{n_5} = \cosh(\sqrt{\omega_2}t)\partial_y.$$

The Noether integrals are the Hamiltonian and the quantities

$$I_{n_2} = \sinh(\sqrt{\omega_1}t)\dot{x} - \sqrt{\omega_1}\cosh(\sqrt{\omega_1}t)x,$$
$$I_{n_3} = \cosh(\sqrt{\omega_1}t)\dot{x} - \sqrt{\omega_1}\sinh(\sqrt{\omega_1}t),$$
$$I_{n_4} = \sinh(\sqrt{\omega_2}t)\dot{y} - \sqrt{\omega_2}\cosh(\sqrt{\omega_2}t)y,$$
$$I_{n_5} = \cosh(\sqrt{\omega_2}t)\dot{y} - \sqrt{\omega_2}\sinh(\sqrt{\omega_2}t)y.$$

Obviously the UDM potential is a particular case of the current general hyperbolic potential. Indeed for $\omega_1 = 2\omega_2$ and with the aid of Eqs. (38) and (47) we fully recover the UDM potential (see Sec. II A)

$$V(\phi) = V_0\left[1 + \cosh^2\left(\frac{3k\epsilon}{8}\phi\right)\right], \tag{52}$$

where $V_0 = \frac{4\omega_2}{3k}$ modulus a constant.

Exponential potential: Here we provide for a first time (to our knowledge) the Lie point and the Noether symmetries of the exponential potential. Indeed from Table 8, line 3 (with $d \neq 0$) of [48] one can immediately see that

$$V_{\text{eff}}(r, \theta) = r^2\tilde{V}(\theta) = r^2 e^{-d\theta}.$$

The corresponding Lie symmetry vector is

$$X_L = 2t\partial_t + \frac{4}{d}(y\partial_x + x\partial_y). \tag{53}$$

Concerning the Noether symmetries from Table 14 of [48] we find that the Noether symmetry of the system for the potential $\tilde{V}(\theta) = e^{-d\theta}$ is

$$X_n = 2t\partial_t + \left(x + \frac{4}{d}y\right)\partial_x + \left(y + \frac{4}{d}x\right)\partial_y. \tag{54}$$

In general the Noether integral for the vector $X_n = 2t\partial_t + \eta^i\partial_i$ is (see Eq. 66 of [48]):

$$I = 2tE - \eta^i\hat{g}_{ij}\dot{x}^i, \tag{55}$$

where $\eta^i = (x + \frac{4}{d}y)\partial_x + (y + \frac{4}{d}x)\partial_y$. After some algebra we compute

$$\eta^i\hat{g}_{ij}\dot{x}^i = \left(x + \frac{4}{d}y \quad y + \frac{4}{d}x\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$
$$= \left(-\left(x + \frac{4}{d}y\right) \quad \left(y + \frac{4}{d}x\right)\right)\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$
$$= -\left(x + \frac{4}{d}y\right)\dot{x} + \left(y + \frac{4}{d}x\right)\dot{y},$$

i.e.,

$$I = 2tE + \left(x + \frac{4}{d}y\right)\dot{x} - \left(y + \frac{4}{d}x\right)\dot{y}, \tag{56}$$

where E is the Hamiltonian. Using $\tilde{V} = e^{-d\theta}$ together with Eq. (38) and (47) we write the potential to its nominal form (see Sec. II A). This is

$$V(\phi) = V_0 \exp\left(-d\sqrt{\frac{3k\epsilon}{8}}\phi\right),\qquad(57)$$

where $V_0 = \frac{8}{3k}$ modulus a constant.[3]

From now on, we focus on the exponential and the UDM potentials because they contain nontrivial integrals of motion, implying the existence of exact analytical solutions (see the next section). In Sec. IV we provide for a first time (to our knowledge) such analytical solutions in the light of either quintessence or phantom scalar field cosmologies that include also a nonrelativistic matter (cold dark matter) component.

IV. ANALYTICAL SOLUTIONS IN THE FLAT SCALAR FIELD COSMOLOGY

In this section, we proceed in an attempt to analytically solve the differential Eqs. (20) and (21). We remind the reader that this is possible because the dynamical system that includes either an exponential or a UDM potential is autonomous and it admits extra Noether symmetries (see previous section). Our aim is to derive the predicted time dependence of the main cosmological functions, namely, $a(t)$ and $\phi(t)$ [and thus of $H(t)$ and $w_\phi(t)$] in the scalar field cosmology.

A. Analytical solutions of the hyperbolic potential

Inserting Eq. (51) into Eqs. (43) and (44) the Lagrangian and the Hamiltonian, respectively, become

$$L = \tfrac{1}{2}(\dot{y}^2 - \dot{x}^2) - \tfrac{1}{2}(\omega_1 x^2 - \omega_2 y^2),\qquad(58)$$

$$E = \tfrac{1}{2}(\dot{y}^2 - \dot{x}^2) + \tfrac{1}{2}(\omega_1 x^2 - \omega_2 y^2).\qquad(59)$$

Technically speaking, in the new coordinate system our dynamical problem is described by two independent hyperbolic oscillators and thus the system is fully integrable. In particular, utilizing the Euler-Lagrange equations in the new coordinate system the corresponding equations of motion can be written as

$$\ddot{x} - \omega_1 x = 0, \qquad \ddot{y} - \omega_2 y = 0,$$

where $\sqrt{\omega_1}$ and $\sqrt{\omega_2}$ are the oscillators' "frequencies" with units of inverse of time. It is routine to perform the integration to find the analytical solutions

$$x(t) = \sinh(\sqrt{\omega_1}t + \theta_1),$$

$$y(t) = \sqrt{\frac{2E + \omega_1}{\omega_2}}\sinh(\sqrt{\omega_2}t + \theta_2),$$

$$a^3(t) = \frac{3}{8}[x^2(t) - y^2(t)]$$

$$= \frac{3}{8}\left[\sinh^2(\sqrt{\omega_1}t + \theta_1) - \frac{2E + \omega_1}{\omega_2}\sinh^2(\sqrt{\omega_2}t + \theta_2)\right],$$

and

$$\phi(t) = \sqrt{\frac{8}{3k\epsilon}}\text{arctanh}\left(\frac{y}{x}\right)$$

$$= \sqrt{\frac{8}{3k\epsilon}}\text{arctanh}\left[\sqrt{\frac{2E + \omega_1}{\omega_2}}\frac{\sinh(\sqrt{\omega_2}t + \theta_2)}{\sinh(\sqrt{\omega_1}t + \theta_1)}\right],$$

where θ_1 and θ_2 are the integration constants of the problem. The constant θ_1 is related to θ_2 because at the singularity ($t = 0$), the scale factor has to be exactly zero. After some algebra, we find that

$$\theta_1 = \ln\left(-\sqrt{\frac{2E + \omega_1}{\omega_2}}\sinh\theta_2 + \sqrt{\frac{2E + \omega_1}{\omega_2}\sinh^2\theta_2 + 1}\right).$$

We would like to remind the reader that the UDM dark energy model is recovered for $\omega_1 = 2\omega_2$. Obviously, we can easily prove that the concordance Λ cosmology is a particular solution[4] of the current hyperbolic potential with $\epsilon = 1$, $\omega_1 = \omega_2 = \omega_\Lambda$ and $(\theta_1, \theta_2) = (0, 0)$.

Finally it is interesting to mention that the UDM cosmological model has been tested against the latest cosmological data (SNIa and BAO) in Basilakos and Lukes [71]. In this paper the authors discussed the evolution of matter perturbations as well as the spherical collapse model. They also compared the UDM scenario with the traditional Λ cosmology and they found that the UDM scalar field model provides an overall (global and local) dynamics which is in a fair agreement with that of the Λ cosmology although there are some differences especially at high redshifts.

B. Analytical solutions of the exponential potential

Now based on the exponential potential of Eq. (57), we extend the analytical solutions found by Russo [69] by taking into account the presence of nonrelativistic matter (cold dark matter), $\rho_m = \frac{|E|}{ka^3} \neq 0$. It is important to emphasize that Russo [69] provided analytical solutions only in the context of quintessence DE ($\epsilon = 1$) with $\rho_m = 0$ while we solve analytically, for a first time (to our knowledge), the current dynamical problem by treating dark energy simultaneously either as quintessence or phantom with $\rho_m \neq 0$. Since for the current potential the (x, y) coordinate system does not lead to an analytical solution we are using the same methodology with that provided by Russo [69], we change variables from (a, ϕ) to (u, v) according to the transformations

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[3]In the special case of $d = 2$, the system admits an additional Lie symmetry $\partial_x + \partial_y$, with Noether integral $I = \dot{x} - \dot{y}$.

[4]In the ΛCDM cosmology the scale factor is $a(t) = a_0\sinh^{2/3}(\omega_\Lambda t)$ where $\omega_\Lambda = 3H_0\sqrt{\Omega_\Lambda}/2$, $a_0 = (\Omega_m/\Omega_\Lambda)^{1/3}$ and $\Omega_\Lambda = 1 - \Omega_m$.

$$u = \sqrt{\frac{3k\epsilon}{8}}\phi + \frac{1}{2}\ln(a^3), \tag{60}$$

$$v = -\sqrt{\frac{3k\epsilon}{8}}\phi + \frac{1}{2}\ln(a^3). \tag{61}$$

Note that we have interchanged (u, v) with respect to those of [69]. Inverting the above equations and using Eq. (57) we get

$$a = e^{(u+v)/3}, \qquad \phi = \frac{1}{2}\sqrt{\frac{8}{3k\epsilon}}(u-v) \tag{62}$$

and

$$V(u, v) = V_0 e^{-d(u-v)/2}. \tag{63}$$

In the new variables (u, v) our Lagrangian (33) is written as follows:

$$L = -e^{(u+v)}\left[\frac{4}{3}\dot{u}\,\dot{v} + kV_0 e^{-2K(u-v)}\right], \qquad K = \frac{d}{4}, \tag{64}$$

where $kV_0 = 8/3$. The next step is to consider a change in the time coordinate as follows:

$$\frac{d\tau}{dt} = \sqrt{\frac{3kV_0}{4}}e^{-K(u-v)}, \tag{65}$$

which implies

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{d\tau}\frac{d\tau}{dt} = v'\sqrt{\frac{3kV_0}{4}}e^{-K(u-v)},$$

$$\dot{u} = \frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt} = u'\sqrt{\frac{3kV_0}{4}}e^{-K(u-v)},$$

where $u' = \frac{du}{d\tau}$ and $v' = \frac{dv}{d\tau}$. Obviously using the latter transformations, Eq. (64) and (65) the action given by Eq. (34) takes the form

$$S = -\sqrt{\frac{4kV_0}{3}}\int d^3x d\tau e^{(u+v)}e^{-K(u-v)}(u'v' + 1). \tag{66}$$

Now varying the action we arrive at

$$u'' + (1 - K)u'^2 - (1 + K) = 0, \tag{67}$$

$$v'' + (1 + K)v'^2 - (1 - K) = 0. \tag{68}$$

Obviously, in the latter equations the variables u, v decouple. In these variables, the Hamiltonian of the system becomes

$$E = e^{(u+v)}e^{-K(u-v)}(u'v' - 1), \tag{69}$$

where $E \neq 0$ (or $\rho_m \neq 0$). As we have already stated the above system of equations has been derived also by [69] in the case of quintessence dark energy ($\epsilon = +1$), and it is solved only for $E = 0$ (see Appendix A). Here we prove

that the same equations are valid also in the case of phantom dark energy in which the scalar field is imaginary, however the potential, the scale factor and the FRLW metric are real as they should.

We conclude that the FRLW metric [using Eqs. (65) and (62)] in the coordinates (τ, x^μ) is

$$ds^2 = -\frac{4}{3kV_0}e^{4K\sqrt{3k\epsilon/8}\phi(\tau)}d\tau^2 + a^2(\tau)dx^i dx_i. \tag{70}$$

Below we provide analytical solutions for the two different cases.

1. Case $K = 1$

In this case the system of Eqs. (67)–(69) becomes

$$u'' - 2 = 0, \qquad v'' + 2v'^2 = 0, \qquad e^{2v}(u'v' - 1) = E,$$

and the solution is

$$u(\tau) = \tau^2 + c_1\tau, \qquad v(\tau) = \tfrac{1}{2}\ln(2c_3\tau),$$

where the constants are related by the constraint

$$E = c_3 c_1 \rightarrow c_3 = \frac{E}{c_1}.$$

Using the latter constraint we simply find

$$u(\tau) = \tau^2 + c_1\tau, \qquad v(\tau) = \frac{1}{2}\ln\left(\frac{2E}{c_1}\tau\right).$$

We note that the solution depends on one arbitrary parameter $c_1 \neq 0$. If we choose $c_1 = E$, then we have the solution

$$u(\tau) = \tau^2 + E\tau, \qquad v(\tau) = \tfrac{1}{2}\ln(2\tau).$$

From these follows

$$a^3(\tau) = \sqrt{2\tau}e^{\tau^2 + E\tau},$$

$$\phi(\tau) = \frac{1}{4}\sqrt{\frac{8}{3k\epsilon}}[2\tau^2 + 2E\tau - \ln(2\tau)].$$

2. Case $K \neq 1$

In this case the solution of the system is

$$u(\tau) = -\frac{1}{2(K-1)}\ln\left[Cc_3^2\mathrm{Sinn}^2\left(\sqrt{|K^2-1|}\tau + \theta_1\right)\right],$$

$$v(\tau) = \frac{1}{2(1+K)}\ln\left[C^{-1}c_1^2\mathrm{Sinn}^2\left(\sqrt{|K^2-1|}\tau + \theta_1\right)\right],$$

where $C = \frac{|K-1|}{1+K}$,

$$\mathrm{Sinn}\,\omega = \begin{cases} \sin\omega & K > 1 \\ \sinh\omega & 0 < K < 1 \end{cases} \tag{71}$$

and θ_1 being the phase constant. Without loosing the generality we can select $\theta_1 = 0$. Also, the two constants of integration satisfy the condition: $E = c_3 c_1$. Next we may choose $c_3 = E \neq 0$ and have the solution

$$u(\tau) = -\frac{1}{2(K-1)} \ln\left[CE^2 \mathrm{Sinn}^2\left(\sqrt{|K^2-1|}\tau\right)\right], \qquad v(\tau) = \frac{1}{2(1+K)} \ln\left[C^{-1}\mathrm{Sinn}^2\left(\sqrt{|K^2-1|}\tau\right)\right].$$

Finally, we transform this solution to the coordinates $a(\tau)$, $\phi(\tau)$. Doing so we obtain

$$a^3(\tau) = C^{-(K/K^2-1)}|E|^{1/(K-1)}\mathrm{Sinn}\left(\sqrt{|K^2-1|}\tau\right)^{-(2/(K^2-1))},$$

$$\phi(\tau) = -\frac{1}{2}\sqrt{\frac{8}{3k\epsilon}}\ln\left[C^{-(1/(K^2-1))}|E|^{(1/(K-1))}\mathrm{Sinn}\left(\sqrt{|K^2-1|}\tau\right)^{(2K/(K^2-1))}\right].$$

It is interesting to mention that in the case of $0 < K < 1$ at late enough times we obtain that $\tau \sim \ln t$ and thus the scale factor evolves as $a(t) \propto t^{2/3\sqrt{1-K^2}}$. The current solution of the scale factor can provide, a recent cosmic acceleration ($\ddot{a}(t) > 0$) for $K \in (\frac{\sqrt{5}}{3}, 1)$.

V. CONCLUSIONS

In this paper we propose to use a theoretical model-independent criterion, based on first integrals of motion, usually named Noether symmetries in order to discriminate the dark energy (quintessence or phantom) models within the context of scalar field FLRW cosmology. This is possible via the geometrical symmetries of the space-time in which both gravity and dark energy live. In particular, following the general methodology of [48] (see also the references therein), the Noether symmetries are computed for nine distinct accelerating cosmological scenarios that contain a homogeneous scalar field associated with different types of potentials. Note that the free parameters of the dark energy models studied here can be constrained by using the current cosmological data. In particular, one has to perform a joint likelihood analysis utilizing, for example, the SNIa data [5], the shift parameter of the CMB [6] and the observed BAOs [72]. Such an analysis is in progress and will be published elsewhere.

The main results of the current paper can be summarized in the following statements (see Secs. III B and IV):

(i) We verify that all the scalar field potentials, studied here, admit the trivial first integral, namely, energy conservation as they should.

(ii) We find that the exponential and the unified dark matter potentials occupy an eminent position in the scalar field potentials hierarchy, being the potentials that admit extra integrals of motion, and therefore appear to be promising candidates for describing the physical properties of dark energy as well as extracting useful cosmological information. The existence of the new Noether integrals can be used to simplify the system of differential equations (equations of motion) as well as to determine the integrability of the system.

(iii) Based on the exponential and hyperbolic potentials we find that the main cosmological functions, such as the scale factor of the Universe, the scalar field,

the Hubble expansion rate and the metric of the FRLW space-time are provided analytically.

In a future work we plan to apply the same approach also to $f(R)$ cosmological models.

APPENDIX A: MATTER DENSITY VERSUS HAMILTONIAN

We remind the reader that if the matter pressure is constant, then the dynamical system described by the general Lagrangian of Eq. (33) is autonomous. Therefore, one can easily prove that $\rho_m = \frac{|E|}{ka^3} - P_m$. Indeed, utilizing Eq. (6) and (35) and we have after some simple algebra that

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K_3}{a^2} = \frac{k}{3}\left(\rho_m + \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi)\right) \Rightarrow a\dot{a}^2 + K_3 a$$

$$= \frac{k}{6}a^3\epsilon\dot{\phi}^2 + a^3\frac{k}{3}\rho_m + \frac{k}{3}a^3 V(\phi)$$

$$\Rightarrow -a\dot{a}^2 + \frac{k}{6}a^3\epsilon\dot{\phi}^2 + \frac{k}{3}a^3[V(\phi) - P_m] - K_3 a$$

$$= -\frac{k}{3}a^3[P_m + \rho_m]$$

or

$$E = -ka^3[P_m + \rho_m]$$

and thus

$$\rho_m = \frac{|E|}{ka^3} - P_m,$$

which satisfies Eq. (11). Note that the inequality $\rho_m \geq 0$ points that $E \leq 0$. The case of nonrelativistic matter $P_m = 0$ implies $\rho_m(a) = \frac{|E|}{ka^3}$.

APPENDIX B: SOLUTIONS OF HARMONIC OSCILLATOR COUPLING WITH A FREE PARTICLE

Here we consider either $\omega_1 \neq 0$, $\omega_2 = 0$ or $\omega_1 = 0$, $\omega_2 \neq 0$. For the latter case one may checks [45]. In both

cases the system is equivalent to a pair of two dynamical systems a simple harmonic oscillator and a free particle. The Lie point and the Noether symmetries of the current system can be found in [50]. As an example for $\omega_1 \neq 0$, $\omega_2 = 0$ Eqs. (58) and (59) become

$$L = \frac{1}{2}(\dot{y}^2 - \dot{x}^2) - \frac{\omega_1 x^2}{2}, \qquad E = \frac{1}{2}(\dot{y}^2 - \dot{x}^2) + \frac{\omega_1 x^2}{2}$$

and

$$\ddot{x} - \omega_1 x = 0, \qquad \ddot{y} = 0.$$

The solution of the above system is

$$x(t) = \sinh(\sqrt{\omega_1}t + \theta_1) \qquad y(t) = \sqrt{2E + \omega_1}t$$

or

$$a^3(t) = \frac{3}{8}[x^2(t) - y^2(t)]$$

$$= \frac{3}{8}[\sinh^2(\sqrt{\omega_1}t + \theta_1) - (2E + \omega_1)t^2],$$

$$\phi(t) = \sqrt{\frac{8}{3k\epsilon}}\text{arctanh}\left(\frac{t\sqrt{2E + \omega_1}}{\sinh(\sqrt{\omega_1}t + \theta_1)}\right).$$

Note that due to $a(0) = 0$ we have $\theta_1 = 0$.

APPENDIX C: EXPONENTIAL POTENTIAL VERSUS EMPTY SPACE $\rho_m = 0$ (OR $E = 0$)

In this appendix we would like to give the reader the opportunity to appreciate the fact that our solutions provided in Sec. IV can be viewed as an extension of those found by Russo [69] for the quintessence ($\epsilon = 1$) dark energy with $\rho_m = 0$ (or $E = 0$). For either quintessence or phantom dark energy, the system of Eqs. (67)–(69), which we have to solve is

$$u'' + (1 - K)u'^2 - (1 + K) = 0,$$

$$v'' + (1 + K)v'^2 - (1 - K) = 0,$$

$$E = (u'v' - 1)e^{(u+v)}e^{-K(u-v)}. \qquad \text{(C1)}$$

Because of the fact that $E = 0$ Eq. (C1) takes the form $u'v' = 1$. Thus we consider the following cases:

Case 1: For $K = 1$ the solution is

$$u(t) = \tau^2, \qquad v(t) = \frac{1}{2}\ln(2\tau)$$

or

$$a^3(\tau) = \sqrt{2\tau}e^{\tau^2}, \qquad \phi(\tau) = \frac{1}{4}\sqrt{\frac{8}{3k\epsilon}}[2\tau^2 - \ln(2\tau)].$$

Case 2: For $K \neq 1$ the solution is

$$u(\tau) = -\frac{1}{(K - 1)}\ln\left[\text{Sinn}\left(\sqrt{|K^2 - 1|}\tau + \theta_1\right)\right],$$

$$v(\tau) = \frac{1}{(K + 1)}\ln\left[\text{Coss}\left(\sqrt{|K^2 - 1|}\tau + \theta_1\right)\right]$$

or (for $\theta_1 = 0$)

$$a^3(\tau) = \frac{\text{Coss}(\sqrt{K^2 - 1}\tau)^{1/(K+1)}}{\text{Sinn}(\sqrt{|K^2 - 1|}\tau)^{1/(K-1)}},$$

$$\phi(\tau) = -\frac{1}{2}\sqrt{\frac{8}{3k\epsilon}}\ln\left[a^3(\tau)\text{Sinn}\left(\sqrt{|K^2 - 1|}\tau\right)^{2/(K-1)}\right],$$

where the quantity Sinn is given by (Eq. (71)) and

$$\text{Coss}\omega = \begin{cases} \cos\omega & K > 1 \\ \cosh\omega & 0 < K < 1. \end{cases}$$

We point out that for $\epsilon = 1$ the current solutions coincide (modulus some constants) those of [69]. In the case of phantom dark energy ($\epsilon = -1$) the scalar field is imaginary, however the potential, the scale factor and the metric of the space-time are real as they should.

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