Determination of χ_c and χ_b polarizations from dilepton angular distributions in radiative decays

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The angular distributions of the decay products in the successive decays $\chi_c(\chi_b) \to \mathrm{J}/\psi(\Upsilon)\gamma$ and $\mathrm{J}/\psi(\Upsilon) \to \ell^+\ell^-$ are calculated as a function of the angular momentum composition of the decaying χ meson and of the multipole structure of the photon radiation, using a formalism independent of production mechanisms and polarization frames. The polarizations of the χ states produced in high-energy collisions can be derived from the dilepton decay distributions of the daughter J/ψ or Υ mesons, with a reduced dependence on the details of the photon reconstruction or simulation. Moreover, this method eliminates the dependence of the polarization measurement on the actual details of the multipole structure of the radiative transition. Problematic points in previous calculations of the χ_c decay angular distributions are identified and clarified.

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I. INTRODUCTION

The existing J/ψ and Υ polarization measurements make no distinction between directly produced states and those resulting from the decay of higher-mass states. J/ψ and Υ mesons coming from χ decays have, in principle, very different polarizations with respect to the directly produced ones. In fact, directly produced χ and directly produced J/ψ or Υ have different angular momentum and parity properties, and originate from different partonic processes. Moreover, the angular momentum composition of the indirectly produced states is influenced by the presence of the accompanying decay photon. Therefore, χ_c and χ_b polarization measurements, together with the knowledge of how these states transmit their polarizations when they decay, are essential in the understanding of the observed J/ψ and Υ polarization patterns. An improved account of feed-down effects in quarkonium polarization measurements, and calculations, can shed new light in the interpretation of the significant discrepancy existing today between the theory predictions and the experimental data [1].

In this paper, we examine how the polarization is transmitted in the decays from P to S quarkonium states. We study the angular distributions of the successive decays $\chi_c(\chi_b) \to \mathrm{J}/\psi(\Upsilon)\gamma$ and $\mathrm{J}/\psi(\Upsilon) \to \ell^+\ell^-$. We discuss the sensitivity of these observable distributions to the angular momentum composition (''polarization'') of the decaying χ meson and their additional dependence on the orbital angular momentum of the photon. As a result of the study, we propose a convenient way of measuring χ polarizations in high-energy experiments, essentially independent of the details regarding the photon detection and of the magnitude of the higher-order multipoles of the radiative transition. This method is valid irrespective of the production process (hadroproduction, photoproduction, etc.). The paper finishes with a critical review of previous calculations of the χ_c decay angular distributions, identifying and clarifying the causes of their seemingly contradictory results.

II. RADIATIVE DECAY AMPLITUDES

Throughout this paper, we generically denote by V the charmonium and bottomonium $^3S_1$ states, J/ψ and Υ, and by χ the ^3P_j states, χ_{cj} and χ_{bj}, with $j = 1, 2$. Without loss of generality for the discussions in this paper, we assume that the ^3P_j state χ_j is produced in a single ''subprocess'' as a given superposition of J_z eigenstates (z being the quantization axis chosen for the χ angular momentum),

$$|\chi_j\rangle = \sum_{m=-j}^{j} b_m |\chi; j, m\rangle, \tag{1}$$

with $\mathbf{J}^2|\chi; j, m\rangle = j(j+1)|\chi; j, m\rangle$ and $\mathrm{J}_z|\chi; j, m\rangle = m|\chi; j, m\rangle$. Notations for axes and angles are shown in Fig. 1(a). The total angular momentum carried by the photon can have any (nonvanishing) value, while its projection along the momentum direction of the γ (and of V), the z' axis, can only be $k' = +1$ or -1, because the orbital component has, by definition, zero projection along this direction. In other words, the photon angular momentum state is an eigenstate of $\mathrm{J}_{z'}$ but, in general, neither of $\mathbf{J}^2$ nor of J_z. It can be represented as a complete expansion over eigenstates of $\mathbf{J}^2$ and J_z, as

$$|\gamma; k'\rangle = \sum_{l=1}^{\infty} \frac{\sqrt{2l+1}}{4\pi} \sum_{k=-l}^{l} \mathcal{D}^l_{kk'}(\Theta, \Phi)|\gamma; l, k\rangle, \tag{2}$$

where $\mathbf{J}^2|\gamma; l, k\rangle = l(l+1)|\gamma; l, k\rangle$, $\mathrm{J}_z|\gamma; l, k\rangle = k|\gamma; l, k\rangle$ and the coefficients $\mathcal{D}^l_{kk'}(\Theta, \Phi)$ [2] are the matrix elements of the rotation corresponding to the change of quantization

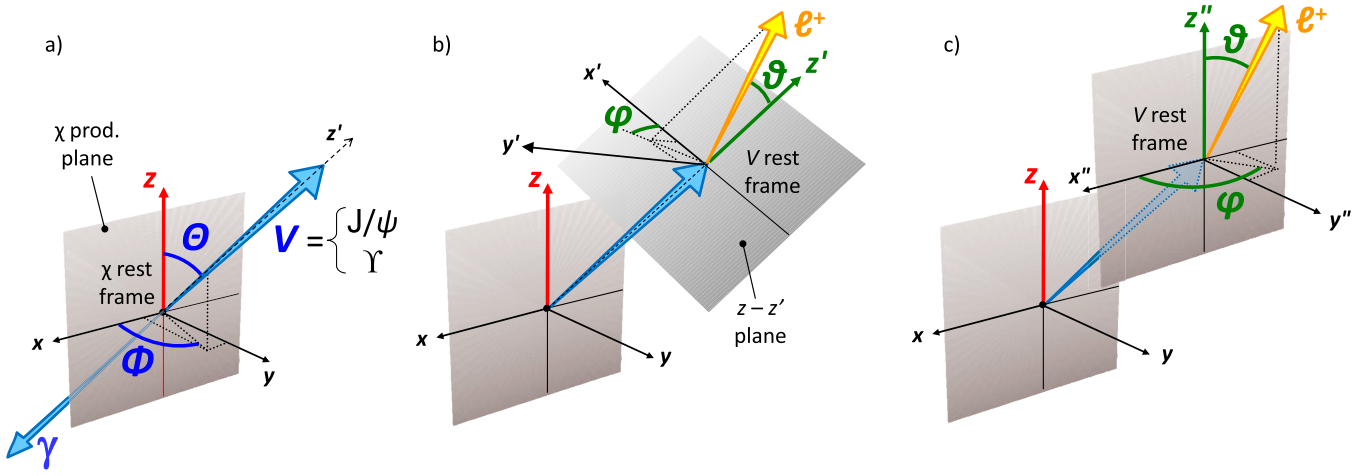

FIG. 1 (color online). Definition of axes and decay angles for $\chi \to V\gamma$ (a) and for $V \to \ell^+\ell^-$ in two options, with the dilepton polarization axis being the V direction in the χ rest frame (b) or parallel to the χ polarization axis (c).

axis from z' ("natural" quantization axis of the photon) to z (χ quantization axis adopted in the measurement).

The amplitude of the radiative transition from the χ_j state to the V state plus a photon having spin projection k' along z' is

$$\mathcal{A}(\chi_j \to V\gamma_{k'}) \propto \sum_{m=-j}^{j} b_m \sum_{l=1}^{\infty} \sqrt{2l+1} \sum_{k=-l}^{l} \mathcal{D}^{l*}_{kk'}(\Theta, \Phi) \times \langle V\gamma_{k'}; 1, m-k, l, k|\mathcal{H}|\chi; j, m\rangle. \quad (3)$$

The matrix element of the elementary transition can be parametrized as

$$\langle V\gamma_{k'}; 1, m-k, l, k|\mathcal{H}|\chi; j, m\rangle = (-1)^{(1-k')l/2} H_{jl} \langle 1, m-k, l, k|j, m\rangle, \quad (4)$$

where we have factored out the k'-dependent sign, determined by imposing that the photon distribution

$$W_j(\Theta, \Phi) = \sum_{k'=\pm 1} |\mathcal{A}(\chi_j \to V\gamma_{k'})|^2 \quad (5)$$

is parity invariant and using the property

$$\mathcal{D}^{l}_{kk'}(\pi - \Theta, \pi + \Phi) = (-1)^{l-k'} e^{2ik'\Phi} \mathcal{D}^{l}_{k-k'}(\Theta, \Phi). \quad (6)$$

The sums in Eq. (3) only include terms in which the Clebsch-Gordan coefficient $\langle 1, m-k, l, k|j, m\rangle$ is well defined, i.e. when

$$1 \le l \le j+1, \qquad |m-k| \le 1. \quad (7)$$

The partial amplitudes H_{jl} denote $\chi_j \to V$ transitions with the emission of a photon of total angular momentum l. In the spectroscopic language they represent electric and magnetic 2^l-pole radiations (dipole, quadrupole, octupole, etc.), indicated with El and Ml, respectively. The two types of transitions differ in their parity properties: the electric 2^l-pole radiation has parity $(-1)^l$, while the magnetic 2^l-pole radiation has parity $(-1)^{l+1}$. Since the ${}^3P_j$ and ${}^3S_1$ quarkonium states have opposite parities, the only allowed transitions are E1 (for all χ states), M2 (for χ_1 and χ_2), and E3 (for the χ_2). Hereafter we use the short notations

$$h_l = H_{1,l} \quad \text{with} \quad l = 1, 2, \qquad g_l = H_{2,l} \quad \text{with} \quad l = 1, 2, 3, \quad (8)$$

with the normalizations

$$h_1^2 + h_2^2 = g_1^2 + g_2^2 + g_3^2 = 1. \quad (9)$$

In short, h_1 and g_1 represent, respectively for χ_1 and χ_2, the relative amplitude of the E1 transition, h_2 and g_2 the corresponding relative amplitudes of the M2 transition, and g_3 the relative amplitude of the E3 transition (only for the χ_2 case). The hierarchies $g_3 < g_2 < g_1$ and $h_2 < h_1$ are expected. In fact, in the generic expansion of the radiation field around a system of oscillating charges in terms of angular momentum eigenfunctions, the l-th term vanishes more and more rapidly, at large distance from the origin, as l increases. This behavior reflects the fact that the wavelength of the emitted photon, $\lambda_\gamma = hc/E_\gamma \simeq hc/(0.4\ \text{GeV}) \simeq 3$ fm, is sizeable with respect to the dimensions of the quarkonium ($\simeq 0.4$–0.7 fm), so that at the typical distance scale $r = \lambda_\gamma$, the electromagnetic field is already only weakly sensitive to the internal charge and current distributions of the radiating object. With respect to the first nonvanishing term, higher multipole terms, produced by more complex charge/current configurations, are therefore foreseen to be increasingly smaller, even if not necessarily as suppressed as in nuclear γ-ray transitions, where the emitted radiation has a wavelength several orders of magnitude larger than the nuclear dimensions. The study of the $c\bar{c}$ radiation multipoles addresses aspects of

TABLE I. Higher-order photon multipoles in $\chi_{c(1,2)} \to J/\psi\gamma$ decays.

Experiment	h_2 [%]	g_2 [%]	g_3 [%]
Crystal Ball [4]	$-0.2^{+0.8}_{-2.0}$	-33^{+11}_{-30}	...
E760 [5]	...	-14 ± 6	0^{+6}_{-5}
E835 [6]	0.2 ± 3.2	$-9.3^{+3.9}_{-4.1}$	$2.0^{+5.6}_{-4.5}$
CLEO [7]	-6.26 ± 0.67	-9.3 ± 1.6	1.7 ± 1.4

the quark model, including the properties of the bound-state wave functions and the electromagnetic properties of the charm quark. For example, the relative contribution of the M2 amplitudes is significantly dependent on the corrections to the charm quark magnetic moment $\mu_c = \frac{2}{3}\frac{e}{2m_c}$ [3]. The existing h_2, g_2, and g_3 measurements, for the χ_c, are shown in Table I. The M2 amplitude contribution is of order 10% for the χ_{c2} and even smaller for the χ_{c1}, although the two most precise χ_{c1} results are incompatible with each other. No experimental information exists for χ_b decays. We will discuss in Sec. V the effects induced by the higher-order multipole contributions on the observable angular distributions.

III. PHOTON DISTRIBUTION

The angular distribution of the photon direction in the χ rest frame, as a function of the χ angular momentum composition $\{b_m\}$ and of the photon multipole amplitudes, is obtained by expanding Eq. (5). The resulting χ_0 distribution is spherically symmetric, reflecting the rotational invariance of the $j = 0$ angular momentum state and the imposed parity invariance of the decay. As for the χ_1 decay, the expression of the angular distribution is

$$W_1(\Theta,\Phi) = \frac{3}{4\pi(3+\lambda_\Theta)}(1 + \lambda_\Theta \cos^2\Theta + \lambda_\Phi \sin^2\Theta \cos 2\Phi + \lambda_{\Theta\Phi} \sin 2\Theta \cos\Phi + \lambda_\Phi^\perp \sin^2\Theta \sin 2\Phi + \lambda_{\Theta\Phi}^\perp \sin 2\Theta \sin\Phi), \quad (10)$$

where

$$\begin{aligned}
\lambda_\Theta &= \frac{1}{D}(1-3\Delta)[2|b_0|^2 - (|b_{+1}|^2 + |b_{-1}|^2)],\\
\lambda_\Phi &= -\frac{2}{D}(1-3\Delta)\,\mathrm{Re}(b_{+1}^* b_{-1}),\\
\lambda_{\Theta\Phi} &= -\frac{\sqrt{2}}{D}(1-3\Delta)\,\mathrm{Re}[b_0^*(b_{+1} - b_{-1})],\\
\lambda_\Phi^\perp &= -\frac{2}{D}(1-3\Delta)\,\mathrm{Im}(b_{+1}^* b_{-1}),\\
\lambda_{\Theta\Phi}^\perp &= \frac{\sqrt{2}}{D}(1-3\Delta)\,\mathrm{Im}[b_0^*(b_{+1} + b_{-1})],
\end{aligned} \quad (11)$$

with

$$\begin{aligned}
D &= 2(1+\Delta)|b_0|^2 + (3-\Delta)(|b_{+1}|^2 + |b_{-1}|^2),\\
\Delta &= -2h_1 h_2.
\end{aligned} \quad (12)$$

The angular distribution of the photon from the χ_2 decay, significantly more complex, is

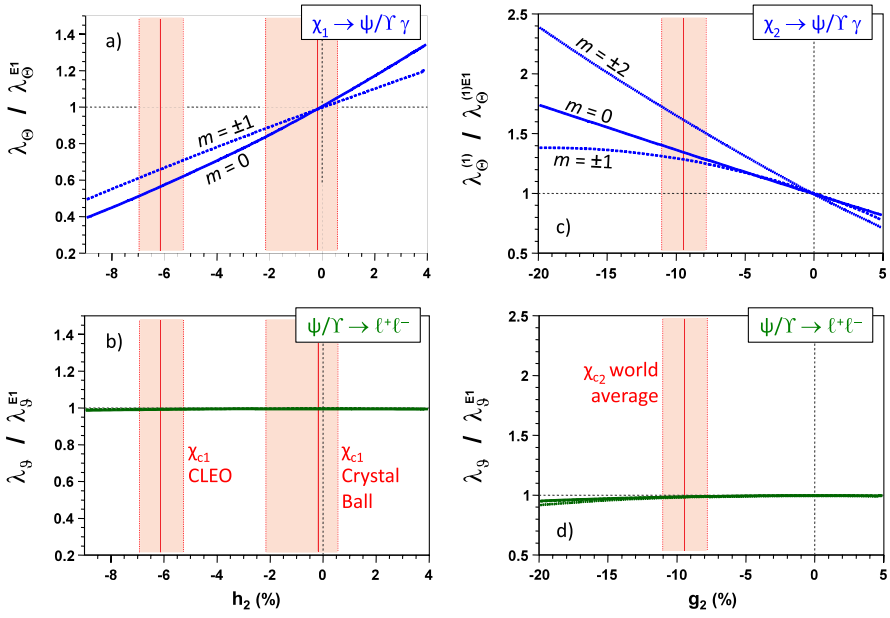

FIG. 2 (color online). Dependence of the parameters λ_Θ (a) and λ_ϑ (b) of the χ_1 photon and dilepton distributions and of the parameters $\lambda_\Theta^{(1)}$ (c) and λ_ϑ (d) of the χ_2 photon and dilepton distributions on the relative contribution of the magnetic quadrupole transition amplitude.

$$W_2(\Theta,\Phi)=\frac{15}{4\pi(15+5\lambda_\Theta^{(1)}+3\lambda_\Theta^{(2)})}(1+\lambda_\Theta^{(1)}\cos^2\Theta+\lambda_\Theta^{(2)}\cos^4\Theta+\lambda_\Phi^{(1)}\sin^2\Theta\cos2\Phi+\lambda_\Phi^{(2)}\sin^4\Theta\cos2\Phi$$
$$+\lambda_\Phi^{(3)}\sin^4\Theta\cos4\Phi+\lambda_\Phi^{(1)\perp}\sin^2\Theta\sin2\Phi+\lambda_\Phi^{(2)\perp}\sin^4\Theta\sin2\Phi+\lambda_\Phi^{(3)\perp}\sin^4\Theta\sin4\Phi+\lambda_{\Theta\Phi}^{(1)}\sin2\Theta\cos\Phi$$
$$+\lambda_{\Theta\Phi}^{(2)}\sin^2\Theta\sin2\Theta\cos\Phi+\lambda_{\Theta\Phi}^{(3)}\sin^2\Theta\sin2\Theta\cos3\Phi+\lambda_{\Theta\Phi}^{(1)\perp}\sin2\Theta\sin\Phi+\lambda_{\Theta\Phi}^{(2)\perp}\sin^2\Theta\sin2\Theta\sin\Phi$$
$$+\lambda_{\Theta\Phi}^{(3)\perp}\sin^2\Theta\sin2\Theta\sin3\Phi), \tag{13}$$

where

$$\lambda_\Theta^{(1)}=-\frac{3}{D}\left[2(1+\Delta_1)|b_0|^2+\left(1-\frac{2}{3}\Delta_1-\frac{5}{3}\Delta_2\right)(|b_{+1}|^2+|b_{-1}|^2)-\left(2+\frac{1}{3}\Delta_1-\frac{5}{3}\Delta_2\right)(|b_{+2}|^2+|b_{-2}|^2)\right],$$
$$\lambda_\Theta^{(2)}=\frac{\Delta}{2}[6|b_0|^2-4(|b_{+1}|^2+|b_{-1}|^2)+|b_{+2}|^2+|b_{-2}|^2],$$
$$\lambda_\Phi^{(1)}=\frac{2}{D}\,\mathrm{Re}[\sqrt{6}(1+\Delta_1)b_0^*(b_{+2}+b_{-2})+(3-2\Delta_1-5\Delta_2)b_{+1}^*b_{-1}],$$
$$\lambda_\Phi^{(2)}=-\Delta\,\mathrm{Re}[\sqrt{6}b_0^*(b_{+2}+b_{-2})-4b_{+1}^*b_{-1}],$$
$$\lambda_\Phi^{(3)}=\Delta\,\mathrm{Re}(b_{+2}^*b_{-2}),$$
$$\lambda_\Phi^{(1)\perp}=\frac{2}{D}\,\mathrm{Im}[-\sqrt{6}(1+\Delta_1)b_0^*(b_{+2}-b_{-2})+(3-2\Delta_1-5\Delta_2)b_{+1}^*b_{-1}], \tag{14}$$
$$\lambda_\Phi^{(2)\perp}=\Delta\,\mathrm{Im}[\sqrt{6}b_0^*(b_{+2}-b_{-2})+4b_{+1}^*b_{-1}],$$
$$\lambda_\Phi^{(3)\perp}=\Delta\,\mathrm{Im}(b_{+2}^*b_{-2}),$$
$$\lambda_{\Theta\Phi}^{(1)}=\frac{1}{D}\,\mathrm{Re}\left[\sqrt{6}\left(1-\frac{2}{3}\Delta_1-\frac{5}{3}\Delta_2\right)b_0^*(b_{+1}-b_{-1})+\left(6+\frac{8}{3}\Delta_1-\frac{10}{3}\Delta_2\right)(b_{+2}^*b_{+1}-b_{-2}^*b_{-1})\right],$$
$$\lambda_{\Theta\Phi}^{(2)}=\Delta\,\mathrm{Re}[\sqrt{6}b_0^*(b_{+1}-b_{-1})-(b_{+2}^*b_{+1}-b_{-2}^*b_{-1})],$$
$$\lambda_{\Theta\Phi}^{(3)}=\Delta\,\mathrm{Re}(b_{+2}^*b_{-1}-b_{-2}^*b_{+1}),$$
$$\lambda_{\Theta\Phi}^{(1)\perp}=\frac{1}{D}\,\mathrm{Im}\left[-\sqrt{6}\left(1-\frac{2}{3}\Delta_1-\frac{5}{3}\Delta_2\right)b_0^*(b_{+1}+b_{-1})+\left(6+\frac{8}{3}\Delta_1-\frac{10}{3}\Delta_2\right)(b_{+2}^*b_{+1}+b_{-2}^*b_{-1})\right],$$
$$\lambda_{\Theta\Phi}^{(2)\perp}=-\Delta\,\mathrm{Im}[\sqrt{6}b_0^*(b_{+1}+b_{-1})+b_{+2}^*b_{+1}+b_{-2}^*b_{-1}],$$
$$\lambda_{\Theta\Phi}^{(3)\perp}=\Delta\,\mathrm{Im}(b_{+2}^*b_{-1}+b_{-2}^*b_{+1}),$$

with

$$D=(10+\Delta_1-\Delta_2)|b_0|^2+(9-\Delta_2)(|b_{+1}|^2+|b_{-1}|^2)$$
$$+\left(6-\frac{1}{2}\Delta_1+\frac{3}{2}\Delta_2\right)(|b_{+2}|^2+|b_{-2}|^2),$$
$$\Delta_1=4g_2^2+6\sqrt{5}g_2g_3-2\sqrt{5}g_1g_2-2g_3^2+14g_1g_3,$$
$$\Delta_2=4g_2^2+4\sqrt{5}g_2g_3+2\sqrt{5}g_1g_2+3g_3^2+4g_1g_3,$$
$$\Delta=5/(3D)\,(\Delta_1+\Delta_2). \tag{15}$$

As shown in Fig. 2, the dependence of the photon distribution on the χ angular momentum configuration is very sensitive to the contribution of the higher photon multipoles. Figure 2(c) shows, in particular, that the polar anisotropy parameter $\lambda_\Theta^{(1)}$, at the average value of g_2 measured for the χ_{c2} (assuming $g_3=0$), is 30% higher than the value expected in the E1-dominance case if the χ_2 polarization state is $m=\pm1$ or 70% higher if $m=\pm2$. This shows that the derivation of the average polarization state in which the χ is produced from the observed photon angular distribution relies crucially on the knowledge of the multipole amplitudes. Seen from the opposite perspective, we see that the so-called E1 approximation ($h_2=g_2=g_3=0$) is clearly not applicable in the calculation of the $\chi\to V\gamma$ decay kinematics expected for a given χ production mechanism.

IV. LEPTON DISTRIBUTION

In the parity-conserving case here considered, the general expression for the angular distribution of the dilepton decay of a vector state is [8]

$$w(\vartheta,\varphi)=\frac{3}{4\pi(3+\lambda_\vartheta)}(1+\lambda_\vartheta\cos^2\vartheta$$
$$+\lambda_\varphi\sin^2\vartheta\cos2\varphi+\lambda_{\vartheta\varphi}\sin2\vartheta\cos\varphi$$
$$+\lambda_\varphi^\perp\sin^2\vartheta\sin2\varphi+\lambda_{\vartheta\varphi}^\perp\sin2\vartheta\sin\varphi), \tag{16}$$

analogous in form to Eq. (10). The traditional choice of axes, adopted in calculations [9,10] and measurements [4–7] of the full decay angular distribution for χ_c mesons produced at low laboratory momentum, is represented in Fig. 1(b), where the V polarization axis, z', is the V direction in the χ rest frame. With respect to this system of axes, any measurement will always find, for instance in the case of the polar anisotropies for χ_1 and χ_2 dileptons (neglecting, for simplicity, the E3 contribution in the latter case), the following values:

$$\lambda_\vartheta^{j=1} = -\frac{1}{3}\left[1 - \frac{16}{3}h_2 + \mathcal{O}(h_2^2)\right],$$
$$\lambda_\vartheta^{j=2} = \frac{1}{13}\left[1 - \frac{80\sqrt{5}}{13}g_2 + \mathcal{O}(g_2^2)\right]. \tag{17}$$

The dilepton distribution in the x', y', z' coordinate system is independent on the χ polarization state. This choice of axes, while suitable for measuring the contribution of the higher-order multipoles, does not provide any information on the polarization of the χ and, hence, on its production mechanism.

We propose here an alternative definition of the V polarization frame, enabling the determination of the χ polarization in high-momentum experiments without the need of measuring the full photon-dilepton kinematic correlations. This definition, shown in Fig. 1(c), ''clones'' the χ polarization frame, defined in the χ rest frame, into the V rest frame, taking the x'', y'', z'' axes to be parallel to the x, y, z axes.

The coefficients of the dilepton distribution can be written as a function of the angular momentum composition of the decaying vector state [8], $|V\rangle = \sum_{n=-1}^{n=+1} a_n |V; 1, n\rangle$, as

$$\lambda_\vartheta = \frac{\mathcal{N} - 3|a_0|^2}{\mathcal{N} + |a_0|^2},$$
$$\lambda_\varphi = \frac{2\,\mathrm{Re}[a_{+1}^* a_{-1}]}{\mathcal{N} + |a_0|^2},$$
$$\lambda_{\vartheta\varphi} = \frac{\sqrt{2}\,\mathrm{Re}[a_0^*(a_{+1} - a_{-1})]}{\mathcal{N} + |a_0|^2},$$
$$\lambda_\varphi^\perp = \frac{2\,\mathrm{Im}[a_{+1}^* a_{-1}]}{\mathcal{N} + |a_0|^2},$$
$$\lambda_{\vartheta\varphi}^\perp = \frac{-\sqrt{2}\,\mathrm{Im}[a_0^*(a_{+1} + a_{-1})]}{\mathcal{N} + |a_0|^2}, \tag{18}$$

where $\mathcal{N} = |a_0|^2 + |a_{+1}|^2 + |a_{-1}|^2$. The partial amplitude of the χ_j decay into a vector state with angular momentum projection n $(= -1,\ 0,\ 1)$ along z'' and a photon with angular momentum projection k' $(= -1,\ 1)$ along z', from Eqs. (3) and (4), is

$$a_n^{(j,k')}(\Theta, \Phi) \propto \sum_{m=-j}^{j} \sum_{l=1}^{j+1} \sum_{k=-l}^{l} \delta_{m-k,n} b_m \sqrt{2l+1}\, \mathcal{D}_{kk'}^{l*}(\Theta, \Phi) \times (-1)^{(1-k')l/2} H_{jl} \langle 1, m-k, l, k | j, m\rangle. \tag{19}$$

Inserting these amplitudes into the expressions of the coefficients in Eq. (18) and averaging over the photon states $k' = \pm 1$ according to the sum rule

$$X = \frac{\frac{f^{(k'=+1)}\mathcal{N}^{(k'=+1)}X^{(k'=+1)}}{3+\lambda_\vartheta^{(k'=+1)}} + \frac{f^{(k'=-1)}\mathcal{N}^{(k'=-1)}X^{(k'=-1)}}{3+\lambda_\vartheta^{(k'=-1)}}}{\frac{f^{(k'=+1)}\mathcal{N}^{(k'=+1)}}{3+\lambda_\vartheta^{(k'=+1)}} + \frac{f^{(k'=-1)}\mathcal{N}^{(k'=-1)}}{3+\lambda_\vartheta^{(k'=-1)}}}, \tag{20}$$

with $X = \lambda_\vartheta$, λ_φ, etc. and $f^{(k'=-1)} = f^{(k'=+1)} = 1/2$ for parity conservation, it is possible to obtain the expression of the full angular distribution $W(\Theta, \Phi, \vartheta, \varphi)$ of the decay process $\chi \to V\gamma \to V\ell^+\ell^-$. In the following discussion, however, we only consider the dilepton distribution, obtained by integrating $W(\Theta, \Phi, \vartheta, \varphi)$ over Θ and Φ.

In the frame defined in Fig. 1(c), the dilepton decay distribution of V mesons originating from χ_0 decays is isotropic. In what concerns the state $|\chi_1\rangle = \sum_{m=-1}^{m=+1} b_m |\chi; 1, m\rangle$, the coefficients of the dilepton angular distribution are:

$$\lambda_\vartheta = \frac{1}{D_1}[2|b_0|^2 - (|b_{+1}|^2 + |b_{-1}|^2)],$$
$$\lambda_\phi = -\frac{2}{D_1}\,\mathrm{Re}(b_{+1}^* b_{-1}),$$
$$\lambda_{\vartheta\phi} = -\frac{\sqrt{2}}{D_1}\,\mathrm{Re}[b_0^*(b_{+1} - b_{-1})],$$
$$\lambda_\phi^\perp = -\frac{2}{D_1}\,\mathrm{Im}(b_{+1}^* b_{-1}),$$
$$\lambda_{\vartheta\phi}^\perp = \frac{\sqrt{2}}{D_1}\,\mathrm{Im}[b_0^*(b_{+1} + b_{-1})], \tag{21}$$

with

$$D_1 = D/(1 - 3\delta),$$
$$D = 2(1+\delta)|b_0|^2 + (3-\delta)(|b_{+1}|^2 + |b_{-1}|^2),$$
$$\delta = \frac{2}{5}h_2^2. \tag{22}$$

The corresponding coefficients for the decay of the state $|\chi_2\rangle = \sum_{m=-2}^{m=+2} b_m |\chi; 2, m\rangle$ are

$$\lambda_\vartheta = -\frac{3}{D_2}[2|b_0|^2 + |b_{+1}|^2 + |b_{-1}|^2 - 2(|b_{+2}|^2 + |b_{-2}|^2)],$$
$$\lambda_\varphi = \frac{2}{D_2}\mathrm{Re}[\sqrt{6}b_0^*(b_{+2} + b_{-2}) + 3b_{+1}^* b_{-1}],$$
$$\lambda_{\vartheta\varphi} = \frac{1}{D_2}\mathrm{Re}[\sqrt{6}b_0^*(b_{+1} - b_{-1}) + 6(b_{+2}^* b_{+1} - b_{-2}^* b_{-1})],$$
$$\lambda_\varphi^\perp = \frac{2}{D_2}\mathrm{Im}[-\sqrt{6}b_0^*(b_{+2} - b_{-2}) + 3b_{+1}^* b_{-1}],$$
$$\lambda_{\vartheta\varphi}^\perp = \frac{1}{D_2}\mathrm{Im}[-\sqrt{6}b_0^*(b_{+1} + b_{-1}) + 6(b_{+2}^* b_{+1} + b_{-2}^* b_{-1})], \tag{23}$$

with

$$D_2 = D/(1-\delta),$$
$$D = 2(5-\delta)|b_0|^2 + (9-\delta)(|b_{+1}|^2 + |b_{-1}|^2) + 2(3+\delta)(|b_{+2}|^2 + |b_{-2}|^2),$$
$$\delta = 2g_2^2 + \frac{5}{7}g_3^2. \tag{24}$$

Without experimental separation between the χ_1 and χ_2 signals (the $\chi_0 \to V\gamma$ contribution is negligible), the dilepton distribution measurement effectively yields the corresponding average polarization parameters, implicitly weighted by $N^{(j=1)}$ and $N^{(j=2)}$, respectively the numbers of reconstructed dileptons coming from χ_1 and χ_2 decays ($X = \lambda_\vartheta$, λ_φ, etc.):

$$X = \frac{\frac{N^{(j=1)}X^{(j=1)}}{3+\lambda_\vartheta^{(j=1)}} + \frac{N^{(j=2)}X^{(j=2)}}{3+\lambda_\vartheta^{(j=2)}}}{\frac{N^{(j=1)}}{3+\lambda_\vartheta^{(j=1)}} + \frac{N^{(j=2)}}{3+\lambda_\vartheta^{(j=2)}}}. \tag{25}$$

V. MEASUREMENT OF χ POLARIZATION AT HIGH MOMENTUM

The formulas obtained in the previous two sections suggest two remarks. First, with the choice of the x'', y'', z'' axes, the dilepton distribution contains as much information as the photon distribution regarding the χ polarization state. The two distributions are even *identical* when higher-order multipoles are neglected, as can be recognized by comparing Eq. (21) with Eq. (11) as well as Eq. (23) with Eq. (14) for $h_2 = g_2 = g_3 = 0$. In this limit, for example, $\lambda_\vartheta = \lambda_\Theta = -1/3$ and $+1$, respectively, for pure $|j, m\rangle = |1, \pm 1\rangle$ and $|1, 0\rangle$ χ states, and $\lambda_\vartheta = \lambda_\Theta^{(1)} = +1$, $-1/3$ and $-3/5$, respectively, for pure $|2, \pm 2\rangle$, $|2, \pm 1\rangle$ and $|2, 0\rangle$ states, while the additional terms of the photon distribution in the χ_2 case ($\lambda_\Theta^{(2)}$, $\lambda_\Phi^{(2)}$, $\lambda_\Phi^{(3)}$, $\lambda_{\Theta\Phi}^{(2)}$, and $\lambda_{\Theta\Phi}^{(3)}$) vanish. Second, the dependence of the dilepton distribution on the higher photon multipoles is negligible, as shown in Figs. 2(b) and 2(d) for λ_ϑ.

The definition of the x, y, z axes (and, therefore, of the x'', y'', z'' axes) uses the momenta of the colliding hadrons *as seen in the χ rest frame*, so that it requires, in general, the knowledge of the photon momentum. However, *for sufficiently high (total) momentum of the dilepton*, the χ and V rest frames coincide and the x'', y'', z'' axes can be approximately defined using only momenta seen in the V rest frame. For example, if the χ polarization axis (z) is defined along the bisector of the beam momenta in the χ rest frame (Collins–Soper frame [11]), the corresponding z'' axis is approximated by the bisector of the beam momenta in the J/ψ or Υ rest frame. The relative error induced by this approximation on the polar anisotropy parameter is

$$\left|\frac{\Delta\lambda_\vartheta}{\lambda_\vartheta}\right| = \mathcal{O}\left[\left(\frac{\Delta M}{p}\right)^2\right], \tag{26}$$

where ΔM is the $\chi - V$ mass difference and p is the *total* laboratory momentum of the dilepton. Therefore, for not-too-small momentum, the frame definition we propose coincides with the frame defined in the measurement of the polarization of inclusively produced J/ψ or Υ mesons (Collins–Soper or helicity, for example). In other words, the measurement of the dilepton distribution at sufficiently high laboratory momentum provides a direct determination of the χ polarization along the chosen polarization axis. This determination is cleaner than the one using the photon distribution in the χ rest frame, because it is independent of the knowledge of the higher-order photon multipoles.

The above-mentioned approximation, in which the system of axes x'', y'', z'' is set without any knowledge of the photon momentum, becomes rapidly invalid as $p \to 0$. The χ_0 case, although of little practical importance in the scope of this paper (the branching ratio of the χ_{c0} decay to J/ψ, for example, is only $\simeq 1\%$), can be used to give a simple illustration of what happens going from high to low p. As discussed above, at high p the polarization of dileptons from χ_0 vanishes in the helicity frame (as well as in any other frame defined ignoring the photon momentum), mirroring the perfect isotropy of the photon emission in the χ_0 rest frame. On the other hand, the $1S$ state coming from the $j = 0$ χ state has an intrinsic spin alignment, always opposite to the one of the photon ($\mathbf{J}_V + \mathbf{J}_\gamma = \mathbf{J}_{\chi_0} = \mathbf{0}$); in other words, a fully transverse $1S$ polarization is observed if the direction of the photon in the χ_0 rest frame [z' in Fig. 1(b)] is taken as reference axis. In the low-p limit, when the χ_0 tends to be produced at rest in the laboratory, that direction tends to coincide with the center-of-mass helicity axis. In short, if we choose the center-of-mass helicity frame, the V polarization equals the zero polarization of the χ_0 only at high momentum, while it changes to fully transverse at low momentum, where it simply reflects the intrinsic photon polarization. This example shows that the possibility to measure the χ polarization from the dilepton distribution ignoring the photon momentum is strictly limited to a kinematic domain where $p \gg \Delta M$. However, the error in Eq. (26) is already

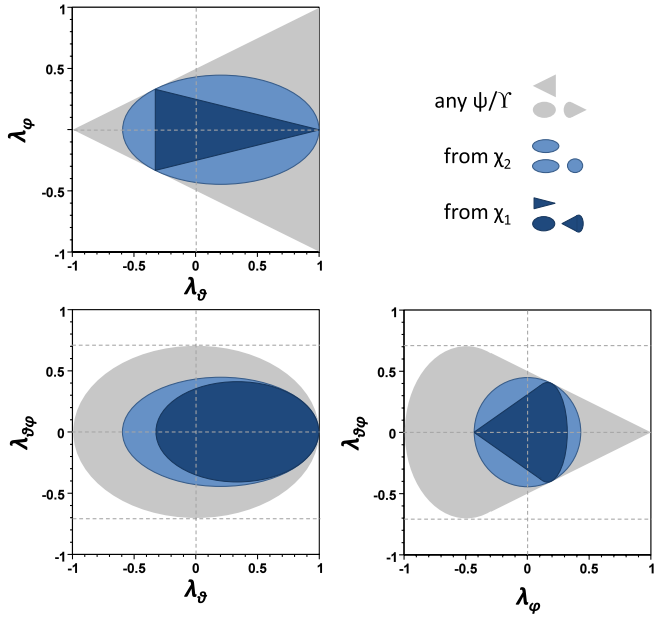

FIG. 3 (color online). Allowed regions for the angular parameters of the dilepton distributions produced by the decay of vector states of any origin (light-shaded [8]), of χ_2 daughters (darker) and of χ_1 daughters (darkest).

as small as 1% when $p > 4$ GeV$/c$, a condition fulfilled, in particular, by essentially all the quarkonium events collected by the LHC experiments.

The parameters of the dilepton distribution at high momentum (Eqs. (21) and (23) with $\delta = 0$) satisfy characteristic inequalities. In the χ_1 case,

$$-\frac{1}{3} \le \lambda_\vartheta \le +1, \qquad |\lambda_\varphi| \le \frac{1-\lambda_\vartheta}{4},$$
$$\frac{9}{4}\left(\lambda_\vartheta - \frac{1}{3}\right)^2 + 6\lambda_{\vartheta\varphi}^2 \le 1, \qquad |\lambda_{\vartheta\varphi}| \le \frac{\sqrt{3}}{2}\left(\lambda_\varphi + \frac{1}{3}\right),$$
$$(6\lambda_\varphi - 1)^2 + 6\lambda_{\vartheta\varphi}^2 \le 1 \quad \text{for } \lambda_\varphi > \frac{1}{9}. \tag{27}$$

In the χ_2 case,

$$\frac{5}{16}\left(\lambda_\vartheta - \frac{1}{5}\right)^2 + \lambda_\varphi^2 + \lambda_{\vartheta\varphi}^2 \le \frac{1}{5}. \tag{28}$$

These inequalities continue to be valid in the presence of a superposition of production processes leading to different angular momentum compositions of the χ_j (see the analogous discussion in Ref. [8] for the direct production of a vector state). The corresponding parameter domains are represented in Fig. 3, compared with the most general constraints valid for vector states, directly or indirectly produced.

VI. COMMENT ON PREVIOUS CALCULATIONS

The angular distributions of the cascade decays $\chi_c \to J/\psi\gamma \to \ell^+\ell^-\gamma$ were calculated in Ref. [9] (denoted as OS in the following) and in Ref. [10] (denoted as RSG in the following) for the specific case of low-energy $p\bar{p}$ collisions, where, due to helicity conservation, the χ_c is only produced in pure J_z eigenstates with eigenvalues $m = \pm 1$ (χ_{c1}) or ± 1, 0 (χ_{c2}). The two calculations use the J/ψ momentum in the χ_c rest frame as quantization axis for the dilepton, as in Fig. 1(b), and provide the full angular distribution of the correlated photon and lepton directions. The result of RSG contradicts the one of OS, pointing to a seemingly wrong sign in the last terms of the χ_{c2} distribution (Eq. (10) of OS, corrected into Eq. (20) of RSG) and of the χ_{c1} distribution (Eq. (15) of OS, Eq. (27) of RSG).

We checked these calculations in two ways, by repeating the steps described in the two papers and by comparing them to our own calculation for the full decay distribution in the special case of pure J_z eigenstates. In the latter case, we have applied a rotation of the lepton variables from the x'', y'', z'' system adopted in our calculation to the x', y', z' system adopted in OS and RSG. We found that, except for an apparent misprint of OS (the fifth line of Eq. (11) in OS has a wrong numerical coefficient, corrected in Eq. (21) of RSG), both calculations are correct. RSG argued that OS used two inconsistent conventions for the reduced rotation matrices d^1_{ij}, adopting one ordering of the indices i and j (the one used in RSG) in the description of the $J/\psi \to \ell^+\ell^-$ process, and the reverse ordering in the description of the $\chi_c \to J/\psi\gamma$ process. We have verified that, instead, the conventions are everywhere consistently used, while RSG did not conform to the calculation of OS and adopted a different definition of the photon angle. OS refers, for the adopted notation, to Ref. [12], where the axes definitions are described in the first figure of the paper. Even if there is no explicit mention in the text, the angle θ in the figure (which we denote by Θ in our Fig. 1) is, unmistakably, the angle formed by the photon momentum with the antiproton direction in the χ_c rest frame, while θ' (which we denote by ϑ in our Fig. 1) is the angle formed by the lepton momentum in the J/ψ rest frame with respect to the J/ψ momentum in the χ_c rest frame. RSG uses the same definition of θ', but an opposite definition of θ: "We will work in the χ_J rest frame with the Z axis taken to be in the direction of ψ. The $\bar{p}$ direction is in the X-Z plane, making an angle θ with the Z axis". As a consequence, when a certain reduced d matrix is used in OS to rotate the quantization axis by an angle θ, the inverse rotation must appear in the calculation of RSG. If $d^1_{ij}(\theta)$ represents a given rotation, the inverse rotation can be denoted either by exchanging i with j (this induced RSG's misinterpretation of the discrepancy) or by replacing θ with $2\pi - \theta$. This explains the different sign in the term proportional to $\sin 2\theta$ resulting from the two calculations. The remaining terms, depending on $\cos^2\theta$, are not sensitive to such a redefinition of the angle.

In short, each of the two calculations is correct, if they are made with the matching angle definition. If, on the

contrary, the definition of θ used by OS is used together with the distributions functions derived in RSG, or vice-versa, a wrong sign appears in the term proportional to $\sin 2\theta$, leading to *unphysical* results. In fact, this artificial change of sign is not reabsorbed in a different definition of sign and/or magnitude of the higher-order multipole amplitudes: already in the E1 approximation, the physical correlation between photon and lepton angles is *substantially* altered by such a mistake. To evaluate the importance of this problem, we assumed the angle definitions of OS and used the formulas derived in RSG, transposing them, by rotation, to the system of axes used in our calculations [Fig. 1(c)]. As a result of this forced mistake, we arrive to a physical result which is almost opposite to the correct one: the lepton distribution, instead of being a perfect clone of the photon distribution (in the E1 limit), becomes a consistently smeared, almost isotropic distribution, for whatever polarization state of the χ (in other words, the domains of the χ_1 and χ_2 dilepton parameters, represented in Fig. 3, are reduced to small areas around the origin).

We have noticed that the measurements of E760 [5] and E835 [6], included in the present world averages of h_2, g_2 and g_3 in the ''Review of Particle Physics'' [13], seem to be affected by this kind of misunderstanding. Both analyses define the photon angle θ as ''the polar angle of the J/ψ with respect to the antiproton'', as in OS, but the formulas are taken from RSG (Table II in the E760 paper and Tables IV–V in the E835 paper reproduce Eqs. (20) and (27) of RSG). On the other hand, the quality of the global fits of the data using the adopted parameterization is rather good and the measurements of the higher-order multipoles are compatible with the CLEO results [7], suggesting that the inconsistency between formulas and angle definitions might simply be an editing mistake in both experimental papers.

VII. CONCLUSIONS

We have derived the expressions of the angular distributions of the radiative decay from a ^3P_J state to a $^3S_1$ state and of the dilepton decay of the latter. No selection rules specific to certain quarkonium production mechanisms have been used and the choice of the polarization frame for the directly produced states has been kept completely general.

We have shown that the χ polarizations can be measured (for not-too-low-momentum experiments) directly from the angular distribution of the *dilepton* decay in the J/ψ or Υ rest frame, with respect to the same kind of system of axes (Collins–Soper, helicity, etc.) adopted in inclusive J/ψ or Υ measurements.

In fact, the dilepton distribution in the J/ψ or Υ rest frame is a clone of the photon distribution in the χ rest frame, *stripped of the contribution of the higher multipoles of photon radiation*.

This represents a significant advantage, given that such contributions—measured to be quite important in the χ_{c2} case, poorly known (due to contradictory measurements) in the χ_{c1} case, and still unmeasured for the bottomonium family—can have a very large impact in the measurement. Furthermore, a simultaneous determination of χ polarization *and* of the multipole parameters is scarcely feasible at hadron colliders.

An additional advantage of this method is that it does not use the photon measurement to reconstruct the event-by-event decay topology. This means that, contrary to previous expectations, the measurement of χ polarization is not intrinsically more challenging than, for instance, the measurement of the χ_{c1}/χ_{c2} cross section ratio (a measurement presently being done by several experiments at the LHC). In both cases the analysis needs to identify an event sample where the J/ψ (or Υ) dilepton is associated to a photon giving an invariant mass of the $\mu^+\mu^-\gamma$ system in the χ mass region. This is usually done using photons reconstructed by the conversion method, given that the tracking of the electron-positron pair gives good enough resolutions to resolve the χ_{c1} and χ_{c2} resonances. Naturally, a larger event sample is needed for a multidimensional angular analysis. But there are no extra difficulties related to photon backgrounds or reconstruction efficiencies depending on the decay angles, specific to the measurement of the polarization (as would be the case using the previously available methods).

It is worth reminding that a certain numerical value of the observable polarization parameters corresponds to very different quantum-mechanical states of the χ_1 and χ_2 (e.g., $\lambda_\vartheta = +1$ can reflect the $J_z = 0$ state of the χ_1 or the $J_z = \pm 2$ state of the χ_2). Therefore, a reliable experimental discrimination between the J/ψ or Υ coming from the decays of these two states is crucial for a proper understanding of χ polarization.

We have also pointed out misunderstandings in previous calculations, which may have affected some of the existing measurements of the higher-order photon multipoles in χ_c decays.

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