Consistent description of kinetic equation with triangle anomaly

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(Received 24 October 2010; revised manuscript received 4 March 2011; published 13 May 2011)

We provide a consistent description of the kinetic equation with a triangle anomaly which is compatible with the entropy principle of the second law of thermodynamics and the charge/energy-momentum conservation equations. In general an anomalous source term is necessary to ensure that the equations for the charge and energy-momentum conservation are satisfied and that the correction terms of distribution functions are compatible to these equations. The constraining equations from the entropy principle are derived for the anomaly-induced leading order corrections to the particle distribution functions. The correction terms can be determined for the minimum number of unknown coefficients in one charge and two charge cases by solving the constraining equations.

I. INTRODUCTION

The experiments at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) provide several pieces of evidence that the quark gluon plasma (QGP) may have been generated and is strongly coupled or nearly a perfect fluid (called sQGP), contrary to the conventional picture of the QGP as a weakly interacting gas of quarks and gluons (for reviews, see e.g. [1–3]). Relativistic hydrodynamics [4–7] is a useful tool to describe the space-time evolution of the fireball formed in heavy ion collisions. The RHIC data for collective flows have been well described by the ideal and dissipative hydrodynamics [8–15].

The correspondence of relativistic hydrodynamics to charged black branes was investigated by AdS/CFT duality [16,17]. A new term associated with the axial anomalies was found in the first order dissipative hydrodynamics (see, e.g., Ref. [18] about holographic hydrodynamics with multiple/non-Abelian symmetries, or Ref. [19] in Sakai-Sugimoto model). Recently the new term has been derived in hydrodynamics with a triangle anomaly [20]. A similar result was also obtained in the microscopic theory of the superfluid [21]. This problem is closely related to the so-called Chiral Magnetic Effect (CME) in heavy ion collisions [22–25]. When two energetic nuclei pass each other a strong magnetic field up to 10^{18} G is formed, which breaks local parity via axial anomaly. This effect may be observed through charge separation. Hydrodynamics in an external background field can be used to pin down the CME in real time simulation. However the anomalous term in the charge current breaks the second law of thermodynamics unless new terms of vorticity and magnetic field are introduced in the charge and entropy currents [20].

In this paper, we try to provide a consistent description of the kinetic equation with a triangle anomaly. We will derive the kinetic equation to the next to leading order as well as the leading order correction to the particle

distribution function arising from anomaly. These results are compatible with the entropy principle of the second law of thermodynamics and the charge/energy-momentum conservation equations.

This paper is organized as follows. In Sec. II, we will derive constraining equations from the entropy principle for the correction terms in distribution functions in the most simple case with one charge and one particle species (without antiparticles). In Sec. III, we will show that an anomaly source term is necessary in general to ensure that the equations for the charge and energy-momentum conservation are satisfied and that the correction terms of distribution functions are compatible to these equations. In Sec. IV and V, we will solve the constraining equations to obtain the correction terms of distribution functions in the one-charge (with antiparticles) and two-charges cases. Finally, we summarize and make conclusions in Sec. VI. We adopt the convention for the metric tensor $g^{\mu\nu} = \text{diag}(+, -, -, -)$.

II. CONSTRAINING DISTRIBUTION FUNCTION WITH ANOMALY COMPATIBLE TO SECOND LAW OF THERMODYNAMICS

In this and the next sections we will consider the most simple case with one charge and one particle species (without antiparticles). The relativistic Boltzmann equation for the on shell phase space distribution $f(x, p)$ in a background electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is given by

$$p^\mu \left(\frac{\partial}{\partial x^\mu} - Q F_{\mu\nu} \frac{\partial}{\partial p_\nu} \right) f(x, p) = \mathcal{C}[f], \qquad (1)$$

where the charge of the particle is $Q = \pm 1$. Here p denotes the on shell 4-momentum satisfying $p^2 = m^2$ where m is the particle mass. We note that $\mathcal{C}[f]$ contains a normal collision term $\mathcal{C}_0[f]$ and a source term from anomaly $\mathcal{C}_A[f]$, $\mathcal{C}[f] = \mathcal{C}_0[f] + \mathcal{C}_A[f]$. We assume that $\mathcal{C}_A[f]$ is at

most of the first order, a small quantity. The necessity for the source term is to make the charge conservation equation hold:

$$\partial_\mu j^\mu = -CE^\mu B_\mu \equiv -CE \cdot B. \qquad (2)$$

Here j^μ is the charge current and $E^\mu = u_\nu F^{\mu\nu}$ and $B_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$ are electric and magnetic field vectors, respectively, where u_μ is the fluid velocity and $\epsilon_{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\alpha\beta} = -1, 1$ for the order of Lorentz indices $(\mu\nu\alpha\beta)$ is an even/odd permutation of (0123). However, the presence of the source term should not influence the energy-momentum conservation:

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} j_\mu. \qquad (3)$$

One can verify that the equilibrium solution of the distribution function,

$$f_0 = \frac{1}{\exp[\beta u_\mu(p^\mu - QF^{\mu\nu}x_\nu) - \beta Q\mu_0] - e}, \qquad (4)$$

satisfies the collisionless Boltzmann equation (1) in an external field for constant $\beta = 1/T$ (T is the local temperature), u_μ and μ_0 (local chemical potential without electromagnetic field). Here $e = 0, \pm 1$ for Boltzmann, Bose and Fermi distributions, respectively. When β, u_μ and μ_0 are not constants but functions of space-time, the Boltzmann equation (1) is not satisfied automatically. Note that we can absorb $-Qx_\nu u_\mu F^{\mu\nu} = Qx \cdot E$ into μ_0 so that f_0 has the form of an equilibrium distribution function:

$$f_0(x, p) = \frac{1}{e^{(u\cdot p - Q\mu)/T} - e}, \qquad (5)$$

where $\mu \equiv \mu_0 - x \cdot E$.

We assume that the distribution function f in presence of an anomaly is a solution of the Boltzmann equation with collision terms in Eq. (1), where β, u_μ and μ are functions of space-time. Generally $f(x, p)$ can be written in the following form:

$$f(x, p) = \frac{1}{e^{(u\cdot p - Q\mu)/T + \chi(x,p)} - e} = f_0(x, p) + f_1(x, p), \qquad (6)$$

where $f_0(x, p)$ is given in Eq. (5) and $f_1(x, p)$ is the first order deviation from it:

$$f_1(x, p) = -f_0(x, p)[1 + ef_0(x, p)]\chi(x, p). \qquad (7)$$

It is known that a magnetic field is closely related to a charge rotation characterized by vorticity. So we introduce into the distribution function terms associated with the vorticity-induced current $\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu\partial^\alpha u^\beta$ and the magnetic field 4-vector B_μ which are assumed to be of the first order, which provide a leading order correction to the particle distribution function. For simplicity we will neglect viscous and diffusive effects throughout the paper, then the ordinary form in the current scheme for $\chi(x, p)$ reads

$$\chi(x, p) = \lambda(p)p \cdot \omega + \lambda_B(p)p \cdot B, \qquad (8)$$

where $\lambda(p)$ and $\lambda_B(p)$ are functions of μ, T, and $u \cdot p$ and have mass dimension -2 and -3, respectively. We will show that $\lambda(p)$ and $\lambda_B(p)$ must depend on momentum otherwise they will contradict the entropy principle from the second law of thermodynamics.

Using Eq. (6) we can decompose the charge and entropy currents and the stress tensor into equilibrium values and the leading order (first order) corrections as $j^\mu = j_0^\mu + j_1^\mu$, $S^\mu = S_0^\mu + S_1^\mu$ and $T^{\mu\nu} = T_0^{\mu\nu} + T_1^{\mu\nu}$ with

$$
\begin{aligned}
j_{0,1}^\mu(x) &= Q\int[dp]p^\mu f_{0,1}(x, p), \\
S_0^\mu(x) &= -\int[dp]p^\mu \psi(f_0), \\
S_1^\mu(x) &= -\int[dp]p^\mu \psi'(f_0)f_1, \\
T_{0,1}^{\mu\nu}(x) &= \int[dp]p^\mu p^\nu f_{0,1}(x, p),
\end{aligned}
\qquad (9)
$$

where we have defined $[dp] \equiv d_g\frac{d^3p}{(2\pi)^3(u\cdot p)}$ (d_g is the degeneracy factor), $\psi(f_0) = f_0\ln(f_0) - e(1 + ef_0) \times \ln(1 + ef_0)$ and $\psi'(f_0) = \ln[f_0/(1 + ef_0)] = -(u \cdot p - Q\mu)/T$. Inserting f_0 into the above formula, we obtain the charge and entropy currents and the stress tensor in equilibrium, $j_0^\mu = nu^\mu$, $S_0^\mu = su^\mu$ and $T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}$, with the energy density ϵ, the pressure P, the particle number density n and the entropy density $s = (\epsilon + P - n\mu)/T$. Using Eqs. (7)–(9), we obtain

$$
\begin{aligned}
j_1^\mu &= \xi\omega^\mu + \xi_B B^\mu, \\
T_1^{\mu\nu} &= DT(u^\mu\omega^\nu + u^\nu\omega^\mu) + D_BT(u^\mu B^\nu + u^\nu B^\mu), \\
S_1^\mu &= -\frac{\mu}{T}(\xi\omega^\mu + \xi_B B^\mu) + (D\omega^\mu + D_B B^\mu),
\end{aligned}
\qquad (10)
$$

where

$$\xi = -QJ_{21}^\lambda \equiv \frac{1}{3}Q\int[dp][(p\cdot u)^2 - m^2]f_0(1 + ef_0)\lambda(p),$$

$$\xi_B = -QJ_{21}^{\lambda_B} \equiv \frac{1}{3}Q\int[dp][(p\cdot u)^2 - m^2]f_0(1 + ef_0)\lambda_B(p),$$

$$D = -\frac{J_{31}^\lambda}{T}$$
$$\equiv \frac{1}{3T}\int[dp][(p\cdot u)^2 - m^2](p\cdot u)f_0(1 + ef_0)\lambda(p),$$

$$D_B = -\frac{J_{31}^{\lambda_B}}{T}$$
$$\equiv \frac{1}{3T}\int[dp][(p\cdot u)^2 - m^2](p\cdot u)f_0(1 + ef_0)\lambda_B(p).$$

$$\qquad (11)$$

On the other hand, ξ, ξ_B, D and D_B as functions of μ and T can be determined by the second law of

thermodynamics or the entropy principle together with Eqs. (2) and (3). We find that $\partial_\mu(su^\mu + S_1^\mu)$ cannot be positive definite unless we make a shift to introduce a new entropy current $\tilde{S}^\mu$ as follows:

$$
\begin{aligned}
\tilde{S}^\mu &= su^\mu + S_1^\mu - (D\omega^\mu + D_B B^\mu) \\
&= su^\mu - \frac{\mu}{T}(\xi\omega^\mu + \xi_B B^\mu) \\
&= \frac{1}{T}(Pu^\mu - \mu j^\mu + u_\lambda T^{\lambda\mu}) - (D\omega^\mu + D_B B^\mu). \quad (12)
\end{aligned}
$$

We will use the thermodynamic relation

$$
\partial_\mu(Pu^\mu) = j_0^\mu \partial_\mu \bar{\mu} - T_0^{\lambda\mu}\partial_\mu u_\lambda \quad (13)
$$

and the identities

$$
\begin{aligned}
u^\mu u^\lambda \partial_\mu \omega_\lambda &= \frac{1}{2}\partial_\mu \omega^\mu, \\
u^\mu u^\lambda \partial_\mu B_\lambda &= \partial_\mu B^\mu - 2\omega^\rho E_\rho, \\
\partial_\mu \omega^\mu &= -\frac{2}{\epsilon + P}(n\omega^\mu E_\mu + \omega^\mu \partial_\mu P), \\
\partial_\mu B^\mu &= 2\omega^\rho E_\rho - \frac{1}{\epsilon + P}(nB_\lambda E^\lambda + B^\mu \partial_\mu P),
\end{aligned}
\quad (14)
$$

to evaluate $\partial_\mu \tilde{S}^\mu$. We have used the shorthand notation $\bar{\mu} \equiv \mu/T$ in Eq. (13). Following the same procedure as in Ref. [20], we obtain

$$
\begin{aligned}
\partial_\mu \tilde{S}^\mu = \omega^\mu &\left[\xi^{SS}\partial_\mu \bar{\mu} - \partial_\mu D + \frac{2D}{\epsilon + P}\partial_\mu P \right] \\
&+ B^\mu\left[\xi_B^{SS}\partial_\mu \bar{\mu} - \partial_\mu D_B + \frac{D_B}{\epsilon + P}\partial_\mu P \right] \\
&+ E \cdot \omega\left[\frac{1}{T}\xi^{SS} + \frac{2nD}{\epsilon + P} - 2D_B \right] \\
&+ E \cdot B\left[\frac{1}{T}\xi_B^{SS} + C\frac{\mu_A}{T} + \frac{nD_B}{\epsilon + P} \right], \quad (15)
\end{aligned}
$$

where we have defined

$$
\xi^{SS} = \frac{DTn}{\epsilon + P} - \xi, \qquad \xi_B^{SS} = \frac{D_B Tn}{\epsilon + P} - \xi_B. \quad (16)
$$

For the constraint $\partial_\mu \tilde{S}^\mu \geq 0$ to hold, we impose that all quantities inside the square brackets should vanish. We finally obtain

$$
\begin{aligned}
D &= \frac{1}{3}C\frac{\mu^3}{T}, \qquad D_B = \frac{1}{2}C\frac{\mu^2}{T}, \\
\xi &= -C\frac{sT\mu^2}{\epsilon + P}, \qquad \xi_B = -C\frac{sT\mu}{\epsilon + P}.
\end{aligned}
\quad (17)
$$

Using Eqs. (16) and (17), one can verify that the values of ξ^{SS} and ξ_B^{SS} are identical to Ref. [20]. The difference between our values in Eq. (17) and those in Ref. [20] arises from the fact that we do not use the Landau frame, while the authors of Ref. [20] do. By equating Eq. (11) and (17), we obtain equations for λ and λ_B

$$
\begin{aligned}
QJ_{21}^\lambda &= -\xi, & J_{31}^\lambda &= -DT, \\
QJ_{21}^{\lambda_B} &= -\xi_B, & J_{31}^{\lambda_B} &= -D_B T.
\end{aligned}
\quad (18)
$$

Equation (18) forms a complete set of constraints for λ and λ_B. We note that λ and λ_B must depend on momentum in general. If λ and λ_B are constants, we would obtain

$$
\frac{\xi}{DT} = \frac{\xi_B}{D_B T} = Q\frac{J_{21}}{J_{31}}, \quad (19)
$$

which contradicts Eq. (17) from the entropy principle.

We can expand $\lambda(p)$ and $\lambda_B(p)$ in powers of $u \cdot p$:

$$
\lambda(p) = \sum_{i=0}\lambda_i(u \cdot p)^i, \qquad \lambda_B(p) = \sum_{i=0}\lambda_i^B(u \cdot p)^i. \quad (20)
$$

So we obtain the following expressions:

$$
J_{n1}^\lambda = \sum_{i=0}\lambda_i J_{i+n,1}, \qquad J_{n1}^{\lambda_B} = \sum_{i=0}\lambda_i^B J_{i+n,1}, \quad (21)
$$

for $n = 2, 3$. Here the functions J_{nq} are integrals defined in Ref. [7,11]:

$$
\begin{aligned}
J_{nq} = (-1)^q \frac{1}{(2q+1)!!}\int \frac{d^3 p}{(2\pi)^3(u \cdot p)}[(u \cdot p)^2 - m^2]^q \\
\times (u \cdot p)^{n-2q}f_0(1 + ef_0). \quad (22)
\end{aligned}
$$

Using Eqs. (20) and (21) in Eq. (18), we can constrain the coefficients λ_i and λ_i^B. If we expand both $\lambda(p)$ and $\lambda_B(p)$ to the first power of $u \cdot p$, we can completely fix the coefficients $\lambda_{0,1}$ and $\lambda_{0,1}^B$ from Eq. (18) since we have two equations for $\lambda_{0,1}$ and two for $\lambda_{0,1}^B$:

$$
\begin{pmatrix} QJ_{21} & QJ_{31} \\ J_{31} & J_{41} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -\xi \\ -DT \end{pmatrix}, \quad (23)
$$

whose solutions to $\lambda_{0,1}$ are

$$
\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \frac{1}{Q(J_{21}J_{41} - J_{31}^2)}\begin{pmatrix} -\xi J_{41} + DTQJ_{31} \\ \xi J_{31} - DTQJ_{21} \end{pmatrix}. \quad (24)
$$

The equations and solutions for $\lambda_{0,1}^B$ are in the same form as Eqs. (23) and (24) with replacements $\lambda_{0,1} \to \lambda_{0,1}^B$, $\xi \to \xi_B$ and $D \to D_B$.

For massless fermions and small $\bar{\mu}$, the results are

$$
\lambda_0 \approx -CG_1\frac{\mu^2}{T^4}, \quad \lambda_1 \approx CG_2\frac{\mu^2}{T^5}, \quad \lambda_0^B \approx \frac{\lambda_0}{\mu}, \quad \lambda_1^B \approx \frac{\lambda_1}{\mu}, \quad (25)
$$

where G_1 and G_2 are two constants, $G_1 \equiv 607500\pi^2\zeta(5)/(d_g G_0)$ and $G_2 \equiv 1260\pi^6/(d_g G_0)$ with $G_0 \equiv 455625\zeta(3)\zeta(5) - 49\pi^8$. We notice that the D terms in Eq. (24) are negligible, so the solutions are proportional to ξ.

III. COLLISION AND ANOMALOUS SOURCE TERMS

In this section we will show that a general form of $\lambda(p)$ and $\lambda_B(p)$ are compatible to the charge and energy-momentum conservation in Eqs. (2) and (3). We will also derive equations for the collision and anomalous source terms. For simplicity we consider the single charge case without antiparticles.

The charge conservation in Eq. (2) can be derived from the Boltzmann equation (1) as

$$
\begin{aligned}
\partial_\mu j^\mu(x) &= \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu \partial_\mu f(x, p) \\
&= \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu F_{\mu\nu} \frac{\partial f}{\partial p_\nu} + \int \frac{d^3 p}{(2\pi)^3 E_p} \mathcal{C}[f] \\
&= \int \frac{d^3 p}{(2\pi)^3 E_p} \mathcal{C}[f],
\end{aligned}
\tag{26}
$$

where have used the identity

$$
\begin{aligned}
\int \frac{d^3 p}{2 E_p} p^\mu F_{\mu\nu} \frac{\partial f}{\partial p_\nu} &= \int d^4 p \, \theta(p_0) \delta(p^2 - m^2) p^\mu F_{\mu\nu} \frac{\partial f}{\partial p_\nu} \\
&= -\int d^4 p \frac{\partial}{\partial p_\nu} \\
&\quad \times [\theta(p_0) \delta(p^2 - m^2) p^\mu F_{\mu\nu}] f = 0.
\end{aligned}
\tag{27}
$$

Then the momentum integral of the collision term must obey

$$
\int \frac{d^3 p}{(2\pi)^3 E_p} \mathcal{C}[f] = -C E \cdot B,
\tag{28}
$$

so that Eq. (2) can hold. The energy and momentum conservation in Eq. (3) can be derived from the Boltzmann equation (1) as

$$
\begin{aligned}
\partial_\mu T^{\mu\nu} &= \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu p^\nu \partial_\mu f \\
&= \int \frac{d^3 p}{(2\pi)^3 E_p} p^\nu p^\alpha F_{\alpha\beta} \frac{\partial f}{\partial p_\beta} + \int \frac{d^3 p}{(2\pi)^3 E_p} p^\nu \mathcal{C}[f] \\
&= F^{\nu\mu} j_\mu + \int \frac{d^3 p}{(2\pi)^3 E_p} p^\nu \mathcal{C}[f],
\end{aligned}
\tag{29}
$$

where we have used

$$
\begin{aligned}
&\int \frac{d^3 p}{2 E_p} p^\nu p^\alpha F_{\alpha\beta} \frac{\partial f}{\partial p_\beta} \\
&= \int d^4 p \, \theta(p_0) \delta(p^2 - m^2) p^\nu p^\alpha F_{\alpha\beta} \frac{\partial f}{\partial p_\beta} \\
&= -\int d^4 p \frac{\partial}{\partial p_\beta} [\theta(p_0) \delta(p^2 - m^2) p^\nu p^\alpha F_{\alpha\beta}] f = F^{\nu\alpha} j_\alpha.
\end{aligned}
\tag{30}
$$

Then we require that the collision term must satisfy

$$
\int \frac{d^3 p}{(2\pi)^3 E_p} p^\nu \mathcal{C}[f] = 0,
\tag{31}
$$

so that Eq. (3) can hold.

We can expand $\mathcal{C}[f]$ to the second order as

$$
\mathcal{C}[f] = \mathcal{C}_0[f_0 + f_1 + f_2] + \mathcal{C}_A[f] \approx \mathcal{C}_1 + \mathcal{C}_2
\tag{32}
$$

where we have used the property $\mathcal{C}_0[f_0] = 0$, and defined

$$
\begin{aligned}
\mathcal{C}_1 &= \frac{d\mathcal{C}_0}{df}\bigg|_{f=f_0} f_1 + \mathcal{C}_{A1}, \\
\mathcal{C}_2 &= \frac{d\mathcal{C}_0}{df}\bigg|_{f=f_0} f_2 + \frac{1}{2} \frac{d^2\mathcal{C}_0}{df^2}\bigg|_{f=f_0} f_1^2 + \mathcal{C}_{A2}.
\end{aligned}
\tag{33}
$$

Note that the general form for the normal part of $\mathcal{C}_1$ is $\frac{d\mathcal{C}_0}{df}\big|_{f=f_0} f_1 = H_\lambda(u \cdot p) p \cdot \omega + H_{\lambda_B}(u \cdot p) p \cdot B$. When inserting the distribution function (6) into Boltzmann equation (1) and using Eq. (32), we obtain the Boltzmann equations to the first and second order:

$$
p^\mu \left(\frac{\partial}{\partial x^\mu} - F_{\mu\nu} \frac{\partial}{\partial p_\nu} \right) f_0 = \mathcal{C}_1,
\tag{34}
$$

$$
p^\mu \left(\frac{\partial}{\partial x^\mu} - F_{\mu\nu} \frac{\partial}{\partial p_\nu} \right) f_1 = \mathcal{C}_2.
\tag{35}
$$

From Eqs. (33) and (34) we can determine the anomalous source term of the first order:

$$
\begin{aligned}
\mathcal{C}_{A1} = &-H_\lambda(u \cdot p) p \cdot \omega - H_{\lambda_B}(u \cdot p) p \cdot B \\
&- f_0(1 + ef_0) p^\mu \partial_\mu [(u \cdot p - \mu_0)/T].
\end{aligned}
\tag{36}
$$

By evaluating the left-hand sides of Eq. (35), we can fix $\mathcal{C}_2$ as follows:

$$
\begin{aligned}
\mathcal{C}_2 = f_0(1 + ef_0)\{&-[(1 + 2ef_0)\lambda \partial_\mu \psi'(f_0) + (\partial_\mu \lambda)] p^\mu p^\nu \omega_\nu - [(1 + 2ef_0)\lambda_B \partial_\mu \psi'(f_0) + (\partial_\mu \lambda_B)] p^\mu p^\nu B_\nu \\
&+ \left[-\beta(1 + 2ef_0)\lambda + \frac{d\lambda}{d(u \cdot p)} \right] p^\mu p^\nu E_\mu \omega_\nu + \lambda p^\mu F_{\mu\nu} \omega^\nu + \left[-\beta(1 + 2ef_0)\lambda_B + \frac{d\lambda_B}{d(u \cdot p)} \right] p^\mu p^\nu E_\mu B_\nu \\
&+ \lambda_B p^\mu F_{\mu\nu} B^\nu - \lambda p^\mu p^\nu \partial_\mu \omega_\nu - \lambda_B p^\mu p^\nu \partial_\mu B_\nu \}.
\end{aligned}
\tag{37}
$$

Taking momentum integrals for Eqs. (34) and (35), we obtain the divergences of charge currents to the first and second order:

$$\partial_\mu j_0^\mu(x) = \int [dp]\mathcal{C}_1 = \int [dp]\mathcal{C}_{A1}, \qquad (38)$$

$$\partial_\mu j_1^\mu(x) = \int [dp]\mathcal{C}_2, \qquad (39)$$

where we have used the property

$$\int [dp]\frac{d\mathcal{C}_0}{df}\Big|_{f=f_0} f_1 = \int [dp][H_\lambda(u \cdot p)p \cdot \omega \\ + H_{\lambda_B}(u \cdot p)p \cdot B] = 0. \qquad (40)$$

With Eq. (9) and identities in Eq. (14), the left-hand side of Eq. (39) can be evaluated as

$$\partial_\mu j_1^\mu(x) = \omega \cdot E\left(2\xi_B - \frac{2n\xi}{\epsilon + P}\right) - B \cdot E\frac{n\xi_B}{\epsilon + P} \\ + \omega^\mu\left(\partial_\mu\xi - \frac{2\xi}{\epsilon + P}\partial_\mu P\right) \\ + B^\mu\left(\partial_\mu\xi_B - \frac{\xi_B}{\epsilon + P}\partial_\mu P\right). \qquad (41)$$

In order for Eq. (2) to be satisfied, it is required that $\partial_\mu j_0^\mu(x)$ must have the form

$$\partial_\mu j_0^\mu(x) = -CB \cdot E - \partial_\mu j_1^\mu(x) \\ = -\omega \cdot E\left(2\xi_B - \frac{2n\xi}{\epsilon + P}\right) - B \cdot E\left(C - \frac{n\xi_B}{\epsilon + P}\right) \\ - \omega^\mu\left(\partial_\mu\xi - \frac{2\xi}{\epsilon + P}\partial_\mu P\right) \\ - B^\mu\left(\partial_\mu\xi_B - \frac{\xi_B}{\epsilon + P}\partial_\mu P\right). \qquad (42)$$

We see that $\partial_\mu j_0^\mu(x)$ is not vanishing but of the second order though it is superficially a first order quantity. This is very important otherwise it would lead to $\partial_\mu j_1^\mu(x) = -CE \cdot B$ and give rise to incompatible results for ξ, ξ_B, D, D_B with Ref. [20]. Therefore we should keep in mind that it is the full charge current that satisfies charge conservation equation (2). We note that the expression of $\partial_\mu j_0^\mu(x)$ in Eq. (42) also gives the momentum integral of the anomalous source term of the first order:

$$\int [dp]\mathcal{C}_{A1} = -\omega \cdot E\left(2\xi_B - \frac{2n\xi}{\epsilon + P}\right) - B \cdot E\left(C - \frac{n\xi_B}{\epsilon + P}\right) \\ - \omega^\mu\left(\partial_\mu\xi - \frac{2\xi}{\epsilon + P}\partial_\mu P\right) \\ - B^\mu\left(\partial_\mu\xi_B - \frac{\xi_B}{\epsilon + P}\partial_\mu P\right). \qquad (43)$$

For the energy and momentum conservation we obtain

$$\partial_\mu T_0^{\mu\nu} - F^{\nu\mu}j_\mu^0 = \int [dp]p^\nu\mathcal{C}_1, \qquad (44)$$

$$\partial_\mu T_1^{\mu\nu} - F^{\nu\mu}j_\mu^1 = \int [dp]p^\nu\mathcal{C}_2. \qquad (45)$$

With Eqs. (9) and (14), we can evaluate the left-hand side of Eq. (45) as

$$\partial_\mu T_1^{\mu\nu} - F^{\nu\mu}j_\mu^1 = \omega^\nu[DT\partial \cdot u + u \cdot \partial(DT)] + B^\nu[D_BT\partial \cdot u + u \cdot \partial(D_BT)] + u^\nu\{\omega \cdot E(2D_BT - \frac{2DTn}{\epsilon + P}) \\ - E \cdot B\frac{D_BTn}{\epsilon + P} + \omega \cdot [\partial(DT) - \frac{2DT}{\epsilon + P}\partial P] + B \cdot [\partial(D_BT) - \frac{D_BT}{\epsilon + P}\partial P]\} + DT(u \cdot \partial\omega^\nu + \omega \cdot \partial u^\nu) \\ + D_BT(u \cdot \partial B^\nu + B \cdot \partial u^\nu) - F^{\nu\mu}(\xi\omega_\mu + \xi_BB_\mu), \qquad (46)$$

where one can verify that each term in the right-hand side is of second order. The left-hand side of Eq. (44) is then given by

$$\partial_\mu T_0^{\mu\nu} - F^{\nu\mu}j_\mu^0 = -(\partial_\mu T_1^{\mu\nu} - F^{\nu\mu}j_\mu^1), \qquad (47)$$

which is also a second order quantity though $p^\nu\mathcal{C}_1 = p^\nu p^\mu(\frac{\partial}{\partial x^\mu} - F_{\mu\sigma}\frac{\partial}{\partial p_\sigma})f_0$ is superficially of first order. One might question the validity of Eq. (14) which follows $\partial_\mu T_0^{\mu\nu} - F^{\nu\mu}j_\mu^0 = 0$, but it is not a problem here since this equation really holds at the first

order or the leading order and is not true at the second order. It is essential that the energy-momentum equation (3) holds for the full quantities $T^{\mu\nu}$ and j_μ, and not for $T_{0,1}^{\mu\nu}$ and $j_\mu^{0,1}$ separately, otherwise the results would be contradictory to those of Ref. [20] following the entropy principle of the second law of thermodynamics.

We now obtain the momentum integral of $p^\nu\mathcal{C}_{1,2}$ from Eqs. (44) and (45) with $\partial_\mu T_{0,1}^{\mu\nu} - F^{\nu\mu}j_\mu^{0,1}$ given by Eqs. (46) and (47).

IV. ONE CHARGE WITH PARTICLE/ANTIPARTICLE

We now consider a one-charge case but add antiparticles to the system. We will calculate $\lambda(p)$ and $\lambda_B(p)$. Since there are particles and antiparticles, we recover the index $Q = \pm 1$ in particle distribution function, $f \to f^Q$ and $f_{0,1} \to f_{0,1}^Q$. In Eq. (9), summations over Q should be added. In Eq. (11) we also have to add the index Q to J_{n1}^λ and $J_{n1}^{\lambda_B}$: $J_{n1}^\lambda \to J_{n1}^{\lambda,Q}$ and $J_{n1}^{\lambda_B} \to J_{n1}^{\lambda_B,Q}$ ($n = 2, 3$), and add summations over Q into the formula of ξ, ξ_B, D, D_B. Inserting f_0^Q into Eq. (9), we obtain the charge and entropy currents and the stress tensor in the equilibrium, $j_0^\mu = nu^\mu$, $S_0^\mu = su^\mu$ and $T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}$, with the particle number density $n \equiv \sum_Q Q n_Q$, the energy density $\epsilon = \sum_Q \epsilon_Q$, the pressure $P = \sum_Q P_Q$, and the entropy density $s = \sum_Q s_Q = (\epsilon + P - n\mu)/T$. The solutions to ξ, ξ_B, D, D_B are the same as in Eq. (17). Then Eq. (18) is modified to

$$J_{21}^{\lambda,+} - J_{21}^{\lambda,-} = -\xi, \qquad J_{31}^{\lambda,+} + J_{31}^{\lambda,+} = -DT,$$
$$J_{21}^{\lambda_B,+} - J_{21}^{\lambda_B,-} = -\xi_B, \qquad J_{31}^{\lambda_B,+} + J_{31}^{\lambda_B,-} = -D_B T. \quad (48)$$

Equation (21) now becomes

$$J_{n1}^{\lambda,Q} = \sum_{i=0} \lambda_i J_{i+n,1}^Q, \qquad J_{n1}^{\lambda_B,Q} = \sum_{i=0} \lambda_i^B J_{i+n,1}^Q. \quad (49)$$

In the case of the minimal number of coefficients we can completely fix the coefficients $\lambda_{0,1}$ by solving following system of equations:

$$\begin{pmatrix} \delta J_{21} & \delta J_{31} \\ \sigma J_{31} & \sigma J_{41} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -\xi \\ -DT \end{pmatrix}, \quad (50)$$

where we have used the shorthand notation, $\delta J_{n1} \equiv J_{n1}^+ - J_{n1}^-$ and $\sigma J_{n1} \equiv J_{n1}^+ + J_{n1}^-$. The solutions to $\lambda_{0,1}$ are

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} -\xi(\sigma J_{41}) + DT(\delta J_{31}) \\ \xi(\sigma J_{31}) - DT(\delta J_{21}) \end{pmatrix}, \quad (51)$$

where $\Delta = (\delta J_{21})(\sigma J_{41}) - (\sigma J_{31})(\delta J_{31})$. The equations and solutions for $\lambda_{0,1}^B$ are in the same form as Eqs. (50) and (51) with replacements $\lambda_{0,1} \to \lambda_{0,1}^B$, $\xi \to \xi_B$ and $D \to D_B$.

For massless fermions and small $\bar\mu$, we have

$$\sigma J_{21} \approx 9\zeta(3)G, \qquad \sigma J_{31} \approx \frac{7\pi^4}{15}GT,$$
$$\sigma J_{41} \approx 225\zeta(5)GT^2, \qquad \delta J_{21} \approx \pi^2 \bar\mu G,$$
$$\delta J_{31} \approx 36\zeta(3)\bar\mu GT, \qquad \delta J_{41} \approx \frac{7\pi^4}{3}\bar\mu GT^2,$$
$$\Delta \approx \frac{\pi^2}{5}G^2 G_0 T^2 \bar\mu, \quad (52)$$

where $G \equiv \frac{d_g T^4}{6\pi^2}$ and $G_0 \equiv -84\pi^2\zeta(3) + 1125\zeta(5)$. The solutions have very simple form:

$$\lambda_0 \approx CG_1 \frac{\mu}{T^3}, \quad \lambda_1 \approx -CG_2\frac{\mu}{T^4}, \quad \lambda_0^B = \frac{\lambda_0}{\mu}, \quad \lambda_1^B = \frac{\lambda_1}{\mu}, \quad (53)$$

where we have used two constants, $G_1 \equiv \frac{6750\pi^2\zeta(5)}{d_g G_0}$, $G_2 \equiv \frac{14}{d_g G_0}$. We notice that the D terms in Eq. (51) are negligible, so the solutions are proportional to ξ. Note that the quantity $\frac{n\mu}{\epsilon+P} \sim \bar\mu^2$ is also small and we have dropped it, since $n\mu \approx \frac{d_g}{6\pi^2}T^4\bar\mu^2$ and $\epsilon + P = \frac{4}{3}\epsilon \approx d_g\frac{7\pi^2}{90}T^4$.

V. WITH TWO CHARGES AND PARTICLE/ANTIPARTICLE

As an example for the case of two charges, we consider adding to the system the chirality or an axial $U(1)$ charge to particles. Then there are two currents, one for each chirality, or equivalently, for the $U(1)/U_A(1)$ charge. For simplicity we assume that there is an anomaly for the axial-charge current but no anomaly for the charge one. There are distribution functions for right-hand and left-hand particles, $f_a^Q(x, p)$ ($a = R, L$), with chemical potentials $\mu_{R,L} = \mu \pm \mu_A$. As an extension to Eq. (8), the corrections $\chi_a(x, p)$ in $f_a^Q(x, p)$ are now $\chi_a(x, p) = \lambda_a p^\mu \omega_\mu + \lambda_{aB} p^\mu B_\mu$. Instead of right-hand and left-hand quantities X_a, we can equivalently use $X = X_R + X_L$ and $X_A = X_R - X_L$, where $X = \lambda$, λ_B, ξ, ξ_B, n, s, ϵ, P, j_μ. The distribution functions $f_a^Q(x, p)$ satisfy two separate Boltzmann equations of the following form:

$$p^\mu\left(\frac{\partial}{\partial x^\mu} - QF_{\mu\nu}\frac{\partial}{\partial p_\nu}\right)f_a^Q(x, p) = \mathcal{C}_{aQ}[f_b^{Q'}]. \quad (54)$$

The $U(1)/U_A(1)$ charge and entropy currents and the stress tensor in equilibrium are given by $j_0^\mu = nu^\mu$, $j_{A0}^\mu = n_A u^\mu$, $S_0^\mu = su^\mu$ and $T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}$, with the particle number density $n \equiv \sum_{aQ} Q n_{aQ}$ ($a = R, L; Q = \pm 1$), the energy density $\epsilon = \sum_{aQ} \epsilon_{aQ}$, the pressure $P = \sum_{aQ} P_{aQ}$, and the entropy density $s = \sum_{aQ} s_{aQ} = (\epsilon + P - \sum_a n_a\mu_a)/T$.

Similar to Eqs. (26) and (28), the $U(1)/U_A(1)$ charge conservation in Eq. (2) can be derived as

$$\partial_\mu j^\mu(x) = \int[dp](\mathcal{C}_{R,+} - \mathcal{C}_{R,+} + \mathcal{C}_{L,+} - \mathcal{C}_{L,+}) = 0, \quad (55)$$

$$\partial_\mu j_A^\mu(x) = \int[dp](\mathcal{C}_{R,+} - \mathcal{C}_{R,-} - \mathcal{C}_{L,+} + \mathcal{C}_{L,+})$$
$$= -CE^\mu B_\mu, \quad (56)$$

where we have used Eq. (27). The energy and momentum conservation equation reads

$$\partial_\mu T^{\mu\nu} - F^{\nu\mu} j_\mu$$

$$= \int [dp] p^\nu (\mathcal{C}_{R,+} + \mathcal{C}_{R,-} + \mathcal{C}_{L,+} + \mathcal{C}_{L,-}) = 0, \quad (57)$$

where we have used Eq. (30).

Similar to Eq. (11), we can express ξ_a, ξ_{aB}, D and D_B in terms of λ_a and λ_{aB} as

$$\xi_a = -(J_{21}^{\lambda_a,+} - J_{21}^{\lambda_a,-}),$$

$$-DT = J_{31}^{\lambda_R,+} + J_{31}^{\lambda_R,-} + J_{31}^{\lambda_L,+} + J_{31}^{\lambda_L,-},$$

$$\xi_{aB} = -(J_{21}^{\lambda_{aB},+} - J_{21}^{\lambda_{aB},-}), \quad (58)$$

$$-D_B T = J_{31}^{\lambda_{RB},+} + J_{31}^{\lambda_{RB},-} + J_{31}^{\lambda_{LB},+} + J_{31}^{\lambda_{LB},-},$$

for $a = R, L$. We have the following first order corrections:

$$j_1^\mu = j_{R1}^\mu + j_{L1}^\mu = \sum_{a=R,L} (\xi_a \omega^\mu + \xi_{aB} B^\mu) = \xi \omega^\mu + \xi_B B^\mu,$$

$$j_{A1}^\mu = j_{R1}^\mu - j_{L1}^\mu = (\xi_R - \xi_L) \omega^\mu + (\xi_{RB} - \xi_{LB}) B^\mu$$

$$= \xi_A \omega^\mu + \xi_{AB} B^\mu,$$

$$T_1^{\mu\nu} = DT(u^\mu \omega^\nu + u^\nu \omega^\mu) + D_B T(u^\mu B^\nu + u^\nu B^\mu),$$

$$S_1^\mu = -\sum_{a=R,L} \frac{\mu_a}{T}(\xi_a \omega^\mu + \xi_{aB} B^\mu) + (D\omega^\mu + D_B B^\mu)$$

$$= -\sum_{i=\text{null},A} \frac{\mu_i}{T}(\xi_i \omega^\mu + \xi_{iB} B^\mu) + (D\omega^\mu + D_B B^\mu). \quad (59)$$

It can be verified that the entropy current in Eq. (59) cannot satisfy $\partial_\mu S^\mu \geq 0$ unless $C = 0$. In order to ensure the positivity of $\partial_\mu S^\mu$ in presence of an anomaly, one would have to substract vector $Q^\mu = D\omega^\mu + D_B B^\mu$ from S^μ. With $U(1)$ and $U_A(1)$ charges, we have

$$\tilde{S}^\mu = S^\mu - Q^\mu = s u^\mu - \sum_{a=R,L} \bar{\mu}_a (\xi_a \omega^\mu + \xi_{aB} B^\mu)$$

$$= \frac{1}{T} \left(Pu^\mu - \sum_{a=R,L} \mu_a j_a^\mu + u_\lambda T^{\lambda\mu} \right) - Q^\mu. \quad (60)$$

In the same way as in Sec. II, the divergence of the entropy current can be evaluated as

$$\partial_\mu \tilde{S}^\mu = \omega^\mu \left[\sum_{a=R,L} \partial_\mu \bar{\mu}_a \left(\frac{n_i T D}{\epsilon + P} - \xi_a \right) - \partial_\mu D + \frac{2D}{\epsilon + P} \partial_\mu P \right]$$

$$+ B^\mu \left[\sum_{a=R,L} \partial_\mu \bar{\mu}_a \left(\frac{n_i T D_B}{\epsilon + P} - \xi_{aB} \right) - \partial_\mu D_B \right.$$

$$\left. + \frac{D_B}{\epsilon + P} \partial_\mu P \right] + E \cdot \omega \left[\sum_{a=R,L} \left(\frac{n_a D}{\epsilon + P} - \frac{\xi_a}{T} \right) \right.$$

$$\left. + \frac{2nD}{\epsilon + P} - 2D_B \right] + E \cdot B \left[\sum_{a=R,L} \left(\frac{n_a D_B}{\epsilon + P} - \frac{\xi_{aB}}{T} \right) \right.$$

$$\left. + C \frac{\mu_A}{T} + \frac{n D_B}{\epsilon + P} \right], \quad (61)$$

following Eqs. (55)–(57). By imposing all quantities inside the square brackets to vanish we can solve ξ_a, ξ_{aB}, D, D_B as follows:

$$\xi = 2C\mu\mu_A \left(1 - \frac{3}{2} \frac{n\mu}{\epsilon + P} \right),$$

$$\xi_A = C\mu^2 \left(1 - \frac{3 n_A \mu_A}{\epsilon + P} \right),$$

$$\xi_B = C\mu_A \left(1 - \frac{2n\mu}{\epsilon + P} \right),$$

$$\xi_{AB} = C\mu \left(1 - \frac{2 n_A \mu_A}{\epsilon + P} \right), \quad (62)$$

$$DT = -C\mu_A \mu^2,$$

$$D_B T = -C\mu_A \mu.$$

It is interesting to observe that for small μ, μ_A, we obtain very simple and symmetric solutions:

$$\xi \approx 2C\mu\mu_A, \qquad \xi_A \approx C\mu^2, \qquad \xi_B \approx C\mu_A,$$

$$\xi_{AB} \approx C\mu, \qquad DT = -C\mu_A \mu^2, \qquad D_B T = -C\mu_A \mu, \quad (63)$$

which is identical to the result of Ref. [23,25]. Here we have assumed that all integral constants are vanishing. Note that Eq. (63) is the result of the entropy principle in the hydrodynamic approach, which was also obtained in Ref. [26–28].

Equivalently we can use $\xi_R = (\xi + \xi_A)/2$ and $\xi_L = (\xi - \xi_A)/2$ to determine λ_a, λ_{aB} ($a = R, L$) via solving a system of Eq. (58), where the first/second line (each has three equations) is for $\lambda_{R,L}/\lambda_{RB,LB}$. For a minimal number of coefficients we can determine the values of these coefficients completely, $\lambda_{R,L}$ and $\lambda_{RB,LB}$ can be expanded to the zeroth or first power of $u \cdot p$. For example, if we expand λ_R to the zeroth power, then we have to expand λ_L to the first power, and vice versa. Suppose we take the former case, $\lambda_R = \lambda_{R0}$ and $\lambda_L = \lambda_{L0} + \lambda_{L1}(u \cdot p)$, we can solve $\lambda_{R,L}$ as

$$\lambda_{R0} = -\xi_R \frac{1}{\delta J_{21}^R},$$

$$\lambda_{L0} = -\frac{1}{\Delta} \xi_L (\sigma J_{41}^L) + \frac{1}{\Delta} \left(DT - \xi_R \frac{\sigma J_{31}^R}{\delta J_{21}^R} \right) (\delta J_{31}^L), \quad (64)$$

$$\lambda_{L1} = \frac{1}{\Delta} \xi_L (\sigma J_{31}^L) - \frac{1}{\Delta} \left(DT - \xi_R \frac{\sigma J_{31}^R}{\delta J_{21}^R} \right) (\delta J_{21}^L),$$

where $\Delta = (\delta J_{21}^L)(\sigma J_{41}^L) - (\delta J_{31}^L)(\sigma J_{31}^L)$. The solutions to $\lambda_{RB,LB}$ take the same form as above with replacements $\lambda_a \to \lambda_{aB}$, $\xi_a \to \xi_{aB}$ and $D \to D_B$ if we assume the same expansion as $\lambda_{R,L}$: $\lambda_{RB} = \lambda_{RB,0}$ and $\lambda_{LB} = \lambda_{LB,0} + \lambda_{LB,1}(u \cdot p)$.

For massless fermions and small $\bar{\mu}_{R,L}$ (or equivalently small $\bar{\mu}$ and $\bar{\mu}_A$) we obtain

$$\sigma J_{21}^{R,L} \approx 9\zeta(3)G, \qquad \sigma J_{31}^{R,L} \approx \frac{7\pi^4}{15}GT,$$

$$\sigma J_{41}^{R,L} \approx 225\zeta(5)GT^2,$$

$$\delta J_{21}^{R,L} \approx \pi^2 \bar{\mu}_{R,L}G, \qquad \delta J_{31}^{R,L} \approx 36\zeta(3)\bar{\mu}_{R,L}GT,$$

$$\delta J_{41}^{R,L} \approx \frac{7\pi^4}{3}\bar{\mu}_{R,L}GT^2,$$

$$\Delta \approx \frac{\pi^2}{5}G^2 G_0 T^2 \bar{\mu}_L, \tag{65}$$

where G and G_0 are the same as in the former section. Then the solutions to $\lambda_{R,L}$ are

$$\lambda_{R0} \approx -\frac{3C}{d_g}\frac{\mu}{T^3}\frac{\mu + 2\mu_A}{\mu + \mu_A},$$

$$\lambda_{L0} \approx \frac{3C}{d_g}\frac{\mu}{T^3}\left(G_4 \frac{\mu + 2\mu_A}{\mu + \mu_A} + G_5 \frac{\mu - 2\mu_A}{\mu - \mu_A}\right),$$

$$\lambda_{L1} \approx \frac{14\pi^4 C}{d_g G_0}\frac{\mu}{T^4}\frac{\mu \mu_A}{\mu^2 - \mu_A^2}, \qquad \lambda_{RB,0} \approx -\frac{3C}{d_g}\frac{1}{T^3}, \tag{66}$$

$$\lambda_{LB,0} \approx \frac{3C}{d_g}\frac{1}{T^3}, \qquad \lambda_{LB,1} \approx \frac{30\pi^2 C}{d_g G_0}\frac{\mu \mu_A}{T^6},$$

where $G_4 \equiv -\frac{1}{G_0}84\pi^2\zeta(3)$ and $G_5 \equiv \frac{1}{G_0}1125\zeta(5)$. The coefficient ratios of $\lambda_{ai}/\lambda_{aB,i}$ ($i = 0, 1$) are proportional to μ times dimensionless factors:

$$\frac{\lambda_{R0}}{\lambda_{RB,0}} \approx \mu \frac{\mu + 2\mu_A}{\mu + \mu_A},$$

$$\frac{\lambda_{L0}}{\lambda_{LB,0}} \approx \mu\left(G_4 \frac{\mu + 2\mu_A}{\mu + \mu_A} + G_5 \frac{\mu - 2\mu_A}{\mu - \mu_A}\right), \tag{67}$$

$$\frac{\lambda_{L1}}{\lambda_{LB,1}} \approx \mu \frac{7}{15}\pi^2 \frac{T^2}{\mu^2 - \mu_A^2}.$$

Note that $\lambda_{LB,1} \ll T\lambda_{LB,0} \sim T\lambda_{RB,0}$, so both $\lambda_{BR}(p)$ and $\lambda_{BL}(p)$ can be constants at small $\bar{\mu}$ and $\bar{\mu}_A$ limit. This property is quite different from the one-charge case in which $\lambda_B(p)$ must have momentum dependence in order to comply with the entropy principle.

VI. DISCUSSIONS AND CONCLUSIONS

We have shown that induced terms related to the vorticity and magnetic field in the charge and entropy currents from a triangle anomaly can be derived in kinetic theory by introducing correction terms to the phase space distribution function at the first order. We demonstrated that the anomalous source terms are necessary to ensure that the equations for the charge and energy-momentum conservation are satisfied and that the correction terms of distribution functions are compatible to these equations.

As examples for the correction terms of distribution functions, we focus on the massless fermionic system in

three cases for small μ/T, with one charge [$U(1)$] and one particle species (without antiparticles), with one charge [$U(1)$] and particles/antiparticles, and with two charges [$U(1) \times U_A(1)$]. In the latter two cases, the coefficients for ω and B terms in distribution functions are found to be proportional to $C\mu/T^3$ and C/T^3, respectively. In the two-charges case the coefficients can be constants or independent of momentum, such a property is impossible for the one-charge case since it is not allowed by the entropy principle.

In the two-charges case, we assumed that there is an anomaly for the axial-charge current but no anomaly for the charge one. The coefficients of correction terms for the charge/axial-charge currents and energy-momentum tensor have a very simple and symmetric form at small μ/T and μ_A/T limit: $\xi \approx 2C\mu\mu_A$, $\xi_A \approx C\mu^2$, $\xi_B \approx C\mu_A$, $\xi_{AB} \approx C\mu$, $DT = -C\mu_A\mu^2$, and $D_B T = -C\mu_A\mu$. This means that similar to the CME an axial anomaly can induce a residual charge current which is proportional to the magnetic field and the axial chemical potential [23].

We have a few comments about our results. In our evaluation of the correction terms of distribution functions, we have assumed that $\lambda(p)$ and $\lambda_B(p)$ are identical to particles and antiparticles. Alternatively, we can assume that they have an opposite sign for particles to antiparticles. We cannot tell which case is correct due to lack of deeper knowledge about these anomalous term at a microscopic level. Similarly we have assumed that $\lambda(p)$ and $\lambda_B(p)$ have different values for right-handed particles from left-handed ones. One can also assume that they have the same or opposite values for right-handed and left-handed particles. In the current framework one cannot tell which is correct. Such a situation is like what happens in an effective theory when many effective candidates point to a unique microscopic theory. We also note that the solutions to $\lambda(p)$ and $\lambda_B(p)$ given in this paper are for the cases where the number of unknown coefficients in $\lambda(p)$ and $\lambda_B(p)$ is equal to that of constraining equations. It is possible that $\lambda(p)$ and $\lambda_B(p)$ can be expanded to higher powers of $(p \cdot u)$ and then have larger number of unknown coefficients than that of constraining equations. In this case the constraining equations just provide constraints for $\lambda(p)$ and $\lambda_B(p)$ from the second law of thermodynamics.

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