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Colliding particles carrying nonzero orbital angular momentum

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Photons carrying nonzero orbital angular momentum (twisted photons) are well-known in optics. Recently, using Compton backscattering to boost optical twisted photons to high energies was suggested. Twisted electrons in the intermediate energy range have also been produced recently. Thus, collisions involving energetic twisted particles seem to be feasible and represent a new tool in high-energy physics. Here we discuss some generic features of scattering processes involving twisted particles in the initial and/or final state. In order to avoid additional complications arising from nontrivial polarization states, we focus here on scalar fields only. We show that processes involving twisted particles allow one to perform a Fourier analysis of the plane-wave cross section with respect to the azimuthal angles of the initial particles. In addition, using twisted states, one can probe the autocorrelation function of the amplitude, which is inaccessible in the plane-wave collisions. Finally, we discuss prospects for experimental study of these effects.

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I. INTRODUCTION

In perturbative quantum field theory, we assume that interaction among the fields can be treated as a perturbation of the free field theory. This perturbation leads to scattering between asymptotically free multiparticle states, which are usually constructed from the plane-wave one-particle states. This choice greatly simplifies the calculations and represents a very accurate approximation to the real experimental situation in virtually all circumstances. However, one can, in principle, choose any complete basis for the one-particle states other than the plane-wave basis, provided that it is still made up of solutions of the free field equations. Such states can carry new quantum numbers absent in the plane-wave choice and, if experimentally realized, they can offer new opportunities in high-energy physics.

Thanks to the progress in optics made in the last two decades, it is now possible to create laser beams carrying nonzero orbital angular momentum (OAM) [1], for a recent review see [2]. The light field in such beams is described via non-plane-wave solutions of the Maxwell equations. Each photon in this light field, which we call a *twisted photon*, carries a nonzero OAM quantized in units of $\hbar$. Several sets of solutions have been investigated, such as Bessel beams or Gauss-Laguerre beams, but in all cases the spatial distribution of the light field is necessarily nonhomogeneous in the sense that the equal phase fronts are not planes but helices. Such states form a complete basis which can be used to describe the initial and final asymptotically free states. Moreover, it is the basis of choice for experimental situations when the initial states are prepared in a state of (more or less) definite OAM.

Twisted photons have been produced in various wavelength domains, from radiowave [3] to optical, with prospects to create a brilliant X-ray beam of twisted light in the keV range [4]. Very recently it was suggested to use the Compton backscattering of twisted optical photons off an ultrarelativistic electron beam to create a beam of high-energy photons with nonzero OAM [5,6]. The technology of Compton backscattering is well established [7], and the high-energy electron beams and the OAM optical laser beams are already available. In addition, in the last months several groups have reported successful creation of twisted electrons, first using phase plates [8] and then with computer-generated holograms [9]. Twisted electrons carried the energy as high as 300 keV and the orbital quantum number up to $\sim$100. With all these achievements, creating high-energy particles in a controlled orbital angular momentum state and colliding them seems now feasible. It is therefore very timely to ask what new insights into the properties of particles and their interactions one can gain with this new degree of freedom.

In this paper we begin this exploration by studying several generic scattering processes involving twisted particles in the initial and/or final states. Namely, we consider three specific cases:

(i) single-twisted scattering: collision of a twisted state with a plane wave,
(ii) double-twisted scattering: collision of two twisted states,
(iii) two-particle decay of an unstable twisted particle.

In the first two cases we assume that the final system X is described by plane waves, while in the last case we consider three choices for the final two-particle state: when both particles are plane waves, when one is twisted, and when both are twisted. The single- and double-twisted scattering will give some hints at new physical opportunities that can be expected in collisions of high-energy particles carrying OAM, while the calculation of a twisted particle decay clarifies various technical details involved in passage from plane waves to twisted states.

The OAM and spin are two forms of angular momentum, and the problem of gauge-invariant separation of these two objects has a long history. It is still being debated both in the optics community, see for example [10], and in the high energy physics community, especially in the context of the notorious proton spin puzzle, [11]. For the problems we consider here it is sufficient to note that all experimental situations which seem to be realizable are well described by the paraxial approximation. It is known that in the paraxial approximation spin can be well separated from the z component of OAM of light, [12]. The same applies to fermions as well. Therefore, incorporation of both spin and OAM degrees of freedom, leading to nontrivial polarization fields, does not seem to pose any problem in the paraxial approximation.

However in the present paper we would like to focus specifically on the OAM component and to understand what new physical opportunities are offered by the nontrivial spatial dependence rather than unusual polarization states. Therefore, we limit ourselves to scattering of *scalar* particles only. In this simple case, all the nonzero angular momentum is definitely due to the orbital part. Additional features arising from the polarization parameter fields will be considered separately.

The paper is organized as follows. In Sec. II we introduce scalar twisted states and describe some of their properties. In Secs. III and IV we derive expressions for the cross section in the single-twisted and double-twisted cases, respectively. Section V gives a thorough discussion of kinematical features arising in the case when the final state contains twisted particles. In Sec. VI we discuss the results obtained and draw our conclusions. In two Appendices we derive some technical results used in the paper.

II. DESCRIBING TWISTED STATES

A. Spatial distribution

As mentioned in the introduction, we focus in this paper on twisted scalar particles with mass M. In their description we follow essentially [5,6].

We represent a state with a nonzero OAM with a Bessel beam-type twisted state. This is a solution of the wave equation in the cylindric coordinates with a definite energy ω and a longitudinal momentum k_z along a fixed axis z, a definite modulus of the transverse momentum $|\mathbf{k}|$ (all transverse momenta will be written in bold) and a definite z projection of OAM. If the plane-wave state $|\mathrm{PW}(\mathbf{k})\rangle$ is

$$|\mathrm{PW}(\mathbf{k})\rangle = e^{-i\omega t+ik_z z}\cdot e^{i\mathbf{k}\mathbf{r}}, \tag{1}$$

then a twisted scalar state $|\kappa, m\rangle$ is defined as the following superposition of plane waves:

$$|\kappa, m\rangle = e^{-i\omega t+ik_z z}\int\frac{d^2\mathbf{k}}{(2\pi)^2}a_{\kappa m}(\mathbf{k})e^{i\mathbf{k}\mathbf{r}},\qquad a_{\kappa m}(\mathbf{k}) = (-i)^m e^{im\phi_k}\sqrt{2\pi}\frac{\delta(|\mathbf{k}|-\kappa)}{\sqrt{\kappa}}. \tag{2}$$

In the coordinate space,

$$|\kappa, m\rangle = e^{-i\omega t+ik_z z}\cdot\psi_{\kappa m}(\mathbf{r}),\qquad \psi_{\kappa m}(\mathbf{r}) = \frac{e^{im\phi_r}}{\sqrt{2\pi}}\sqrt{\kappa}J_m(\kappa r). \tag{3}$$

Here, following [5] we call κ the conical momentum spread, m is the z projection of OAM, and the dispersion relation is $k^\mu k_\mu = \omega^2 - k_z^2 - \kappa^2 = M^2$. We note in passing that the average values of the components of the four-momentum carried by a twisted state are

$$\langle k^\mu\rangle = (\omega, \mathbf{0}, k_z), \tag{4}$$

so that $\langle k^\mu\rangle\langle k_\mu\rangle = M^2+\kappa^2$, which is larger than the true mass of the particle squared.

The transverse spatial distribution is normalized according to

$$\int d^2\mathbf{r}\,\psi^*_{\kappa' m'}(\mathbf{r})\psi_{\kappa m}(\mathbf{r}) = \delta_{m,m'}\sqrt{\kappa\kappa'}\int rdrJ_m(\kappa r)J_m(\kappa' r) = \delta_{m,m'}\delta(\kappa-\kappa'). \tag{5}$$

The plane wave can be recovered from the twisted states as follows:

$$|\mathrm{PW}(\mathbf{k}=0)\rangle = \lim_{\kappa\to 0}\sqrt{\frac{2\pi}{\kappa}}|\kappa, 0\rangle, \tag{6}$$

$$|\mathrm{PW}(\mathbf{k})\rangle = \sqrt{\frac{2\pi}{\kappa}}\sum_{m=-\infty}^{+\infty} i^m e^{-im\phi_k}|\kappa, m\rangle,\qquad \kappa = |\mathbf{k}_\perp|. \tag{7}$$

If needed, these two cases can be written as a single expression:

$$|\mathrm{PW}(\mathbf{k})\rangle = \lim_{\kappa\to|\mathbf{k}|}\sqrt{\frac{2\pi}{\kappa}}\sum_{m=-\infty}^{+\infty} i^m e^{-im\phi_k}|\kappa, m\rangle. \tag{8}$$

From these expressions one sees that the twisted states with different m and κ represent nothing but another basis for the transverse wave functions.

B. Density of states

When calculating cross sections and decay rates, we need to integrate the transition probability over the phase space of the final particles. When calculating the density of states, we consider a large but finite volume and count how many mutually orthogonal states with prescribed boundary conditions can be squeezed inside. In the present case due to the cylindrical symmetry of the problem, we choose a cylinder of a large radius R and a length L_z. In the case of plane waves we have

$$dn_{\rm PW} = \pi R^2 L_z \frac{dk_z d^2\mathbf{k}}{(2\pi)^3}. \quad (9)$$

The full number of states with transverse momenta up to κ_0 and longitudinal momenta $|k_z| \le k_{z0}$ is $k_{z0}L_z \cdot R^2\kappa_0^2/4\pi$.

To count the number of twisted states $|\kappa, m\rangle$ in the same volume, we specify the boundary condition, e.g. $\psi_{\kappa m}(r = R) = 0$, which makes κ discrete such that $\kappa_i R$ is the i-th root of the Bessel function J_m. Then we note that the position of the first root of the Bessel function $J_m(x)$ is always at $x > m$, and as m grows $x \to m$. For a given κ, the maximal m for which the wave can still be contained inside the cylindrical volume is $m_{\rm max} = \kappa R$, which has a very natural quasiclassical interpretation.

If m is small and not growing with R, then one can use the well-known asymptotic form of the Bessel functions to count the number of states:

$$dn_{\rm tw} = \frac{R d\kappa L_z dk_z \Delta m}{2\pi^2}. \quad (10)$$

Here, Δm is written instead of just 1 to signal the presence of a discrete running parameter m.

If m is not restricted to small values, this asymptotic form of $J_m(x)$ cannot be used since it requires $m^2 \lesssim x$. Instead, the so-called approximation by tangents can be used, which gives the following density of states:

$$dn_{\rm tw} = \sqrt{m_{\rm max}^2 - m^2}\frac{d\kappa}{\kappa}\frac{\Delta m}{\pi}\frac{L_z dk_z}{2\pi}. \quad (11)$$

In the limit $m \ll m_{\rm max} \equiv \kappa R$ this expression reproduced (10). Alternatively, one can calculate the radial part of the density of states via the adiabatic invariant as suggested in [6]. The number of radial excitations n_r for a fixed m is

$$n_r = \int_{m/\kappa}^{R} \frac{k_r(r)dr}{\pi}, \qquad k_r(r) = \sqrt{\kappa^2 - \frac{m^2}{r^2}}. \quad (12)$$

The density of states is then given by

$$dn_r = \frac{dn_r}{d\kappa}d\kappa = \sqrt{m_{\rm max}^2 - m^2}\frac{d\kappa}{\kappa\pi}. \quad (13)$$

One important remark is in order. Effectively, switching from the plane wave to twisted state basis for the final particles implies replacement,

$$d^2\mathbf{k} \to 4\sqrt{1 - \frac{m^2}{m_{\rm max}^2}}\kappa d\kappa \frac{\Delta m}{m_{\rm max}}. \quad (14)$$

Note that the contribution of each "partial wave" with a fixed m vanishes in the infinite volume limit as $1/R$. However, the number of partial waves grows $\propto R$, and in order to get a nonvanishing result for a physical observable, one must integrate over the full available m interval up to $m_{\rm max}$. This remains true even if the transverse momenta stay small, and it is related to the fact that the plane wave contains contributions from all impact parameters with respect to any axis noncollinear to its propagation direction.

Another expression one needs for the probability calculations is the normalization constants for the one-particle states. A usual plane-wave one-particle state is normalized to $2E \cdot V$; to renormalize it to one particle per the entire volume, the plane wave should be multiplied by $N_{\rm PW}$, with

$$N_{\rm PW}^2 = \frac{1}{2EV}, \qquad V = \pi R^2 L_z. \quad (15)$$

For a twisted state the corresponding normalization factor $N_{\rm tw}$ is

$$N_{\rm tw}^2 = \frac{1}{2E}\frac{\pi\kappa}{\sqrt{m_{\rm max}^2 - m^2}L_z}, \quad (16)$$

which in the small-m case simplifies to

$$N_{\rm tw}^2 \approx \frac{1}{2E}\frac{\pi}{RL_z}, \quad (17)$$

also derived in [6]. Note however that even in the general case the product of the normalization constant squared and the density of states for each final twisted particle is simplified as

$$N_{\rm tw}^2 dn_{\rm tw} = \frac{d\kappa dk_z \Delta m}{2E \cdot 2\pi}. \quad (18)$$

C. Flux and the cross section

The definitions of the flux factor and the cross section have to be reevaluated when a collision of non-plane-wave states is considered, which involves subtle issues described in [6]. By definition, the cross section is the transition probability per unit time divided by flux. In the plane-wave case, when the four-momenta of colliding particles are fixed, both the flux and probability are constant across any chosen plane, so the proportionality between them holds locally. For a head-on collision one gets $j_{\rm PW} = (|v_1| + |v_2|)/V$, which together with the energies of the incoming particles and one volume factor combine to the familiar Lorentz invariant expression

$$I_{\rm PW} = \sqrt{(pk)^2 - p^2k^2}. \quad (19)$$

This formula can of course be used in any frame, including cases when the collision is not head-on.

In the single-twisted case, when a twisted state collides with a plane wave, both the flux and the transition probability are not constant but change across the transverse plane. As noted in [6], one therefore needs to redefine the notion of the cross section to adapt to this situation. One introduces the *averaged cross section* defined as the transition probability integrated over all $\mathbf{r}$ divided by the flux again integrated over all $\mathbf{r}$. While the definition of the former quantity is clear (this is what we calculate in the following two Sections), the proper definition of the integrated flux is more intricate and apparently not unique. A definition of the flux should however be correlated with a definition of the (generalized) luminosity, so that the observable event rate remains uniquely defined.

In [6] the forward electron-photon collision was considered (the 3-momenta of the initial plane-wave electrons were directed exactly along the axis z), and the following procedure was suggested: the total flux is just the sum of the z components of the two fluxes (note that we actually assume the absolute values of the fluxes):

$$\langle j\rangle = j_z^e + \langle j_z^\gamma\rangle = \frac{v+\cos\alpha_k}{V}, \tag{20}$$

where $\tan\alpha_k = \kappa/k_z$. We think that a more appropriate definition of flux should take into account not only the z components of the individual fluxes but also the relative lateral motion of the two waves. Indeed, the sole purpose of calculating the flux factor is, classically speaking, to derive the volume swept by one particle in the "gas" of opposing particles and find how many "attempts" at collision are made per unit time. Therefore, we propose the following general definition of the integrated flux factor $I_{\rm tw}$ for the single-twisted case:

$$I_{\rm tw} = \int \frac{d\phi_k}{2\pi} I_{\rm PW}(\mathbf{k},\mathbf{p}). \tag{21}$$

Note that it is well defined for any transverse momentum $\mathbf{p}$ of the opposing plane wave. In the specific case considered above, this definition gives $\langle j\rangle = (1+v\cos\alpha_k)/V$, which differs from (20). This definition can also be generalized to the double-twisted case:

$$I_{2\,\rm tw} = \int \frac{d\phi_k}{2\pi}\frac{d\phi_p}{2\pi} I_{\rm PW}(\mathbf{k},\mathbf{p}). \tag{22}$$

III. SINGLE-TWISTED CROSS SECTION

We start with a usual $2\to n$ collision in which both incoming particles are described by plane waves with definite four-momenta k and p. The final system X is also treated as a collection of plane waves with the total momentum p_X. The invariant amplitude of this process is denoted by $\mathcal{M}(\mathbf{k},\mathbf{p})$, where the transverse momenta of the initial particles are indicated explicitly.

The cross section of this process is calculated according to the standard rules. When squaring the scattering matrix element

$$S_{\rm PW} = i(2\pi)^4\delta^{(4)}(k+p-p_X)\cdot\mathcal{M}(\mathbf{k},\mathbf{p}), \tag{23}$$

we reinterpret the square of the delta-function as

$$[\delta^{(4)}(k+p-p_X)]^2 = \delta^{(4)}(k+p-p_X)\cdot\frac{\pi R^2 L_z T}{(2\pi)^4}. \tag{24}$$

Using the plane-wave normalization factors (15) for all the initial and final particles, we get the cross section

$$d\sigma_{\rm PW}(\mathbf{k},\mathbf{p}) = \frac{(2\pi)^4\delta(E_i-E_f)\delta(p_{zi}-p_{zf})\delta^{(2)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_X)}{4I_{\rm PW}} \times|\mathcal{M}(\mathbf{k},\mathbf{p})|^2\cdot d\Gamma_X. \tag{25}$$

For future convenience the delta-function is explicitly broken into the longitudinal and transverse parts. As usual, we can extract from the final phase space integration measure $d\Gamma_X$ the integral over the total transverse momentum $\mathbf{p}_X$, $d\Gamma \equiv d^2\mathbf{p}_X d\Gamma'_X$, and write the cross section as

$$d\sigma_{\rm PW}(\mathbf{k},\mathbf{p}) = \frac{(2\pi)^4\delta(E_i-E_f)\delta(p_{zi}-p_{zf})}{4I_{\rm PW}} \times|\mathcal{M}(\mathbf{k},\mathbf{p})|^2\cdot d\Gamma'_X. \tag{26}$$

Now we recalculate the cross section for the case when the first particle is in the twisted state $|\kappa,m\rangle$. We apply prescription (2) to the S-matrix element:

$$S_{\rm tw} = \int\frac{d^2\mathbf{k}}{(2\pi)^2}a_{\kappa m}(\mathbf{k})S_{\rm PW}(\mathbf{k},\mathbf{p}), \tag{27}$$

as originally suggested in [5]. The plane-wave S-matrix (23) contains a delta-function of transverse momenta, which makes it possible to simplify the square of the twisted S-matrix element as

$$\begin{aligned}|S_{\rm tw}|^2 &= \int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{k}'}{(2\pi)^2}a_{\kappa m}(\mathbf{k})a^*_{\kappa m}(\mathbf{k}')S_{\rm PW}(\mathbf{k},\mathbf{p})S^*_{\rm PW}(\mathbf{k}',\mathbf{p})\\ &\propto \int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{k}'}{(2\pi)^2}a_{\kappa m}(\mathbf{k})a^*_{\kappa m}(\mathbf{k}')\delta^{(2)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_X)\\ &\quad\times\delta^{(2)}(\mathbf{k}'+\mathbf{p}-\mathbf{p}_X)\mathcal{M}(\mathbf{k},\mathbf{p})\mathcal{M}^*(\mathbf{k}',\mathbf{p})\\ &= \int\frac{d^2\mathbf{k}}{(2\pi)^4}a_{\kappa m}(\mathbf{k})a^*_{\kappa m}(\mathbf{k})\delta^{(2)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_X)|\mathcal{M}(\mathbf{k},\mathbf{p})|^2.\end{aligned} \tag{28}$$

The square of $a_{\kappa m}(\mathbf{k})$ contains a radial delta-function squared, which is reinterpreted as

$$[\delta(\kappa-|\mathbf{k}|)]^2 = \delta(\kappa-|\mathbf{k}|)\cdot\delta(0)\to\delta(\kappa-|\mathbf{k}|)\frac{R}{\pi}. \tag{29}$$

This prescription comes from the observation that at large but finite R and at $\kappa = |\mathbf{k}|$ the radial delta-function is regularized as

$$\delta(0) = \int_0^\infty rdr[J_m(\kappa r)]^2 \to \int_0^R rdr[J_m(\kappa r)]^2 \approx \frac{R}{\pi}, \tag{30}$$

see [5,6]. Therefore, with all the normalization factors (15) and (17) the single-twisted cross section takes form

$$d\sigma_{\rm tw} = \int \frac{d^2\mathbf{k}}{2\pi} \frac{I_{\rm PW}(\mathbf{k},\mathbf{p})}{I_{\rm tw}} \frac{\delta(\kappa - |\mathbf{k}|)}{\kappa} \cdot d\sigma_{\rm PW}(\mathbf{k},\mathbf{p}) = \int \frac{d\phi_k}{2\pi} \frac{I_{\rm PW}(\mathbf{k},\mathbf{p})}{I_{\rm tw}} d\sigma_{\rm PW}(\mathbf{k},\mathbf{p}), \quad (31)$$

see Section II C for the definition of the fluxes. Note that in the paraxial approximation one can replace the ratio of the fluxes by the unity.

A couple of remarks concerning this result are in order. In the usual case of a plane-wave collision, the total initial and, therefore, final momenta are fixed. This is highlighted by the presence of the transverse delta-function in (25) and by implicit correlations among transverse momenta of the final particles inside $d\Gamma'_X$ in (26). In the process involving a twisted particle, the total final momentum is not fixed, making the final particles less correlated. Although all the plane waves $|\mathrm{PW}(\mathbf{k})\rangle$ which constitute the initial twisted state $|\kappa, m\rangle$ are summed up coherently at the amplitude level, the final states with different momenta do not interfere, and this breaks the coherence. As a result, the twisted particle cross section is expressed via an *incoherent* superposition of the cross sections induced by each initial plane wave. This is precisely what (31) displays.

The previous paragraph contains one additional subtlety. When saying that the final states with different momenta do not interfere, we implicitly assume that the final state is detected by a usual detector, which measures the linear momentum but not the OAM. On the contrary, if the final particles were detected by a hypothetical "coherent detector" sensitive to a coherent superposition of final states with distinct momenta, or if the production of particles in the reaction we consider is followed by another process which is OAM-selective, then the coherence would be restored. Thus, the coherence is not actually destroyed in the scattering process, but remains hidden.

The second remark concerns the angular region contributing to the integral (31). In the fully differential case, that is when we fix the momenta of all the final particles, the transverse delta-function inside $d\sigma_{\rm PW}$ assures that there is only one value of ϕ that contributes to the integral. At this level of consideration, representing the cross section as an angular integral might look somewhat misleading. However if this delta-function is killed by the integration over all allowed $\mathbf{p}_X$, or equivalently by integrating over one of the final particles, then the integral receives contributions from all angles ϕ_k. Note that the angular integral representation is also justified even in the fully differential case if the detector resolution of the final particles' momenta is worse than κ.

Let us also show a slightly different derivation of (31). We first explicitly perform the integration in (27), which is effectively killed by the transverse part of the delta-function, keeping the scattering amplitude essentially unchanged:

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} a_{\kappa m}(\mathbf{k}) \delta^{(2)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_X) \cdot \mathcal{M}(\mathbf{k},\mathbf{p}) = \frac{(-i)^m}{(2\pi)^{3/2}} e^{im\phi_q} \frac{\delta(\kappa - q)}{\sqrt{\kappa}} \cdot \mathcal{M}(\mathbf{q},\mathbf{p}), \quad (32)$$

where $\mathbf{q} \equiv \mathbf{p}_X - \mathbf{p}$. Note that after the integration the transverse momentum $\mathbf{k}$ cannot appear in the amplitude any more. However, since the other particles are plane waves with well-defined momenta, the vector $\mathbf{q}$ with modulus q and azimuthal angle ϕ_q is also well defined and can be used instead of $\mathbf{k}$ everywhere. Squaring (32), we again encounter the square of the radial delta-function which is reinterpreted as in (29). The cross section then becomes

$$d\sigma_{\rm tw} = \frac{(2\pi)^4 \delta(E_i - E_f)\delta(p_{zi} - p_{zf}) |\mathcal{M}(\mathbf{q},\mathbf{p})|^2}{4 I_{\rm tw}} \cdot d\Gamma_X \cdot \frac{\delta(\kappa - q)}{2\pi\kappa}. \quad (33)$$

The integral over the overall transverse momentum present in $d\Gamma_X$ kills the radial delta function:

$$\int d^2\mathbf{p}_X \frac{\delta(\kappa - q)}{2\pi\kappa} = \int d^2\mathbf{q} \frac{\delta(\kappa - q)}{2\pi\kappa} = \int \frac{d\phi_q}{2\pi}, \quad (34)$$

and one recovers the result (31).

One can make several observations concerning (31):

(i) The cross section is m independent.

(ii) The initial twisted particles effectively perform the angular averaging of the plane-wave cross section.

(iii) There is no smallness associated with nonzero m.

(iv) There is no small factor associated with small κ.

Let us now consider the case when the twisted particle is not in an eigenstate of the OAM operator, but in a superposition of states $|\kappa, m\rangle$ with equal κ but different m. For example, consider the twisted state of the form $a|\kappa, m\rangle + a'|\kappa, m'\rangle$, with $|a|^2 + |a'|^2 = 1$. Repeating the same calculation we encounter an interference term in the cross section:

$$d\sigma^{\Delta m}_{\rm tw} = \int \frac{d\phi_k}{2\pi} \frac{I_{\rm PW}(\mathbf{k},\mathbf{p})}{I_{\rm tw}} \cos(\Delta m \phi_k + \alpha) d\sigma_{\rm PW}(\mathbf{k},\mathbf{p}), \quad (35)$$

where $\Delta m = m - m'$ and α is the relative phase between the two complex coefficients a and a'. The cross section for such an initial state takes the following form:

$$d\sigma = d\sigma_{\rm tw} + 2|aa'| d\sigma^{\Delta m}_{\rm tw}. \quad (36)$$

Results (31) and (35) mean that if the initial particle can be prepared in a twisted state with an adjustable superposition of different m, a Fourier analysis of the cross section with respect to the initial azimuthal angle can be performed. In principle, the same analysis can be done with plane waves, but it would require making several experiments with different angles of the initial particle **k** and then extracting the Fourier components via the partial wave analysis. From the experimental view, it is likely that systematics of the two schemes can be different, which makes them complementary to each other.

IV. DOUBLE-TWISTED CROSS SECTION

Let us now consider collision of two initial twisted particles. In this work, we do not aim at a systematic study of this case, but rather outline some new features that can be expected in such circumstances. Therefore, we consider the simplest setup, in which the two colliding particles are described by twisted states $|\kappa, m\rangle$ and $|\eta, n\rangle$ defined with respect to the same quantization axis z:

$$|\kappa, m\rangle = \int \frac{d^2\mathbf{k}}{(2\pi)^2} a_{\kappa m}(\mathbf{k})|\mathrm{PW}_1(\mathbf{k})\rangle, \quad |\eta, n\rangle = \int \frac{d^2\mathbf{p}}{(2\pi)^2} a_{\eta n}(\mathbf{p})|\mathrm{PW}_2(\mathbf{p})\rangle, \tag{37}$$

with the same functional form of the projectors a as before. Here, the subscripts 1 and 2 refer to the first and second colliding particles. The "double-twisted" version of (27) is

$$S_{2\,\mathrm{tw}} = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{p}}{(2\pi)^2} a_{\kappa m}(\mathbf{k}) a_{\eta n}(\mathbf{p}) S_{\mathrm{PW}}(\mathbf{k}, \mathbf{p}), \tag{38}$$

and its square is proportional to

$$\int \frac{d^2\mathbf{k} d^2\mathbf{p} d^2\mathbf{k}' d^2\mathbf{p}'}{(2\pi)^8} a_{\kappa m}(\mathbf{k}) a_{\eta n}(\mathbf{p}) a^*_{\kappa m}(\mathbf{k}') a^*_{\eta n}(\mathbf{p}') \times \delta^{(2)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_X)\delta^{(2)}(\mathbf{k}'+\mathbf{p}'-\mathbf{p}_X)\mathcal{M}(\mathbf{k},\mathbf{p})\mathcal{M}^*(\mathbf{k}',\mathbf{p}'). \tag{39}$$

Let us consider the kinematical restrictions imposed by the delta functions entering this expression and compare them with the result $\mathbf{k} = \mathbf{k}'$ found in the previous Section. Here, too, the moduli of the momenta are fixed and pairwise equal: $|\mathbf{k}| = \kappa = |\mathbf{k}'|$ and $|\mathbf{p}| = \eta = |\mathbf{p}'|$. Besides, the two pairs of momenta sum up to a well-defined $\mathbf{p}_X$. For each pair there are *two* possibilities satisfying these conditions, shown in Fig. 1, which are mirror reflections of each other with respect to the direction of $\mathbf{p}_X$. Therefore, the integral (39) receives contributions from two kinematical configurations:

$$\begin{aligned} \text{direct } \mathbf{k}' &= \mathbf{k}, \qquad \mathbf{p}' = \mathbf{p}, \\ \text{reflected } \mathbf{k}' &= \mathbf{k}^* \equiv -\mathbf{k} + 2(\mathbf{k}\mathbf{n}_X)\mathbf{n}_X, \\ \mathbf{p}' &= \mathbf{p}^* \equiv -\mathbf{p} + 2(\mathbf{p}\mathbf{n}_X)\mathbf{n}_X, \end{aligned} \tag{40}$$

with $\mathbf{n}_X \equiv \mathbf{p}_X/|\mathbf{p}_X|$. As we will see below, in contrast to the "single-twisted" case, here the existence of two configurations for any given $\mathbf{p}_X \neq 0$ leads to a residual coherence between different plane-wave components in the twisted state.

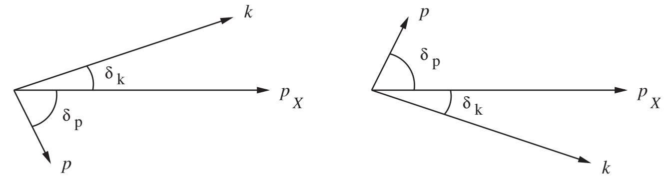

FIG. 1. Two kinematical configurations of the transverse momenta **k** and **p** of fixed absolute values that sum up to the vector $\mathbf{p}_X$.

Our goal now is to see whether the double-twisted cross section can be expressed as an angle-averaged plane-wave cross section similar to (31). To this end let us first study a generic expression

$$J = \int d\phi_k d\phi_p \delta^{(2)}(\mathbf{k} + \mathbf{p} - \mathbf{p}_X) \cdot f(\mathbf{k}, \mathbf{p}), \tag{41}$$

where $|\mathbf{k}| = \kappa$ and $|\mathbf{p}| = \eta$ are fixed and $f(\mathbf{k}, \mathbf{p})$ is a continuous function of the incoming particles' momenta. Because of the delta-function, the integral J receives contributions only from two points, shown in Fig. 1. At these points the azimuthal angles ϕ_k and ϕ_p take specific values:

$$\phi_k = \phi_X \pm \delta_k, \qquad \phi_p = \phi_X \mp \delta_p, \tag{42}$$

where ϕ_X is the azimuthal angle of $\mathbf{p}_X$ and

$$\delta_k = \arccos\left(\frac{\mathbf{p}_X^2 + \kappa^2 - \eta^2}{2|\mathbf{p}_X|\kappa}\right), \quad \delta_p = \arccos\left(\frac{\mathbf{p}_X^2 - \kappa^2 + \eta^2}{2|\mathbf{p}_X|\eta}\right) \tag{43}$$

are functions of the absolute values of the momenta. Let us denote the values of $f(\mathbf{k}, \mathbf{p})$ at these two points as f_+ and f_-, respectively. In Appendix A we derive the following result

$$J = \frac{f_+ + f_-}{2\Delta} = \int d\phi_k d\phi_p \delta^{(2)}(\mathbf{k} + \mathbf{p} - \mathbf{p}_X) \cdot \frac{f_+ + f_-}{2}, \tag{44}$$

where Δ is the area of the triangle of sides $|\mathbf{p}_X|$, κ, η. It allows us to rewrite the square of the integral J as

$$\begin{aligned} |J|^2 &= \int d\phi_k d\phi_p \delta^{(2)}(\mathbf{k} + \mathbf{p} - \mathbf{p}_X) f(\mathbf{k}, \mathbf{p}) \frac{f_+^* + f_-^*}{2\Delta} \\ &= \frac{1}{4\Delta}\int d\phi_k d\phi_p \delta^{(2)}(\mathbf{k} + \mathbf{p} - \mathbf{p}_X)[|f_+|^2 + |f_-|^2 + 2\,\mathrm{Re}(f_+ f_-^*)] \\ &= \frac{1}{2\Delta}\int d\phi_k d\phi_p \delta^{(2)}(\mathbf{k} + \mathbf{p} - \mathbf{p}_X)[|f(\mathbf{k}, \mathbf{p})|^2 + \mathrm{Re} f(\mathbf{k}, \mathbf{p}) f^*(\mathbf{k}^*, \mathbf{p}^*)], \end{aligned} \tag{45}$$

where $\mathbf{k}^*$ and $\mathbf{p}^*$ are given by (40).

These results can be directly applied to the integral (39) if we note that it has the form of $|J|^2$ with the function

$$f(\mathbf{k},\mathbf{p}) = \frac{1}{(2\pi)^3\sqrt{\kappa\eta}} e^{im\phi_k + in\phi_p}\mathcal{M}(\mathbf{k},\mathbf{p}).$$

The integral (39) then takes the form

$$\frac{1}{(2\pi)^6 \sin(\delta_k+\delta_p)}\int d\phi_k d\phi_p \delta^{(2)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_X) \times \{|\mathcal{M}(\mathbf{k},\mathbf{p})|^2 + \mathrm{Re}[e^{2im(\phi_k-\phi_X)+2in(\phi_p-\phi_X)}\mathcal{M}(\mathbf{k},\mathbf{p}) \times \mathcal{M}^*(\mathbf{k}^*,\mathbf{p}^*)]\}. \tag{46}$$

Bringing together all the normalization coefficients, we finally arrive at the following representation for the double-twisted cross section:

$$d\sigma_{2\,\mathrm{tw}} = \frac{1}{8\pi\sin(\delta_k+\delta_p)}\int d\phi_k d\phi_p \frac{I_{\mathrm{PW}}(\mathbf{k},\mathbf{p})}{I_{2\mathrm{tw}}} \times [d\sigma_{\mathrm{PW}}(\mathbf{k},\mathbf{p}) + d\sigma'(\mathbf{k},\mathbf{p})], \tag{47}$$

where the flux $I_{2\,\mathrm{tw}}$ is given by (22) and

$$d\sigma'(\mathbf{k},\mathbf{p}) = \frac{(2\pi)^4\delta(E_i-E_f)\delta(p_{zi}-p_{zf})}{4I_{\mathrm{PW}}} \times \mathrm{Re}[e^{2im(\phi_k-\phi_X)+2in(\phi_p-\phi_X)}\mathcal{M}(\mathbf{k},\mathbf{p}) \times \mathcal{M}^*(\mathbf{k}^*,\mathbf{p}^*)]\cdot d\Gamma'_X. \tag{48}$$

We see that the "direct" contribution yields the standard cross section, while the "reflected" contribution gives rise to the novel quantity $\sigma'(\mathbf{k},\mathbf{p})$. This quantity describes the *autocorrelation* of the amplitude and is absent in the plane-wave case.

Let us also see how (47) simplifies in the case when $\mathcal{M}(\mathbf{k},\mathbf{p}) = \mathcal{M}(\mathbf{k}^*,\mathbf{p}^*)$. The novel quantity reduces to the usual cross section, $d\sigma' = \cos[2(m\delta_k - n\delta_p)]d\sigma_{\mathrm{PW}}$, and we obtain

$$d\sigma_{2\,\mathrm{tw}} = \frac{\cos^2(m\delta_k - n\delta_p)}{4\pi\sin(\delta_k+\delta_p)} \times \int d\phi_k d\phi_p \frac{I_{\mathrm{PW}}(\mathbf{k},\mathbf{p})}{I_{2\,\mathrm{tw}}} d\sigma_{\mathrm{PW}}(\mathbf{k},\mathbf{p}). \tag{49}$$

It is only in this case that the double-twisted cross section is expressed via the angular integral of the plane-wave cross section.

As in the single-twisted case, we note that if all the final momenta were fixed and if the detector had an infinitely good momentum resolution, there would be just two points contributing to the angular integral (47). If at least one of these conditions is broken, the integrand extends over the full integration range.

Similarly to the single-twisted case, the double-twisted cross section stays finite even when the conical momentum spreads κ and η are very small. However in contrast to the single-twisted case, the cross section now depends also on the orbital angular momenta m and n. This dependence comes solely from the correlation term in (47). If the cross section is averaged with a sufficiently smooth function of the initial or final momenta, then in the limit of large m and n the correlation term gets suppressed by rapid oscillations, and the double-twisted cross section stays approximately m and n independent.

Certainly, one can also consider double-twisted cross sections with twisted states in superposition of different m and n. We leave a detailed study of this situation for a future work.

Finally we reiterate the point that for this particular version of double-twisted scattering we chose the simplest possible setup, in which both twisted states are defined with respect to the same z axis. One can also study what would happen if two different axes, either parallel or not, were used. Answering this question requires a more elaborate formalism.

V. TWO-PARTICLE DECAY OF A TWISTED SCALAR

A. Preliminary remarks

The previous two Sections were devoted to the cases when the twisted particles appeared only in the initial state, while the final state X was assumed to be describable with the plane waves. Let us now consider the case when at least one of the final particles is also twisted. This is exactly the situation that arises in the original suggestion of [5,6] to use Compton backscattering of the twisted optical photons to produce final high-energy OAM photons. The key question here is to what extent the twisted parameters of the initial state $|\kappa, m\rangle$ determine the parameters of the final twisted state $|\kappa', m'\rangle$. Here again both twisted states are defined with respect to the same z axis. In the case of strictly backward scattering the conservation of the orbital momentum ensures that $\kappa' = \kappa$, $m' = m$, [5]. In this Section, we study how this result changes for the non-forward scattering. Note that we use the term "forward" for any process with zero transverse momentum transfer, i.e. both for strictly forward and strictly backward kinematics.

In order to focus on the final state kinematics and to avoid possible complications coming from a nontrivial matrix element, we would like to answer the above question in the context of the simplest possible problem: the decay of a twisted scalar particle with mass M into a pair of massless distinguishable particles due to the cubic interaction $g\cdot\Phi\phi_1\phi_2$. The momenta of the initial and final particles are p, k_1 and k_2, respectively. We will calculate the decay width in the center of mass frame defined by $p_z = 0$. This is not the true rest frame because due to the transverse motion a twisted particle is never at rest. To make the presentation more pedagogical, we will first calculate the decay rate when both particles in the final state are plane waves, then for the plane wave plus twisted final state, and finally for the case when both final particles are twisted. Although the total decay width must be the same in all these cases, the differential decay rates will be rather different.

B. Two plane waves

Again, let us first recall the standard calculation for the case when all the particles including the initial one are plane waves. The S-matrix is given as usual by $S = i(2\pi)^4\delta^{(4)}(p - k_1 - k_2)\cdot g$. With the plane-wave normalization coefficients for all the particles, the differential decay rate for a particle at rest is

$$d\Gamma = \frac{(2\pi)^4 g^2 \delta^{(4)}(p-k_1-k_2)VT}{T}\cdot (N_{\rm PW}^2)^3\cdot dn_{\rm PW}(k_1) \times dn_{\rm PW}(k_2) = \frac{g^2}{(2\pi)^2}\frac{\delta(M-\omega_1-\omega_2)}{8M\omega_1\omega_2}d^3k_1, \qquad (50)$$

so that the total width is

$$\Gamma = \frac{g^2}{16\pi M}. \qquad (51)$$

Now we repeat this calculation for the initial twisted state $|\kappa, m\rangle$, while keeping the plane-wave basis of the final particles. The S-matrix is

$$S = i(2\pi)^4 g\delta(E-\omega_1-\omega_2)\delta(k_{1z}+k_{2z}) \times \int\frac{d^2\mathbf{k}}{(2\pi)^2}a_{\kappa m}(\mathbf{k})\delta^{(2)}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \qquad (52)$$

$$= i(2\pi)^4 g\delta(E-\omega_1-\omega_2)\delta(k_{1z}+k_{2z}) \times\frac{(-i)^m}{(2\pi)^{3/2}}e^{im\phi_{12}}\frac{\delta(\kappa-k_{12})}{\sqrt{\kappa}}, \qquad (53)$$

where $k_{12}\equiv\sqrt{\mathbf{k}_1^2+\mathbf{k}_2^2+2|\mathbf{k}_1||\mathbf{k}_2|\cos(\phi_1-\phi_2)}$ and ϕ_{12} is the angle of the 2D vector $\mathbf{k}_{12}$ with respect to some axis x. Similarly to the single scattering cross section, this phase factor is inessential and m disappears in the decay rate.

The square of $\delta(\kappa - k_{12})$ is treated as in (29), and with the appropriate normalization factors taken into account, the decay rate has the form

$$d\Gamma = \frac{(2\pi)^3 g^2}{8E\omega_1\omega_2\cdot T}\delta(E-\omega_1-\omega_2)\delta(k_{1z}+k_{2z}) \times TL_z\frac{\delta(\kappa-k_{12})}{\kappa}\frac{R}{\pi}\cdot\frac{1}{V^2}\frac{\pi}{RL_z}\cdot\frac{Vd^3k_1}{(2\pi)^3}\frac{Vd^3k_2}{(2\pi)^3} = \frac{g^2}{(2\pi)^3}\frac{\delta(\kappa-k_{12})}{\kappa}\frac{\delta(E-\omega_1-\omega_2)}{8E\omega_1\omega_2}dk_z d^2\mathbf{k}_1 d^2\mathbf{k}_2. \qquad (54)$$

For the transverse integral we write

$$\int d\phi_2\frac{\delta(\kappa-k_{12})}{\kappa} = 2\int d\phi_2\delta[\kappa^2-\mathbf{k}_1^2-\mathbf{k}_2^2 - 2|\mathbf{k}_1||\mathbf{k}_2|\cos(\phi_1-\phi_2)] = \frac{1}{\Delta}, \qquad (55)$$

where Δ is the area of the triangle with sides κ, $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$, see Appendix A. As usual, the energy delta function can be killed by the k_z integration

$$\int dk_z\frac{\delta(E-\omega_1-\omega_2)}{\omega_1\omega_2} = \frac{2}{Ek_z^*}, \qquad (56)$$

where

$$k_z^* = \frac{1}{2E}\sqrt{E^4+\mathbf{k}_1^4+\mathbf{k}_2^4-2(E^2\mathbf{k}_1^2+E^2\mathbf{k}_2^2+\mathbf{k}_1^2\mathbf{k}_2^2)}. \qquad (57)$$

The decay rate becomes

$$d\Gamma = \frac{g^2}{4\pi^2}\frac{1}{4E^2k_z^*}\frac{|\mathbf{k}_1|d|\mathbf{k}_1||\mathbf{k}_2|d|\mathbf{k}_2|}{\Delta}. \qquad (58)$$

The integration region over $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$ is defined by the requirement that, in addition to (A2), the longitudinal momentum k_z^* is well defined, which cuts the rectangular shape shown in Fig. 2. It is conveniently described with variables

$$x = \frac{|\mathbf{k}_1|-|\mathbf{k}_2|}{\kappa},\qquad z = \frac{|\mathbf{k}_1|+|\mathbf{k}_2|}{\kappa},\qquad x\in[-1,1], \qquad z\in[1, z_{\max}],\qquad z_{\max}\equiv\frac{E}{\kappa}>1. \qquad (59)$$

In these variables

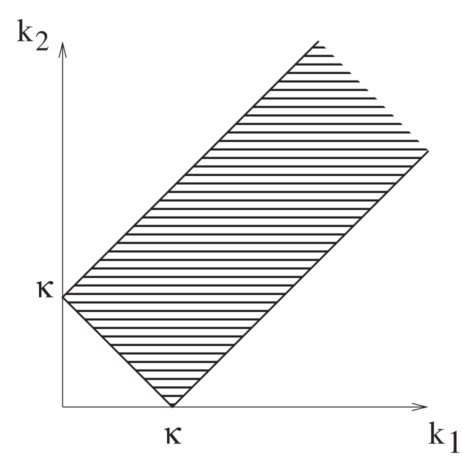

FIG. 2. The allowed kinematical region of the values of $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$ for a fixed κ defined by the "triangle rules" (A2).

$$\Delta = \frac{\kappa^2}{4}\sqrt{(z^2-1)(1-x^2)},$$
$$k_z^* = \frac{E}{2}\sqrt{\left(1-\frac{x^2}{z_{\max}^2}\right)\left(1-\frac{z^2}{z_{\max}^2}\right)}, \tag{60}$$

and the decay rate takes the form

$$d\Gamma = \frac{g^2}{16\pi^2 E}\cdot\frac{(z^2-x^2)dzdx}{[(z_{\max}^2-z^2)(z_{\max}^2-x^2)(z^2-1)(1-x^2)]^{1/2}}. \tag{61}$$

This integral can be taken exactly, and it gives the decay width of the twisted scalar particle

$$\Gamma = \frac{g^2}{16\pi E}, \tag{62}$$

which is a very natural result. In the limit $\kappa \to 0$, we recover the plane-wave decay width (51).

Of course, one could also arrive at (62) just by using our previous results concerning the single-twisted cross section modified to the case of one initial particle. However the detailed derivation given here is needed to understand what changes when the final state includes twisted particles.

C. Twisted state plus plane wave

Let us now describe the final state as twisted state plus a plane wave:

$$|\kappa, m\rangle \to |\kappa_1, m_1\rangle + |\mathrm{PW}(\mathbf{k}_2)\rangle. \tag{63}$$

Since the full decay width cannot depend on the basis we choose for the final particles, we must recover the same result (62) in this basis. In addition to that, we also want to know how the final twisted state parameters κ_1 and m_1 are related to the initial state parameters κ and m.

The S-matrix is now

$$S = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{k}_1}{(2\pi)^2} a^*_{\kappa_1 m_1}(\mathbf{k}_1) a_{\kappa m}(\mathbf{k}) S_{\mathrm{PW}}. \tag{64}$$

It can be written as

$$S = i(2\pi)^4 g\delta(E-\omega_1-\omega_2) \times \delta(k_{z1}+k_{z2})\cdot \mathcal{I}_{m,m_1}(\kappa,\kappa_1,\mathbf{k}_2), \tag{65}$$

where the master integral $\mathcal{I}_{m,m_1}(\kappa,\kappa_1,\mathbf{k}_2)$ is defined as

$$\mathcal{I}_{m,m_1}(\kappa,\kappa_1,\mathbf{k}_2) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{k}_1}{(2\pi)^2} a^*_{\kappa_1 m_1}(\mathbf{k}_1) a_{\kappa m}(\mathbf{k}) \times \delta^{(2)}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2). \tag{66}$$

Let us first calculate this integral in the strictly forward case $\mathbf{k}_2 = 0$:

$$\begin{aligned}\mathcal{I}_{m,m_1}(\kappa,\kappa_1,0) &= \frac{i^{m_1-m}}{(2\pi)^3}\int d^2\mathbf{k}\, d^2\mathbf{k}_1 e^{im\phi - im_1\phi_1}\frac{\delta(|\mathbf{k}|-\kappa)}{\sqrt{\kappa}} \times\frac{\delta(|\mathbf{k}_1|-\kappa_1)}{\sqrt{\kappa_1}}\delta^{(2)}(\mathbf{k}-\mathbf{k}_1)\\ &= \frac{i^{m_1-m}}{(2\pi)^3}\sqrt{\kappa\kappa_1}\int d\phi d\phi_1 e^{im\phi-im_1\phi_1}\cdot 2\delta(\mathbf{k}^2-\mathbf{k}_1^2) \times\delta(\phi-\phi_1)\\ &= \frac{1}{(2\pi)^2}\delta(\kappa-\kappa_1)\delta_{m,m_1},\end{aligned} \tag{67}$$

which was first obtained in [5]. This result implies that the twisted state quantum numbers are transferred from the initial to the final twisted particle without any change. The differential decay rate is

$$\begin{aligned}d\Gamma &= \frac{g^2}{(2\pi)^2}\frac{\delta(E-\omega_1-\omega_2)\delta(k_{1z}+k_{2z})}{8E\omega_1\omega_2} \times\delta_{m,m_1}\delta(\kappa-\kappa_1)\cdot d\kappa_1 dk_{1z} d^3k_2\\ &= \frac{g^2}{(2\pi)^2}\frac{\delta(E-\omega_1-\omega_2)}{8E\omega_1\omega_2}d^3k_2 = \frac{g^2}{(2\pi)^2}\frac{d^2\mathbf{k}_2}{4E^2 k_z^*},\end{aligned} \tag{68}$$

and we stress that this result is applicable only at $\mathbf{k}_2 = 0$.

In the general nonforward case the master integral (66) can be rewritten as

$$\mathcal{I}_{m,m_1}(\kappa,\kappa_1,\mathbf{k}_2) = \frac{i^{m_1-m}}{(2\pi)^3}\sqrt{\kappa\kappa_1}\int d\phi d\phi_1 e^{im\phi-im_1\phi_1}\cdot\delta^{(2)}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2). \tag{69}$$

There are two ways to look at this integral. First, using the results of Appendix A we obtain

$$\mathcal{I}_{m,m_1}(\kappa,\kappa_1,\mathbf{k}_2) = \frac{i^{m_1-m}}{(2\pi)^3}\sqrt{\kappa\kappa_1}e^{i(m-m_1)\phi_2}\frac{\cos[m_1\delta_1-(m-m_1)\delta_2]}{\Delta}, \tag{70}$$

where Δ is the same as in (A7) with $\mathbf{k}_1^2$ replaced by κ_1^2. Additionally, we can also rewrite

$$\delta^{(2)}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) = \frac{1}{(2\pi)^2}\int d^2\mathbf{r}e^{i\mathbf{r}\mathbf{k}-i\mathbf{r}\mathbf{k}_1-i\mathbf{r}\mathbf{k}_2} \tag{71}$$

and represent the master integral as an integral over a triple product of Bessel functions which is useful in certain circumstances:

$$\mathcal{I}_{m,m_1}(\kappa,\kappa_1,\mathbf{k}_2) = \frac{i^{m_1-m}e^{i(m-m_1)\phi_2}}{(2\pi)^2}\sqrt{\kappa\kappa_1}\cdot\int_0^\infty rdrJ_m(\kappa r)J_{m_1}(\kappa_1 r) \times J_{m-m_1}(|\mathbf{k}_2|r), \tag{72}$$

where ϕ_2 is the azimuthal angle of $\mathbf{k}_2$. Note that although we are considering the case with two twisted particles (one in the initial and one in the final state), an integral over three Bessel functions arises automatically. A comparison of (72) with (70) gives the result for the integral of the triple Bessel-function product. See also [13] for some mathematics involved in evaluation of this and similar integrals.

Let us also check the $\mathbf{k}_2 \to 0$ limit of the result (70). When $|\mathbf{k}_2| \ll \kappa$, the distribution over κ_1 spans from $\kappa - |\mathbf{k}_2|$ to $\kappa + |\mathbf{k}_2|$. The angle $\delta_1 \to 0$, while δ_2 can still be arbitrary. However, in the $\mathbf{k}_2 \to 0$ limit, only the $m - m_1 = 0$ term survives due to $J_{m-m_1}(|\mathbf{k}_2|r)$, so that the cosine in (70) approaches unity. The analysis of Δ shows that

$$\lim_{|\mathbf{k}_2|\to 0} \frac{\sqrt{\kappa\kappa_1}}{\Delta} = 2\pi\delta(\kappa - \kappa_1), \tag{73}$$

so that one indeed recovers the strictly forward result for the master integral given in (67). Note that although we wrote simply $|\mathbf{k}_2| \to 0$, we imply a very specific combination of this limit with the limit $R \to 0$, an issue to be discussed in Sec. VI B.

We continue the calculation of the decay rate in the nonforward case. After integration over k_z, the decay rate can be written as

$$d\Gamma = 4\pi^3 g^2 \frac{1}{4E^2 k_z^*} \cdot |\mathcal{I}_{m,m_1}(\kappa, \kappa_1, \mathbf{k}_2)|^2 \cdot \frac{d\kappa_1}{R} d^2\mathbf{k}_2. \tag{74}$$

Note that this decay rate is differential not only in κ_1 and $\mathbf{k}_2$ but also in the discrete variable m_1; the full decay width includes integrals over momenta and a summation over all possible m_1's.

A close inspection shows that the immediate integration over κ_1 or $\mathbf{k}_2$ cannot be done due to singularities of $|\mathcal{I}|^2$ along the boundaries of the kinematically allowed region. In contrast to the plane waves in the final state, the denominator now contains Δ^2 instead of just Δ. Therefore, in terms of variables x and z one encounters singularities of the form

$$\int_{-1}^{1} \frac{dx}{1 - x^2} \quad \text{and} \quad \int_{1}^{z_{\max}} \frac{dz}{z^2 - 1}.$$

Clearly, this is an artefact of the infinite radial integration range in (72). If instead we take R to be large but finite, we expect that a trick similar to (29) should be at work, namely, that after regularization $|\mathcal{I}|^2$ would yield R times a less singular function.

This trick does not seem to work for each m_1 separately. However, as we prove in Appendix B, it works for $|\mathcal{I}|^2$ summed over all possible m_1 values. In the limit $R \to \infty$ we obtain

$$\sum_{m_1=-\infty}^{+\infty} |\mathcal{I}_{m,m_1}(\kappa, \kappa_1, \mathbf{k}_2)|^2 = \frac{1}{(2\pi)^5} \frac{R\kappa_1}{\pi} \frac{1}{\Delta}, \tag{75}$$

with the same Δ as before. The regularization parameter R then disappears from the result, and the decay rate reads

$$d\Gamma = \frac{g^2}{4\pi^2} \frac{1}{4E^2 k_z^*} \cdot \frac{\kappa_1 d\kappa_1 |\mathbf{k}_2| d|\mathbf{k}_2|}{\Delta}. \tag{76}$$

Comparing (76) with the previous results (68) and (58) leads us to two conclusions.

(i) The transition from the strictly forward to the nonforward cross section/decay rate consists in replacement

$$\frac{\delta(\kappa - \kappa_1)}{\kappa_1} \to \frac{1}{2\pi\Delta}. \tag{77}$$

(ii) The result (76) for a twisted particle in the final state coincides with the result (58) for the case when both final particles are plane waves.

Although these conclusions were drawn for the specific process we consider, it represents a universal kinematical feature characteristic to all processes involving nonforward scattering of twisted states.

D. Two twisted states

For completeness, let us also recalculate the decay rate in the basis when both final particles are described by twisted states: $|\kappa_1, m_1\rangle$ and $|\kappa_2, m_2\rangle$. We remind that all twisted states are defined with respect to the same common z axis. It turns out that this calculation closely follows the case of twisted state plus plane wave just considered. This is not surprising because the appearance of the triple-Bessel integral highlights the fact that when two particles (one in the initial and one in the final state) are twisted, the third one is automatically projected from the plane wave onto an appropriately defined twisted state as well.

The S-matrix takes the form

$$S = ig(2\pi)^{3/2}\delta(E - \omega_1 - \omega_2)\delta(k_{z1} + k_{z2}) \cdot \delta_{m,m_1+m_2}\sqrt{\kappa\kappa_1\kappa_2} \times \int r dr J_m(\kappa r) J_{m_1}(\kappa_1 r) J_{m_2}(\kappa_2 r), \tag{78}$$

and we again encounter the triple-Bessel-function integral. The decay rate is written as

$$d\Gamma = \frac{g^2}{4\pi^2} \frac{1}{4E^2 k_z^*} \cdot \frac{\kappa_1 d\kappa_1 \kappa_2 d\kappa_2}{\Delta}. \tag{79}$$

This expression is identical to (76) up to the obvious replacement $|\mathbf{k}_2| \to \kappa_2$; its integration over all conical momenta spreads κ_1, κ_2 gives again (62).

VI. DISCUSSION

A. Potential uses of the twisted cross sections

In this paper we represented the cross sections for the single-twisted and double-twisted processes via the angular integrals (31) and (47). These are rather unconventional quantities, as they involve averaging of the plane-wave cross sections over the azimuthal angles of the *initial* particles at fixed final momenta. A further study is needed to see if they can be related to the more conventional cross sections, in which the initial momenta are fixed but final state azimuthal angles are integrated out.

Using twisted states in superpositions of different orbital angular momenta, one can perform, in principle, a Fourier analysis of the differential cross section as the function of the initial azimuthal angles at fixed final momenta, see (35). This might be a useful complementary tool for the study of various azimuthal asymmetries.

Perhaps, the most intriguing feature we have found arises in the double-twisted cross section. We showed that this cross section is sensitive not only to the plane-wave cross section, but also to a quantity describing the *autocorrelation* of the plane-wave amplitude. Such quantity is absent in the conventional plane-wave scattering.

Note that it is not the first time the amplitude autocorrelation appears in particle physics. In [14], where finite beam size effects were analyzed, the product of amplitudes at different momentum space points was also involved. The essential novelty of the present case is the extra angular factors in (48), which offer a more detailed probe of the amplitude.

Finally, let us stress once again that experimental study of these features does not require large conical momentum spread κ and is feasible even for κ in the eV range, which was already realized for energetic electrons in [8,9] and which was proposed in [5] for high-energy twisted photons. It is the nontrivial angular dependence that is at work. However we stress that in order to probe the autocorrelation function of the amplitude with the momentum offset $\mathbf{q}$, one would need conical momentum spread $\kappa \sim |\mathbf{q}|$.

It is natural to ask whether the new degree of freedom offered by twisted particles, and, in particular, photons, can be used to gain more insight into the structure of hadrons. Indeed, the structure of the proton remains one of the hottest topics in QCD. One of the questions that might be particularly relevant for the present study is the contribution of the quark/gluon orbital momentum to the proton spin, see e.g. reviews [11]. It is, therefore, very interesting to know whether the OAM of the initial photon can be transferred to partons, allowing us to probe the partonic composition of the proton in a novel way.

Without going into detail, let us make some preliminary observations that can be inferred from the results of the present work.

Let us first note that the definition of a twisted state (2) involves an angular functional: it picks up a plane-wave quantity as a function of the initial azimuthal angle ϕ, weights it with $e^{im\phi}$ and integrates over all angles. This might give an impression that the twisted state is a perfect Fourier analyser at level of amplitude. This impression is wrong because this functional acts not on the amplitudes itself, but on the S-matrix element, which includes a transverse delta-function. Equation (32) shows that this functional partially kills the delta-function, while the amplitude remains unchanged. The orbital angular momentum is just transferred to the overall motion of the final system instead of exciting internal degrees of freedom. This somewhat pessimistic conclusion is supported by the result of this paper that the single-twisted cross section is just an azimuthal integral of the plane-wave cross section.

The situation might change if both the initial photon and proton were twisted. As we showed above, in this case a residual coherence between two kinematical configurations survives, which leads to an additional term in the cross section proportional to the autocorrelation function of the amplitude. This term has a nontrivial azimuthal structure and might represent a new probe of the proton. This issue certainly deserves a dedicated study.

B. Properties of the final twisted particle in a nonforward scattering

Let us also discuss in detail what the results of Sec. V C tell us about the values of κ_1 and m_1. The differential decay rate (76) shows that at large $|\mathbf{k}_2| \gg \kappa$ the conical momentum spread κ_1 is limited to the interval from $|\mathbf{k}_2| - \kappa$ to $|\mathbf{k}_2| + \kappa$ with an inverse square root singularity at the endpoints. This singularity is integrable, so that the entire interval more or less homogeneously contributes to the integral. Since $|\mathbf{k}_2|$ can be as high as E, the total decay width is therefore dominated by large $\kappa_1 \gg \kappa$. In a more complicated process, the decay rate or the cross section will include the amplitude squared which can serve as a cutoff function. For example, in the Compton scattering one expects that κ_1 up to $\sim m_e$ will contribute to the cross section.

Although we found a result for the decay rate summed over all m_1, we can trace the main m_1 region from the intermediate formulas. The result is that essentially all m_1 values from minus to plus infinity are important for the decay rate, which is in a strong contrast to the strictly forward result $m = m_1$.

Indeed, the distribution over final m_1 can be seen in (74), where the master integral $\mathcal{I}_{m,m_1}$ is given by (70). For generic transverse momenta, growth of m_1 leads to oscillations of the cosine function with constant amplitude. Thus, when averaging over the entire κ_1 interval, one can approximate cosine squared by $1/2$, and the dependence on m_1 drops off. This result holds for any nonzero transverse momentum transfer $|\mathbf{k}_2| \to 0$.

Since the forward and nonforward m_1 distributions are so dramatically different, a natural question arises whether there is a continuous transition from the nonforward to the forward scattering. The answer to this question involves an accurate treatment of two limits: $|\mathbf{k}_2| \to 0$ and $R \to \infty$. Let us keep R large but finite, and set $|\mathbf{k}_2| \to 0$; then the transition is smooth. Looking at the triple-Bessel representation of the master integral (72) with the upper limit replaced by R, one sees that the result will begin to significantly decrease only when the position of the first node of the last Bessel function $J_{m-m_1}(|\mathbf{k}_2|r)$ falls outside of the integration range, that is for $|\mathbf{k}_2| \lesssim |m - m_1|/R$, where $m_1 \neq m$. If $m_1 = m$, then at $|\mathbf{k}_2| \ll 1/R$ the last Bessel function can be approximated by the unity. Therefore, the limit $|\mathbf{k}_2| \to 0$ taken in (73) implies that

$$|\mathbf{k}_2| \to 0 \quad \text{and} \quad R \to \infty \quad \text{provided that } |\mathbf{k}_2|R \ll 1.$$

If instead $R \to \infty$ at fixed $|\mathbf{k}_2|$, then the transition of nonforward to forward results is discontinuous at $|\mathbf{k}_2| = 0$.

In [5] it is claimed with the specific example of the Compton cross section that if the transverse momentum transfer is small compared to κ (in our notation, finite $|\mathbf{k}_2| \ll \kappa$ at infinite R), then the m_1 dependence has a narrow distribution peaked at $m_1 = m$. Our analysis does not support this claim.

Looking back at the formalism used, we can conclude that our result, that all m_1 values essentially contribute to the decay rate/cross section, just reflects the unfortunate choice of the same common axis z for all the twisted states appearing in the process. It does not give a clue of how twisted the final particles are with respect to their own propagation axes defined by the average values of the components of the 3-momentum operator. Indeed, even a simple nonforward plane wave, when expanded in the basis of twisted states, contains all partial waves, see (7). Nevertheless it carries a zero orbital angular momentum with respect to its own direction of propagation. Therefore, it appears that a more physically reasonable quantity is the *"orbital helicity"*, projection of orbital angular momentum on the axis of motion. The relevant question is then how this "orbital helicity", not the OAM with respect to a fixed axis, is transferred from the initial to the final twisted state. We postpone this question for future studies.

C. Conclusions

Orbital angular momentum is a new degree of freedom, which can be used in high-energy physics to gain more insight into properties of particles and their interactions. In this paper, focusing on the scalar case, we studied high-energy collisions in which initial and/or final particles were described by twisted states, i.e. carried a nonzero OAM. We derived expressions for the cross sections for the single-twisted and double-twisted cases as well as for the two-particle decay rate of a twisted scalar, and observed a number of remarkable features.

The single-twisted differential cross section is represented via the plane-wave cross section averaged over the azimuthal angle of one of the incoming particles. If the initial twisted particle is prepared in a superposition state with different orbital quantum numbers, a Fourier analysis of the plane-wave cross section can be performed. The expression we found for the double-twisted cross section is more intriguing, as it involves not only the plane-wave cross section, but also the autocorrelation function of the amplitude. We stress that these features do not rely on a large conical momentum spread in the twisted states, and their experimental study looks feasible even with today's technology.

When analyzing the decay rate of a twisted particle, we established the procedure that allows one to pass from the plane wave to the twisted particle basis for the final state. These results can now be used, for example, to investigate the Compton backscattering of the twisted photons in the nonforward region, which was missing in the original suggestion [5,6].

Analyzing parameters of the final twisted state, we discovered that there is no smooth transition from the nonforward to forward scattering. We attribute this result to the infinite transverse size of twisted states and to the deficiency of our formalism in which all the twisted states are defined with respect to the same z axis. We expect that the problem will disappear if the orbital angular momentum projection is calculated not on the fixed reaction axis but on the direction of the outgoing twisted particle. Incorporation of this "orbital helicity" into the present formalism remains to be done.

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APPENDIX A: RESTRICTED ANGULAR INTEGRALS

Here we derive some properties of the restricted angular integrals which appear in the paper. The basic integral has the following form

$$J_0 = \int_0^{2\pi} d\phi_1 d\phi_2 \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \tag{A1}$$

and corresponds to a situation when two transverse momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ of fixed absolute values sum up to a fixed momentum $\mathbf{k}$ with modulus $|\mathbf{k}| = \kappa$ and azimuthal angle ϕ_k.

Clearly the integral can be nonzero only if it is possible at all to form a triangle with sides κ, $|\mathbf{k}|_1$, and $|\mathbf{k}_2|$, that is, when $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$ satisfy the "triangle rules":

$$\kappa \le |\mathbf{k}_1| + |\mathbf{k}_2|, \qquad |\mathbf{k}_1| \le \kappa + |\mathbf{k}_2|, \qquad |\mathbf{k}_2| \le \kappa + |\mathbf{k}_1|. \tag{A2}$$

This allowed region on the $(|\mathbf{k}_1|, |\mathbf{k}_2|)$ plane has the form of a stripe shown in Fig. 2. If all moduli are fixed, the integral (A1) receives contributions only from two points (see Fig. 1 in the main text). Let us introduce

$$\delta_1 = \arccos\left(\frac{\kappa^2 + \mathbf{k}_1^2 - \mathbf{k}_2^2}{2\kappa|\mathbf{k}_1|}\right), \qquad \delta_2 = \arccos\left(\frac{\kappa^2 + \mathbf{k}_2^2 - \mathbf{k}_1^2}{2\kappa|\mathbf{k}_2|}\right). \tag{A3}$$

Then, the two configurations of transverse momenta correspond to

$$\phi_k - \phi_1 = \pm\delta_1, \qquad \phi_k - \phi_2 = \mp\delta_2, \tag{A4}$$

so that the signs of $\phi_k - \phi_1$ and $\phi_k - \phi_2$ are always opposite.

To evaluate the integral itself, we rewrite the delta-function as

$$\delta^{(2)}(\mathbf{k}_1 - (\mathbf{k} - \mathbf{k}_2)) = 2\delta[\mathbf{k}_1^2 - (\mathbf{k} - \mathbf{k}_2)^2] \times \delta(\phi_1 - \phi_{\mathbf{k}-\mathbf{k}_2}). \tag{A5}$$

The ϕ_1 integration is then eliminated, and we obtain

$$J_0 = 2\int d\phi_2 \delta[\mathbf{k}_1^2 - \kappa^2 - \mathbf{k}_2 + 2\kappa|\mathbf{k}_2|\cos(\phi_2 - \phi_k)] = \frac{1}{\Delta}, \tag{A6}$$

where

$$\Delta = \frac{1}{4}\sqrt{2(\mathbf{k}_1^2\mathbf{k}_2^2 + \kappa^2\mathbf{k}_1^2 + \kappa^2\mathbf{k}_2^2) - \kappa^4 - \mathbf{k}_1^4 - \mathbf{k}_2^4} \tag{A7}$$

is the area of the triangle with sides κ, $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$. When needed, this area can be also rewritten as

$$\Delta = \frac{1}{2}|\mathbf{k}_1||\mathbf{k}_2|\sin(\delta_1 + \delta_2). \tag{A8}$$

Let us now consider an integral similar to (A1) but with an extra angular dependence in the integrand:

$$J = \int_0^{2\pi} d\phi_1 d\phi_2 \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \cdot f(\phi_1, \phi_2) \tag{A9}$$

with a regular function $f(\phi_1, \phi_2)$. Here again the momenta must satisfy the triangle rules (A2) and the integral receives contribution only from two points (A4) in the (ϕ_1, ϕ_2) space. Denoting the values of $f(\phi_1, \phi_2)$ at these two points as f_+ and f_-, respectively, we obtain

$$J = \frac{f_+ + f_-}{2\Delta}. \tag{A10}$$

We can also rewrite this result as

$$J = \int_0^{2\pi} d\phi_1 d\phi_2 \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \cdot \frac{f_+ + f_-}{2}, \tag{A11}$$

which means that in these circumstances the delta-function plays the role of a functional that maps a test function $f(\phi_1, \phi_2)$ to its symmetrized value $(f_+ + f_-)/2$.

In the particular case of $f = \exp(im_1\phi_1 + im_2\phi_2)$, (A10) takes the form

$$J = e^{i(m_1+m_2)\phi_k}\frac{\cos(m_1\delta_1 - m_2\delta_2)}{\Delta}. \tag{A12}$$

APPENDIX B: REGULARIZATION OF $|\mathcal{I}|^2$

Here we calculate the large-R behavior of the m_1 sum of the squares of the triple-Bessel integral:

$$\sum_{m_1=-m_{\max}}^{+m_{\max}}\left[\int_0^R rdrJ_m(\kappa r)J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r)\right]^2, \tag{B1}$$

which appears in the decay rate (74). Evaluation of the integral itself with $R \to \infty$ performed in the main text shows that it can be nonzero only if κ, κ_1, κ_2 satisfy the triangle rules (A2), i.e. a triangle with these sides can be constructed. Since m describes the initial state, we take it small and not growing with R: $m \ll m_{1\max} = \kappa_1 R$, while m_1 can extend up to $m_{1\max}$. The final κ_1, κ_2 can be much larger than κ.

Since the expression (B1) is regularized with large but finite R, the summation and integration can be interchanged:

$$\int rdrr'dr'J_m(\kappa r)J_m(\kappa r') \times \sum_{m_1=-m_{1\max}}^{+m_{1\max}} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r)J_{m_1}(\kappa_1 r')J_{m-m_1}(\kappa_2 r'). \tag{B2}$$

Thanks to the properties of the Bessel functions, only m_1's up to $\min(\kappa_{1,2}r, \kappa_{1,2}r')$ are effectively contributing to this sum; for larger m_1 values, the Bessel functions strongly decrease. But $r, r' \le R$, which means that the limits on the summation can in fact be safely extended to infinity. Then, the sum of the product of four Bessel functions is treated in the following way:

$$\sum_{m_1=-\infty}^{+\infty} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r)J_{m_1}(\kappa_1 r')J_{m-m_1}(\kappa_2 r')$$
$$= \sum_{m_1,m_1'=-\infty}^{+\infty} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r)J_{m_1'}(\kappa_1 r')J_{m-m_1'}(\kappa_2 r')\cdot \delta_{m_1,m_1'}$$
$$= \sum_{m_1,m_1'=-\infty}^{+\infty} \frac{1}{2\pi}\int_0^{2\pi} d\alpha e^{i(m_1-m_1')\alpha} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r)J_{m_1'}(\kappa_1 r')J_{m-m_1'}(\kappa_2 r')$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\alpha \left[\sum_{m_1=-\infty}^{+\infty} e^{im_1\alpha} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r)\right]\left[\sum_{m_1'=-\infty}^{+\infty} e^{-im_1'\alpha} J_{m_1'}(\kappa_1 r')J_{m-m_1'}(\kappa_2 r')\right]. \qquad \text{(B3)}$$

The first sum in the square brackets is calculated as follows:

$$\sum_{m_1=-\infty}^{+\infty} e^{im_1\alpha} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r) = \frac{(-i)^m}{(2\pi)^2}\int d\phi_1 d\phi_2 e^{i\kappa_1 r\cos\phi_1 + i\kappa_2 r\cos\phi_2} \sum_{m_1=-\infty}^{+\infty} e^{im_1\alpha + im_1\phi_1 + i(m-m_1)\phi_2}$$
$$= \frac{(-i)^m}{2\pi} e^{im\alpha}\int d\phi_1 e^{im\phi_1} e^{i\kappa_1 r\cos\phi_1 + ik_2 r\cos(\phi_1+\alpha)}. \qquad \text{(B4)}$$

The combination of angles and momenta inside the exponential can be expressed as

$$\kappa_1\cos\phi_1 + \kappa_2\cos(\phi_1+\alpha) = |\mathbf{k}_1 + \mathbf{k}_2|_\alpha \cos(\phi_1 + \delta\phi), \qquad \text{(B5)}$$

where

$$|\mathbf{k}_1+\mathbf{k}_2|_\alpha \equiv \sqrt{\kappa_1^2+\kappa_2^2+2\kappa_1\kappa_2\cos\alpha}, \quad \tan\delta\phi = \frac{\kappa_2\sin\alpha}{\kappa_1+\kappa_2\cos\alpha}. \qquad \text{(B6)}$$

Geometrically, $|\mathbf{k}_1+\mathbf{k}_2|_\alpha$ is the norm of the sum of two vectors of moduli κ_1 and κ_2 and the relative azimuthal angle α. Therefore, (B4) is

$$\sum_{m_1=-\infty}^{+\infty} e^{im_1\alpha} J_{m_1}(\kappa_1 r)J_{m-m_1}(\kappa_2 r) = e^{im(\alpha-\delta\phi)} J_m(|\mathbf{k}_1+\mathbf{k}_2|_\alpha r). \qquad \text{(B7)}$$

This expression can be viewed as a 2D generalization of the well-known addition formula for the Bessel functions

$$\sum_{m_1=-\infty}^{+\infty} J_{m_1}(x)J_{m-m_1}(y) = J_m(x+y).$$

Now, the second sum differs only by $\alpha \to -\alpha$ (or, alternatively, complex conjugation) and $r \to r'$. Therefore, the summation (B3) is simplified to

$$\frac{1}{2\pi}\int_0^{2\pi} d\alpha J_m(|\mathbf{k}_1+\mathbf{k}_2|_\alpha r)J_m(|\mathbf{k}_1+\mathbf{k}_2|_\alpha r'). \qquad \text{(B8)}$$

Note that this integral involves only the Bessel functions of small order. We now plug this expression into (B2) and get

$$\frac{1}{2\pi}\int_0^{2\pi} d\alpha\left[\int rdr J_m(\kappa r)J_m(|\mathbf{k}_1+\mathbf{k}_2|_\alpha r)\right] \times\left[\int r'dr' J_m(\kappa r')J_m(|\mathbf{k}_1+\mathbf{k}_2|_\alpha r')\right]. \qquad \text{(B9)}$$

As usual, we extend the integration range in one of the integrals to infinity, which gives a delta function, and then we use it on the second integral calculated up to R. The original expression (B1) then becomes

$$\frac{1}{2\pi}\int_0^{2\pi} d\alpha \frac{\delta(\kappa - |\mathbf{k}_1+\mathbf{k}_2|_\alpha)}{\kappa}\frac{R}{\pi\kappa} = \frac{R}{2\pi^2\kappa}\cdot\frac{1}{\Delta}, \qquad \text{(B10)}$$

where Δ is, as always, the area of the triangle with sides κ, κ_1, κ_2.

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