PHYSICAL REVIEW D **83,** 087502 (2011)

Massless scalar field and solar-system experiments

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(Received 14 February 2011; published 7 April 2011)

The solution of Einstein's field equations with the energy-momentum tensor of a massless scalar field is known as the Fisher solution. It is well known that this solution has a naked singularity due to the "charge" Σ of the massless scalar field. Here I obtain the radial null geodesic of the Fisher solution and use it to confirm that there is no black hole. In addition, I use the parametrized post-Newtonian formalism to show that the Fisher spacetime predicts the same effects on solar-system experiments as the Schwarzschild one does, as long as we impose a limit on Σ. I show that this limit is not a strong constraint and we can even take values of Σ bigger than M. By using the exact formula of the redshift and some assumptions, I evaluate this limit for the experiment of Pound and Snider [Phys. Rev. **140**, B788 (1965)]. It turns out that this limit is $\Sigma < 5.8 \times 10^3$ m.

DOI: 10.1103/PhysRevD.83.087502 PACS numbers: 04.20.−q, 04.80.Cc, 04.20.Dw

I. INTRODUCTION

Despite the fact that no scalar field has ever been observed in nature, physicists have used it in many different theories. The interest in these fields appears in both classical and quantum field theories. As an example, we have the Higgs boson, which plays an important role in the standard model and it is expected to be found soon. One also uses scalar fields to confine particles in hypersurfaces in models with extra dimensions [1], to explain the accelerated expansion of the Universe [2–4] and so on. There are many roles that a scalar field can play. One of them is played by a massless scalar field that makes the event horizon disappear in a spherically symmetric solution of general relativity.

The spherically symmetric solution of Einstein's equation (with zero cosmological constant) with an energy-momentum tensor of a massless scalar field (EMS) has no black hole [5,6]. This solution was first obtained by Fisher [7] and rediscovered by many other authors [5,8,9], such as Wyman, whose version became well known in the literature [10]. It was proved later that this solution is the most general static and spherically symmetric solution to the EMS [9]. In Ref. [5], the authors obtain many other solutions with scalar fields, namely, the massless scalar field with electric charge, with a rotating body, and a static spherically symmetric solution with a conformally invariant scalar field. The generalization of the Fisher solution to a d-dimensional spacetime was given in [11], and studied in detail in [12].

The aim of this paper is to obtain the exact solution of the radial null geodesics and prove that, for a long range of values of the "massless scalar charge" Σ, the four-dimensional Fisher solution yields the same results to solar-system experiments as the Schwarzschild one. This article is organized as follows. In Sec. II, the Fisher solution is presented. In Sec. III, I show the possibility of round-trip travel for a light signal which goes from an observer to the center of a spherical body, which is treated as a pointlike particle, and is reflected back. This result is used to relate the proper time measured by the observer to his or her radial coordinate. In Sec. IV the solar-system experiments are analyzed by using the parametrized post-Newtonian (PPN) formalism. In addition, the spectral shift predicted by the Fisher solution is calculated and compared with the one predicted by the Schwarzschild solution for the Pound-Snider experiment [13]. Our results are summarized in Sec. V. Except for some few details, the notation and conventions used throughout this article are basically the same as those of Ref. [12].

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II. THE GENERALIZED FISHER SOLUTION

The Einstein field equations with a massless scalar field φ and a massive body can be written as [5]

$$G_{\mu\nu} = -2\left(\varphi_\mu \varphi_\nu - \frac{1}{2} g_{\mu\nu}\varphi^\lambda \varphi_\lambda\right), \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, and φ_μ stands for $\partial\varphi/\partial x^\mu$. Furthermore, the scalar field satisfies $\Box\varphi = 0$, where $\Box$ is the d'Alembertian in a curved spacetime.

The static, spherically symmetric, and asymptotically flat solution of Eq. (1) can be written in the form [12]

$$ds^2 = W^S dt^2 - W^{-S} dr^2 - r^2 W^{1-S} d\Omega^2, \tag{2}$$

$$\varphi(r) = \frac{\Sigma}{2\eta} \ln|W(r)|, \tag{3}$$

where $\eta = \sqrt{M^2 + \Sigma^2}$ and $S = M/\eta$ are constants, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The constant $M \geq 0$ is the mass of the body, while Σ can be interpreted as the scalar charge. Since M is always positive, we have $S \in [0, 1]$. The function $W(r)$ is given by

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$$W(r) = 1 - \frac{r_0}{r}, \qquad (4)$$

where $r_0 = 2\eta$. In Ref. [12] the authors take into account two disconnected parts of the Fisher manifold: the Fisher universe $r \in (0, r_0)$, and the Fisher spacetime $r \in (r_0, \infty)$. However, I shall consider only the former, and so the radial coordinate r in the metric (2) takes values between r_0 and ∞, where r_0 is the center of the body. In this case, we have $W \in (0, 1)$.

III. RADIAL NULL GEODESICS

In the case of the Schwarzschild metric one may associate the coordinates with the empirical distances directly because the error made can be neglected. Because of this, the coordinate positions of a coordinate system may differ from other equally good, but the values predicted for the experiments are basically the same (see, e.g., Ref. [14], p. 1107). This discrepancy clearly depends on the value of the mass M, and big values of M, compared with the distances used, would increase the error. As we are interested in the effects caused by the metric (2), we have to be very careful about the identification of r with those distances. In order to avoid imposing constraints on the values we may choose for η, at least while calculating the spectral shift, let us obtain an exact relation between the proper time $\Delta\tau$ measured by an observer and his or her coordinate position. To do so, we consider that the observer sends a light signal toward the center of a spherical body, which is characterized by $r = r_0$, and the signal is reflected back. It should be clear that we are treating the spherical body as a pointlike particle, which allows us to use the exterior solution (2) during the whole trajectory.

The radial null geodesics allow us to relate the proper time $\Delta\tau$ of the observer with his or her radial coordinate r_1 through the integral

$$\Delta\tau = 2\sqrt{W^S(r_1)} \int_{r_0}^{r_1} dr[W(r)]^{-S}. \qquad (5)$$

To solve this integral, it is more convenient to change the variable of integration to W. By doing so, we get

$$\Delta\tau = 2r_0 W_1^{S/2} \int_0^{W_1} dW(1-W)^{-2} W^{-S}, \qquad (6)$$

where $W_1 = W(r_1)$. At first sight, one might be tempted to express this integral in terms of the incomplete beta function $B_x(p, q)$ directly. However, this is not possible because q cannot be negative (see, e.g., [15], p. 523), as would be necessary in this case. It turns out to be more suitable to express this integral in terms of the hypergeometric function ${}_2F_1(a, b; c; x)$. By performing the substitution $u = W/W_1$, one sees that (6) can be written in the following form (see, e.g., [16], p. 558),

$$\Delta\tau = \frac{2r_0 W^{1-S/2}}{1-S} {}_2F_1(2, 1-S; 2-S; W), \qquad (7)$$

where the index 1 has been dropped from W.

By using the ratio test one can easily prove that the series which represents the hypergeometric function (see, e.g., Ref. [16], p. 556) converges for the case in Eq. (7), and so $\Delta\tau$ is finite. The finiteness of $\Delta\tau$ is in agreement with the fact that there is no black hole in this model. In the Schwarzschild case, treating the body as a pointlike particle is equivalent to putting it inside its Schwarzschild radius, and then the light signal would never come back.

The formula (7) is very useful to get the values of the coordinate position of an observer by means of experimental data, at least for values of Σ that are not "too close to zero" (or equivalently $S \neq 1$) [17]. Of course, no one is able to send a light signal to the center of either a planet or the Sun and those bodies are not pointlike particles. Nevertheless, it is reasonable to assume that $\Delta\tau$ is basically twice the empirical distance. As an example, consider $\eta = \sqrt{5}M$, where M is the Earth's mass, and $\Delta\tau = 2 \times 6.37 \times 10^6$ m (twice the Earth's mean radius). The resultant radial coordinate is $r = 6.369\,999\,8 \times 10^6$ m.

IV. SOLAR-SYSTEM EXPERIMENTS

A. PPN formalism

The PPN formalism consists in expanding the metric of any conceivable metric theory to the lowest-order corrections provided by this theory to the Newtonian one. In doing so, one obtains for the kind of solution we are dealing with the following PPN metric [18],

$$ds^2 = \left[1 - \frac{2M}{r_i} + \frac{2\beta M^2}{r_i^2}\right]dt^2 - \left[1 + \frac{2\gamma M}{r_i}\right](dr_i^2 + r_i^2 d\Omega^2), \qquad (8)$$

where r_i is the radial isotropic coordinate, and γ, β are one of the PPN parameters. These parameters are directly related to the type of experiment performed. For instance, the gravitational spectral shift is independent of γ, while the gravitational deflection of light and the time delay do depend on it. In the Schwarzschild case one has $\gamma = \beta = 1$.

In order to use this formalism for the Fisher spacetime, we have to write the metric given by (2) in an isotropic coordinate system. The relation between the radial isotropic coordinate r_i and r is already known (see, e.g., Ref. [12], footnote 21). This relation, which is invertible, allows us to write (2) in the following form,

$$ds^2 = W^S dt^2 - 16\frac{W^{1-S}}{(1+\sqrt{W})^4}[dr_i^2 + r_i^2 d\Omega^2], \qquad (9)$$

TABLE I. The values of r_0/r_i for some choices of $\Delta\tau/2$, the mass M, and η.

$\Delta\tau/2(10^8$ m)	M	η	(r_0/r_i)
0.0637	Earth	$10^4 M$	10^{-5}
6.96	Sun	$10M$	4×10^{-5}
459	Sun	$10^2 M$	10^{-5}

TABLE II. The values of $2|\Delta\nu/\nu|$ for $\Delta\tau_1/2$ equal to the Earth's radius, while $\Delta\tau_2/2 = \Delta\tau_1/2 + 22.5$, and some values of η. The mass M is the Earth's mass.

$\eta(10^4$ m)	$2|\Delta\nu/\nu|\times10^{15}$
0.58	4.915
1.58	4.900
4.02	4.863

where W is still a function of r. By using $r(r_i)$, we can expand the coefficient of dt^2 to the second order of $\varepsilon = r_0/r_i$ and the other to just the first order, which leads us to

$$ds^2 = \left(1 - \frac{2M}{r_i} + \frac{2M^2}{r_i^2}\right)dt^2 - \left(1 + \frac{2M}{r_i}\right)(dr_i^2 + r_i^2 d\Omega^2). \tag{10}$$

This is exactly the Schwarzschild metric in the isotropic coordinate, and so we can conclude that the scalar field does not affect the solar-system experiments for $r_i \gg r_0$. However, this does not mean that we can choose any value of Σ without changing the predictions. This is so because r_0 increases with η or, equivalently, with Σ. Therefore, one has to choose a value for η, obtain the value of the coordinate r [by using Eq. (7), for example], and only then see if $r_0 \ll r_i(r)$ is satisfied. Some values of r_0/r_i are given in Table I. One can numerically check that the value of r_0/r_i decreases as η does, and so does the error too. As Table I suggests, we may take large values for η without causing any significant change in the experimental predictions. For example, in the case of M being the Earth's mass, we can take $\Sigma \approx \eta = 10^4 M \approx 44.4$ m. As another example, we can take $\Delta\tau/2$ as being the perihelion of Mercury, M as the Sun's mass, and $\eta = 10^2 M \approx 1.5\times10^5$ m. As we shall see next, the value of η can be much larger than the one given in the first example without changing the prediction of the redshift.

B. Spectral shift

Roughly speaking, the spectral shift experiment consists basically in sending a light signal with a certain frequency from one point to another and seeing if there is any change in the frequency. We can represent this change by $\Delta\nu/\nu$, where we take ν as being the frequency measured by the observer who sends the signal and the other, say, $\bar{\nu}$, the one measured by the observer who receives the signal. When $\Delta\nu$ is negative, the light is shifted to the red (redshift).

Given a metric $g_{\mu\nu}$, the gravitational redshift for a static spacetime can be calculated through (see, e.g., Ref. [19], p. 202)

$$\frac{\Delta\nu}{\nu} = \sqrt{\frac{g_{00}(x_1^\alpha)}{g_{00}(x_2^\alpha)}} - 1, \tag{11}$$

where x_1^α and x_2^α are the positions of the observer who sends and the one who receives the signal, respectively.

To compare the formula (11) with the value measured in the Pound-Snider experiment [13], one needs to take the difference between the blueshift and the redshift, which gives twice the magnitude of (11). The value measured in this experiment was (0.9990 ± 0.0076) times 4.905×10^{-15}. In turn, the value predicted by the Schwarzschild solution is 4.92×10^{-15} [20]. With the help of the formula (7), one finds that the spectral shift starts to deviate from the one predicted by the Schwarzschild case when $\eta \geq 5.8\times10^3$ m; it gives the exact value of the experiment for $\eta = 1.58\times10^4$ m and is outside the margin of error for $\eta > 4.02\times10^4$ m (see Table II). The degree of accuracy of these results depends on how accurate the association of $\Delta\tau/2$ with the empirical distances is. Perhaps one would overcome this problem by using a nonradial geodesic that allowed the direct measurement of $\Delta\tau$, which does not seem to be an easy task.

V. CONCLUSIONS

As we already know the presence of a massless scalar field prevents the formation of black holes in the static and spherical symmetric solution of the EMS, no matter how small the value of its charge Σ is. This allowed me to relate the proper time of an observer to his or her radial coordinate by using a radial null geodesic only. It was also possible to verify that the difference between $\Delta\tau/2$ and the radial coordinate r is negligible for values of η not much bigger than M.

With the assumption that r_0 is much smaller than r, which is not a strong constraint, we saw that the Fisher spacetime agrees with the solar-system experiments. In general, all the results are in good agreement with these experiments for $\eta \leq 10^4$ m. We can even take $\eta = 1.47\times10^5$ m without changing the result predicted by the Schwarzschild solution for distances of the order of 4.59×10^{10} m (see Table I). In this sense, this result generalizes the statement made at the end of Ref. [5], where the authors state, without proof, that the effect of Σ on the precession of the perihelion is undetectable at least when $\Sigma^2 \ll M^2$.

ACKNOWLEDGMENTS

I would like to thank CNPq for financial support. Thanks also go to B. de Lima Bernardo for some words of advice.

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