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Refined matrix models from BPS counting

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We construct a free fermion and matrix model representation of refined Bogomol'nyi-Prasad-Sommerfeld generating functions of D2 and D0 branes bound to a single D6 brane, in a class of toric manifolds without compact four-cycles. In appropriate limit we obtain a matrix model representation of refined topological string amplitudes. We consider a few explicit examples which include a matrix model for the refined resolved conifold, or equivalently five-dimensional $U(1)$ gauge theory, as well as a matrix representation of the refined MacMahon function. Matrix models which we construct have ordinary unitary measure, while their potentials are modified to incorporate the effect of the refinement.

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I. INTRODUCTION

The purpose of this paper is to provide a free fermion and matrix model representation of refined topological string amplitudes and, more generally, refined Bogomol'nyi-Prasad-Sommerfeld (BPS) counting functions in a system of D2- and D0-branes bound to a single D6-brane, in toric Calabi-Yau manifolds without compact four-cycles. Such a putative free fermion representation is interesting, as it would extend earlier results on wall-crossing in the D6-D2-D0 system to the refined case. The motivation for finding a matrix model representation is as follows. In the nonrefined case connections between such systems and matrix models are known from several perspectives. General relations between topological strings, gauge theories, and matrix models were postulated by Dijkgraaf and Vafa [1], and related to $\mathcal{N} = 2$ theories in [2]. The Chern-Simons matrix model for the conifold and generalizations to lens spaces were considered in [3,4]. Explicit representations of partition functions of gauge theories and topological string theories on corresponding Calabi-Yau manifolds have been found in [5–7]. A matrix model representation of partition functions on general toric manifolds has been found in [8,9]. Matrix models encoding wall-crossing phenomena for a class of toric manifolds without compact four-cycles have been constructed in [10–12]. While partition functions of four-dimensional gauge theories can be encoded in Hermitian matrix models, a generalization to five-dimensional theories, and more generally, topological strings on toric manifolds, amounts to considering unitary matrix models [4–6,8,10]. All these relations gained new interest with the formulation of the general matrix model solution in terms of the topological recursion [13], and the related remodeling conjecture postulated in the context of topological string theory [14]. One might therefore wonder if the relation between matrix models and topological strings, and more generally BPS counting, extends to the refined case as well. We also stress that the world-sheet definition of the refined topological string theory is still not well understood, and the hope that matrix model reformulation might give some hint in this context is also an important motivation for this work.

Yet another motivation to study refinement from the matrix model perspective arises from the AGT conjecture [15]: as proposed in [16], partition functions of four-dimensional, $\mathcal{N} = 2$ theories can be encoded in so-called beta-deformed, Hermitian matrix models. Certain aspects of this statement were tested in [17–27]. In particular, the appearance of the beta-deformed measure from the Nekarsov partition functions has been demonstrated also for both four- and five-dimensional gauge theories and certain topological string theories in [28], however only to the leading order. On the other hand, the formalism of the topological recursion for Hermitian models has been extended to the beta-deformed case [29]. Therefore, one might hope that the refined topological string theories could be encoded in unitary, beta-deformed matrix models. However, as explained and demonstrated explicitly in [30,31], this turns out not to be true even in the simple example of the resolved conifold. Nonetheless, due to deep consequences of the topological recursion [13], finding some matrix model representation of refined partition functions would be quite desirable; such matrix models would presumably arise as some deformation of a certain class of already known unitary matrix models. This is the task we cope with in this paper, not only from the viewpoint of topological string amplitudes, but also more generally in the context of BPS counting and wall-crossing phenomena. The refined matrix models which we find involve matrices of infinite size and have ordinary, unitary measure, while their potentials are modified in a way which encodes the refinement. We stress this is opposite to the beta-deformed models, whose measure is modified; however, potentials are the same in both the refined and nonrefined cases. One immediate advantage of our result is the fact that the topological recursion for models with

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undeformed measure [13,14] is much simpler and tractable than in the beta-deformed case [29], and could be readily applied to gain more insight into properties of refined amplitudes.

We recall that there are various definitions of refinement whose physical equivalence is not quite clear; however, the agreement of the resulting exact solutions is a strong argument for an underlying common, general structure. In all these so-called refined theories a dependence on a single parameter, such as the string coupling g_s or the background $\hbar$ in gauge theories, is replaced by a dependence on two parameters, customarily denoted ϵ_1 and ϵ_2. In the context of gauge theory, refined amplitudes arose from their formulation in the Ω background [32]. In the case of topological strings on noncompact, toric manifolds, refinement was introduced in terms of refined BPS counting, reformulated combinatorially in terms of the refined topological vertex [33,34], and shown to agree with gauge theory results in the Ω background in [35,36]. From the viewpoint of the AGT conjecture, refined amplitudes are encoded in relevant conformal blocks of two-dimensional conformal field theory, and the corresponding beta-deformed matrix models are characterized by the Vandermonde determinant raised to the power $\beta = -\epsilon_1/\epsilon_2$. In the context of wall-crossing and BPS counting in a system of D6-D2-D0–branes on toric manifolds, following and in parallel with nonrefined developments in [37–44], refined amplitudes were considered from physical and mathematical perspectives in [45,46]. Among multitude chambers in which (refined) generating functions of D6-D2-D0 bound states are known, there is a special chamber in which they agree with topological string amplitudes on the same Calabi-Yau manifold, and, in particular, the agreement with the refined topological vertex calculation was shown in [46]. This is this last formulation of the refinement on which our derivation is based.

To find refined matrix models we follow a strategy which extends a nonrefined presentation of [10].[1] First, generalizing the results of [43], we construct a free fermion representation of crystals representing the refined BPS states in question. This allows us to write the refined BPS generating functions $Z_n^{\rm ref}$ in a chamber specified by n as

$$Z_n^{\rm ref} = \langle \Omega_+^{\rm ref} | \bar{W}_n^{\rm ref} | \Omega_-^{\rm ref} \rangle, \tag{1}$$

where $|\Omega_\pm^{\rm ref}\rangle$ are states representing a manifold in question, and $\bar{W}_n^{\rm ref}$ are wall-crossing operators which determine a chamber of interest. Then we turn these fermionic correlators into a unitary matrix model form. The refined character of fermionic correlators results in a modified form of matrix model potentials. Similar to [5,6,10], our potentials have nontrivial string coupling dependence to all orders. While our results are valid in all chambers, in the so-called commutative chamber we obtain a matrix model representation of refined topological string amplitudes.

To briefly exemplify our results, we recall first that the refined topological string amplitude for the resolved conifold with the Kähler parameter Q [or equivalently five-dimensional, $U(1)$ gauge theory] is given by

$$\mathcal{Z}_{\rm top}^{\rm ref} = M(t_1, t_2) \prod_{k,l=0}^{\infty} (1 - Q t_1^{k+1} t_2^l), \tag{2}$$

where $t_1 = e^{-\epsilon_1}$, $t_2 = e^{\epsilon_2}$, and $M(t_1, t_2) = \prod_{k,l=0}^{\infty} (1 - t_1^{k+1} t_2^l)^{-1}$ is the refined MacMahon function. To find a matrix model representation of $\mathcal{Z}_{\rm top}^{\rm ref}$, we first construct a general refined BPS generating function in the form (1), where in the case of the conifold, n is a single integer. We then translate such a fermionic correlator into a matrix model form, and in the $n \to \infty$ limit, which corresponds to the so-called commutative chamber, we find the matrix model representation (written in terms of eigenvalues $z_k = e^{iu_k}$) of the refined topological string amplitude

$$\mathcal{Z}_{\rm top}^{\rm ref} = \int \mathcal{D}U \prod_k \prod_{j=0}^{\infty} \frac{(1 + z_k t_1^{j+1})(1 + t_2^j / z_k)}{(1 + t_2^j Q / z_k)},$$

where $\mathcal{D}U$ is the ordinary unitary measure [see (18)]. To the leading order the above integrand gives rise to the following potential:

$$V(u;\beta) = \tfrac{1}{2} u^2 - (1 - \beta^{-1}) \mathrm{Li}_2(-e^{iu}) - \mathrm{Li}_2(-Q e^{-iu}) + \mathcal{O}(g_s, \beta). \tag{3}$$

In what follows we also present matrix models associated with other chambers of the Kähler moduli space. We can also immediately note that in the limit $Q \to 0$, the above result reduces to a matrix model representation of the refined MacMahon function, with the exact integrand given by a deformed theta function, which in the genus expansion gives a β deformation of the Gaussian potential of the Chern-Simons matrix model [such that both the dilogarithm term and the $\mathcal{O}(g_s, \beta)$ corrections vanish for $\beta = 1$]. In the main text we discuss in more detail other explicit results for $\mathbb{C}^3$, the conifold, or the resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity. Similar to [12] we postulate a relation of those refined matrix integrands to open BPS amplitudes.

The paper is organized as follows. In Sec. II we recall definitions and basic properties of refined BPS invariants and introduce relevant notation. In Sec. III we extend the formalism of [43] to the refined case and present a fermionic representation of refined generating functions. In Sec. IV we turn these refined fermionic results into matrix models and describe their properties. Section V contains a discussion.

[1]Our results were obtained independently and before an overlapping work [47,48] appeared.

II. REFINED WALL-CROSSING IN THE D6-D2-D0 SYSTEM

Refined degeneracies of D2- and D0-branes bound to a D6-brane on a Calabi-Yau manifold X can be encoded in a generating function

$$Z_n^{\mathrm{ref}}(q,Q)=\sum_{\alpha,\gamma}\Omega^{\mathrm{ref}}_{\alpha,\gamma}(n;y)q^{\alpha}Q^{\gamma},$$

with the D0-brane charge represented by $\alpha\in\mathbb{Z}$, the D2-brane charge represented by $\gamma\in H_2(X,\mathbb{Z})$, and a chamber in the Kähler moduli space specified by (possibly a set of parameters) n. Let $\mathcal{H}_{\alpha,\gamma}(n)$ denote a space of BPS states with given charges α, γ and asymptotic values of moduli corresponding to a chamber n; J_3 denotes a generator of the spatial rotation group. For fixed charges α, γ and a choice of chamber n, refined degeneracies

$$\Omega^{\mathrm{ref}}_{\alpha,\gamma}(n;y)=\mathrm{Tr}_{\mathcal{H}_{\alpha,\gamma}(n)}(-y)^{2J_3} \tag{4}$$

are interesting invariants if X does not possess complex structure deformations, which is the case for noncompact, toric manifolds which we consider in this paper. These invariants were argued in [45] to agree with motivic Donaldson-Thomas invariants of [49], and in the case of the resolved conifold, the corresponding BPS generating functions were derived using the refined wall-crossing formula, and encoded in a refined crystal model. From a mathematical viewpoint, and in terms of dimer models, such an analysis was extended to quite a general class of toric manifolds without compact four-cycles in [46], and shown therein to agree, in the commutative chamber, with refined topological vertex computations. For $y=1$ all these invariants reduce to ordinary nonrefined invariants, whose generating functions were encoded in dimer or crystal models in [39–41], and represented in the free fermion formalism in [43,44]. In the next section, based on definitions of BPS generating functions in terms of dimers or crystals constructed in [45,46], we will extend such a free fermion formalism to the refined models.

Before proceeding we present in more detail a class of manifolds we are interested in. Similar to [10,43], we consider toric, noncompact Calabi-Yau manifolds without compact four-cycles, whose toric diagrams arise from a triangulation of a strip. Such diagrams consist of $N+1$ vertices, and there are N $\mathbb{P}^1$'s in the geometry, with Kähler parameters denoted $Q_p=e^{-T_p}$, $p=1,\ldots,N$. To each vertex in the diagram we associate a type $\tau_i=\pm1$. If the local neighborhood of $\mathbb{P}^1$, represented by an interval between vertices i and $i+1$, is $\mathcal{O}(-2)\oplus\mathcal{O}$, then $\tau_{i+1}=\tau_i$; if this neighborhood is of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ type, then $\tau_{i+1}=-\tau_i$. We choose the type of the first vertex as $\tau_1=+1$.

We also need to introduce relevant notation for refined quantities. In the nonrefined case the string coupling g_s is related to the D0-brane charge as $q=e^{-g_s}$. The refinement is encoded in an additional parameter β. Instead of g_s and β, it is more convenient to use a pair of parameters

$$\epsilon_1=\sqrt{\beta}g_s,\qquad \epsilon_2=-\frac{g_s}{\sqrt{\beta}},$$

so that $\beta=-\frac{\epsilon_1}{\epsilon_2}$, $\epsilon_1\epsilon_2=-g_s^2$. We often use the exponentiated counterparts

$$t_1=e^{-\epsilon_1},\qquad t_2=e^{\epsilon_2},$$

and also introduce

$$g_sB=\epsilon_1+\epsilon_2=g_s\Big(\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\Big).$$

The variable y in (4) can be expressed as $y=t_1/q=q/t_2$, so that $y^2=t_1/t_2=q^B$. In this notation the nonrefined limit $y=1$ corresponds to $\beta=1$, for which $\epsilon_1=-\epsilon_2=g_s$ and $t_1=t_2=q$ and $B=0$.

With the above notation we can present some explicit BPS generating functions whose matrix model representation we are going to find. The simplest manifold one can consider is $\mathbb{C}^3$, for which one gets the refined MacMahon function [33] (see Fig. 1),

$$Z^{\mathbb{C}^3}=M(t_1,t_2)=\prod_{k,l=0}^{\infty}\frac{1}{1-t_1^{k+1}t_2^{l}}. \tag{5}$$

In this case there is no Kähler parameter, and therefore there are no interesting wall-crossing phenomena.

We note that one could consider a more general family of refinements parametrized by δ, such that $M_\delta(t_1,t_2)=\prod_{k,l=0}^{\infty}(1-t_1^{k+1+((\delta-1)/2)}t_2^{l-((\delta-1)/2)})^{-1}$. For simplicity, in what follows we choose the value $\delta=1$ (note that in [45] another choice $\delta=0$ was made).

The resolved conifold provides a basic nontrivial example of wall-crossing, with a set of chambers parametrized by an integer n (in the refined case one might also consider additional *invisible* walls, which we do not discuss here). Corresponding refined generating functions were computed in [45] using a refined wall-crossing formula, and in the chamber labeled by $n-1$, they read

$$Z^{\mathrm{conifold}}_{n-1}=M(t_1,t_2)^2\Big(\prod_{k,l=0}^{\infty}(1-Qt_1^{k+1}t_2^{l})\Big)\times\Big(\prod_{k\geq1,l\geq0,k+l\geq n}(1-Q^{-1}t_1^{k}t_2^{l})\Big). \tag{6}$$

In the commutative chamber $n\to\infty$ the terms in the last set of brackets do not contribute anymore and the BPS generating function is simply related to the refined topological string amplitude given in (2),

$$Z^{\mathrm{conifold}}_{\infty}=M(t_1,t_2)Z^{\mathrm{ref}}_{\mathrm{top}}.$$

On the other hand, in the noncommutative chamber $n=0$, the refined generating function is given by the modulus square of the refined topological string amplitude.

For a resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity there is also a discrete set of chambers parametrized by an integer n, and the corresponding BPS generating functions read

$$Z_{n-1}^{\mathbb{C}^3/\mathbb{Z}_2} = M(t_1, t_2)^2 \left(\prod_{k,l=0}^{\infty} (1 - Q t_1^{k+1} t_2^l)^{-1} \right) \times \left(\prod_{k\geq 1, l\geq 0, k+l\geq n} (1 - Q^{-1} t_1^k t_2^l)^{-1} \right). \quad (7)$$

It is harder to write down generating functions for an arbitrary chamber of an arbitrary geometry of our interest. However, this can be done for the noncommutative chamber of arbitrary geometry, where—similar to the nonrefined case—the BPS generating function is given by the modulus square of the refined topological string amplitude,

$$Z_0^{\mathrm{ref}} = |Z_{\mathrm{top}}^{\mathrm{ref}}|^2 \equiv Z_{\mathrm{top}}^{\mathrm{ref}}(Q_i) Z_{\mathrm{top}}^{\mathrm{ref}}(Q_i^{-1}). \quad (8)$$

The (instanton part of the) refined topological string amplitude is given by [33,35]

$$Z_{\mathrm{top}}^{\mathrm{ref}}(Q_i) = M(t_1, t_2)^{((N+1)/2)} \prod_{k,l=0}^{\infty} \prod_{1\leq i<j\leq N+1} \times (1 - (Q_i Q_{i+1} \cdots Q_{j-1}) t_1^{k+1} t_2^l)^{-\tau_i \tau_j}, \quad (9)$$

with the notation introduced above.

III. REFINED WALL-CROSSING AND FREE FERMIONS

The problem of counting bound states of D6-D2-D0-branes for local toric Calabi-Yau manifolds without compact four-cycles has been formulated in the free fermion formalism in [43,44]. Among many advantages of such a representation is its immediate relation to melting crystals, as well as to matrix models, which was exploited in [10,12]. Here we wish to extend such a free fermion formalism to capture refined BPS invariants, as defined in [45,46].

We consider first statistical models of colored pyramids. In the nonrefined case [43], to a geometry consisting of N $\mathbb{P}^1$'s, one associates a crystal which is sliced into layers in $N+1$ colors, denoted $q_0, q_1, q_2, \ldots, q_N$. In the nonrefined case, parameters $q_1, \ldots, q_N$ encode Kähler parameters of the geometry $Q_1, \ldots, Q_N$, while the product $\prod_{i=0}^N q_i$ is mapped to (possibly the inverse of) $q = e^{-g_s}$. In the refined case the assignment of colors is more subtle, as it must take into account a refinement of a single parameter q into t_1 and t_2 introduced above. In particular, in the noncommutative chamber $q_{i\neq 0}$ are mapped (up to a sign, as in the nonrefined case) to Q_i; however, we will have to replace q_0 by two refined colors $q_0^{(1)}$ or $q_0^{(2)}$, so that $t_i = q_0^{(i)} q_1 \cdots q_N$, for $i = 1, 2$. The simplest case of $\mathbb{C}^3$ refined plane partitions, discussed also in [33], is shown in Fig. 1. For other manifolds, in other chambers we will find a more complicated assignment of colors.

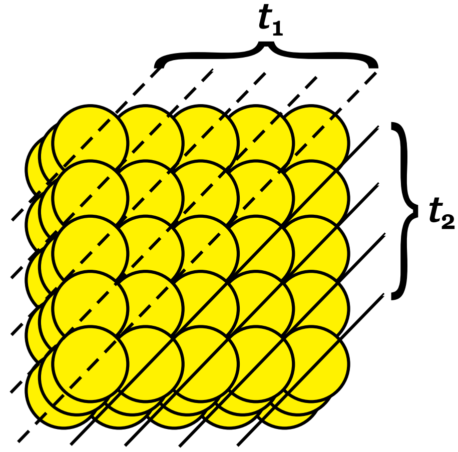

FIG. 1 (color online). Refined plane partitions which count D6-D0 bound states in $\mathbb{C}^3$, as seen from the bottom (i.e. a negative direction of the z axis). Stones in each layer which intersects a dashed or solid line have weight t_1 or t_2, respectively. The resulting generating function is the refined MacMahon function $M(t_1, t_2)$.

In [43] the structure and coloring of a given crystal, corresponding to a particular toric geometry, was encoded in fermionic states $|\Omega_\pm\rangle$, so that the generating function of BPS invariants could be written as a superposition of two such states (with additional insertion of wall-crossing operators in nontrivial chambers). In this section we construct refined states $|\Omega_\pm^{\mathrm{ref}}\rangle$ with similar properties. In the noncommutative chamber the states which we construct are such that

$$Z_0^{\mathrm{ref}} = \langle \Omega_+^{\mathrm{ref}} | \Omega_-^{\mathrm{ref}} \rangle. \quad (10)$$

We also construct a refined version of wall-crossing operators $\bar{W}_n^{\mathrm{ref}}$, such that the BPS generating function in the nth chamber can be written as

$$Z_n^{\mathrm{ref}} = \langle \Omega_+^{\mathrm{ref}} | \bar{W}_n^{\mathrm{ref}} | \Omega_-^{\mathrm{ref}} \rangle. \quad (11)$$

In Sec. III A below we construct states $|\Omega_\pm^{\mathrm{ref}}\rangle$ for an arbitrary manifold in a class of our interest. In Sec. III B we construct states $|\Omega_\pm^{\mathrm{ref}}\rangle$ and wall-crossing operators $\bar{W}_n^{\mathrm{ref}}$ for all chambers of the resolved conifold and a resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity. We follow the conventions used in [10,12,43], which are summarized, for convenience, in the Appendix.

A. Arbitrary geometry—noncommutative chamber

In this section we construct fermionic states $|\Omega_\pm^{\mathrm{ref}}\rangle$, which allow us to write the BPS generating functions in the noncommutative chamber as claimed in (10). Similar to the nonrefined case, the states $|\Omega_\pm^{\mathrm{ref}}\rangle$ are constructed from an interlacing pattern of vertex $\Gamma_\pm^{\tau_i}$ and weight operators. As the refinement does not modify the three-dimensional shape of the corresponding crystal, the assignment of vertex operators is the same as in the nonrefined case

[43] and can be similarly read off from the toric diagram. In particular, to the ith vertex in the toric diagram (of type τ_i given above) we associate a vertex operator $\Gamma_{\pm}^{\tau_i}(x)$, such that

$$\Gamma_{\pm}^{\tau_i=+1}(x) = \Gamma_{\pm}(x), \qquad \Gamma_{\pm}^{\tau_i=-1}(x) = \Gamma'_{\pm}(x).$$

Examples of this assignment for $\mathbb{C}^3$, the conifold, and a resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity are shown in Fig. 2.

The structure which is modified in the refined case is the assignment of colors, which are encoded in the weight operators. A product of $N+1$ such operators $\Gamma_{\pm}^{\tau_i}(x)$ is interlaced with weight operators in the following way. We introduce N operators $\hat{Q}_i$ representing colors q_i, for $i = 1, \ldots, N$, and, in addition, two other colors $q_0^{(1)}$ and $q_0^{(2)}$, which are eigenvalues of $\hat{Q}_0^{(1)}$ and $\hat{Q}_0^{(2)}$. Operators $\hat{Q}_1, \ldots, \hat{Q}_N$ are associated with $\mathbb{P}^1$ in the toric diagram, and we define

$$\hat{Q}^{(i)} = \hat{Q}_0^{(i)}\hat{Q}_1 \cdots \hat{Q}_N, \qquad t_i = q_0^{(i)} q_1 \cdots q_N, \qquad \text{for } i = 1, 2. \tag{12}$$

Now we introduce

$$\bar{A}_+(x) = \Gamma_+^{\tau_1}(x)\hat{Q}_1\Gamma_+^{\tau_2}(x)\hat{Q}_2 \cdots \Gamma_+^{\tau_N}(x)\hat{Q}_N\Gamma_+^{\tau_{N+1}}(x)\hat{Q}_0^{(1)},$$
$$\bar{A}_-(x) = \Gamma_-^{\tau_1}(x)\hat{Q}_1\Gamma_-^{\tau_2}(x)\hat{Q}_2 \cdots \Gamma_-^{\tau_N}(x)\hat{Q}_N\Gamma_-^{\tau_{N+1}}(x)\hat{Q}_0^{(2)}.$$

Commuting all $\hat{Q}_i$'s to the left or right, we also introduce

$$\begin{aligned} A_+(x) &= (\hat{Q}^{(1)})^{-1}\bar{A}_+(x) \\ &= \Gamma_+^{\tau_1}(xt_1)\Gamma_+^{\tau_2}\left(\frac{xt_1}{q_1}\right)\Gamma_+^{\tau_3}\left(\frac{xt_1}{q_1q_2}\right)\cdots\Gamma_+^{\tau_{N+1}}\left(\frac{xt_1}{q_1q_2\cdots q_N}\right), \\ A_-(x) &= \bar{A}_-(x)(\hat{Q}^{(2)})^{-1} \\ &= \Gamma_-^{\tau_1}(x)\Gamma_-^{\tau_2}(xq_1)\Gamma_-^{\tau_3}(xq_1q_2)\cdots\Gamma_-^{\tau_{N+1}}(xq_1q_2\ldots q_N). \end{aligned}$$

When the argument of any of these operators is $x = 1$, we often use a simplified notation

$$\bar{A}_\pm \equiv \bar{A}_\pm(1), \qquad A_\pm \equiv A_\pm(1).$$

Finally, we can associate to a given toric manifold two states

$$\begin{aligned} \langle\Omega_+^{\rm ref}| &= \langle 0|\ldots\bar{A}_+(1)\bar{A}_+(1)\bar{A}_+(1) \\ &= \langle 0|\ldots A_+(t_1^2)A_+(t_1)A_+(1), \\ |\Omega_-^{\rm ref}\rangle &= \bar{A}_-(1)\bar{A}_-(1)\bar{A}_-(1)\ldots|0\rangle \\ &= A_-(1)A_-(t_2)A_-(t_2^2)\ldots|0\rangle, \end{aligned}$$

where $|0\rangle$ is the fermionic Fock vacuum.

Our first claim is that the refined BPS generating function can be written as

$$Z_0^{\rm ref} = \langle\Omega_+^{\rm ref}|\Omega_-^{\rm ref}\rangle \equiv Z_{\rm top}(Q_i)Z_{\rm top}(Q_i^{-1}), \tag{13}$$

with $Z_{\rm top}(Q_i)$ given in (9), and under the following identification between q_i parameters which enter a definition of $|\Omega_{\pm}^{\rm ref}\rangle$ and string parameters $Q_i = e^{-T_i}$ (for $i = 1, \ldots, N$),

$$q_i = (\tau_i\tau_{i+1})Q_i,$$

and with refined parameters $t_{1,2}$ identified as in (12). This result, in the special case of $\mathbb{C}^3$, conifold, and $\mathbb{C}^3/\mathbb{Z}_2$ geometries, reproduces formulas (5)–(7).

To prove (13) for general geometry, we first note that commuting operators $A_+(x)$ with $A_-(y)$,

$$A_+(x)A_-(y) = A_-(y)A_+(x)C(x, y),$$

gives rise to a factor

$$\begin{aligned} C(x, y) = \frac{1}{(1-t_1xy)^{N+1}} &\prod_{1\le i<j\le N+1} \\ &\times\Big((1-(\tau_i\tau_j)xyt_1(q_iq_{i+1}\ldots q_{j-1})) \\ &\times\left(1-\frac{(\tau_i\tau_j)xyt_1}{q_iq_{i+1}\ldots q_{j-1}}\right)\Big)^{-\tau_i\tau_j}. \end{aligned}$$

Now we write the states $|\Omega_{\pm}^{\rm ref}\rangle$ in terms of $A_\pm$ operators, and commute $\Gamma_\pm$ within each pair of A_+ and A_- separately,

$$\begin{aligned} Z_0^{\rm ref} &= \langle\Omega_+^{\rm ref}|\Omega_-^{\rm ref}\rangle = \langle 0|\left(\prod_{i=0}^{\infty}A_+(t_1^i)\right)\left(\prod_{j=0}^{\infty}A_-(t_2^j)\right)|0\rangle \\ &= \prod_{i,j=0}^{\infty}C(t_1^i, t_2^j). \end{aligned}$$

This last product reproduces the modulus square of the refined topological string partition function in (13) and therefore proves the claim (10).

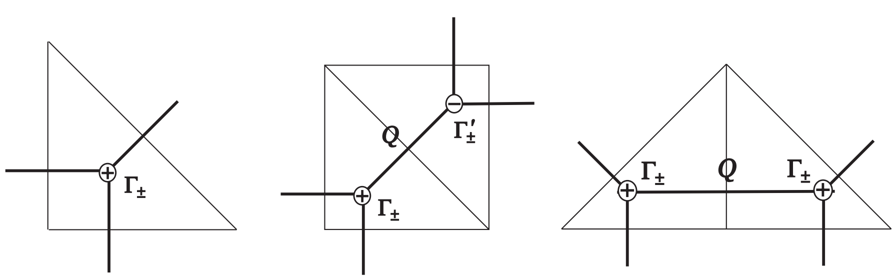

FIG. 2. Toric diagrams and assignment of vertex operators in the case of $\mathbb{C}^3$ (left diagram), the conifold (middle diagram), and a resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity (right diagram). A sign ⊕ or ⊖ in each vertex denotes a corresponding type $\tau_i = \pm 1$.

B. Conifold and $\mathbb{C}^3/\mathbb{Z}_2$—all chambers

A fermionic representation can also be extended to nontrivial chambers. Even though this can be done for general geometry without compact four-cycles, for simplicity we restrict our considerations to the case of a conifold and a resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity, which involve just one Kähler parameter $Q_1 \equiv Q$. In those cases, in a chamber labeled by $n-1$, we find the following representation of the BPS generating function,

$$Z^{\mathrm{ref}}_{n-1} = \langle \Omega^{\mathrm{ref}}_+ | \bar{W}^{\mathrm{ref}}_{n-1} | \Omega^{\mathrm{ref}}_- \rangle, \tag{14}$$

where $\bar{W}^{\mathrm{ref}}_{n-1}$ represents the appropriate wall-crossing operator. In both these cases the toric diagram has two vertices, the first one of type $\tau_1 = 1$ and the second one now denoted $\tau \equiv \tau_2$ and $\tau = \mp 1$, respectively, for the conifold and $\mathbb{C}^3/\mathbb{Z}_2$. A crystal associated with the expression (14) has n stones in the top row and can be sliced into interlacing single-colored layers. The assignment of colors is analogous to the pyramid model discussed in [45,46] (however, our convention is slightly different, and corresponds to integer and nonsymmetric, rather than half-integer and symmetric, powers of $t_{1,2}$ in [45]). The pyramid crystal for the conifold is shown in Fig. 3. The coloring and weights for $\mathbb{C}^3/\mathbb{Z}_2$ are the same as for the conifold, even though the plane-partition shape of the $\mathbb{C}^3/\mathbb{Z}_2$ crystal is different than (though very analogous to) the pyramidlike conifold crystal; see Fig. 4. Using the notation introduced above, the assignment of colors is determined as follows.

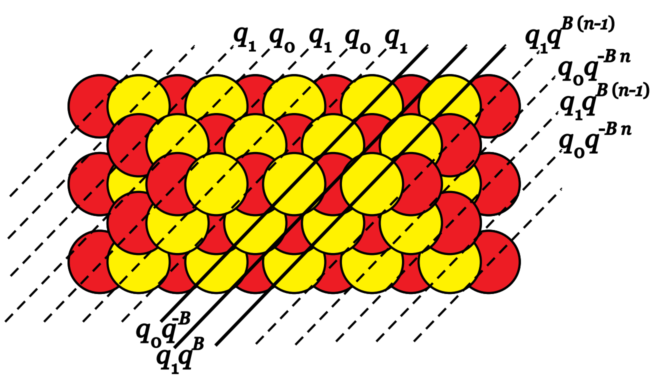

FIG. 3 (color online). Refined pyramid crystal for the conifold, in the chamber corresponding to n stones in the top row. Along each slice (as indicated by broken or solid lines) all stones have the same color, assigned as follows. On the left side (along broken lines), each light (yellow) and dark (red) slice has color denoted q_0 and q_1, respectively. Moving to the right, in the intermediate region (along solid lines), a color of each new light or dark slice is modified by, respectively, a $q^{\mp B}$ factor (with respect to the previous light or dark slice). On the right side (again along broken lines), each light or dark slice has again the same color, respectively, $q_0 q^{-Bn}$ or $q_1 q^{B(n-1)}$. The assignment of colors in the intermediate region (along solid lines) interpolates between constant assignments on the left and right sides of the pyramid.

All stones on one side of the crystal are encoded in the state

$$\langle \Omega^{\mathrm{ref}}_+ | = \langle 0 | \dots (\Gamma_+(1)\hat{Q}_1 \Gamma^{\tau}_+(1)\hat{Q}_0)(\Gamma_+(1)\hat{Q}_1 \Gamma^{\tau}_+(1)\hat{Q}_0).$$

The Kähler parameter Q, as well as the parameter t_1, are determined as

$$q_1 = \tau Q t_1^{1-n}, \qquad q_0 = \tau \frac{t_1^n}{Q}, \qquad \text{so that} \quad q_0 q_1 = t_1.$$

Then the extended crystal, which has $n-1$ additional stones in the top row, is constructed by an insertion of the operator

$$\begin{aligned} \bar{W}^{\mathrm{ref}}_{n-1} = &\Big(\Gamma_-(1)\hat{Q}_1 \Gamma^{\tau}_+(1)\hat{Q}_0 \widehat{q^{-B}}\Big) \\ &\times \Big(\Gamma_-(1)\hat{Q}_1 \widehat{q^{B}} \Gamma^{\tau}_+(1)\hat{Q}_0 \widehat{q^{-2B}}\Big) \cdots\cdots \\ &\times \Big(\Gamma_-(1)\hat{Q}_1 \widehat{q^{(n-2)B}} \Gamma^{\tau}_+(1)\hat{Q}_0 \widehat{q^{(1-n)B}}\Big). \end{aligned}$$

This operator consists of $n-1$ terms of the form $(\Gamma_-(1)\hat{Q}_1 \widehat{q^{iB}} \Gamma^{\tau}_+(1)\hat{Q}_0 \widehat{q^{-(i+1)B}})$ for $i = 0, \dots, n-2$, where in each consecutive dark or light slice of stones we insert one additional operator $\widehat{q^{\pm B}}$, which changes the weight of each stone in this slice by $q^{\pm B} = (t_1/t_2)^{\pm 1}$ (with respect to the previous slice of the same light or dark color).

Finally, all stones on the right side of the crystal have, again, the same light or dark color, and the corresponding state reads

$$\begin{aligned} |\Omega^{\mathrm{ref}}_-\rangle = &\Big(\Gamma_-(1)\hat{Q}_1 \widehat{q^{(n-1)B}} \Gamma^{\tau}_-(1)\hat{Q}_0 \widehat{q^{-nB}}\Big) \\ &\times \Big(\Gamma_-(1)\hat{Q}_1 \widehat{q^{(n-1)B}} \Gamma^{\tau}_-(1)\hat{Q}_0 \widehat{q^{-nB}}\Big) \dots |0\rangle. \end{aligned}$$

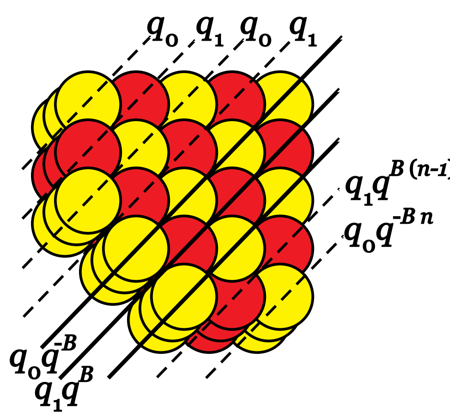

FIG. 4 (color online). Refined pyramid crystal for the resolution of $\mathbb{C}^3/\mathbb{Z}_2$ singularity, in the chamber corresponding to n stones in the top row, as seen from the bottom (i.e. a negative direction of the z axis). Even though the three-dimensional shape of the crystal is different than in the conifold case, the assignment of colors is the same; see Fig. 3.

Therefore, the varying weights in the middle range (along solid lines in Figs. 3 and 4) interpolate between fixed weights of light and dark stones on two external sides of a crystal.

We can now commute away all weight operators in the above expressions, using relations from the Appendix. This leads to the representation

$$Z_{n-1}^{\mathrm{ref}} = \langle 0| \Big(\prod_{k=1}^{\infty} \Gamma_+(t_1^k) \Gamma_+^{\tau}(t_1^k/q_1) \Big) \Big(\prod_{i=0}^{n-2} \Gamma_-(t_2^i) \Gamma_+^{\tau}(q_1^{-1} t_1^{-i}) \Big) \times \Big(\prod_{k=0}^{\infty} \Gamma_-(t_2^{n-1+k}) \Gamma_-^{\tau}(tQt_2^k) \Big) |0\rangle.$$

Finally, commuting all vertex operators, we find

$$Z_{n-1}^{\mathrm{ref}} = M(t_1, t_2)^2 \prod_{k=1,l=0}^{\infty} (1 - Q t_1^k t_2^l)^{-\tau} \prod_{k\geq 1, l\geq 0, k+l\geq n}^{\infty} \times (1 - Q^{-1} t_1^k t_2^l)^{-\tau}, \tag{15}$$

where $\tau = \mp 1$, respectively, for the conifold and $\mathbb{C}^3/\mathbb{Z}_2$. This indeed reproduces (6) and (7), and agrees (up to the half-integer convention of $t_{1,2}$) with the results of [45].

IV. MATRIX MODELS FROM FERMIONS

Once the generating function of Donaldson-Thomas invariants is written in the fermionic formalism, it can be turned into the matrix model form upon inserting an appropriately chosen identity operator in the correlator (14),

$$Z_{n-1}^{\mathrm{ref}} = \langle \Omega_+^{\mathrm{ref}} | \mathbb{I} \bar{W}_{n-1}^{\mathrm{ref}} | \Omega_-^{\mathrm{ref}} \rangle. \tag{16}$$

The identity operator $\mathbb{I}$ is represented by the complete set of states $|R\rangle\langle R|$ (representing two-dimensional partitions). We can use orthogonality relations of $U(\infty)$ characters χ_R, and the representation of these characters in terms of Schur functions $\chi_R = s_R(\vec{z})$ for $\vec{z} = (z_1, z_2, z_3, \ldots)$, to write

$$\begin{aligned} \mathbb{I} = \sum_R |R\rangle\langle R| &= \sum_{P,R} \delta_{P^t R^t} |P\rangle\langle R| \\ &= \int \mathcal{D}U \sum_{P,R} s_{P^t}(\vec{z}) \overline{s_{R^t}(\vec{z})} |P\rangle\langle R| \\ &= \int \mathcal{D}U \Big(\prod_k \Gamma'_-(z_k)|0\rangle \Big) \Big(\langle 0| \prod_k \Gamma'_+(z_k^{-1}) \Big), \end{aligned} \tag{17}$$

where $\mathcal{D}U$ denotes the unitary measure written in terms of eigenvalues,

$$\mathcal{D}U = \prod_k du_k \prod_{k<j} |z_k - z_j|^2, \qquad z_k = e^{iu_k}. \tag{18}$$

The identity operator in the above form can be inserted into (16), which results in an expression involving only vertex operators $\Gamma_{\pm}^{(\pm 1)}$. Then we can commute vertex operators away, again using relations from the Appendix, which leads to a matrix model with the unitary measure $\mathcal{D}U$. In the noncommutative chamber all factors arising from commuting these $\Gamma_{\pm}^{(\pm 1)}$ operators depend on z_k and contribute just to the matrix model potentials. In other chambers additional factors may arise which are independent of z_k, and which, in a chamber labeled by n, contribute to some overall factor f_n. Thus, in general, we write the Donaldson-Thomas generating function as a matrix model in the form

$$Z_n^{\mathrm{ref}} = f_n \int \mathcal{D}U \prod_k e^{-(\sqrt{\beta}/g_s) V(z_k;\beta)}, \tag{19}$$

and it is convenient to introduce a factor $\sqrt{\beta}$ in front of the potential $V(z;\beta)$, or work with a rescaled coupling $g_s \beta^{-1/2}$.

In (16) the identity operator has been inserted in a specific location. In fact, there is large freedom of where this insertion should be chosen, which leads to various forms of a matrix integrand. In [12] it has been shown that those various integrands can be identified with open BPS generating functions in various open chambers. We will also comment on a possible similar interpretation of refined integrands in what follows. However, let us first restrict ourselves to a specific choice (16) and discuss resulting matrix models. We use the following notation for a deformation of a theta function,

$$\Theta(z; t_1, t_2) = \prod_{j=0}^{\infty} (1 + z t_1^{j+1})(1 + t_2^j/z),$$

to express certain integrands of matrix models that we come across.

A. Arbitrary geometry—noncommutative chamber

As the first explicit example, we find a matrix model representation of the refined BPS generating function in the noncommutative chamber. We start with the expression (16) with no $\bar{W}_{n-1}^{\mathrm{ref}}$ insertion, and use the form of $|\Omega_{\pm}^{ref}\rangle$ derived in Sec. III A. Performing the computation described above we get, in the noncommutative chamber for general geometry, the following matrix model:

$$Z_0^{\mathrm{ref}} = \int \mathcal{D}U \prod_k \prod_{l=0}^{N} \Theta\Big(\frac{\tau_{l+1} z_k}{q_1 \cdots q_l}; t_1, t_2 \Big)^{\tau_{l+1}};$$

i.e. we identify $e^{-\frac{\sqrt{\beta}}{g_s} V(z;\beta)} \equiv \prod_{l=0}^{N} \Theta(\tau_{l+1} z (q_1 \cdots q_l)^{-1}; t_1, t_2)^{\tau_{l+1}}$. The product over l runs over all vertices, and in this chamber we identify Kähler parameters Q_p with weights q_p via $q_p = (\tau_p \tau_{p+1}) Q_p$.

Some special cases of the above result include the following:

(i) For $\mathbb{C}^3$ the generating function $Z^{ref} = M(t_1, t_2)$ is given by the refined MacMahon function (5), and we find that the corresponding potential is a refined theta function,

$$e^{-(\sqrt{\beta}/g_s)V(z;\beta)} = \prod_{j=0}^{\infty}(1+zt_1^{j+1})(1+t_2^j/z) = \Theta(z;t_1,t_2). \quad (20)$$

(ii) For the conifold the noncommutative generating function $Z_0^{\rm conifold}$ determined from (6) gives rise to a matrix model with the following potential term,

$$e^{-(\sqrt{\beta}/g_s)V(z;\beta)} = \prod_{j=0}^{\infty}\frac{(1+zt_1^{j+1})(1+t_2^j/z)}{(1+zt_1^{j+1}/Q)(1+\frac{Qt_2^j}{z})} = \frac{\Theta(z;t_1,t_2)}{\Theta(z/Q;t_1,t_2)}.$$

(iii) For $\mathbb{C}^3/\mathbb{Z}_2$ the noncommutative generating function $Z_0^{\mathbb{C}^3/\mathbb{Z}_2}$ determined from (7) gives rise to a matrix model with the following potential term,

$$e^{-(\sqrt{\beta}/g_s)V(z;\beta)} = \prod_{j=0}^{\infty}(1+zt_1^{j+1})(1+t_2^j/z) \times(1+zt_1^{j+1}/Q)\left(1+\frac{Qt_2^j}{z}\right) = \Theta(z;t_1,t_2)\Theta(z/Q;t_1,t_2).$$

B. $\mathbb{C}^3$ matrix model

Let us consider the simplest refined matrix model, corresponding to $\mathbb{C}^3$ geometry, with the exact potential given in (20). Similar to [5,50], one might expect that its behavior is governed by the leading order term in the potential. Using the asymptotics

$$\log\prod_{i=1}^{\infty}(1-zq^i) = -\frac{1}{g_s}\sum_{m=0}^{\infty}\mathrm{Li}_{2-m}(z)\frac{B_m g_s^m}{m!}$$

this leading behavior reads

$$e^{-(\sqrt{\beta}/g_s)V(z;\beta)} = e^{-(\sqrt{\beta}/g_s)[-(1/2)(\log z)^2-(1-\beta^{-1})\mathrm{Li}_2(-z)+\mathcal{O}(g_s,\beta)]}. \quad (21)$$

The first, quadratic term in the potential is the same as in the nonrefined case. The terms involving $\mathrm{Li}_2(-z)$, as well as all higher order terms $\mathcal{O}(g_s,\beta)$, vanish for $\beta=1$. Therefore, for $\beta=1$, we obtain a Chern-Simons matrix model which indeed is known to give rise to the MacMahon function in the $N\to\infty$ limit [4,10]. For general β, a resolvent $\omega(p)$ for a unitary model with the above potential can be found using the Migdal integral, as discussed in detail in [10,51]. This requires bringing the measure into a Hermitian Vandermonde form, which introduces an additional $T\log z$ term in the matrix potential, with the 't Hooft parameter $T=(g_s\beta^{-1/2})N$. For the lowest order terms of the potential arising from (21), this leads to

$$\partial_z V(z;\beta) = \frac{T-\log z-(1-\beta^{-1})\log(z+1)}{z}. \quad (22)$$

Assuming a one-cut solution of the matrix model, and in terms of the rescaled coupling $g_s\beta^{-1/2}$, the Migdal resolvent is then given by[2]

$$\omega(p) = \frac{1}{2T}\oint\frac{dz}{2\pi i}\frac{\partial_z V(z)}{p-z}\frac{\sqrt{(p-a_-)(p-a_+)}}{\sqrt{(z-a_-)(z-a_+)}},$$

so that the endpoints of the cuts a_- and a_+ are encircled counterclockwise by the integration contour. Moreover, one has to impose the following consistency condition on the resolvent,

$$\lim_{p\to\infty}\omega(p) = \frac{1}{p},$$

which imposes certain conditions on the endpoints of the cut $a_\pm$. We find that for the potential (22) these conditions take the form

$$\frac{2}{\sqrt{a_-}+\sqrt{a_+}}\left(\frac{2}{\sqrt{a_-+1}+\sqrt{a_++1}}\right)^{(1-\beta^{-1})} = e^{T/2}, \quad (23)$$

$$\frac{\sqrt{a_-}+\sqrt{a_+}}{\sqrt{a_-a_+}}\left(\frac{\sqrt{a_-}+\sqrt{a_+}}{\sqrt{(a_-+1)a_+}+\sqrt{(a_++1)a_-}}\right)^{(1-\beta^{-1})} = 2e^{-T/2}. \quad (24)$$

For the nonrefined case $\beta=1$ these equations simplify and can be exactly solved [10]. For arbitrary β the cut endpoints found in [10] get corrections in $(1-\beta^{-1})$,

$$a_\pm = -1+2e^{-T}\pm 2ie^{-T/2}\sqrt{1-e^{-T}}+\mathcal{O}(1-\beta^{-1}),$$

which leads to a β-deformed spectral curve. To find these corrections $\mathcal{O}(1-\beta^{-1})$ in the exact form does not appear to be easy, and it would be interesting to compare the resulting curve with the quantum curve of the beta-deformed formalism of [29]. In particular, they both give rise to the same result in the four-dimensional limit [31], so understanding a discrepancy of the five-dimensional results is an important issue. It would also be interesting to find the partition function for the above model with the

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[2] A useful result [10] in such computations is $\frac{1}{2T}\oint\frac{dz}{2\pi i}\frac{\log(z+c)}{z(p-z)}\times\frac{\sqrt{(p-a)(p-b)}}{\sqrt{(z-a)(z-b)}} = -\frac{1}{2pT}\log\left(\frac{\sqrt{(a+c)(b-p)}-\sqrt{(b+c)(a-p)}}{(p+c)(\sqrt{b-p}-\sqrt{a-p})}\right)^2 - \frac{\sqrt{(p-a)(p-b)}}{2pT\sqrt{ab}}\log\left(\frac{\sqrt{(a+c)b}-\sqrt{(b+c)a}}{c(\sqrt{a}-\sqrt{b})}\right)^2$. This arises from contour integrals around poles $z=0$ and $z=p$, as well as along the branch cut of the logarithm $(-\infty,-c)$ which is found using $\int\frac{dx}{(x-p)\sqrt{(x-a)(x-b)}} = -\frac{1}{\sqrt{(p-a)(p-b)}}\log\frac{(\sqrt{(x-a)(b-p)}-\sqrt{(x-b)(a-p)})^2}{(p-x)\sqrt{(p-a)(p-b)}}$.

finite 't Hooft coupling T, and verify if it is related to the refined conifold topological string amplitude, as is indeed the case in the nonrefined case.

As already mentioned before, we can also obtain more general matrix models by inserting the identity operator in various places in the fermionic representation of the BPS function. In particular, inserting it at position k in a string of $\bar{A}_-$ operators in (13) in $\mathbb{C}^3$ case, we get the following representation,

$$Z_k^{\rm ref} = f_k \int \mathcal{D}U \prod_{j=0}^{\infty}(1 + z t_1^{j+1})(1 + t_1^k t_2^j/z),$$

with the prefactor $f_k = M(t_1, t_2)\prod_{j=k}^{\infty}\prod_{i=0}^{\infty}(1 - t_1^j t_2^i)$. In the nonrefined case in [12], the above integrand with identification $t_1 = t_2 = q$ was related to an open BPS generating function in an open chamber labeled by k. It would be interesting to extend such an interpretation to the refined case, too. In particular, we note that in the limit $k \to \infty$, which should correspond to the ordinary open topological string amplitude, the above integrand indeed reduces to one particular form of a refined brane partition function in $\mathbb{C}^3$ computed in [33].

C. Conifold—all chambers

Using the representation (16) and fermionic results found in Sec. III B, we find the following matrix model for the conifold in the nth chamber [corresponding to a pyramid with $(n + 1)$ stones on top],

$$\begin{aligned} Z_n^{\rm ref} &= M(t_1, t_2)^2 \prod_{k=1,l=0}^{\infty}(1 - Q t_1^k t_2^l) \prod_{k\geq 1, l\geq 0, k+l\geq n+1}^{\infty} \times (1 - Q^{-1} t_1^k t_2^l) \\ &= f_n(q, Q) \int \mathcal{D}U \prod_k \prod_{j=0}^{\infty} \frac{(1 + z_k t_1^{j+1})(1 + t_2^j/z_k)}{(1 + z_k t_1^{j+n+1}/Q)(1 + t_2^j Q/z_k)}, \end{aligned}$$

where

$$f_n(q, Q) = \bigg(\prod_{i=1}^{n}\prod_{k=0}^{\infty}\frac{1}{1 - t_1^i t_2^k}\bigg)\bigg(\prod_{i=1}^{n}\prod_{j=n+1-i}^{\infty}(1 - t_1^i t_2^j/Q)\bigg).$$

We can again write equations for the cut endpoints, analogous to (23) and (24), which would lead to a β deformation of the general solution found in [10]; we defer solving these equations to a future work. We also note that in the limit of the commutative chamber, $n \to \infty$, we get $f_\infty = M(t_1, t_2)$. Therefore, in the commutative chamber we get a matrix model representation of the refined topological string conifold amplitude

$$\begin{aligned} \mathcal{Z}_{\rm top}^{\rm ref} &= M(t_1, t_2) \prod_{k,l=0}^{\infty}(1 - Q t_1^{k+1} t_2^l) \\ &= \int \mathcal{D}U \prod_k \prod_{j=0}^{\infty} \frac{(1 + z_k t_1^{j+1})(1 + t_2^j/z_k)}{(1 + t_2^j Q/z_k)}. \end{aligned}$$

In this case the lowest order potential is a modification of the $\mathbb{C}^3$ potential (21) by a Q-dependent dilogarithm term

$$V(z;\beta) = -\tfrac{1}{2}(\log z)^2 - (1 - \beta^{-1})\mathrm{Li}_2(-z) - \mathrm{Li}_2(-Q/z) + \mathcal{O}(g_s, \beta), \qquad (25)$$

as already advocated in (3). In the limit $Q \to 0$ the above topological string partition function becomes just the refined MacMahon function, and the matrix integral consistently reproduces the $\mathbb{C}^3$ result (20).

We can again obtain the whole family of matrix models by inserting the identity operator in various locations. Inserting it at position k in a string of $\bar{A}_-$ operators in (13), we get a matrix representation

$$Z_{n,k}^{\rm ref} = f_{n,k}(q, Q) \int \mathcal{D}U \prod_l \prod_{j=0}^{\infty} \times \frac{(1 + z_l t_1^{j+1})(1 + t_2^j t_1^k/z_l)}{(1 + z_l t_1^{j+n+1}/Q)(1 + t_2^j Q t^k/z_l)} \qquad (26)$$

with a more complicated prefactor $f_{n,k}(q, Q)$. It would be interesting, generalizing the results of [12], to relate the above integrand to the refined open BPS states.

D. $\mathbb{C}^3/\mathbb{Z}_2$—all chambers

The results for $\mathbb{C}^3/\mathbb{Z}_2$ arise similarly as those for the conifold. Using the representation (16) and fermionic construction from Sec. III B, we find

$$\begin{aligned} Z_n &= M(t_1, t_2)^2 \prod_{k=1,l=0}^{\infty}(1 - Q t_1^k t_2^l)^{-1} \prod_{k\geq 1, l\geq 0, k+l\geq n}^{\infty} \times (1 - Q^{-1} t_1^k t_2^l)^{-1} \\ &= f_n(q, Q) \int \mathcal{D}U \prod_k \prod_{j=0}^{\infty}(1 + z_k t_1^{j+1})(1 + t_2^j/z_k) \times (1 + z_k t_1^{j+n+1}/Q)(1 + t_2^j Q/z_k), \end{aligned}$$

where

$$f_n(q, Q) = \bigg(\prod_{i=1}^{n}\prod_{k=0}^{\infty}\frac{1}{1 - t_1^i t_2^k}\bigg)\bigg(\prod_{i=1}^{n}\prod_{j=n+1-i}^{\infty}\frac{1}{1 - t_1^i t_2^j/Q}\bigg).$$

In particular, in the commutative chamber $n \to \infty$ we get again $f_\infty = M(t_1, t_2)$. Therefore, in the commutative chamber we get a matrix model representation of the refined topological string amplitude

$$\mathcal{Z}^{\mathrm{ref}}_{\mathrm{top}} = M(t_1, t_2) \prod_{k=1,l=0}^{\infty} \frac{1}{1 - Q t_1^k t_2^l} = \int \mathcal{D}U \prod_k \prod_{j=0}^{\infty} (1 + z_k t_1^{j+1})(1 + t_2^j / z_k)(1 + t_2^j Q / z_k).$$

In the limit $Q \to 0$ we again recover the refined MacMahon function, as well as the expected integrand of the $\mathbb{C}^3$ matrix model (20). In this case it is also straightforward to find more general matrix models, analogous to (26), which would presumably be related to refined open amplitudes.

V. DISCUSSION

In this paper we have found a free fermion, as well as a unitary matrix model representation of refined BPS generating functions of D0- and D2-branes bound to a single D6-brane, and, in particular, topological string amplitudes, in toric Calabi-Yau manifolds without compact four-cycles. We mainly considered explicit examples of $\mathbb{C}^3$, conifold, and $\mathbb{C}^3/\mathbb{Z}_2$ geometries, as well as an arbitrary geometry in the noncommutative chamber; however, generalization to other chambers for manifolds in this class is straightforward. A general consequence of our results is the fact that refined generating functions, at least for the class of manifolds which we considered, have nice properties of ordinary matrix model expressions [13,14], such as integrability, symplectic invariance of associated free-energy coefficients F_g, automatic appearance of the whole family of differentials W_n^g, etc. One advantage of our representation is that these properties are much better understood for ordinary matrix models, rather than for matrix models for beta ensembles [29], which in fact are known *not* to reproduce the refined topological string amplitudes [30,31]. It is also important to understand a difference between these two beta deformations. As follows from the results of [31], in the case of the conifold [or five-dimensional $U(1)$ gauge theory], the four-dimensional gauge theory limits of both deformations agree. Understanding the origin of a discrepancy in five-dimensional deformation should lead to interesting new insights.

There are many other questions which require further investigation. First, a nontrivial task is to find spectral curves of our models. As we discussed, these would be β deformation of curves found in a nonrefined case in [10]. Having known such curves would allow us to apply the topological recursion to recover quantities W_n^g and F_g explicitly from the matrix model perspective. This appears nontrivial, in particular, due to all order g_s corrections to our potentials. However, these corrections arise from terms involving quantum dilogarithms. Potentials which involve quantum dilogarithms were considered also in [5,50], where it was shown that higher g_s essentially do not modify resulting invariants, and one can effectively consider a leading order contribution to the potential, similar to (21) and (25) in our case. It would be interesting to confirm if an analogous phenomenon takes place for the potentials which we consider.

Furthermore, it would be interesting to extend our discussion to the open string case, on one hand refining the discussion in [12] and providing an M-theory derivation of putative open BPS generating functions, and on the other, relating W_n^g to brane amplitudes in matrix models in the topological string limit. In particular, this should provide a deeper understanding of nontrivial prefactors in intermediate chambers.

It would of course be interesting to extend our results to toric manifolds with compact four-cycles, in particular, those related by geometric engineering to gauge theories. This might be possible by considering more involved crystal models, such as those in [52].

Among other questions, it is interesting what our matrix models compute for the finite size of matrices N. It was shown in [10] that in the nonrefined case finite N engineers more complicated toric manifolds with an additional two-cycle (as is already the case in the Chern-Simons matrix models [4], where a finite ’t Hooft coupling encodes the size of the single $\mathbb{P}^1$ of the resolved conifold). In particular, it is tempting to speculate whether the matrix model (20) with finite N would also provide the refined conifold topological string partition function.

It would also be interesting to understand the issues of holomorphic anomaly and modularity and make contact with discussions in [30,53,54], and more generally with the extensive literature on refined invariants.

We hope that continuing this line of research would be a rewarding experience.

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APPENDIX: FREE FERMION FORMALISM

In this appendix we summarize the free fermion formalism used in [10,12,43]. We consider the Heisenberg algebra $[\alpha_m, \alpha_{-n}] = n\delta_{m,n}$ and define the vertex operators

$$\Gamma_{\pm}(x) = e^{\sum_{n>0} (x^n/n)\alpha_{\pm n}}, \qquad \Gamma'_{\pm}(x) = e^{\sum_{n>0} ((-1)^{n-1} x^n/n)\alpha_{\pm n}},$$

which satisfy the commutation relations

$$\Gamma_+(x)\Gamma_-(y) = \frac{1}{1-xy}\Gamma_-(y)\Gamma_+(x),$$
$$\Gamma'_+(x)\Gamma'_-(y) = \frac{1}{1-xy}\Gamma'_-(y)\Gamma'_+(x),$$

$$\Gamma'_+(x)\Gamma_-(y) = (1+xy)\Gamma_-(y)\Gamma'_+(x),$$
$$\Gamma_+(x)\Gamma'_-(y) = (1+xy)\Gamma'_-(y)\Gamma_+(x).$$

These operators act on fermionic states $|\mu\rangle$, corresponding to two-dimensional partitions μ, as

$$\Gamma_-(x)|\mu\rangle = \sum_{\lambda\succ\mu} x^{|\lambda|-|\mu|}|\lambda\rangle, \quad \Gamma_+(x)|\mu\rangle = \sum_{\lambda\prec\mu} x^{|\mu|-|\lambda|}|\lambda\rangle, \tag{A1}$$

$$\Gamma'_-(x)|\mu\rangle = \sum_{\lambda^t\succ\mu^t} x^{|\lambda|-|\mu|}|\lambda\rangle, \quad \Gamma'_+(x)|\mu\rangle = \sum_{\lambda^t\prec\mu^t} x^{|\mu|-|\lambda|}|\lambda\rangle, \tag{A2}$$

where $\prec$ is the interlacing relation. We also consider various weight operators $\hat{Q}_g$, with eigenvalues representing colors and denoted q_g, such that

$$\hat{Q}_g|\lambda\rangle = q_g^{|\lambda|}|\lambda\rangle,$$

and their commutation relations with vertex operators read

$$\Gamma_+(x)\hat{Q}_g = \hat{Q}_g\Gamma_+(xq_g), \qquad \Gamma'_+(x)\hat{Q}_g = \hat{Q}_g\Gamma'_+(xq_g), \tag{A3}$$

$$\hat{Q}_g\Gamma_-(x) = \Gamma_-(xq_g)\hat{Q}_g, \qquad \hat{Q}_g\Gamma'_-(x) = \Gamma'_-(xq_g)\hat{Q}_g. \tag{A4}$$

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