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Fermionic condensate in a conical space with a circular boundary and magnetic flux

S. Bellucci,[1,*] E. R. Bezerra de Mello,[2,†] and A. A. Saharian[3,‡]

[1]*INFN, Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy*
[2]*Departamento de Física, Universidade Federal da Paraíba 58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil*
[3]*Department of Physics, Yerevan State University, Alex Manoogian Street, 0025 Yerevan, Armenia*

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The fermionic condensate is investigated in a (2 + 1)-dimensional conical spacetime in the presence of a circular boundary and a magnetic flux. It is assumed that on the boundary the fermionic field obeys the MIT bag boundary condition. For irregular modes, we consider a special case of boundary conditions at the cone apex, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. The fermionic condensate is a periodic function of the magnetic flux with the period equal to the flux quantum. For both exterior and interior regions, the fermionic condensate is decomposed into boundary-free and boundary-induced parts. Two integral representations are given for the boundary-free part for arbitrary values of the opening angle of the cone and magnetic flux. At distances from the boundary larger than the Compton wavelength of the fermion particle, the condensate decays exponentially, with the decay rate depending on the opening angle of the cone. If the ratio of the magnetic flux to the flux quantum is not a half-integer number for a massless field the boundary-free part in the fermionic condensate vanishes, whereas the boundary-induced part is negative. For half-integer values of the ratio of the magnetic flux to the flux quantum, the irregular mode gives a nonzero contribution to the fermionic condensate in the boundary-free conical space.

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I. INTRODUCTION

Field theoretical models in 2 + 1 dimensions exhibit a number of interesting effects, such as parity violation, flavor symmetry breaking, and fractionalization of quantum numbers (see Refs. [1–7]). An important aspect is the possibility of giving a topological mass to the gauge bosons without breaking gauge invariance. Field theories in 2 + 1 dimensions provide simple models in particle physics, and related theories also arise in the long-wavelength description of certain planar condensed matter systems, including models of high-temperature superconductivity. An interesting application of Dirac theory in 2 + 1 dimensions recently appeared in nanophysics. In a sheet of hexagons from the graphite structure, known as graphene, the long-wavelength description of the electronic states can be formulated in terms of the Dirac-like theory of massless spinors in (2 + 1)-dimensional spacetime with the Fermi velocity playing the role of the speed of light (for a review see Ref. [8]). One-loop quantum effects induced by the nontrivial topology of graphene made cylindrical and toroidal nanotubes have been recently considered in Ref. [9]. The vacuum polarization in graphene with a topological defect is investigated in Ref. [10] within the framework of a long-wavelength continuum model.

The interaction of a magnetic flux tube with a fermionic field gives rise to a number of interesting phenomena, such as the Aharonov-Bohm effect, parity anomalies, formation of a condensate, and generation of exotic quantum numbers. For background Minkowski spacetime, the combined effects of the magnetic flux and boundaries on the vacuum energy have been studied in Refs. [11,12]. In Ref. [13], we have investigated the vacuum expectation value of the fermionic current induced by the vortex configuration of a gauge field in a (2 + 1)-dimensional conical space with a circular boundary. On the boundary the fermionic field obeys the MIT bag boundary condition. Continuing in this line of investigation, in the present paper we evaluate the fermionic condensate for the same bulk and boundary geometries. The fermionic condensate is among the most important quantities that characterize the properties of the quantum vacuum. Although the corresponding operator is local, due to the global nature of the vacuum, this quantity carries important information about the global properties of the background spacetime. The fermionic condensate plays an important role in the models of dynamical breaking of chiral symmetry (see Ref. [14] for the chiral symmetry breaking in Nambu-Jona-Lasino and Gross-Neveu models on background of a curved spacetime with nontrivial topology). Note that the combined effects of the topology and boundaries on the polarization of the vacuum were studied in Refs. [15–18] for the cases of scalar, electromagnetic and fermionic fields. In these papers, a cylindrical boundary is considered in the geometry of a cosmic string, assuming that the boundary is coaxial with the string. The case of a scalar field was considered in an arbitrary number of spacetime dimensions, whereas the problems for the electromagnetic and fermionic fields were studied in four dimensional

*bellucci@lnf.infn.it
†emello@fisica.ufpb.br
‡saharian@ysu.am

spacetime. The fermionic condensate in de Sitter spacetime with toroidally compactified spatial dimensions has been recently investigated in Refs. [19].

From the point of view of the physics in the region outside the conical defect core, the geometry considered in the present paper can be viewed as a simplified model for the nontrivial core. This model presents a framework in which the influence of the finite core effects on physical processes in the vicinity of the conical defect can be investigated. In particular, it enables us to specify conditions under which the idealized model with the core of zero thickness can be used. The corresponding results may shed light upon features of finite core effects in more realistic models, including those used for defects in crystals and superfluid helium. In addition, the problem considered here is of interest as an example with combined topological and boundary-induced quantum effects, in which the vacuum characteristics can be found in closed analytic form.

The results obtained in the present paper can be applied for the evaluation of the fermionic condensate in graphitic cones. Graphitic cones are obtained from the graphene sheet if one or more sectors are excised. The opening angle of the cone is related to the number of sectors removed, N_c, by the formula $2\pi(1 - N_c/6)$, with $N_c = 1, 2, \ldots, 5$ (for the electronic properties of graphitic cones see, e.g., [20] and references therein). All these angles have been observed in experiments [21]. Note that the fermionic condensate in cylindrical and toroidal carbon nanotubes has been investigated in Ref. [9] within the framework of the Dirac-like theory for the electronic states in graphene sheet.

The organization of the paper is as follows. In the next section, we evaluate the fermionic condensate (FC) in a boundary-free conical space with an infinitesimally thin magnetic flux placed at the apex of the cone. A special case of boundary conditions at the cone apex is considered, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. Two integral representations are provided for the renormalized FC. A simple expression is found for the special case of the magnetic flux. In Sec. III, we consider the FC in the region inside a circular boundary with the MIT bag boundary condition. The condensate is decomposed into boundary-free and boundary-induced parts. A rapidly convergent integral representation for the latter is obtained. A similar investigation for the region outside a circular boundary is presented in Sec. IV. A special case with half-integer values of the ratio of the magnetic flux to the quantum one is discussed in Sec. V. The main results are summarized in Sec. VI.

II. FERMIONIC CONDENSATE IN THE BOUNDARY-FREE GEOMETRY

Let us consider a two-component spinor field ψ on the background of a $(2+1)$-dimensional conical spacetime. The corresponding line element is given by the expression

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - r^2 d\phi^2, \tag{2.1}$$

where $r \geq 0$, $0 \leq \phi \leq \phi_0$, and the points (r, ϕ) and $(r, \phi + \phi_0)$ are to be identified. In the discussion below, in addition to ϕ_0, we use the notation

$$q = 2\pi/\phi_0. \tag{2.2}$$

In the presence of the external electromagnetic field with the vector potential A_μ, the dynamics of the field is governed by the Dirac equation

$$i\gamma^\mu(\nabla_\mu + ieA_\mu)\psi - m\psi = 0, \quad \nabla_\mu = \partial_\mu + \Gamma_\mu, \tag{2.3}$$

where $\gamma^\mu = e^\mu_{(a)}\gamma^{(a)}$ are the 2×2 Dirac matrices in polar coordinates and $e^\mu_{(a)}$, $a = 0, 1, 2$, is the basis tetrad. The operator of the covariant derivative in Eq. (2.3) is defined by the relation

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^\nu_{(a)}e_{(b)\nu;\mu}, \tag{2.4}$$

where ";" means the standard covariant derivative for vector fields. In $(2+1)$-dimensional spacetime there are two inequivalent irreducible representations of the Clifford algebra. Here we choose the flat space Dirac matrices in the form $\gamma^{(0)} = \sigma_3$, $\gamma^{(1)} = i\sigma_1$, $\gamma^{(2)} = i\sigma_2$, with σ_l being Pauli matrices. In the second representation the gamma matrices can be taken as $\gamma^{(0)} = -\sigma_3$, $\gamma^{(1)} = -i\sigma_1$, $\gamma^{(2)} = -i\sigma_2$. The corresponding results for the second representation are obtained by changing the sign of the mass, $m \to -m$. Note that there is no other 2×2 matrix which anticommutes with all $\gamma^{(a)}$, and, hence, we have no chiral symmetry that would be broken by a mass term in two-dimensional representation.

Our interest in the present paper is the FC, $\langle 0|\bar{\psi}\psi|0\rangle = \langle\bar{\psi}\psi\rangle$, with $|0\rangle$ being the vacuum state, in the conical space with a circular boundary. Here and in what follows $\bar{\psi} = \psi^\dagger\gamma^0$ is the Dirac adjoint and the dagger denotes Hermitian conjugation. We assume the magnetic field configuration corresponding to a infinitely thin magnetic flux located at the apex of the cone. This will be implemented by considering the vector potential $A_\mu = (0, 0, A)$ for $r > 0$. The quantity A is related to the magnetic flux Φ by the formula $A = -\Phi/\phi_0$.

First, we consider the FC in a boundary-free conical space. It can be evaluated by using the mode-sum formula

$$\langle\bar{\psi}\psi\rangle = \sum_\sigma \bar{\psi}^{(-)}_\sigma\psi^{(-)}_\sigma, \tag{2.5}$$

where $\{\psi^{(+)}_\sigma, \psi^{(-)}_\sigma\}$ is a complete set of positive and negative energy solutions to the Dirac equation specified by quantum numbers σ. As it is well known, the theory of von Neumann deficiency indices leads to a one-parameter family of allowed boundary conditions in the background of an Aharonov-Bohm gauge field [22]. Here we consider a special case of boundary conditions at the cone apex, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. The FC for other boundary conditions on the cone apex are evaluated in a way similar to that described below. The contribution of the regular modes is the same for all boundary conditions and the results differ by the parts related to the irregular modes.

In the boundary-free conical space the eigenspinors are specified by the set $\sigma = (\gamma, j)$ of quantum numbers with $0 \leq \gamma < \infty$ and $j = \pm 1/2, \pm 3/2, \ldots$. For $j \neq -e\Phi/2\pi$, the corresponding normalized negative-energy eigenspinors have the form [13]

$$\psi^{(-)}_{(0)\gamma j} = \left(\gamma \frac{E+m}{2\phi_0 E}\right)^{1/2} e^{-iqj\phi + iEt} \begin{pmatrix} \frac{\gamma \epsilon_j e^{-iq\phi/2}}{E+m} J_{\beta_j+\epsilon_j}(\gamma r) \\ J_{\beta_j}(\gamma r) e^{iq\phi/2} \end{pmatrix}, \tag{2.6}$$

where $E = \sqrt{\gamma^2 + m^2}$, $J_\nu(x)$ is the Bessel function. The order of the Bessel function in (2.6) is given by the expression

$$\beta_j = q|j + \alpha| - \epsilon_j/2, \qquad q = 2\pi/\phi_0, \tag{2.7}$$

with

$$\alpha = eA/q = -e\Phi/2\pi, \tag{2.8}$$

and we have defined

$$\epsilon_j = \begin{cases} 1, & j > -\alpha \\ -1, & j < -\alpha \end{cases}. \tag{2.9}$$

The expression for the positive energy eigenspinor is found from (2.6) by using the relation $\psi^{(+)}_{\gamma j} = \sigma_1 \psi^{(-)*}_{\gamma j}$, where the asterisk means complex conjugate. Here we assume that the parameter α is not a half integer. The special case of half integer α will be considered separately in Sec. V.

Substituting the eigenspinors (2.6) into the mode-sum (2.5), for the FC in a boundary-free conical space one finds

$$\langle \bar\psi \psi \rangle_0 = \frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \frac{\gamma}{E} [(E-m) J^2_{\beta_j+\epsilon_j}(\gamma r) - (E+m) J^2_{\beta_j}(\gamma r)], \tag{2.10}$$

where $\sum_j$ means the summation over $j = \pm 1/2, \pm 3/2, \ldots$. Of course, the expression on the right-hand side of this formula is divergent and needs to be regularized. We introduce a cutoff function $e^{-s\gamma^2}$ with the cutoff parameter $s > 0$. At the end of calculations the limit $s \to 0$ is taken. The corresponding regularized expectation value is presented in the form

$$\langle \bar\psi \psi \rangle_{0,\mathrm{reg}} = \frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \gamma e^{-s\gamma^2} [J^2_{\beta_j+\epsilon_j}(\gamma r) - J^2_{\beta_j}(\gamma r)] - \frac{qm}{4\pi} \sum_j \int_0^\infty d\gamma \frac{\gamma e^{-s\gamma^2}}{\sqrt{\gamma^2+m^2}} [J^2_{\beta_j+\epsilon_j}(\gamma r) + J^2_{\beta_j}(\gamma r)]. \tag{2.11}$$

The γ integral in the first term on the right-hand side is expressed in terms of the modified Bessel function $I_\nu(x)$. In the second term we use the relation

$$\frac{1}{\sqrt{\gamma^2+m^2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty dt e^{-(\gamma^2+m^2)t^2} \tag{2.12}$$

and change the order of integrations. After the evaluation of the γ integral, the regularized FC is presented in the form

$$\langle \bar\psi \psi \rangle_{0,\mathrm{reg}} = \frac{q e^{-r^2/2s}}{8\pi s} \sum_j [I_{\beta_j+\epsilon_j}(r^2/2s) - I_{\beta_j}(r^2/2s)] - \frac{qm e^{m^2 s}}{2(2\pi)^{3/2}} \sum_j \int_0^{r^2/2s} dx \frac{x^{-1/2} e^{-m^2 r^2/2x - x}}{\sqrt{r^2 - 2xs}} \times [I_{\beta_j+\epsilon_j}(x) + I_{\beta_j}(x)]. \tag{2.13}$$

Before further considering the FC for the general case of the parameters characterizing the conical structure and the magnetic flux, we study a special case, which allows us to obtain a simple expression.

A. Special case

In the special case with q being an integer and

$$\alpha = 1/2q - 1/2, \tag{2.14}$$

the orders of the modified Bessel functions in Eq. (2.13) become integer numbers: $\beta_j = q|n|$, $j = n + 1/2$. The series over n is summarized explicitly by using the formula [23]

$$\sum_{n=0}^{\infty}{}' I_{qn}(x) = \frac{1}{2q} \sum_{k=0}^{q-1} e^{x\cos(2\pi k/q)}, \tag{2.15}$$

where the prime means that the term $n = 0$ should be halved. For the regularized FC we find the expression[1]

[1]Under the condition (2.14), the induced fermionic current in a higher-dimensional cosmic string spacetime has been analyzed in [24].

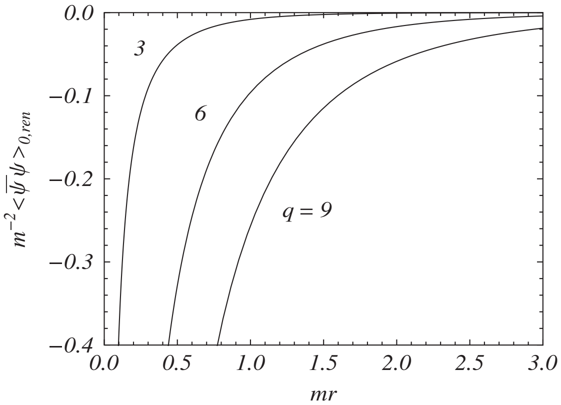

FIG. 1. The fermionic condensate in a boundary-free conical space, as a function of mr, for the special case of integer values of q, with the magnetic flux defined by Eq. (2.14).

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{reg}} = -\frac{1}{4\pi s}\sum_{k=1}^{q-1}\sin^2(\pi k/q)e^{-2(r^2/2s)\sin^2(\pi k/q)} - \frac{me^{m^2 s}}{(2\pi)^{3/2}}\sum_{k=0}^{q-1}\cos^2(\pi k/q) \times\int_0^{r^2/2s}dx\frac{x^{-1/2}e^{-m^2r^2/2x}}{\sqrt{r^2-2xs}}e^{-2x\sin^2(\pi k/q)}. \tag{2.16}$$

The first term on the right-hand side of this formula vanishes in the limit $s\to 0$. In the second term, the only divergent contribution in the limit $s\to 0$ comes from the $k=0$ term. This term coincides with the regularized FC in the Minkowski spacetime, in the absence of the magnetic flux. Subtracting this contribution and taking the limit $s\to 0$, for the renormalized FC, we find

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} = -\frac{m}{4\pi r}\sum_{k=1}^{q-1}\frac{\cos^2(\pi k/q)}{\sin(\pi k/q)}e^{-2mr\sin(\pi k/q)}. \tag{2.17}$$

Note that the renormalized FC vanishes for a massless field and for a massive field in a conical space with $q=2$. For other cases the FC is negative. As expected, it decays exponentially at distances larger than the Compton wavelength of the fermionic particle. In Fig. 1, the FC is plotted versus mr for different values of q. The corresponding values of the parameter α are found from Eq. (2.14).

B. General case

For the general case of the parameters q and α, as it is seen from (2.13), the regularized FC is expressed in terms of the series

$$\mathcal{I}(q,\alpha,z) = \sum_j I_{\beta_j}(z). \tag{2.18}$$

We present the parameter α, related to the magnetic flux by Eq. (2.8), in the form

$$\alpha = \alpha_0 + n_0, \qquad |\alpha_0| < 1/2, \tag{2.19}$$

with being n_0 an integer number. Now, Eq. (2.18) is written as

$$\mathcal{I}(q,\alpha,z) = \sum_{n=0}^{\infty}[I_{q(n+\alpha_0+1/2)-1/2}(z) + I_{q(n-\alpha_0+1/2)+1/2}(z)], \tag{2.20}$$

which explicitly shows the independence of the series on n_0. Note that, for the second series appearing in the expression of the FC, we have

$$\sum_j I_{\beta_j+\epsilon_j}(z) = \mathcal{I}(q,-\alpha_0,z). \tag{2.21}$$

From these relations we conclude that the FC depends on α_0 alone, and, hence, it is a periodic function of α with period 1.

In terms of the function (2.18), the expression (2.13) for the regularized FC is written as

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{reg}} = -\frac{qe^{-r^2/2s}}{8\pi s}\sum_{\delta=\pm1}\delta\mathcal{I}(q,\delta\alpha_0,r^2/2s) - \frac{qme^{m^2 s}}{2(2\pi)^{3/2}} \times\int_0^{r^2/2s}dx\frac{x^{-1/2}e^{-m^2r^2/2x-x}}{\sqrt{r^2-2xs}}\sum_{\delta=\pm1}\mathcal{I}(q,\delta\alpha_0,x). \tag{2.22}$$

For $2p < q < 2p+2$, with p being an integer, we use the representation [13]

$$\mathcal{I}(q,\alpha_0,z) = \frac{e^z}{q} + \mathcal{J}(q,\alpha_0,z), \tag{2.23}$$

with the notation

$$\mathcal{J}(q,\alpha_0,z) = -\frac{1}{\pi}\int_0^\infty dy\frac{e^{-z\cosh y}f(q,\alpha_0,y)}{\cosh(qy)-\cos(q\pi)} + \frac{2}{q}\sum_{l=1}^{p}(-1)^l\cos[2\pi l(\alpha_0-1/2q)]e^{z\cos(2\pi l/q)}. \tag{2.24}$$

The function in the integrand is defined by the expression

$$f(q,\alpha_0,y) = \cos[q\pi(1/2-\alpha_0)]\cosh[(q\alpha_0+q/2-1/2)y] - \cos[q\pi(1/2+\alpha_0)] \times\cosh[(q\alpha_0-q/2-1/2)y]. \tag{2.25}$$

In the case $q=2p$, the term

$$-(-1)^{q/2}\frac{e^{-z}}{q}\sin(q\pi\alpha_0) \tag{2.26}$$

should be added to the right-hand side of Eq. (2.24). For $1 \leq q < 2$, the last term on the right-hand side of Eq. (2.24) is absent.

In the limit $s \to 0$, the only divergent contributions to the functions $e^{-r^2/2s}\mathcal{I}(q, \pm\alpha_0, r^2/2s)/s$ come from the first term in the right-hand side of Eq. (2.23). The contribution of this term to the FC does not depend on α_0, and, consequently, the divergences are cancelled in the evaluation of the first term in the right-hand side of (2.22). This term vanishes in the limit $s \to 0$, and, hence, it does not contribute to the renormalized FC. Substituting (2.23) into the second term in the right-hand side of Eq. (2.22), we see that the only divergent contribution comes from the term e^z/q. This contribution does not depend on the opening angle of the cone and on the magnetic flux. It coincides with the corresponding quantity in the Minkowski spacetime, in the absence of the magnetic flux. Subtracting the Minkowskian part and taking the limit $s \to 0$ for the renormalized FC, we find

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} = -\frac{qm}{2(2\pi)^{3/2}r}\int_0^\infty dx x^{-1/2}e^{-m^2r^2/2x-x} \times \sum_{\delta=\pm1}\mathcal{J}(q, \delta\alpha_0, x). \quad (2.27)$$

Note that in the case $q = 2p$ the contribution of the additional term (2.26) to the renormalized FC vanishes.

By taking into account Eq. (2.24), the integration over x in Eq. (2.27) is performed explicitly and one finds the following formula

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} = \frac{m}{2\pi r}\left\{-\sum_{l=1}^{p}(-1)^l\frac{\cot(\pi l/q)}{e^{2mr\sin(\pi l/q)}}\cos(2\pi l\alpha_0) + \frac{q}{4\pi}\int_0^\infty dy\frac{e^{-2mr\cosh(y/2)}}{\cosh(y/2)}\frac{\sum_{\delta=\pm1}f(q, \delta\alpha_0, y)}{\cosh(qy)-\cos(q\pi)}\right\}, \quad (2.28)$$

where p is an integer defined by $2p \leq q < 2p + 2$. Note that the sum in the integrand may be written in the form

$$\sum_{\delta=\pm1}f(q, \delta\alpha_0, y) = -2\sinh(y/2)\sum_{\delta=\pm1}\cos[q\pi(1/2 + \delta\alpha_0)] \times \sinh[q(1/2 - \delta\alpha_0)y]. \quad (2.29)$$

For integer q and for the parameter α given by the special value (2.14), from (2.28) we obtain the result (2.17). At distances larger than the Compton wavelength of the spinor particle, $mr \gg 1$, the FC is suppressed by the factor e^{-2mr} for $1 \leq q \leq 2$ and by the factor $e^{-2mr\sin(\pi/q)}$ for $q > 2$. In the latter case, the main contribution comes from the first term in the figure braces of the right-hand side in Eq. (2.28):

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} \approx \frac{m\cos(2\pi\alpha_0)}{2\pi r}\frac{\cot(\pi/q)}{e^{2mr\sin(\pi/q)}}, \quad mr \gg 1. \quad (2.30)$$

In the special case when the magnetic flux is absent, we have $\alpha_0 = 0$ and the general formula (2.28) simplifies to

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} = -\frac{m}{2\pi r}\left\{\sum_{l=1}^{p}(-1)^l\frac{\cot(\pi l/q)}{e^{2mr\sin(\pi l/q)}} + \frac{2q}{\pi}\cos(q\pi/2) \times \int_0^\infty dx\frac{\sinh(qx)\tanh(x)e^{-2mr\cosh x}}{\cosh(2qx) - \cos(q\pi)}\right\}. \quad (2.31)$$

In this case the FC is only a consequence of the conical structure of the space. For odd values of the parameter q the second term in the figure braces vanishes and for the FC we have the simple formula

$$\langle 0|\bar{\psi}\psi|0\rangle_{0,\mathrm{ren}} = -\frac{m}{2\pi r}\sum_{l=1}^{(q-1)/2}(-1)^l\frac{\cot(\pi l/q)}{e^{2mr\sin(\pi l/q)}}. \quad (2.32)$$

Another limiting case corresponds to the magnetic flux in background of Minkowski spacetime. In this case, taking $q = 1$, from Eq. (2.28) we find

$$\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} = -\frac{m\sin(\pi\alpha_0)}{2\pi^2 r}\int_0^\infty dx\frac{\sinh x}{\cosh^2 x}\frac{\sinh(2\alpha_0 x)}{e^{2mr\cosh x}}, \quad (2.33)$$

and the FC is negative for $\alpha_0 \neq 0$.

In Fig. 2, the fermionic condensate is plotted as a function of the magnetic flux for a massive fermionic field in conical spaces with $\phi_0 = \pi$ (left plot) and $\phi_0 = \pi/2$ (right plot). Note that for $q = 2$ the first term in figure braces of (2.28) vanishes and the second term contains the factor $\cos(2\pi\alpha_0)$. Consequently, in this case the FC vanishes at $\alpha_0 = \pi/4$.

An alternative expression for the FC is obtained by using the formula [13]

$$\mathcal{I}(q, \alpha_0, x) = A(q, \alpha_0, x) + \frac{2}{q}\int_0^\infty dz I_z(x) - \frac{4}{\pi q}\int_0^\infty dz\,\mathrm{Re}\left[\frac{\sinh(z\pi)K_{iz}(x)}{e^{2\pi(z+i|q\alpha_0-1/2|)/q}+1}\right], \quad (2.34)$$

with $A(q, \alpha_0, x) = 0$ for $|\alpha_0 - 1/2q| \leq 1/2$, and

$$A(q, \alpha_0, x) = \frac{2}{\pi}\sin[\pi(|q\alpha_0 - 1/2| - q/2)] \times K_{|q\alpha_0-1/2|-q/2}(x) \quad (2.35)$$

for $1/2 < |\alpha_0 - 1/2q| < 1$. Substituting the representation (2.34) into the expression (2.22) for the regularized FC, we see that the part with the second term on the right-hand side of (2.34) does not depend on the opening angle of

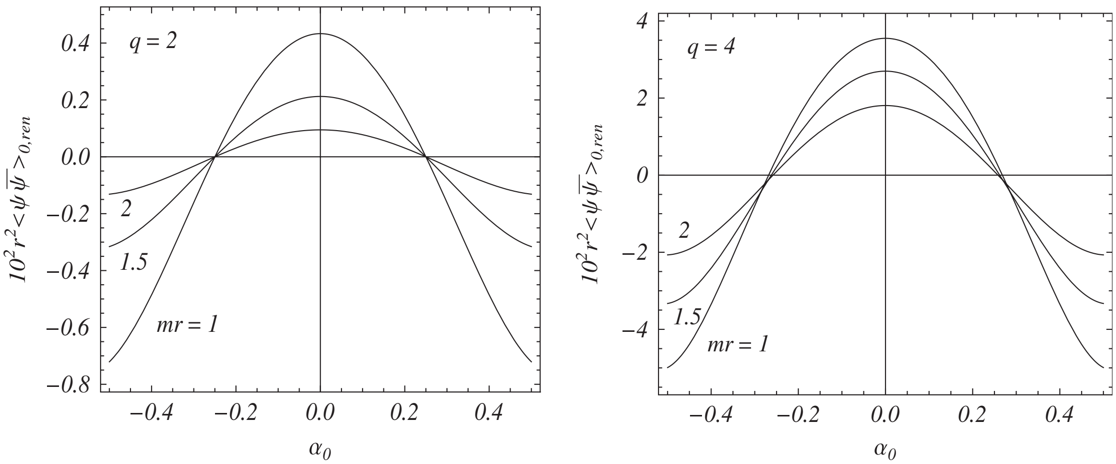

FIG. 2. The FC as a function of the magnetic flux for a massive fermionic field in boundary-free conical spaces with $q = 2$ (left plot) and $q = 4$ (right plot).

the cone and on the magnetic flux. It is the same as in the Minkowski bulk in the absence of the magnetic flux, and, hence, it should be subtracted in the renormalization procedure. Subtracting the part corresponding to $q = 1$ and $\alpha_0 = 0$, in the remaining part the limit $s \to 0$ can be taken directly. The first term on the right-hand side of Eq. (2.22) vanishes in this limit, and for the renormalized FC we find the representation

$$\langle \bar{\psi}\psi\rangle_{0,\text{ren}} = -\frac{2m}{(2\pi)^{5/2}r}\int_0^\infty dx x^{-1/2}e^{-m^2r^2/2x-x} \times [qB(q(|\alpha_0| - 1/2) + 1/2, x) - 2\int_0^\infty dz K_{iz}(x)h(q, \alpha_0, z)]. \tag{2.36}$$

In this formula we have used the notations

$$h(q, \alpha_0, z) = \sum_{\delta=\pm 1} \text{Re}\left[\frac{\sinh(z\pi)}{e^{2\pi(z+i|q\delta\alpha_0-1/2|)/q}+1} + \frac{\sinh(z\pi)}{e^{2\pi z}-1}\right] \tag{2.37}$$

and

$$B(y, x) = \begin{cases} 0, & y \leq 0, \\ \sin(\pi y)K_y(x) & y > 0. \end{cases} \tag{2.38}$$

The representation (2.36) is valid for conical spaces with $q < 4$. For special values $q = 2$ and $\alpha_0 = 1/4$, by taking into account that $h(2, 1/4, z) = 0$, we see that the FC defined by (2.36) vanishes.

In the case of a magnetic flux in the background of the Minkowski spacetime ($q = 1$), we find

$$\langle \bar{\psi}\psi\rangle_{0,\text{ren}} = -\frac{2m\sin(\pi|\alpha_0|)}{(2\pi)^{5/2}r}\int_0^\infty dx x^{-1/2}e^{-m^2r^2/2x-x} \times \left[K_{\alpha_0}(x) - 4\sin(\pi|\alpha_0|) \times \int_0^\infty dz \frac{K_{iz}(x)\cosh(\pi z)}{\cosh(2\pi z) - \cos(2\pi\alpha_0)}\right]. \tag{2.39}$$

For a conical space in the absence of the magnetic flux the general formula reduces to

$$\langle \bar{\psi}\psi\rangle_{0,\text{ren}} = \frac{4m}{(2\pi)^{5/2}r}\int_0^\infty dx x^{-1/2}e^{-m^2r^2/2x-x}\int_0^\infty dz K_{iz}(x) \times \frac{\cosh(\pi z)\cos(\pi/q) + \cosh[\pi z(2/q-1)]}{\cosh(2\pi z/q) + \cos(\pi/q)}. \tag{2.40}$$

For $q = 2$ the integral over z is evaluated explicitly (see, for instance, [23]), and we get a simple expression $\langle \bar{\psi}\psi\rangle_{0,\text{ren}} = (m/\pi)^2 \int_1^\infty dt K_1(2mrt)/t$. Recall that for odd values of q we have the simple formula (2.32). For the second representation of the Clifford algebra the renormalized FC in a boundary-free conical space changes the sign.

We can generalize the results given above for a more general situation where the spinor field ψ obeys quasiperiodic boundary condition along the azimuthal direction

$$\psi(t, r, \phi + \phi_0) = e^{2\pi i\chi}\psi(t, r, \phi), \tag{2.41}$$

with a constant parameter χ, $|\chi| \leq 1/2$. With this condition, the exponential factor in the expression for the eigenspinors (2.6) has the form $e^{-iq(n+\chi)\phi+iEt}$. The corresponding expression for the eigenfunctions is obtained from that given above with the parameter α defined by

$$\alpha = \chi - e\Phi/2\pi. \tag{2.42}$$

The same replacement generalizes the expression of the FC for the case of a field with periodicity condition (2.41).

In general, the fermionic modes in the background of the magnetic vortex are divided into two classes, regular and irregular (square integrable) ones. In the problem under consideration, for given q and α, the irregular mode corresponds to the value of j for which $q|j+\alpha| < 1/2$. If we present the parameter α in the form (2.19), then the irregular mode is present if $|\alpha_0| > (1-1/q)/2$. This mode corresponds to $j = -n_0 - \mathrm{sgn}(\alpha_0)/2$. Note that, in a conical space, under the condition $|\alpha_0| \leq (1-1/q)/2$, there are no square integrable irregular modes. As we have already mentioned, there is a one-parameter family of allowed boundary conditions for irregular modes. These modes are parametrized by the angle θ, $0 \leq \theta < 2\pi$ (see Ref. [22]). For $|\alpha_0| < 1/2$, the boundary condition, used in deriving eigenspinors (2.6), corresponds to $\theta = 3\pi/2$. If α is a half integer, the irregular mode corresponds to $j = -\alpha$ and for the corresponding boundary condition one has $\theta = 0$. Note that in both cases there are no bound states.

III. FERMIONIC CONDENSATE INSIDE A CIRCULAR BOUNDARY

In this section, we consider the change in the FC induced by a circular boundary concentric with the apex of the cone. We assume that the field obeys the MIT bag boundary condition on the circle with radius a:

$$(1 + in_\mu \gamma^\mu)\psi|_{r=a} = 0, \tag{3.1}$$

where n_μ is the outward oriented normal (with respect to the region under consideration) to the boundary. For the interior region, $n_\mu = \delta_\mu^1$. In this region, the negative-energy eigenspinors are given by the expression [13]

$$\psi_{\gamma j}^{(-)} = \varphi_0 e^{-iqj\phi + iEt} \begin{pmatrix} \frac{\epsilon_j \gamma e^{-iq\phi/2}}{E+m} J_{\beta_j+\epsilon_j}(\gamma r) \\ e^{iq\phi/2} J_{\beta_j}(\gamma r) \end{pmatrix}, \tag{3.2}$$

with the same notations as in Eq. (2.6). From the boundary condition at $r = a$, we find that the eigenvalues of γ are solutions of the equation

$$J_{\beta_j}(\gamma a) - \frac{\gamma \epsilon_j J_{\beta_j+\epsilon_j}(\gamma a)}{\sqrt{\gamma^2 + m^2} + m} = 0. \tag{3.3}$$

For a given β_j, Eq. (3.3) has an infinite number of solutions, which we denote by $\gamma a = \gamma_{\beta_j,l}$, $l = 1, 2, \ldots$. The normalization coefficient in Eq. (3.2) is given by the expression

$$\varphi_0^2 = \frac{y T_{\beta_j}(y)}{2\phi_0 a^2} \frac{\mu + \sqrt{y^2+\mu^2}}{\sqrt{y^2+\mu^2}}, \tag{3.4}$$

with the notations $\mu = ma$ and

$$T_{\beta_j}(y) = \frac{y}{J_{\beta_j}^2(y)} \left[y^2 + (\mu - \epsilon_j \beta_j)\left(\mu + \sqrt{y^2+\mu^2}\right) - \frac{y^2}{2\sqrt{y^2+\mu^2}} \right]^{-1}. \tag{3.5}$$

Substituting the eigenspinors (3.2) into the mode-sum formula

$$\langle \bar\psi \psi \rangle = \sum_j \sum_{l=1}^\infty \bar\psi_{\gamma j}^{(-)} \psi_{\gamma j}^{(-)}, \tag{3.6}$$

for the FC we find

$$\langle \bar\psi \psi \rangle = \frac{q}{4\pi a^2} \sum_j \sum_{l=1}^\infty y T_{\beta_j}(y) \left[\left(1 - \frac{\mu}{\sqrt{y^2+\mu^2}}\right) J_{\beta_j+\epsilon_j}^2(yr/a) - \left(1 + \frac{\mu}{\sqrt{y^2+\mu^2}}\right) J_{\beta_j}^2(yr/a) \right], \tag{3.7}$$

with $y = \gamma_{\beta_j,l}$. Here we assume that a cutoff function is introduced without explicitly writing it. The specific form of this function is not important for the discussion below.

For the summation of the series over l in Eq. (3.7), we use the summation formula (see [25,26])

$$\sum_{l=1}^\infty f(\gamma_{\beta_j,l}) T_\beta(\gamma_{\beta_j,l}) = \int_0^\infty dx f(x) - \frac{1}{\pi} \int_0^\infty dx \left[e^{-\beta_j \pi i} f(x e^{\pi i/2}) \times \frac{K_{\beta_j}^{(+)}(x)}{I_{\beta_j}^{(+)}(x)} + e^{\beta_j \pi i} f(x e^{-\pi i/2}) \frac{K_{\beta_j}^{(+)*}(x)}{I_{\beta_j}^{(+)*}(x)} \right], \tag{3.8}$$

where the asterisk means complex conjugate. In this formula, for a given function $F(x)$, we use the notation

$$F^{(+)}(x) = \begin{cases} xF'(x) + \left(\mu + \sqrt{\mu^2 - x^2} - \epsilon_j \beta_j\right) F(x), & x < \mu, \\ xF'(x) + \left(\mu + i\sqrt{x^2 - \mu^2} - \epsilon_j \beta_j\right) F(x), & x \geq \mu. \end{cases} \tag{3.9}$$

Note that for $x < \mu$ one has $F^{(+)*}(x) = F^{(+)}(x)$. The ratio of the combinations of the modified Bessel functions in Eq. (3.8) may be presented in the form

$$\frac{K_{\beta_j}^{(+)}(x)}{I_{\beta_j}^{(+)}(x)} = \frac{W_{\beta_j,\beta_j+\epsilon_j}^{(+)}(x) + i\sqrt{1 - \mu^2/x^2}}{x[I_{\beta_j}^2(x) + I_{\beta_j+\epsilon_j}^2(x)] + 2\mu I_{\beta_j}(x) I_{\beta_j+\epsilon_j}(x)}, \tag{3.10}$$

with the notation defined by

$$W^{(\pm)}_{\beta_j,\beta_j+\epsilon_j}(x) = x[I_{\beta_j}(x)K_{\beta_j}(x) - I_{\beta_j+\epsilon_j}(x)K_{\beta_j+\epsilon_j}(x)] \pm \mu[I_{\beta_j+\epsilon_j}(x)K_{\beta_j}(x) - I_{\beta_j}(x)K_{\beta_j+\epsilon_j}(x)]. \quad (3.11)$$

The notation with the lower sign will be used below.

Applying to the series over l in Eq. (3.7) the summation formula and comparing with Eq. (2.10), we see that the term in the FC corresponding to the first integral in the right-hand side of Eq. (3.8) coincides with the condensate in a boundary-free conical space. As a result, the FC is presented in the decomposed form

$$\langle\bar\psi\psi\rangle = \langle\bar\psi\psi\rangle_{0,\mathrm{ren}} + \langle\bar\psi\psi\rangle_{\mathrm{b}}, \quad (3.12)$$

where $\langle\bar\psi\psi\rangle_{\mathrm{b}}$ is the part induced by the circular boundary. For the function $f(x)$ corresponding to Eq. (3.7), in the second term on the right-hand side of Eq. (3.8), the part of the integral over the region $(0, \mu)$ vanishes. Consequently, the boundary-induced contribution for the FC in the region inside the circle is given by the expression

$$\langle\bar\psi\psi\rangle_{\mathrm{b}} = \frac{q}{2\pi^2}\sum_j \int_m^\infty dx x\left\{m\frac{I^2_{\beta_j}(xr) - I^2_{\beta_j+\epsilon_j}(xr)}{\sqrt{x^2-m^2}} \times \mathrm{Re}[K^{(+)}_{\beta_j}(xa)/I^{(+)}_{\beta_j}(xa)] - [I^2_{\beta_j}(xr) + I^2_{\beta_j+\epsilon_j}(xr)]\,\mathrm{Im}[K^{(+)}_{\beta_j}(xa)/I^{(+)}_{\beta_j}(xa)]\right\}. \quad (3.13)$$

The real and imaginary parts appearing in this equation are easily obtained from Eq. (3.10). Note that under the change $\alpha \to -\alpha$, $j \to -j$, we have $\beta_j \to \beta_j + \epsilon_j$, $\beta_j + \epsilon_j \to \beta_j$. From here it follows that the real/imaginary part in Eq. (3.13) is an odd/even function under this change. Now, from Eq. (3.13), we see that the boundary-induced part in the FC is an even function of α. For points away from the circular boundary and the cone apex, the boundary-induced contribution is finite and the renormalization is reduced to that for the boundary-free geometry. This contribution is a periodic function of the parameter α with the period equal to 1. So, if we present this parameter in the form (2.19) with n_0 being an integer, then the FC depends on α_0 alone.

In the case of a massless field, the expressions for the boundary-induced part in the FC takes the form

$$\langle\bar\psi\psi\rangle_{\mathrm{b}} = -\frac{q}{2\pi^2a^2}\sum_j\int_0^\infty dz\frac{I^2_{\beta_j}(zr/a) + I^2_{\beta_j+\epsilon_j}(zr/a)}{I^2_{\beta_j}(z) + I^2_{\beta_j+\epsilon_j}(z)}. \quad (3.14)$$

As it is seen, this part is always negative. We would like to point out that the boundary-induced FC does not vanish for a massless filed. The corresponding boundary-free part vanishes, and, hence, for a massless field $\langle\bar\psi\psi\rangle = \langle\bar\psi\psi\rangle_{\mathrm{b}}$.

Various special cases of the general formula (3.13) can be considered. In the absence of the magnetic flux one has $\alpha = 0$ and the contributions of the negative and positive values of j to the FC coincide. The corresponding formulas are obtained from (3.13) and (3.14), making the replacements

$$\sum_j \to 2\sum_{j=1/2,3/2,\ldots}, \quad \beta_j \to qj - 1/2,\ \beta_j + \epsilon_j \to qj + 1/2. \quad (3.15)$$

In the case $q = 1$, we obtain the FC induced by the magnetic flux and a circular boundary in the Minkowski spacetime. And finally, in the simplest case $\alpha = 0$ and $q = 1$ one has $\langle\bar\psi\psi\rangle_{0,\mathrm{ren}} = 0$, and the expression (3.13) gives the FC induced by a circular boundary in the Minkowski bulk:

$$\langle\bar\psi\psi\rangle = \frac{1}{\pi^2a^2}\sum_{n=0}^\infty\int_\mu^\infty\frac{dx}{I_n^2(x) + I^2_{n+1}(x) + 2\mu I_n(x)I_{n+1}(x)/x} \times\left\{\mu\frac{W^{(+)}_{n,n+1}(x)}{\sqrt{x^2-\mu^2}}[I_n^2(xr/a) - I^2_{n+1}(xr/a)] - \sqrt{1-\mu^2/x^2}[I_n^2(xr/a) + I^2_{n+1}(xr/a)]\right\}, \quad (3.16)$$

where the function $W^{(+)}_{n,n+1}(x)$ is defined by Eq. (3.10).

Now we turn to the investigation of the FC in asymptotic regions of the parameters. For large values of the circle radius, we replace the modified Bessel functions in Eq. (3.13) with xa in their arguments, by asymptotic expansions for large values of the argument. In the case of a massive field, the dominant contribution to the integral comes from the integration range near the lower limit. In the leading order, one has

$$\langle\bar\psi\psi\rangle_{\mathrm{b}} \approx \frac{qm^2e^{-2ma}}{8\sqrt{\pi}(ma)^{3/2}}\sum_j\epsilon_j[\beta_jI^2_{\beta_j+\epsilon_j}(mr) - (\beta_j+\epsilon_j)I^2_{\beta_j}(mr)], \quad (3.17)$$

and for a fixed value of the radial coordinate, the boundary-induced FC is exponentially small.

For a massless field, assuming $r/a \ll 1$, we expand the modified Bessel function in the numerator of the integrand in Eq. (3.14) in powers of r/a. The dominant contribution comes from the term $j = 1/2$ for $\alpha_0 < 0$ and from the term $j = -1/2$ for $\alpha_0 > 0$. To the leading order we find

$$\langle\bar\psi\psi\rangle_{\mathrm{b}} \approx -\frac{q}{2\pi^2a^2}\frac{(r/2a)^{2q_\alpha-1}}{\Gamma^2(q_\alpha+1/2)} \times\int_0^\infty dz\frac{z^{2q_\alpha-1}}{I^2_{q_\alpha+1/2}(z) + I^2_{q_\alpha-1/2}(z)}, \quad (3.18)$$

where q_α is defined by the relation

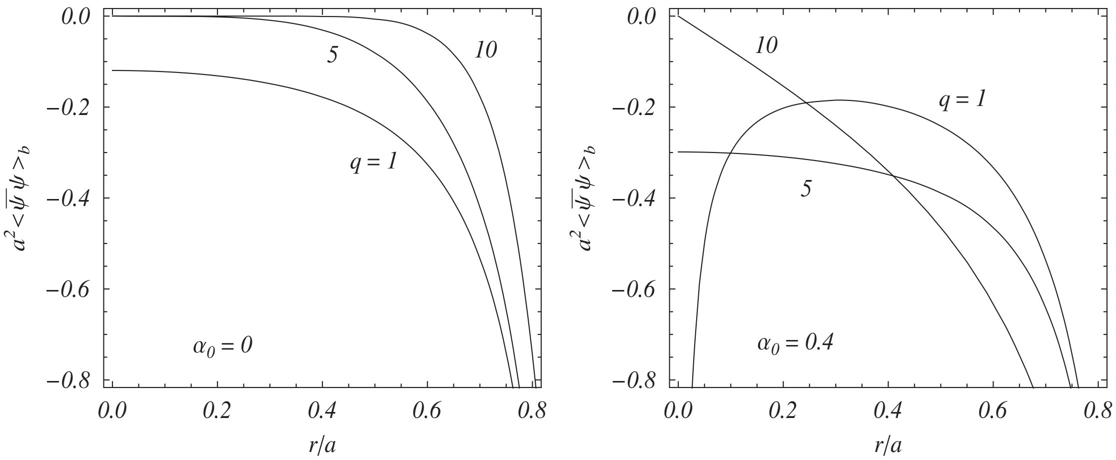

FIG. 3. The FC inside a circular boundary as a function of the radial coordinate for a massless fermionic field.

$$q_\alpha = q(1/2 - |\alpha_0|). \tag{3.19}$$

Hence, for a massless field the FC decays as $a^{-(2q_\alpha+1)}$.

For points near the apex of the cone, $r \to 0$, we use the expansion of the modified Bessel function for small values of the argument. The leading term in the boundary-induced FC takes the form

$$\langle\bar{\psi}\psi\rangle_{\rm b} \approx \frac{q}{2\pi^2 a^2}\frac{(r/2a)^{2q_\alpha-1}}{\Gamma^2(q_\alpha+1/2)}\int_\mu^\infty dz \frac{z^{2q_\alpha}}{\sqrt{z^2-\mu^2}} \times \frac{\mu W^{(+)}_{q_\alpha-1/2,q_\alpha+1/2}(z)-(z^2-\mu^2)/z}{z[I^2_{q_\alpha-1/2}(z)+I^2_{q_\alpha+1/2}(z)]+2\mu I_{q_\alpha-1/2}(z)I_{q_\alpha+1/2}(z)}. \tag{3.20}$$

Note that for a massless field this expression reduces to Eq. (3.18). As it is seen, in the limit $r \to 0$ the boundary-induced part vanishes when $|\alpha_0| < 1/2 - 1/(2q)$ and diverges for $|\alpha_0| > 1/2 - 1/(2q)$. Notice that in the former case the irregular mode is absent and the divergence in the latter case comes from the irregular mode. For the magnetic vortex in the background Minkowski spacetime, the boundary-induced contribution diverges as $r^{-2|\alpha_0|}$. In the case $|\alpha_0| = 1/2 - 1/(2q)$, corresponding to $q_\alpha = 1/2$, the boundary-induced FC tends to a finite limiting value.

The boundary-induced part in the FC diverges on the circle. For points near the circle the main contribution to Eq. (3.14) comes from large values of j. Introducing a new integration variable $y = z/\beta_j$, we use the uniform asymptotic expansion for the modified Bessel function for large values of the order. To the leading order in the expansion over $(1 - r/a)$ one finds the behavior

$$\langle\bar{\psi}\psi\rangle_{\rm b} \approx -\frac{1}{8\pi(a-r)^2}. \tag{3.21}$$

This leading term does not depend on the opening angle of the cone and on the magnetic flux. It coincides with the corresponding term for the FC in the geometry of a circle in (2 + 1)-dimensional Minkowski spacetime. This asymptotic behavior is well seen in Fig. 3, where the dependence of the FC on the radial coordinate is presented for a massless fermionic field for various values of the parameter q. The left/right plot corresponds to the value of the parameter $\alpha_0 = 0/\alpha_0 = 0.4$. Note that, in accordance with the asymptotic analysis given above, for $\alpha_0 = 0.4$ the FC diverges at the cone apex for $q < 5$, vanishes for $q > 5$, and takes a finite value for $q = 5$. In particular, for $q = 10$ one has $\langle\bar{\psi}\psi\rangle \propto r$ in the limit $r \to 0$. These properties are well seen from the right plot of Fig. 3.

In Fig. 4, we present the condensate for a massless fermionic field inside a circular boundary as a function of the magnetic flux. The graphs are plotted for $r/a = 0.5$ and for several values of the opening angle of the conical space. Recall that for a massless field the boundary-free part in the FC vanishes.

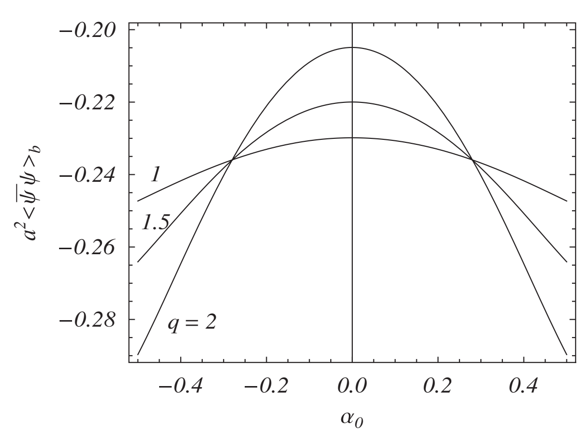

FIG. 4. The FC for a massless field inside a circular boundary as a function of α_0.

IV. FERMIONIC CONDENSATE IN THE EXTERIOR REGION

In the region outside a circular boundary the negative-energy eigenspinors, obeying the boundary condition (3.1) with $n_\mu = -\delta^1_\mu$, have the form [13]

$$\psi^{(-)}_{\gamma j}(x) = c_0 e^{-iqj\phi+iEt}\begin{pmatrix} \frac{\gamma\epsilon_j e^{-iq\phi/2}}{E+m} g_{\beta_j,\beta_j+\epsilon_j}(\gamma a, \gamma r) \\ g_{\beta_j,\beta_j}(\gamma a, \gamma r)e^{iq\phi/2}\end{pmatrix}, \quad (4.1)$$

with the function

$$g_{\nu,\rho}(x, y) = \bar{Y}^{(-)}_\nu(x)J_\rho(y) - \bar{J}^{(-)}_\nu(x)Y_\rho(y), \quad (4.2)$$

and $Y_\nu(x)$ being the Neumann function. The barred notation in Eq. (4.2) is defined by the relation

$$\bar{F}^{(-)}_{\beta_j}(z) = -\epsilon_j z F_{\beta_j+\epsilon_j}(z) - \left(\sqrt{z^2+\mu^2}+\mu\right)F_{\beta_j}(z), \quad (4.3)$$

with $F = J, Y$ and $\mu = ma$. The normalization coefficient is given by the expression

$$c_0^2 = \frac{2E\gamma}{\phi_0(E+m)}[\bar{J}^{(-)2}_{\beta_j}(\gamma a) + \bar{Y}^{(-)2}_{\beta_j}(\gamma a)]^{-1}. \quad (4.4)$$

The positive-energy eigenspinors are found with the help of the relation $\psi^{(+)}_{\gamma n} = \sigma_1 \psi^{(-)*}_{\gamma n}$. Note that for the region under consideration the conical singularity is excluded by the boundary and all modes described by eigenspinors (4.1) are regular.

Substituting the eigenspinors into the mode-sum formula (2.5), the FC is written in the form

$$\langle\bar{\psi}\psi\rangle = \frac{q}{4\pi}\sum_j \int_0^\infty d\gamma \frac{\gamma}{E} \times \frac{(E-m)g^2_{\beta_j,\beta_j+\epsilon_j}(\gamma a,\gamma r) - (E+m)g^2_{\beta_j,\beta_j}(\gamma a,\gamma r)}{\bar{J}^{(-)2}_{\beta_j}(\gamma a) + \bar{Y}^{(-)2}_{\beta_j}(\gamma a)}. \quad (4.5)$$

As before, we assume the presence of a cutoff function which makes the expression on the right-hand side of Eq. (4.5) finite. Similar to the interior region, the FC outside a circular boundary may be written in the decomposed form (3.12).

In order to find an explicit expression for the boundary-induced part, we note that the boundary-free part is given by Eq. (2.10). For the evaluation of the difference between the total FC and the boundary-free part, we use the identity

$$\frac{g^2_{\beta_j,\lambda}(x,y)}{\bar{J}^{(-)2}_{\beta_j}(x) + \bar{Y}^{(-)2}_{\beta_j}(x)} - J^2_\lambda(y) = -\frac{1}{2}\sum_{l=1,2}\frac{\bar{J}^{(-)}_{\beta_j}(x)}{\bar{H}^{(-,l)}_{\beta_j}(x)}H^{(l)2}_\lambda(y), \quad (4.6)$$

with $\lambda = \beta_j, \beta_j + \epsilon_j$, and with $H^{(l)}_\nu(x)$ being the Hankel function. For the boundary-induced part in the FC, we find the expression

$$\langle\bar{\psi}\psi\rangle_{\rm b} = -\frac{q}{8\pi}\sum_j\sum_{l=1,2}\int_0^\infty d\gamma\frac{\gamma}{E}\frac{\bar{J}^{(-)}_{\beta_j}(\gamma a)}{\bar{H}^{(-,l)}_{\beta_j}(\gamma a)} \times [(E-m)H^{(l)2}_{\beta_j+\epsilon_j}(\gamma r) - (E+m)H^{(l)2}_{\beta_j}(\gamma r)]. \quad (4.7)$$

In the complex plane γ, the integrand of the term with $l = 1$ ($l = 2$) decays exponentially in the limit $\mathrm{Im}(\gamma) \to \infty$ [$\mathrm{Im}(\gamma) \to -\infty$] for $r > a$. By using these properties, we rotate the integration contour in the complex plane γ by the angle $\pi/2$ for the term with $l = 1$ and by the angle $-\pi/2$ for the term with $l = 2$. The integrals over the segments $(0, im)$ and $(0, -im)$ of the imaginary axis cancel each other. Introducing the modified Bessel functions, the boundary-induced part in the FC is presented in the form

$$\langle\bar{\psi}\psi\rangle_{\rm b} = \frac{q}{2\pi^2}\sum_j\int_m^\infty dz z\left\{m\frac{K^2_{\beta_j}(zr) - K^2_{\beta_j+\epsilon_j}(zr)}{\sqrt{z^2-m^2}} \times \mathrm{Re}[I^{(-)}_{\beta_j}(za)/K^{(-)}_{\beta_j}(za)] - [K^2_{\beta_j}(zr) + K^2_{\beta_j+\epsilon_j}(zr)]\,\mathrm{Im}[I^{(-)}_{\beta_j}(za)/K^{(-)}_{\beta_j}(za)]\right\}, \quad (4.8)$$

where

$$F^{(-)}(z) = zF'(z) - \left(\mu + i\sqrt{z^2-\mu^2} + \epsilon_j\beta_j\right)F(z). \quad (4.9)$$

By using the definition (4.9), the ratio in the integrand of Eq. (4.8) can be written in the form

$$\frac{I^{(-)}_{\beta_j}(x)}{K^{(-)}_{\beta_j}(x)} = \frac{W^{(-)}_{\beta_j,\beta_j+\epsilon_j}(x) + i\sqrt{1-\mu^2/x^2}}{x[K^2_{\beta_j}(x) + K^2_{\beta_j+\epsilon_j}(x)] + 2\mu K_{\beta_j}(x)K_{\beta_j+\epsilon_j}(x)}, \quad (4.10)$$

with the notation $W^{(-)}_{\beta_j,\beta_j+\epsilon_j}(x)$ defined by Eq. (3.11). Now the real and imaginary parts appearing in Eq. (4.8) are easily obtained from Eq. (4.10). By taking into account that under the change $\alpha \to -\alpha$, $j \to -j$, one has $\beta_j \to \beta_j + \epsilon_j$, $\beta_j + \epsilon_j \to \beta_j$, we conclude that the real/imaginary part in Eq. (4.10) is an odd/even function under this change. Now, from Eq. (4.8) it follows that the boundary-induced part in the FC is an even function of α. This function is periodic with the period equal to 1.

For a massless field the expression for the boundary-induced part in the FC simplifies to

$$\langle\bar{\psi}\psi\rangle_{\rm b} = -\frac{q}{2\pi^2a^2}\sum_j\int_0^\infty dz\frac{K^2_{\beta_j}(zr/a) + K^2_{\beta_j+\epsilon_j}(zr/a)}{K^2_{\beta_j}(z) + K^2_{\beta_j+\epsilon_j}(z)}. \quad (4.11)$$

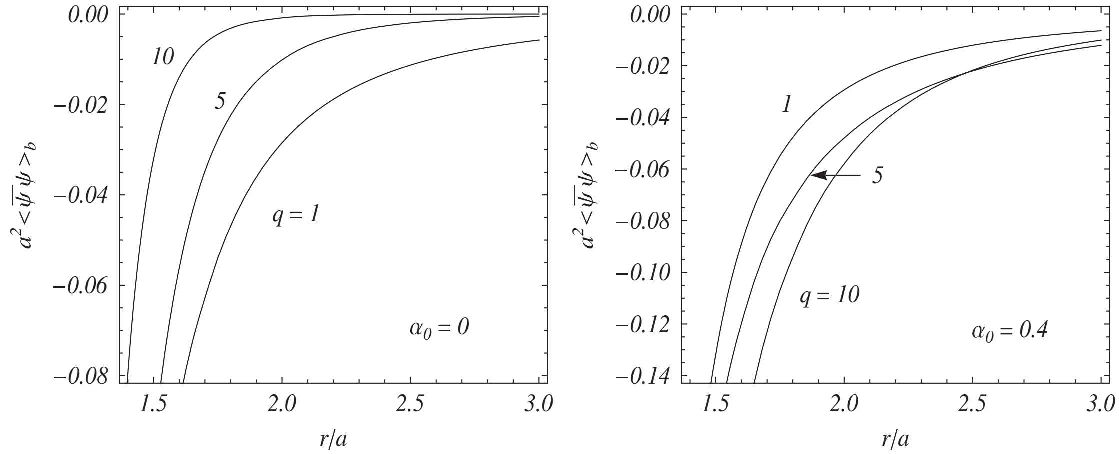

FIG. 5. The FC outside a circular boundary as a function on the radial coordinate for a massless fermionic field.

As in the case of the interior region, the boundary-induced FC does not vanish for a massless field. The corresponding boundary-free part vanishes, and, hence, in this case we have $\langle\bar{\psi}\psi\rangle = \langle\bar{\psi}\psi\rangle_{\rm b}$. When the magnetic flux is absent, $\alpha = 0$, the corresponding expression for the boundary-induced part is obtained from Eq. (4.8) by the replacements (3.15). In particular, for the circle in the Minkowski bulk the formula for the fermionic condensate is obtained from Eq. (3.16) by the interchange $I \rightleftarrows K$, replacing $W^{(+)}_{n,n+1}(x) \to W^{(-)}_{n,n+1}(x)$.

Now let us consider the behavior of the boundary-induced part in the FC in the asymptotic regions of the parameters. First, we consider the limit $a \to 0$ for fixed values of r. By taking into account the asymptotics of the modified Bessel functions for small values of the arguments, to the leading order we find the expression

$$\langle\bar{\psi}\psi\rangle_{\rm b} \approx \frac{q(a/2r)^{2q_\alpha}}{\pi^2 r^2 \Gamma^2(q_\alpha + 1/2)} \int_{mr}^{\infty} dz \frac{z^{2q_\alpha}}{\sqrt{z^2 - m^2 r^2}} \times [(2m^2 r^2 - z^2) K^2_{q_\alpha - 1/2}(z) - z^2 K^2_{q_\alpha + 1/2}(z)], \quad (4.12)$$

with the notation (3.19). For a massless field the integral in (4.12) is evaluated in terms of the gamma function, and one has

$$\langle\bar{\psi}\psi\rangle_{\rm b} \approx -\frac{q\Gamma(q_\alpha + 1)\Gamma(2q_\alpha + 1/2)}{2\pi r^2 \Gamma^3(q_\alpha + 1/2)} \left(\frac{a}{2r}\right)^{2q_\alpha}. \quad (4.13)$$

Hence, in the limit $a \to 0$ and for fixed values of r, the boundary-induced part in FC vanishes as a^{2q_α}.

For a massive field, at large distances from the boundary, under the condition $mr \gg 1$, the main contribution to the integral in Eq. (4.8) comes from the region near the lower limit of the integration. In the leading order we find

$$\langle\bar{\psi}\psi\rangle_{\rm b} \approx -\frac{qe^{-2mr}}{4\pi r^2} \sum_j {\rm Im}[I^{(+)}_{\beta_j}(ma)/K^{(+)}_{\beta_j}(ma)], \quad (4.14)$$

and the boundary-induced FC is exponentially suppressed. For a massless field, the asymptotic at large distances is given by Eq. (4.13) and the boundary-induced condensate decays as $r^{-2q_\alpha - 2}$. For points near the circle, the main contribution to (4.11) comes from large values of j. By using the uniform asymptotic expansion for the Macdonald function for large values of the order, to the leading order one finds $\langle\bar{\psi}\psi\rangle_{\rm b} \approx -[8\pi(r-a)^2]^{-1}$. The leading term in the asymptotic expansion does not depend on the opening angle of the cone and on the magnetic flux. The dependence of the FC outside a circular boundary on the radial coordinate is presented in Fig. 5 for a massless field for various values of the parameter q. The left/right plot corresponds to the value of the parameter $\alpha_0 = 0/\alpha_0 = 0.4$.

In Fig. 6, the fermionic condensate is plotted for a massless field outside a circular boundary as a function of the magnetic flux. The graphs are plotted for $r/a = 1.5$ and for several values of the opening angle of the conical space. For the exterior region there are no irregular modes and the FC is a continuous function of α at half-integer

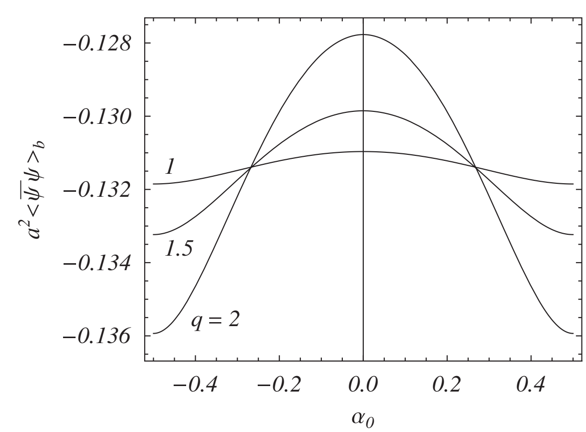

FIG. 6. The FC outside a circular boundary as a function of the magnetic flux.

values. In particular, its derivative vanishes at these points. Note that this is not the case for the interior region.

V. HALF-INTEGER VALUES OF THE PARAMETER α

In this section, we consider the FC for half-integer values of the parameter α. In this case, for the boundary-free geometry the eigenspinors with $j \neq -\alpha$ are still given by Eqs. (2.6). For the eigenspinor corresponding to the special mode with $j = -\alpha$ one has [13]

$$\psi^{(-)}_{(0)\gamma,-\alpha}(x) = \left(\frac{E+m}{\pi\phi_0 rE}\right)^{1/2} e^{iq\alpha\phi+iEt}\begin{pmatrix} \frac{\gamma e^{-iq\phi/2}}{E+m}\sin(\gamma r-\gamma_0) \\ e^{iq\phi/2}\cos(\gamma r-\gamma_0)\end{pmatrix}, \tag{5.1}$$

where $\gamma_0 = \arccos[\sqrt{(E-m)/2E}]$. As we have noted, for half-integer values of α the mode with $j = -\alpha$ corresponds to the irregular mode. The contribution of the modes with $j \neq -\alpha$ to the FC is the same as before. Special consideration is needed for the mode with $j = -\alpha$ only. For the contribution of this mode to the FC one has

$$\langle\bar\psi\psi\rangle_{0,j=-\alpha} = \int_0^\infty d\gamma\, \bar\psi^{(-)}_{(0)\gamma,-\alpha}\psi^{(-)}_{(0)\gamma,-\alpha} = -\frac{q}{2\pi^2 r} \times \int_0^\infty d\gamma \frac{m+\gamma\sin(2\gamma r) - m\cos(2\gamma r)}{\sqrt{\gamma^2+m^2}}. \tag{5.2}$$

The part with the last term in the numerator is finite, whereas the part with the first two terms is divergent. As before, in order to deal with this divergence, we introduce the cutoff function $e^{-s\gamma^2}$. The integral in the right-hand side of Eq. (5.2) is expressed in terms of the Macdonald function.

For half-integer values of α, it can be easily seen that for the series in the contribution of the modes with $j \neq -\alpha$ one has

$$\sum_{j\neq-\alpha} I_{\beta_j}(x) = \sum_{j\neq-\alpha} I_{\beta_j+\epsilon_j}(x) = \sum_{n=1}^\infty [I_{qn-1/2}(x) + I_{qn+1/2}(x)]. \tag{5.3}$$

Summing the contributions from the mode with $j = -\alpha$ and from the modes $j \neq -\alpha$, for the regularized FC we find the expression

$$\langle\bar\psi\psi\rangle_{0,\mathrm{reg}} = -\frac{qme^{m^2 s}}{(2\pi)^{3/2}}\sum_{n=1}^\infty \int_0^{r^2/2s} dx \frac{x^{-1/2}e^{-m^2r^2/2x-x}}{\sqrt{r^2-2xs}} \times [I_{qn-1/2}(x) + I_{qn+1/2}(x)] - \frac{qm}{4\pi^2 r}[e^{m^2s/2}K_0(m^2s/2) + 2K_1(2mr) - 2K_0(2mr)]. \tag{5.4}$$

After the summation over n by using the formula given in Sec. II, we find the following representation

$$\langle\bar\psi\psi\rangle_{0,\mathrm{reg}} = -\frac{m}{2\pi}\left\{\frac{e^{m^2 s}}{\sqrt{2\pi}}\int_0^{r^2/2s} dx \frac{x^{-1/2}e^{-m^2r^2/2x}}{\sqrt{r^2-2xs}} + \frac{1}{r}\sum_{l=1}^p \frac{\cot(\pi l/q)}{e^{2mr\sin(\pi l/q)}} + \frac{q}{2\pi r} \times \int_0^\infty dy \frac{\sinh(y/2)\sinh(qy)}{\cosh(qy)-\cos(q\pi)}\frac{e^{-2mr\cosh(y/2)}}{\cosh(y/2)}\right\} - \frac{qm}{2\pi^2 r}[K_1(2mr) - K_0(2mr)] + o(s), \tag{5.5}$$

where $2p \leq q < 2p+2$. The first term in the figure braces of this expression corresponds to the contribution coming from the Minkowski spacetime part. It is subtracted in the renormalization procedure, and for the renormalized FC in a boundary-free conical space one finds

$$\langle\bar\psi\psi\rangle_{0,\mathrm{ren}} = -\frac{qm}{4\pi^2 r}\int_0^\infty dy \frac{\sinh(y/2)\sinh(qy)}{\cosh(qy)-\cos(q\pi)} \times \frac{e^{-2mr\cosh(y/2)}}{\cosh(y/2)} - \frac{m}{2\pi r} \times \sum_{l=1}^p \frac{\cos(\pi l/q)}{\sin(\pi l/q)} e^{-2mr\sin(\pi l/q)} - \frac{qm}{2\pi^2 r}[K_1(2mr) - K_0(2mr)]. \tag{5.6}$$

As before, the FC is a periodic function of α with the period 1. Note that, in the case under consideration, the renormalized FC in a boundary-free conical space does not vanish for a massless field:

$$\langle\bar\psi\psi\rangle_{0,\mathrm{ren}} = -\frac{q}{4\pi^2 r^2}, \quad m = 0. \tag{5.7}$$

This corresponds to the contribution of the irregular mode.

Now we consider the region inside a circle with radius a. The contribution of the modes with $j \neq -\alpha$ is given by Eq. (3.13), where now the summation goes over $j \neq -\alpha$. For the evaluation of the contribution coming from the mode with $j = -\alpha$, we note that the negative-energy eigenspinor for this mode has the form [13]

$$\psi^{(-)}_{\gamma,-\alpha}(x) = \frac{b_0}{\sqrt{r}} e^{iq\alpha\phi+iEt}\begin{pmatrix} \frac{\gamma e^{-iq\phi/2}}{E+m}\sin(\gamma r-\gamma_0) \\ e^{iq\phi/2}\cos(\gamma r-\gamma_0)\end{pmatrix}, \tag{5.8}$$

where γ_0 is defined after Eq. (5.1). From boundary condition (3.1) it follows that the eigenvalues of γ are solutions of the equation

$$m\sin(\gamma a) + \gamma\cos(\gamma a) = 0. \tag{5.9}$$

We denote the positive roots of this equation by $\gamma_l = \gamma a$, $l = 1, 2, \ldots$. From the normalization condition, for the coefficient in Eq. (5.8) one has

$$b_0^2 = \frac{E+m}{aE\phi_0}[1-\sin(2\gamma a)/(2\gamma a)]^{-1}. \tag{5.10}$$

Using Eq. (5.8), for the contribution of the mode under consideration to the FC we find

$$\langle\bar{\psi}\psi\rangle_{j=-\alpha} = -\frac{1}{a\phi_0 r} \times \sum_{l=1}^{\infty} \frac{\mu + \gamma_l \sin(2\gamma_l r/a) - \mu\cos(2\gamma_l r/a)}{\sqrt{\gamma_l^2+\mu^2}[1-\sin(2\gamma_l)/(2\gamma_l)]}, \tag{5.11}$$

where $\mu = ma$ and the presence of a cutoff function is assumed. For the summation of the series in Eq. (5.11), we use the Abel-Plana-type formula

$$\sum_{l=1}^{\infty} \frac{\pi f(\gamma_l)}{1-\sin(2\gamma_l)/(2\gamma_l)} = -\frac{\pi f(0)/2}{1/\mu+1} + \int_0^\infty dz f(z) - i\int_0^\infty dz \frac{f(iz)-f(-iz)}{\frac{z+\mu}{z-\mu}e^{2z}+1}. \tag{5.12}$$

The latter is obtained from the summation formula given in [27] (see also [26]) taking $b_1 = 0$ and $b_2 = -1/\mu$. For the functions $f(z)$ corresponding to Eq. (5.11) one has $f(0) = 0$. The second term on the right-hand side of (5.12) gives the part corresponding to the boundary-free geometry. As a result, the FC is presented in the form

$$\langle\bar{\psi}\psi\rangle_{j=-\alpha} = \langle\bar{\psi}\psi\rangle_{0,j=-\alpha} + \langle\bar{\psi}\psi\rangle_{\mathrm{b},j=-\alpha}, \tag{5.13}$$

where the boundary-induced part is given by the expression

$$\langle\bar{\psi}\psi\rangle_{\mathrm{b},j=-\alpha} = \frac{q}{\pi^2 r} \times \int_m^\infty dx \frac{m - x\sinh(2xr) - m\cosh(2xr)}{\sqrt{x^2-m^2}(\frac{x+m}{x-m}e^{2ax}+1)}. \tag{5.14}$$

The contribution of the modes $j \neq -\alpha$ remains the same and is obtained from the corresponding expressions given above for non-half-integer values of α by the direct substitution $\alpha = 1/2$.

Expression (5.14) for the boundary-induced part is simplified for a massless field

$$\langle\bar{\psi}\psi\rangle_{\mathrm{b},j=-\alpha} = -\frac{q}{4\pi^2 r^2}\left[\frac{\pi r/a}{\sin(\pi r/a)} - 1\right]. \tag{5.15}$$

Note that this part is finite at the circle center. By taking into account Eq. (5.7) and adding the contribution coming from the modes with $j \neq -\alpha$, for the total FC one finds

$$\langle\bar{\psi}\psi\rangle = -\frac{q}{4\pi a r\sin(\pi r/a)} - \frac{q}{\pi^2 a^2} \times \sum_{n=1}^{\infty}\int_0^\infty dz \frac{I_{qn-1/2}^2(zr/a) + I_{qn+1/2}^2(zr/a)}{I_{qn-1/2}^2(z) + I_{qn+1/2}^2(z)}. \tag{5.16}$$

The expression on the right-hand side is always negative. The first term dominates near the cone apex. Near the boundary this term behaves as $(1-r/a)^{-1}$, whereas the second term behaves like $(1-r/a)^{-2}$. Hence, the latter dominates near the circle.

In the region outside a circular boundary there are no irregular modes and the FC is a continuous function of the parameter α at half-integer values. The corresponding expression is obtained taking the limit $\alpha_0 \to 1/2$: $\langle\bar{\psi}\psi\rangle = \lim_{\alpha_0\to 1/2}[\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}} + \langle\bar{\psi}\psi\rangle_{\mathrm{b}}]$, where the separate terms are given by expressions (2.28) and (4.8). However, note that the limiting values of the separate terms $\langle\bar{\psi}\psi\rangle_{0,\mathrm{ren}}$ and $\langle\bar{\psi}\psi\rangle_{\mathrm{b}}$, defined by these expressions, do not coincide with the boundary-free and boundary-induced parts of the FC at half-integer values of α.

VI. CONCLUSION

In this paper, we have investigated the FC in a $(2+1)$-dimensional conical spacetime with a circular boundary in the presence of a magnetic flux. The case of a massive fermionic field is considered with the MIT bag boundary condition on the circle. As the first step, we have considered a conical space without boundaries and with a special case of boundary conditions at the cone apex, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. For the evaluation of the FC the direct summation over the modes is used with the spinorial eigenfunctions (2.6). If the ratio of the magnetic flux to the flux quantum is not a half-integer number, the regularized FC with the exponential cutoff function is given by expression (2.13). A simple expression for the renormalized FC, Eq. (2.17), is obtained in the special case when the parameter q is an integer and is related to the parameter α by Eq. (2.14). In this special case, the renormalized FC vanishes for a massless field and for a massive field in a conical space with $q = 2$ and is negative for other cases.

For the general case of the parameters q and α, a convenient expression for the regularized FC is obtained by using the integral representation (2.23) for the series involving the modified Bessel function. This formula allows us to extract explicitly the part in FC corresponding to the Minkowski spacetime in the absence of the magnetic flux. Subtracting this part, for the renormalized FC we derived formula (2.28). At distances larger than the Compton wavelength of the spinor particle, $mr \gg 1$, the FC is suppressed by the factor e^{-2mr} for $1 \le q < 2$ and by the factor $e^{-2mr\sin(\pi/q)}$ for $q \ge 2$. In the special case when the magnetic flux is absent, the general formula simplifies to Eq. (2.31). Another limiting case corresponds to the magnetic flux in the background of the Minkowski spacetime with the renormalized FC given by Eq. (2.33). An alternative expression for the FC is obtained by using the integral representation (2.34) for the series involving the modified Bessel function. This leads to the expression (2.36) for the renormalized FC. In the special cases of a magnetic flux in the background of the Minkowski spacetime and for a conical space in the absence of the magnetic flux, the general formula reduces to Eqs. (2.39) and (2.40), respectively.

In Sec. III, we have considered the FC inside a circular boundary concentric with the apex of the cone. The corresponding eigenspinors are given by the expression (3.2) and the eigenvalues of the quantum number γ are solutions of Eq. (3.3). The mode sum for the FC contains a series over these solutions. For the summation of this series we have used the Abel-Plana-type formula (3.8). This allows us to decompose the FC into the boundary-free and boundary-induced parts, Eq. (3.12), with the boundary-induced part given by Eq. (3.13). The asymptotic near the cone apex is given by Eq. (3.20). In this limit the boundary-induced part vanishes when $|\alpha_0| < 1/2 - 1/(2q)$ and diverges for $|\alpha_0| > 1/2 - 1/(2q)$. In the former case the irregular mode is absent, and the divergence in the latter case comes from the irregular mode. The boundary-induced FC diverges on the circle. The leading term in the asymptotic expansion over the distance from the boundary is given by Eq. (3.21). This term does not depend on the opening angle of the cone and on the magnetic flux and coincides with the corresponding term for the FC in the geometry of a circle in (2 + 1)-dimensional Minkowski spacetime.

The region outside a circular boundary is considered in Sec. IV. The boundary-induced part of the FC in this region is given by Eq. (4.8). This expression is obtained from the corresponding formula for the interior region by the interchange of the modified Bessel functions I and K. For a massless field, the general formula is simplified to Eq. (4.11) and the boundary-induced part is negative. In the limit when the circle radius tends to zero, $a \to 0$, and for a fixed value of r, the boundary-induced part in FC vanishes as a^{2q_α}. At large distances from the boundary, for a massive field, the asymptotic behavior is given by Eq. (4.14) and the boundary-induced FC is exponentially suppressed. For a massless field, the asymptotic at large distances is given by Eq. (4.13) and the boundary-induced condensate decays as $r^{-2q_\alpha-2}$.

The special case of the magnetic flux corresponding to half-integer values of the parameter α is discussed in Sec. V. For this case, the contribution of the mode with $j = -\alpha$ should be considered separately. The renormalized FC in the boundary-free geometry is given by Eq. (5.6) and does not vanish in the massless limit. In the region inside a circular boundary, the contribution of the special mode with $j = -\alpha$ to the FC is given by Eq. (5.14) and is finite at the circle center. For a massless fermionic field, the total FC inside a circular boundary is given by Eq. (5.16) and is negative. In the region outside a circular boundary, the FC is a continuous function of the parameter α at half-integer values and the corresponding expression is obtained from that in Sec. IV taking the limit $\alpha_0 \to 1/2$.

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[1] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983); R. Jackiw, Phys. Rev. D **29**, 2375 (1984); A. N. Redlich, Phys. Rev. D **29**, 2366 (1984); M. B. Paranjape, Phys. Rev. Lett. **55**, 2390 (1985); D. Boyanovsky and R. Blankenbecler, Phys. Rev. D **31**, 3234 (1985); R. Blankenbecler and D. Boyanovsky, Phys. Rev. D **34**, 612 (1986).
[2] T. Jaroszewicz, Phys. Rev. D **34**, 3128 (1986).
[3] E. G. Flekkøy and J. M. Leinaas, Int. J. Mod. Phys. A **6**, 5327 (1991).
[4] H. Li, D. A. Coker, and A. S. Goldhaber, Phys. Rev. D **47**, 694 (1993).
[5] V. P. Gusynin, V. A. Miransky, and L. A. Shovkovy, Phys. Rev. D **52**, 4718 (1995); R. R. Parwani, Phys. Lett. B **358**, 101 (1995).
[6] Yu. A. Sitenko, Phys. At. Nucl. **60**, 2102 (1997); Phys. Rev. D **60**, 125017 (1999).
[7] G. V. Dunne, *Topological Aspects of Low Dimensional Systems* (Springer, Berlin, 1999).
[8] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009).

[9] S. Bellucci and A. A. Saharian, Phys. Rev. D **79**, 085019 (2009); **80**, 105003 (2009); S. Bellucci, A. A. Saharian, and V. M. Bardeghyan, Phys. Rev. D **82**, 065011 (2010).
[10] Yu. A. Sitenko and N. D. Vlasii, Low Temp. Phys. **34**, 826 (2008).
[11] S. Leseduarte and A. Romeo, Commun. Math. Phys. **193**, 317 (1998).
[12] C. G. Beneventano, M. De Francia, K. Kirsten, and E. M. Santangelo, Phys. Rev. D **61**, 085019 (2000); M. De Francia and K. Kirsten, Phys. Rev. D **64**, 065021 (2001).
[13] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and V. M. Bardeghyan, Phys. Rev. D **82**, 085033 (2010).
[14] I. L. Buchbinder and E. N. Kirillova, Int. J. Mod. Phys. A **4**, 143 (1989); E. Elizalde, S. D. Odintsov, and Yu. I. Shil'nov, Mod. Phys. Lett. A **9**, 931 (1994); E. Elizalde, S. Leseduarte, and S. D. Odintsov, Phys. Rev. D **49**, 5551 (1994); Phys. Lett. B **347**, 33 (1995); D. K. Kim and G. Koh, Phys. Rev. D **51**, 4573 (1995); E. Elizalde and S. D. Odintsov, Phys. Rev. D **51**, 5990 (1995); E. Elizalde, S. Leseduarte, S. D. Odintsov, and Yu. I. Shil'nov, Phys. Rev. D **53**, 1917 (1996).
[15] I. Brevik and T. Toverud, Classical Quantum Gravity **12**, 1229 (1995).
[16] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and A. S. Tarloyan, Phys. Rev. D **74**, 025017 (2006).
[17] E. R. Bezerra de Mello, V. B. Bezerra, and A. A. Saharian, Phys. Lett. B **645**, 245 (2007).
[18] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and A. S. Tarloyan, Phys. Rev. D **78**, 105007 (2008).
[19] A. A. Saharian, Classical Quantum Gravity **25**, 165012 (2008); E. R. Bezerra de Mello and A. A. Saharian, J. High Energy Phys. 12 (2008) 081.
[20] P. E. Lammert and V. H. Crespi, Phys. Rev. Lett. **85**, 5190 (2000); A. Cortijo and M. A. H. Vozmediano, Nucl. Phys. B **763**, 293 (2007); Yu. A. Sitenko and N. D. Vlasii, Nucl. Phys. B **787**, 241 (2007); C. Furtado, F. Moraes, and A. M. M. Carvalho, Phys. Lett. A **372**, 5368 (2008); A. Jorio, G. Dresselhaus, and M. S. Dresselhaus, *Carbon Nanotubes: Advanced Topics in the Synthesis, Structure, Properties and Applications* (Springer, Berlin, 2008).
[21] A. Krishnan *et al.*, Nature (London) **388**, 451 (1997); S. N. Naess, A. Elgsaeter, G. Helgesen, and K. D. Knudsen, Sci. Tech. Adv. Mater. **10**, 065002 (2009).
[22] P. de Sousa Gerbert and R. Jackiw, Commun. Math. Phys. **124**, 229 (1989); P. de Sousa Gerbert, Phys. Rev. D **40**, 1346 (1989); Yu. A. Sitenko, Ann. Phys. (N.Y.) **282**, 167 (2000).
[23] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.
[24] E. R. Bezerra de Mello, Classical Quantum Gravity **27**, 095017 (2010).
[25] A. A. Saharian and E. R. Bezerra de Mello, J. Phys. A **37**, 3543 (2004).
[26] A. A. Saharian, *The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect* (Yerevan State University Publishing House, Yerevan, 2008); , Report No. ICTP/2007/082; arXiv:0708.1187.
[27] A. Romeo and A. A. Saharian, J. Phys. A **35**, 1297 (2002).