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Galilean conformal mechanics from nonlinear realizations

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We apply the nonlinear realizations method for constructing new Galilean conformal mechanics models. Our starting point is the Galilean conformal algebra which is a nonrelativistic contraction of its relativistic counterpart. We calculate Maurer-Cartan one-forms, examine various choices of the relevant coset spaces, and consider the geometric inverse Higgs-type constraints which reduce the number of the independent coset parameters and, in some cases, provide dynamical equations. New Galilean conformally invariant actions are derived in arbitrary space-time dimension $D = d + 1$ (no central charges), as well as in the special dimension $D = 2 + 1$ with one exotic central charge. We obtain new classical mechanics models which extend the standard ($D = 0 + 1$) conformal mechanics in the presence of d nonvanishing space dimensions.

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I. INTRODUCTION

In recent years, there has been an increasing interest in the applications of renowned AdS/CFT correspondence [1–3] to nonrelativistic conformal field theory [4,5] (see also [6–9] and references therein). In the AdS/CFT framework, an important role is played by the (super)conformal quantum mechanics as the simplest counterpart of the higher-dimensional (super)conformal field theories. Keeping in mind an extension to the nonrelativistic case, it is desirable to consider various nonrelativistic versions of the (super)conformal mechanics. The study of such models would allow us to gain deeper insights into the physical and mathematical aspects of nonrelativistic conformal symmetry and can be used in the analysis of the corresponding (super)strings and field theories.

The basic aim of the present paper is to construct, at the classical level, several new mechanical models invariant under Galilean conformal symmetries. Our main tool will be the systematic use of the universal geometric method of nonlinear realizations [10].

It is known that the de Alfaro, Fubini, and Furlan (AFF) conformal mechanics [11], as well as its supersymmetric extensions [12–14], can be adequately described in the framework of nonlinear realizations of the $D = 0 + 1$ conformal group $\mathrm{SL}(2,\mathrm{R}) \sim \mathrm{SO}(1,2)$ [15] and its supersymmetric extensions [14,16]. An important part of these geometric techniques is the covariant reduction of the number of (super)conformal group parameters by means of the inverse Higgs mechanism [17], which singles out the dynamical variables. The inverse Higgs constraints can be derived in the geometrically transparent way, using the formalism of the Maurer-Cartan (MC) one-forms on the suitably chosen cosets of the symmetry group.

In this paper we apply the MC method to the Galilean conformal (GC) group. The GC group extends the one-dimensional conformal symmetry of [11] to the conformal symmetry of the $D = d + 1$–dimensional nonrelativistic space-time, with $d \geq 1$ being the number of space dimensions. For a long time, since it was proposed in [18,19], the name of nonrelativistic conformal symmetry was attributed to the Schrödinger symmetries, which provide the covariance of the Schrödinger equation describing a nonrelativistic massive particle.[1] However, the corresponding Schrödinger algebra does not require mass parameters to vanish and does not contain the nonrelativistic counterpart of the conformal spatial accelerations. An alternative candidate for the nonrelativistic conformal symmetry algebra is the Galilean conformal algebra (GCA), and it is the symmetry we shall deal with in this paper. It can be obtained by a contraction of the $\frac{(d+2)(d+3)}{2}$-dimensional relativistic conformal algebra $o(d+1,2)$, in such a way that the number of generators is preserved [21–26].[2]

GCA in d space dimensions has the following semidirect sum structure:

$$
\begin{aligned}
&\mathcal{C}^{(d)} = (o(2,1) \oplus o(d)) \ltimes \mathcal{A}^{(3d)} \qquad (d \geq 2),\\
&\mathcal{C}^{(1)} = o(2,1) \ltimes \mathcal{A}^{(3)}, \qquad \mathcal{C}^{(0)} = o(2,1).
\end{aligned}
\tag{1.1}
$$

[1] It should be mentioned that the Schrödinger algebra is simply related to the Lie algebra description of the invariance of the heat equation, proposed first in the nineteenth century [20].

[2] To avoid possible confusion, let us note that the term "nonrelativistic conformal symmetries" is sometimes used for the infinite-dimensional conformal isometries of nonrelativistic space-time [6,22,26,27]. Nonrelativistic conformal symmetries arise due to the degeneracy of the Galilean space-time metric and have no relativistic counterpart. Here we will not deal with this type of conformal symmetry.

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Here, $o(2,1)$ describes the conformal symmetries on the worldline and was employed in [11,15], $o(d)$ generates the space rotations, and $\mathcal{A}^{(3d)}$ represents the $3d$-dimensional Abelian subalgebra of space translations, Galilean boosts, and nonrelativistic constant accelerations. We see that the symmetry of standard conformal mechanics is given by $\mathcal{C}^{(0)} = o(2,1) \subset \mathcal{C}^{(d)}$ for $d > 0$; besides, it is clear that $\mathcal{C}^{(d)}$ includes as a subalgebra the centerless Galilean $\frac{(d+1)(d+2)}{2}$-dimensional algebra in d space dimensions:

$$\mathcal{G}^{(d)} = (o(1,1) \oplus o(d)) \ltimes \tilde{\mathcal{A}}^{(2d)} \qquad (d \geq 2),$$
$$G^{(1)} = o(1,1) \ltimes \tilde{\mathcal{A}}^{(2)}, \tag{1.2}$$

where the Abelian subalgebra $o(1,1)$ describes the time translations and $\tilde{\mathcal{A}}^{(2d)}$ is formed by the space translations and Galilean boosts; i.e., we get that $\mathcal{G}^{(d)} \subset \mathcal{C}^{(d)}$. Note that the algebra (1.2) admits an extension by one central charge for any d and by two central charges for the special case of $d = 2$ (see below); however, only the latter case, and with only one central charge, can be promoted to a central extension of the GCA for $d = 2$. We also add that the semidirect sum structure represented by the formulas (1.1) and (1.2) reflects as well the semidirect product decomposition of the corresponding centerless Galilean and Galilean conformal groups. We will denote these groups as $\hat{\mathcal{G}}^{(d)}$ and $\hat{\mathcal{C}}^{(d)}$, respectively.

We begin our paper by recalling, in Sec. II, a few known examples of the application of the techniques of MC one-forms and inverse Higgs constraints to mechanical systems in order to derive the relevant dynamical equations of motion and invariant actions. We consider the following models of classical mechanics:

(i) Massive free nonrelativistic particle model (for any d, we use $\mathcal{G}^{(d)}$ with one central charge M) [28,29];
(ii) Massive free nonrelativistic particle model with higher-order Chern-Simons-type term (for $d = 2$, we use $\mathcal{G}^{(2)}$ with two central charges) [30];
(iii) Standard conformal mechanics model [for $d = 0$, with $\mathcal{C}^{(0)} = o(2,1)$] [11,15].

In cases (i).and (ii) we shall use the MC one-forms for centrally extended Lie algebras. In particular, the MC one-forms associated with the central charge generators will be used for the geometric construction of the invariant actions [28,29,31].

In Sec. III we consider GCA for arbitrary d, calculate the corresponding MC one-forms, and propose the choice of inverse Higgs constraints and GC-covariant dynamical equations, which leads to the extensions of standard AFF conformal mechanics model [11,15]. We consider four examples of cosets for GCA. The canonical choice [with the stability subalgebra $o(d)$] is shown to lead, after imposing the properly chosen inverse Higgs constraints, to new GC-covariant field equations for arbitrary d.

In Sec. IV we propose the actions dynamically generating the GC-covariant inverse Higgs constraints. For arbitrary $D = d + 1$ we propose two new extensions of the AFF model [11] and, for $D = 2 + 1$, an extension of the conformal dynamics considered in [32].

Brief conclusions are collected in Sec. V.

II. NONLINEAR REALIZATIONS METHOD IN CLASSICAL MECHANICS: ILLUSTRATIVE EXAMPLES

In this section we recall some known examples of the nonlinear realizations, for the standard Galilei and exotic Galilei groups and for the one-dimensional conformal group $\mathcal{C}^{(0)}$ leading to standard conformal mechanics. We demonstrate that the dynamics of considered systems is completely determined by imposing the appropriate conditions on the MC one-forms. Some of these conditions, the inverse Higgs constraints, are algebraic equations eliminating part of the original coset variables. Other covariant conditions imposed on the MC one-forms are the dynamical equations of motion. Invariant actions are also constructed by making use of the MC one-forms. In nonconformal cases the correct action is obtained from the MC one-forms associated with the generators of central charges. In the case of conformal mechanics, both the algebraic constraints and the equations of motion follow from the action which is linear in MC one-forms.

A. Galilei group in arbitrary space-time dimension D

The centrally extended Galilei algebra in $D = d+1$–dimensional space-time is spanned by the generator of the time translation H, the space translation generators P_i, $i = 1, \ldots, d$, the boosts B_i, the $o(d)$ rotation generators $J_{ij} = -J_{ji}$, and the central charge M describing a nonrelativistic mass. The full set of commutation relations consists of $o(d)$ Lie algebra, the relations

$$[H, P_i] = 0, \qquad [H, B_i] = iP_i, \tag{2.1}$$

$$[P_i, P_j] = 0, \quad [B_i, P_j] = i\delta_{ij}M, \quad [B_i, B_j] = 0, \tag{2.2}$$

and the commutators of $o(d)$ generators J_{ij} with the vector generators P_i, B_i.

Let us consider the nonlinear realization of the centrally extended Galilei group in the coset with O(d) as the stability subgroup [28,29]. We choose the following explicit parametrization of the coset

$$G = e^{itH} e^{ix_k P_k} e^{iv_k B_k} e^{i\varphi M}. \tag{2.3}$$

The left-invariant MC one-forms defined by the general relation

$$G^{-1}dG = i(\omega_H H + \omega_{P,k} P_k + \omega_{B,k} B_k + \omega_M M) \tag{2.4}$$

are given by the following explicit expressions:

$$\omega_H = dt, \qquad \omega_{P,i} = dx_i - v_i dt,$$
$$\omega_{B,i} = dv_i, \qquad \omega_M = d\varphi + v_i dx_i - \frac{1}{2} v_i v_i dt. \tag{2.5}$$

The group variables in (2.3) describe the mechanical system with the Hamiltonian H and the trajectories in the extended phase space $x_i = x_i(t)$, $v_i = v_i(t)$, $\varphi = \varphi(t)$. The fields $v_i(t)$ can be covariantly eliminated by imposing the algebraic inverse Higgs constraints

$$\omega_{P,i} = 0 \qquad \Rightarrow \qquad v_i = \dot{x}_i. \tag{2.6}$$

The equations of motion for the remaining physical variables $x_i(t)$ are represented by the constraints

$$\omega_{B,i} = 0 \qquad \Rightarrow \qquad \ddot{x} = 0. \tag{2.7}$$

Both the algebraic inverse Higgs constraints (2.6) and dynamical equations (2.7) can be derived as the Euler-Lagrange equations from the first-order action [28,29]

$$S_m = m \int \omega_M = m \int dt \Big[\dot{\varphi} + v_i \dot{x}_i - \frac{1}{2} v_i v_i \Big]. \tag{2.8}$$

After inserting the constraints (2.6) in the action (2.8) we obtain (up to a total derivative under the integral) the standard action for the massive particle

$$S_m = \frac{1}{2} m \int dt \dot{x}_i \dot{x}_i. \tag{2.9}$$

B. Exotic Galilei group in $D = 2 + 1$

The $D = 2 + 1$–dimensional space-time is special because in this case we can add to the set H, P_i, B_i, J_{ij}, M the second (exotic) central charge Θ and consider exotic Galilei algebra with the additional nonvanishing commutators

$$[B_i, B_j] = i\epsilon_{ij}\Theta. \tag{2.10}$$

The MC one-forms

$$\tilde{G}^{-1} d\tilde{G} = i(\omega_H H + \omega_{P,k} P_k + \omega_{B,k} B_k + \omega_M M + \omega_\Theta \Theta),$$
$$\tilde{G} = G e^{i\phi\Theta}, \tag{2.11}$$

are given by (2.5) and the additional expression

$$\omega_\Theta = d\phi + \frac{1}{2} \epsilon_{ij} v_i dv_j \tag{2.12}$$

for the MC one-form corresponding to the exotic central charge. Using this one-form, we can generalize (2.8) and consider the action [31]

$$S_{m,\theta} = m \int \omega_M + \theta \int \omega_\Theta$$
$$= m \int dt \Big[\dot{\varphi} + v_i \dot{x}_i - \frac{1}{2} v_i v_i \Big]$$
$$+ \frac{1}{2} \theta \int dt [2\dot{\phi} + \epsilon_{ij} v_i \dot{v}_j], \tag{2.13}$$

where both θ and m are constant. Inserting the inverse Higgs constraints (2.6),[3] we get, modulo a total derivative, the action for the $D = 2 + 1$ massive particle with the higher-order Chern-Simons-type term, which was proposed in [30]:

$$S_{m,\theta} = \frac{1}{2} m \int dt \dot{x}_i \dot{x}_i + \frac{1}{2} \theta \int dt \epsilon_{ij} \dot{x}_i \ddot{x}_j. \tag{2.14}$$

C. $D = 0 + 1$ conformal mechanics

Following [15], the AFF conformal mechanics [11] can be obtained by applying the MC method to the one-dimensional conformal algebra $\mathcal{C}^{(0)} = o(2, 1)$:

$$[D, H] = -iH, \quad [K, H] = -2iD, \quad [D, K] = iK. \tag{2.15}$$

We choose the exponential parametrization for the group $\hat{\mathcal{C}}^{(0)} = \mathrm{SO}(2, 1)$,

$$\hat{\mathcal{C}}^{(0)} \equiv G_0 = e^{itH} e^{izK} e^{iuD}, \tag{2.16}$$

and obtain the following left-covariant MC one-forms

$$G_0^{-1} dG_0 = i(\omega_H H + \omega_K K + \omega_D D), \tag{2.17}$$

with

$$\omega_H = e^{-u} dt, \qquad \omega_K = e^u (dz + z^2 dt), \qquad \omega_D = du - 2z dt. \tag{2.18}$$

In the conformal mechanics model [11], as in the construction of unitary representations of the group SO(2,1) [33], one is led to the choice of the following basis in the $o(2, 1)$ algebra,

$$R^\pm = \frac{1}{2}(\gamma K \pm \gamma^{-1} H), \quad D; \qquad [R^+, R^-] = iD,$$
$$[D, R^\pm] = iR^\mp, \tag{2.19}$$

where γ is a constant with the mass dimension, so that $R^\pm$ are dimensionless. The MC one-forms related to the generators $R^\pm$ are, respectively,

$$\omega_R^\pm = \gamma^{-1} \omega_K \pm \gamma \omega_H. \tag{2.20}$$

The dynamics of AFF conformal mechanics is obtained by imposing the following constraints [15]

$$\omega_D = 0, \tag{2.21a}$$
$$\omega_R^- = 0 \tag{2.21b}$$

on the one-dimensional coset fields $z(t)$ and $u(t)$. From the inverse Higgs constraint (2.21a) it follows that

$$z = \frac{1}{2} \dot{u}, \tag{2.22}$$

[3]If $\theta \neq 0$ the constraints (2.6) can be obtained from (2.13) as on-shell conditions, i.e., as a consequence of field equations.

while the dynamical constraint (2.21b) leads to the equation of motion

$$\ddot{\rho} = \gamma^2 \rho^{-3} \tag{2.23}$$

for the single independent variable $\rho = e^{u/2}$.

The standard AFF conformal mechanics action [11]

$$S_0 = \int dt(\dot{\rho}^2 - \gamma^2 \rho^{-2}), \tag{2.24}$$

which generates the equation of motion (2.23), in the formalism of MC one-forms can be rewritten as [15]

$$S_0 = -\gamma \int \omega_R^+ = -\int dt[e^u(\dot{z} + z^2) + \gamma^2 e^{-u}]. \tag{2.25}$$

We see that the action (2.25) is specified by the remaining nonvanishing MC one-form in $o(2,1)$. Both the kinematical constraint (2.22) ($\omega_D = 0$) and the dynamical equation (2.23) ($\omega_R^- = 0$) are the equations of motion following from the action (2.25).

Note that Eqs. (2.21a) and (2.21b) define a class of geodesics on the SO(1,2) group manifold, described by the one-parameter compact subgroup with the generator R^+ [15]. Only such a class leads to the standard conformal mechanics with good quantum properties [11], as opposed to any other nontrivial choice of the constraints [for example, the choice of $\omega_R^+ = 0$ instead of (2.21b). This is the reason why in our further considerations we will use only the constraints (2.21a) and (2.21b).

Let us make brief comments on the Hamiltonian formulation of the model (2.25), which will be useful later in the consideration of other GC-invariant actions.

The definitions of the momenta yield the second class constraints

$$p_u \approx 0, \qquad p_z + e^u \approx 0. \tag{2.26}$$

These constraints allow us to eliminate the phase space variables (p_z, p_u). The Dirac brackets for the surviving pair of the phase space variables (u, z) and the Hamiltonian take the form

$$\{u, z\}_D = e^{-u}, \qquad H = e^u z^2 + \gamma^2 e^{-u}. \tag{2.27}$$

After introducing the variables $\rho = e^{u/2}$, $p_\rho = 2e^{u/2}z$ which possess the standard canonical brackets

$$(u,z)\colon \{u,z\}_D = e^{-u} \quad \Rightarrow \quad (\rho, p_\rho)\colon \{\rho, p_\rho\}_D = 1, \tag{2.28}$$

we obtain that the system (2.25) is described by the Hamiltonian

$$H = \frac{1}{4} p_\rho^2 + \gamma^2 \rho^{-2}, \tag{2.29}$$

which follows from the action (2.24).

III. ALGEBRAIC DESCRIPTION OF GALILEAN CONFORMAL SYMMETRY

A. Galilean conformal algebra and corresponding MC one-forms

1. Arbitrary D

GCA $\mathcal{C}^{(d)}$ in $D = d + 1$ defined by Eq. (1.1) is obtained by adding to the $\mathcal{C}^{(0)} = o(2,1)$ algebra (2.15) the Lie algebra of space rotations $o(d)$ which commutes with $\mathcal{C}^{(0)}$,

$$[J_{ij}, J_{kl}] = i(\delta_{ik}J_{jl} - \delta_{il}J_{jk} + \delta_{jl}J_{ik} - \delta_{jk}J_{il}), \tag{3.1}$$

$$[J_{ij}, H] = [J_{ij}, D] = [J_{ij}, K] = 0, \tag{3.2}$$

as well as the $3d$-dimensional Abelian subalgebra $\mathcal{A}^{(3d)}$ spanned by the generators P_i, B_i, and F_i with the following commutators:

$$\begin{aligned} &[H,P_k]=0, && [H,F_k]=2iB_k, && [H,B_k]=iP_k,\\ &[K,P_k]=-2iB_k, && [K,F_k]=0, && [K,B_k]=-iF_k,\\ &[D,P_k]=-iP_k, && [D,F_k]=iF_k, && [D,B_k]=0, \end{aligned} \tag{3.3}$$

$$[J_{ij}, \mathcal{A}_{a,k}] = i(\delta_{ik}\mathcal{A}_{a,j} - \delta_{jk}\mathcal{A}_{a,i}), \tag{3.4}$$

$$[\mathcal{A}_{a,i}, \mathcal{A}_{b,j}] = 0. \tag{3.5}$$

Here, $\mathcal{A}_{1,i} = P_i$, $\mathcal{A}_{2.i} = B_i$, and $\mathcal{A}_{3,i} = F_i$.

The generators of GCA enlarge the algebras (2.1), (2.2), and (2.15), considered in the previous section. We recall that the operators B_i generate the Galilean boosts and the nonrelativistic energy operator H generates the Galilean time translations. The operators F_i generate constant nonrelativistic accelerations, and their presence implies that the central charge M [introduced in (2.2)] should be put equal to zero for any $D = d + 1$. For $d = 2$ one can still add the central charge Θ as in (2.10), without breaking any Jacobi identity of the full GCA algebra $\mathcal{C}^{(2)}$.

We choose the coset $\mathcal{K}^{(d)} = \hat{\mathcal{C}}^{(d)}/\mathcal{H}$, where $\hat{\mathcal{C}}^{(d)}$ is the GC group with the algebra (1.1) and $\mathcal{H} = \mathrm{SO}(d)$. We call this coset *canonical* and use for it the following parametrization:

$$\mathcal{K}^{(d)} = G_0 e^{ix_kP_k} e^{if_kF_k} e^{iv_kB_k}, \tag{3.6}$$

where $\hat{\mathcal{C}}^{(0)} = G_0$ is defined in (2.16).

The left-covariant MC one-forms are defined, as usual, by

$$\begin{aligned}\mathcal{K}^{(d)-1} d\mathcal{K}^{(d)} = i(&\omega_H H + \omega_K K + \omega_D D + \omega_{P,k}P_k\\ &+ \omega_{F,k}F_k + \omega_{B,k}B_k).\end{aligned} \tag{3.7}$$

The forms ω_H, ω_K, and ω_D are the same as in (2.18), while the remaining Cartan forms read

$$\omega_{P,i} = dx_i + x_i\omega_D - v_i\omega_H, \tag{3.8}$$

$$\omega_{F,i} = df_i - f_i\omega_D + v_i\omega_K, \quad (3.9)$$

$$\omega_{B,i} = dv_i + 2x_i\omega_K - 2f_i\omega_H. \quad (3.10)$$

Rewriting (3.7) in the form

$$\mathcal{K}^{(d)-1}d\mathcal{K}^{(d)} = i\omega_H(H + \mathcal{D}zK + \mathcal{D}uD + \mathcal{D}x_kP_k + \mathcal{D}f_kF_k + \mathcal{D}v_kB_k) \quad (3.11)$$

we are left with the worldline density E,

$$\omega_H = dtE, \qquad E = e^{-u}, \quad (3.12)$$

and the covariant time derivatives

$$\begin{aligned} \mathcal{D}z &= e^{2u}(\dot{z} + z^2), \\ \mathcal{D}u &= e^u(\dot{u} - 2z), \\ \mathcal{D}x_i &= e^u\dot{x}_i + x_i\mathcal{D}u - v_i, \\ \mathcal{D}f_i &= e^u\dot{f}_i - f_i\mathcal{D}u + v_i\mathcal{D}z, \\ \mathcal{D}v_i &= e^u\dot{v}_i + 2x_iDz - 2f_i. \end{aligned} \quad (3.13)$$

The infinitesimal transformations of the coset parameters, generated by the constant coset group elements

$$\mathcal{K}^{(d)}(\varepsilon) = e^{iaH}e^{ibK}e^{icD}e^{ia_kP_k}e^{ib_kF_k}e^{ic_kB_k}, \quad (3.14)$$

are as follows:[4]

$$\begin{aligned} &\delta t = a + bt^2 + ct \equiv \alpha(t), \qquad \delta z = b(1 - 2tz) - cz, \\ &\delta u = 2bt + c, \qquad \delta x_i = e^{-u}[a_i + t^2b_i + tc_i], \\ &\delta f_i = e^u[z^2a_i + (1 - tz)^2b_i - z(1 - tz)c_i], \\ &\delta v_i = -2za_i + 2t(1 - tz)b_i + (1 - 2tz)c_i. \end{aligned} \quad (3.15)$$

The forms (2.18), (3.8), (3.9), and (3.10) are invariant with respect to the transformations (3.15) and are covariant under the SO(d) transformations, which act as the standard rotations of the vector index i.

In our further consideration, by analogy with the basis (2.19) in the $o(2,1)$ algebra, we shall use the following new basis in the Abelian subalgebra $\mathcal{A}^{(3d)}$:

$$A_i^{\pm} = \frac{1}{2}(\gamma F_i \pm \gamma^{-1}P_i), \qquad B_i. \quad (3.16)$$

The commutation relations between the generators (3.16) and (2.19) are as follows:

$$\begin{aligned} &[R^{\pm}, A_k^{\pm}] = 0, \quad [R^{\pm}, A_k^{\mp}] = \pm iB_k, \quad [R^{\pm}, B_k] = -iA_k^{\mp}, \\ &[D, A_k^{\pm}] = iA_k^{\mp}, \quad [D, B_k] = 0. \end{aligned} \quad (3.17)$$

The explicit expressions for the corresponding MC one-forms

$$\omega_{A,i}^{\pm} = \gamma^{-1}\omega_{F,i} \pm \gamma\omega_{P,i}, \qquad \omega_{B,i} \quad (3.18)$$

are

$$\begin{aligned} \omega_{A,i}^{\pm} &= d\mathcal{X}_i^{\mp} - \mathcal{X}_i^{\mp}\omega_D + v_i\omega_R^{\mp}, \\ \omega_{B,i} &= dv_i + \mathcal{X}_i^{+}\omega_R^{-} - \mathcal{X}_i^{-}\omega_R^{+}, \end{aligned} \quad (3.19)$$

where we introduced new group variables

$$\mathcal{X}_i^{\mp} = \pm\gamma x_i + \gamma^{-1}f_i. \quad (3.20)$$

The covariant derivatives of the new vector coset variables (3.20) are

$$\begin{aligned} \mathcal{D}\mathcal{X}_i^{\mp} &= e^u\dot{\mathcal{X}}_i^{\pm} - \mathcal{X}_i^{\mp}\mathcal{D}u - \gamma^{-1}v_i(\mathcal{D}z \mp \gamma^2), \\ \mathcal{D}v_i &= e^u\dot{v}_i - \gamma^{-1}\mathcal{X}_i^{-}(\mathcal{D}z + \gamma^2) \\ &\quad + \gamma^{-1}\mathcal{X}_i^{\mp}(\mathcal{D}z - \gamma^2). \end{aligned} \quad (3.21)$$

2. Exotic $D = 2 + 1$ case with central charge Θ

If $D = 2 + 1$, the central charge Θ can be added [see (2.10)]. It appears in the following commutators:

$$[B_i, B_j] = i\epsilon_{ij}\Theta, \qquad [P_i, F_j] = -2i\epsilon_{ij}\Theta. \quad (3.22)$$

The parametrization of the coset $\tilde{\mathcal{K}}^{(2)} = \tilde{\mathcal{C}}^{(2)}/\mathrm{SO}(2)$, where $\tilde{\mathcal{C}}^{(2)}$ is the centrally extended GC group for $d = 2$, can be chosen as

$$\tilde{\mathcal{K}}^{(2)} = G_0e^{ix_kP_k}e^{if_kF_k}e^{iv_kB_k}e^{i\phi\Theta}, \quad (3.23)$$

where $G_0 = \hat{\mathcal{C}}^{(0)}$ and $k = 1, 2$.

The left-covariant MC one-forms are defined as

$$\tilde{\mathcal{K}}^{(2)-1}d\tilde{\mathcal{K}}^{(2)} = i(\omega_HH + \omega_KK + \omega_DD + \omega_{P,k}P_k + \omega_{F,k}F_k + \omega_{B,k}B_k + \omega_\Theta\Theta). \quad (3.24)$$

All "noncentral" one-forms are given by the old expressions (2.18), (3.8), (3.9), and (3.10), whereas ω_Θ is

$$\omega_\Theta = d\phi - 2\epsilon_{ij}f_i\omega_{P,j} + \frac{1}{2}\epsilon_{ij}v_i\omega_{B,j} + \epsilon_{ij}v_i(f_j\omega_H + x_j\omega_K). \quad (3.25)$$

The MC one-form ω_Θ will be used in Sec. IV for the construction of new GC-invariant action.

To summarize, we observe that, before imposing the inverse Higgs constraints, our mechanical system is spanned by the trajectories

$$\begin{aligned} &z = z(t), \qquad u = u(t), \qquad x_k = x_k(t), \\ &f_k = f_k(t), \qquad v_k = v_k(t), \end{aligned} \quad (3.26)$$

describing the motion in the coset $\mathcal{K}^{(d)}$, and, in the specific $D = 2 + 1$ case, in the coset $\tilde{\mathcal{K}}^{(2)}$ with the extra coordinate $\phi(t)$. In the next subsection, we propose the natural covariant constraints on the MC one-forms which permit us to eliminate a part of the functions (3.26).

[4] We use the formula $i\mathcal{K}^{(d)-1}(\varepsilon \cdot T)\mathcal{K}^{(d)} = \mathcal{K}^{(d)-1}\delta\mathcal{K}^{(d)} + \delta h$, where T are coset generators and δh defines induced transformations of the stability subgroup $h_{\mathrm{ind}} = 1 + \delta h$ (see [34]).

B. Inverse Higgs constraints and field equations

In this subsection and in Sec. IV we shall consider possible extensions of the AFF conformal mechanics which are covariant under the Galilean conformal symmetry with $d \neq 0$. Here we start our analysis at the level of equations of motion, leaving aside the existence of relevant Lagrangians. The choice of independent dynamical degrees of freedom is specified by the choice of cosets and the appropriate inverse Higgs constraints. The dynamical equations are also formulated as constraints imposed on the MC one-forms. In such a way the resulting dynamics is by construction covariant under the GC group transformations. Possible choices of actions for such systems will be considered in Sec. IV.[5]

1. Canonical case: Coset $\mathcal{K}^{(d)}$ with the stability subalgebra $o(d)$

We postulate that the constraints of standard conformal mechanics (2.21) remain valid also in the presence of additional vectorial variables which appear if $d \neq 0$. In terms of the covariant derivatives defined by (3.13), Eqs. (2.21a) and (2.21b) take the form

$$\mathcal{D}u = 0, \qquad (3.27a)$$

$$\mathcal{D}z = \gamma^2. \qquad (3.27b)$$

The remaining MC one-forms (3.19) are given by the expressions

$$\omega^+_{A,i} = d\mathcal{X}^+_i, \qquad (3.28)$$

$$\omega^-_{A,i} = d\mathcal{X}^-_i + v_i\omega^+_R = d\mathcal{X}^-_i + 2\gamma v_i\omega_H, \qquad (3.29)$$

$$\omega_{B,i} = dv_i - \mathcal{X}^-_i\omega^+_R = dv_i - 2\gamma\mathcal{X}^-_i\omega_H. \qquad (3.30)$$

We use here the "conformal" basis (3.17) and (3.18), since in this case the variable $\mathcal{X}^+_i$ decouples from other vector variables $\mathcal{X}^-_i$, v_i. It enters only into the one-form $\omega^+_{A,i}$, whereas the other two MC one-forms contain only $\mathcal{X}^-_i$, v_i.

Besides (2.21), we also impose the following additional constraints,

$$\omega_{B,i} = 0, \qquad (3.31a)$$

$$\omega^-_{A,i} = 0, \qquad (3.31b)$$

which yield the equations

$$\rho^2\dot{v}_i - 2\gamma X^-_i = 0, \qquad (3.32a)$$

$$\rho^2\dot{X}^-_i + 2\gamma v_i = 0, \qquad (3.32b)$$

where $\rho = e^{u/2}$. After eliminating v_i by the inverse Higgs constraint (3.32b), we obtain the following new dynamical second-order equations,

$$\rho^2\frac{d}{dt}(\rho^2\dot{\mathcal{X}}^-_i) + 4\gamma^2\mathcal{X}^-_i = 0, \qquad (3.33)$$

for the trajectory functions $\mathcal{X}^-_i(t)$.

Equations of motions for $\mathcal{X}^+_i$ are defined by the constraints on the MC one-forms $\omega^+_{A,i}$. For instance, the admissible GC covariant constraints are $\omega^+_{A,i} = 0$, which lead to the constant, time-independent vector $\mathcal{X}^+_i$. It is more interesting to look at the case when the equations of motion for $\mathcal{X}^+_i$ are of the second order in time derivative. Such equations are

$$\frac{d}{dt}(\rho^2\dot{\mathcal{X}}^+_i) = 0. \qquad (3.34)$$

These equations, like Eqs. (3.32) and (3.33), are covariant under the GC transformations (3.15): the variations of (3.34) are proportional to the equation of motion for the dilaton (2.23). For example,

$$\delta\left\{\frac{d}{dt}(\rho^2\dot{\mathcal{X}}^+_i)\right\} = -\dot{\alpha}\frac{d}{dt}(\rho^2\dot{\mathcal{X}}^+_i) - \gamma^{-1}\frac{d}{dt}[\rho^3(\ddot{\rho} - \gamma^2\rho^{-3})\delta v_i], \qquad (3.35)$$

where α and δv_i are defined in (3.15).

The GC covariance of Eqs. (3.32), (3.33), and (3.34) becomes manifest after rewriting them using the covariant derivatives (3.21). Modulo Eqs. (3.27a) and (3.27b), Eqs. (3.32a) and (3.32b) can be written equivalently as

$$\mathcal{D}v_i = 0, \qquad (3.36a)$$

$$\mathcal{D}\mathcal{X}^-_i = 0, \qquad (3.36b)$$

while (3.33) and (3.34) as

$$\mathcal{D}\mathcal{D}\mathcal{X}^-_i - 2\gamma\mathcal{D}v_i = 0, \qquad (3.37a)$$

$$\mathcal{D}\mathcal{D}\mathcal{X}^+_i = 0, \qquad (3.37b)$$

where the covariant derivative acting on $\mathcal{D}\mathcal{X}^\mp_i$ is just $\mathcal{D} = E^{-1}\partial_t$: $\mathcal{D}\mathcal{D}\mathcal{X}^\pm_i = \rho^2\partial_t(\mathcal{D}\mathcal{X}^\pm_i)$.

We see that our extended conformal mechanics is described by the dynamical variables ρ and $\mathcal{X}^\pm_i$. The variable ρ still obeys the standard equation (2.23), but now it is coupled to the vectorial coset variables $\mathcal{X}^\pm_i$ via Eqs. (3.33) and (3.34).

There is another dynamical system which is still invariant under the GC symmetry but contains a smaller number of degrees of freedom. Namely, using the dynamical equations (3.34) we can consider the system in which, instead of the full vector $\mathcal{X}^-_i$, only its covariant projection

$$X \equiv \mathcal{X}^-_i\mathcal{D}\mathcal{X}^+_i \qquad (3.38)$$

appears. Taking into account Eqs. (3.27a) and (3.27b), we obtain that $X = \rho^2\mathcal{X}^-_i\dot{\mathcal{X}}^+_i$. Equation (3.33) leads to the following dynamical equation for X:

$$\rho^2\frac{d}{dt}(\rho^2\dot{X}) + 4\gamma^2 X = 0. \qquad (3.39)$$

[5]Different ways of eliminating the auxiliary coset fields by the inverse Higgs effect and an issue of deriving the relevant constraints as equations of motion from some actions were discussed in a recent paper [35].

The action for such a system encompassing the dynamical variables ρ, $\mathcal{X}_i^+$, and X will be presented in Sec. IV. Note that Eq. (3.39) is the projection of Eq. (3.37a) on the covariantly constant vector $\mathcal{D}\mathcal{X}_i^+$ [see (3.37b)]. With Eqs. (3.27a) and (3.27b) taken into account, Eq. (3.39) can be equivalently rewritten in the manifestly covariant form as

$$\mathcal{D}\mathcal{D}X - 2\gamma\mathcal{D}V = 0, \tag{3.40}$$

where $V \equiv v_i\mathcal{D}\mathcal{X}_i^+$, $\mathcal{D}V = \mathcal{D}v_i\mathcal{D}\mathcal{X}_i^+$, and $\mathcal{D}X = \mathcal{D}\mathcal{X}_i^-\mathcal{X}_i^+$.

2. Three noncanonical cosets

The noncanonical cosets are obtained by including some of the $o(2,1)$ generators H, K, D into the stability subgroup. This gives rise to reducing the set of the primary coset fields (3.26).

(i) *GCA coset* $\mathcal{K}_1^{(d)}$ *with the stability subalgebra* $[o(d)\oplus K\oplus D]$.—This case is obtained by setting $z = u = 0$ in the formulas (2.18), (3.8), (3.9), (3.10), and (3.26). We obtain the following reduced MC one-forms:

$$\omega_H = dt, \qquad \omega_{P,i} = dx_i - v_i dt, \qquad \omega_{F,i} = df_i, \qquad \omega_{B,i} = dv_i - 2f_i dt. \tag{3.41}$$

Imposing the inverse Higgs conditions

$$\omega_{P,i} = 0, \qquad \omega_{B,i} = 0, \tag{3.42}$$

we can express v_i and f_i in terms of x_i:

$$v_i = \dot{x}_i, \qquad f_i = \frac{1}{2}\dot{v}_i = \frac{1}{2}\ddot{x}_i. \tag{3.43}$$

Further, the additional covariant constraint

$$\omega_{F,i} = 0 \tag{3.44}$$

results in the following dynamical equation for x_i:

$$\dddot{x}_i = 0. \tag{3.45}$$

In the $D = 2+1$ case, Eq. (3.45) coincides with the dynamics of the exotic model considered in [32], but here the same equation is obtained for arbitrary $D = d+1$.

(ii) *GCA coset* $\mathcal{K}_1^{(d)}$ *with the stability subalgebra* $[o(d)\oplus K]$.—This case is obtained by setting $z = 0$ in (2.18), (3.8), (3.9), (3.10), and (3.26). The MC one-forms read

$$\omega_H = e^{-u}dt, \qquad \omega_D = du, \tag{3.46}$$

$$\omega_{P,i} = e^{-u}[d(e^u x_i) - v_i dt], \qquad \omega_{F,i} = e^u d(e^{-u}f_i), \qquad \omega_{B,i} = dv_i - 2f_i e^{-u}dt. \tag{3.47}$$

Inverse Higgs conditions $\omega_{P,i} = 0$, $\omega_{B,i} = 0$ in (3.42) express v_i and f_i in terms of x_i as

$$v_i = (e^u x_i)^{\cdot}, \qquad f_i = \frac{1}{2}e^u\dot{v}_i. \tag{3.48}$$

From the condition $\omega_{F,i} = 0$ we get the dynamical equations

$$\dddot{y}_i = 0 \tag{3.49}$$

for $y_i \equiv e^u x_i$. Thus, after the redefinition $x_i \to y_i$, the vector sector coincides with the one obtained in case (i).

Note that we can impose the additional condition $\omega_D = 0$, which implies that the residual variable u becomes a constant.

(iii) *GCA coset* $\mathcal{K}_3^{(d)}$ *with the stability subalgebra* $[o(d)\oplus D]$.—In this case we set $u = 0$ in the MC one-forms for the canonical coset (3.6). We obtain

$$\omega_H = dt, \qquad \omega_K = dz + z^2 dt, \tag{3.50}$$

$$\omega_{P,i} = dx_i - 2zx_i dt - v_i dt, \tag{3.51}$$

$$\omega_{F,i} = df_i + 2zf_i dt + v_i(dz + z^2 dt), \tag{3.52}$$

$$\omega_{B,i} = dv_i + 2x_i(dz + z^2 dt) - 2f_i dt. \tag{3.53}$$

In this case the conditions (3.42) express v_i and f_i in terms of x_i as

$$v_i = \dot{x}_i - 2zx_i, \quad f_i = \frac{1}{2}\dot{v}_i + (\dot{z} + z^2)x_i \tag{3.54}$$

and also lead to the field equations (3.45), which leaves the decoupled variable z arbitrary. The minimal formulation corresponds to adding the constraint $\omega_K = 0$. In this case we obtain the following dynamical equation for z:

$$\dot{z} + z^2 = 0. \tag{3.55}$$

Thus, in all three cases (i), (ii), and (iii), the vector variables decouple and describe the motion with constant acceleration given by Eq. (3.45).

IV. LAGRANGIAN GC-INVARIANT MODELS

A. New actions for arbitrary *D*

Here we consider GC-invariant actions for arbitrary D, without central charge. We present two GC-invariant models. In one of them the Lagrangian is bilinear in the covariant derivatives of the vector coset variables and the other model is described by the action which resembles the well-known Brink-Schwarz action.

1. The actions bilinear in covariant derivatives

We consider the following general class of extended AFF actions,

$$S_1 = \int dt E m_{ab} \mathcal{D}Y_i^a \mathcal{D}Y_i^b, \tag{4.1}$$

where $Y_i^a = (x_i, v_i, f_i)$, $a = 1, 2, 3$, and m_{ab} is a constant matrix. The manifest GC invariance of this action is obvious. Note that, following the Volkov's proposal [10], this action can be equivalently rewritten in a geometric way as $\int \frac{m_{ab}\omega_i^a \omega_i^b}{\omega_H}$, where ω_i^a are the MC one-forms corresponding to the variables Y_i^a.

Let us study in more detail the example with

$$S_1 = \int dt L_1 = \frac{1}{2}\int dt E \mathcal{D}\mathcal{X}_i^+ \mathcal{D}\mathcal{X}_i^+, \tag{4.2}$$

where

$$\begin{aligned} \mathcal{D}\mathcal{X}_i^+ &= e^u[\dot{\mathcal{X}}_i^+ - \mathcal{X}_i^-(\dot{u} - 2z) \\ &\quad + \gamma^{-1} v_i(e^u(\dot{z} + z^2) - \gamma^2 e^{-u})], \\ E &= e^{-u}. \end{aligned} \tag{4.3}$$

One can show that the action (4.2) describes the dynamical system introduced in Sec. III A 1 from purely geometric considerations. The equations of motion following from (4.2) are

$$\delta v_i\colon\ e^u(\dot{z} + z^2) - \gamma^2 e^{-u} = 0, \tag{4.4}$$

$$\delta \mathcal{X}_i^-\colon\ \dot{u} - 2z = 0, \tag{4.5}$$

$$\delta \mathcal{X}_i^+\colon\ \dot{\mathcal{P}}_i^+ = 0, \tag{4.6}$$

$$\delta u\colon\ P_i^+(e^u \dot{\mathcal{X}}_i^- + 2\gamma v_i) + \frac{1}{2}\mathcal{P}_i^+\mathcal{P}_i^+ = 0, \tag{4.7}$$

$$\delta z\colon\ \mathcal{P}_i^+(e^u \dot{v}_i - 2\gamma \mathcal{X}_i^-) = 0, \tag{4.8}$$

for $\mathcal{D}\mathcal{X}_i^+ \neq 0$. Here

$$\mathcal{P}_i^+ = \mathcal{D}\mathcal{X}_i^+ \tag{4.9}$$

is the momentum conjugate to $\mathcal{X}_i^+$. Equations (4.4) and (4.5) provide the relation

$$\mathcal{P}_i^+ = e^u \dot{\mathcal{X}}_i^+. \tag{4.10}$$

Equations (4.4) and (4.5) are the equations of standard conformal mechanics [see (2.22) and (2.23)]. Thus, the action (4.2) reproduces as well the equations of motion for the standard conformal mechanics sector. Eliminating v_i from Eqs. (4.7) and (4.8), we obtain

$$\mathcal{P}_i^+\left[\frac{d}{dt}(\rho^2 \dot{\mathcal{X}}_i^-) + 4\gamma^2\rho^{-2}\mathcal{X}_i^-\right] = 0. \tag{4.11}$$

This is the projection of the field equations (3.33) for $\mathcal{X}_i^-$ on $\mathcal{P}_i^+$; i.e., we obtained precisely Eq. (3.39). Finally, the formulas in Eq. (4.6) are the equations of motion (3.34) for $\mathcal{X}_i^+$.

Thus the model with the action (4.2) amounts to one of the dynamical systems described in Sec. III A by constrained MC one-forms. This particular system is represented by the geometric variables ρ, $\mathcal{X}_i^+$, and X with the equations of motion (2.23), (3.34), and (3.39). At present it is not known whether one can define the off-shell action for the system in which all $\mathcal{X}_i^-$ are dynamical and are described by Eq. (3.33).

Eliminating auxiliary variable z by the algebraic equation (4.5) and introducing the new variable $\rho = e^{u/2}$, we obtain an equivalent Lagrangian

$$L_1 = \frac{1}{2}\rho^2[\dot{\mathcal{X}}_i^+ + \gamma^{-1} v_i(\rho\ddot{\rho} - \gamma^2\rho^{-2})]^2. \tag{4.12}$$

The equations of motion following from (4.12) can be identified with those derived from (4.2).

Let us now consider the Hamiltonian formulation of the system described by the action (4.2). The definitions of the momenta lead to the primary constraints

$$\mathcal{P}_{vi} \approx 0, \qquad \mathcal{P}_i^- \approx 0, \tag{4.13}$$

$$\begin{aligned} \mathcal{F}_u &\equiv p_u + \mathcal{X}_i^-\mathcal{P}_i^+ \approx 0, \\ \mathcal{F}_z &\equiv p_z - \gamma^{-1}e^u v_i \mathcal{P}_i^+ \approx 0, \end{aligned} \tag{4.14}$$

where the canonical pairs are

$$\begin{aligned} \{\mathcal{X}_i^\pm, \mathcal{P}_j^\pm\}_P &= \delta_{ij}, & \{v_i, \mathcal{P}_{v_j}\}_P &= \delta_{ij}, \\ \{z, p_z\}_P &= 1, & \{u, p_u\}_P &= 1. \end{aligned} \tag{4.15}$$

The canonical Hamiltonian is

$$H_1 = \frac{1}{2}e^{-u}\mathcal{P}_i^+\mathcal{P}_i^+ - [2z\mathcal{X}_i^- + \gamma^{-1}v_i(e^u z^2 - \gamma^2 e^{-u})]\mathcal{P}_i^+. \tag{4.16}$$

From the explicit form of nonvanishing Poisson brackets of the constraints (4.13) and (4.14)

$$\begin{aligned} \{\mathcal{F}_u, \mathcal{P}_i^-\}_P &= \mathcal{P}_i^+, \\ \{\mathcal{F}_z, \mathcal{P}_{v_i}\}_P &= -\gamma^{-1}e^u\mathcal{P}_i^+, \\ \{\mathcal{F}_u, \mathcal{F}_z\}_P &= \gamma^{-1}e^u v_i P_i^+, \end{aligned} \tag{4.17}$$

we see that the constraints (4.13) and (4.14) are the mixture of first and second class constraints. A simple analysis shows that the considered system is described by the second class constraints $\mathcal{F}_u \approx 0$, $\mathcal{F}_z \approx 0$, $\mathcal{P}_{v_i}\mathcal{P}_i^+ \approx 0$, $\mathcal{P}_i^-\mathcal{P}_i^+ \approx 0$ and the first class constraints given by the components of $\mathcal{P}_{v_i} \approx 0$ and $\mathcal{P}_i^- \approx 0$ orthogonal to $\mathcal{P}_i^+$. Using the gauge freedom generated by the first class constraints we can eliminate the components of v_i, $\mathcal{P}_{v_i}$, $\mathcal{X}_I^-$, and $\mathcal{P}_i^-$ orthogonal to $\mathcal{P}_i^+$.

Remaining phase space variables are $\tilde{\mathcal{X}}_i^+$, $\mathcal{P}_i^+$, u, p_u, z, p_z and the projections

$$V \equiv v_i \mathcal{P}_i^+, \qquad P_V; \qquad X \equiv \mathcal{X}_i^- \mathcal{P}_i^+, \qquad P_X \quad (4.18)$$

of v_i, $\mathcal{P}_{v_i}$, $\mathcal{X}_i^-$, $\mathcal{P}_i^-$. The expressions for the new variables $\tilde{\mathcal{X}}_i^+$, P_V, and P_X can be given explicitly. It is important to note that, if we introduce Dirac brackets (DBs), for the remaining variables they coincide with canonical Poisson brackets. The remaining second class constraints take the form

$$P_V \approx 0, \qquad P_X \approx 0, \qquad \mathcal{F}_u \equiv p_u + X \approx 0,$$
$$\mathcal{F}_z \equiv p_z - \gamma^{-1} e^u V \approx 0. \qquad (4.19)$$

Introducing DBs for the second class constraints (4.19) and eliminating the variables p_u, p_z, P_V, and P_X, we are left with the variables u, z, V, and X with the following nonvanishing DBs

$$\{u, X\}_D = -1, \qquad \{z, V\}_D = \gamma e^{-u},$$
$$\{V, X\}_D = V, \qquad \{\mathcal{X}_i^+, \mathcal{P}_j^+\}_D = \delta_{ij} \qquad (4.20)$$

and the Hamiltonian

$$H_1 = \frac{1}{2} e^{-u} \mathcal{P}_i^+ \mathcal{P}_i^+ - 2zX - \gamma^{-1}(e^u z^2 - \gamma^2 e^{-u})V. \qquad (4.21)$$

The set of equations (4.20) and (4.21), determines the dynamics of our model (4.2) in phase space. We can check that the equations of motion generated by the Hamiltonian (4.20), $\dot{u} = \{u, H\}_D$, etc., coincide with Eqs. (4.4), (4.5), (4.6), (4.7), and (4.8).

Introducing the variables

$$\rho \equiv e^{u/2}, \qquad p_\rho \equiv -2e^{-u/2} X,$$
$$y \equiv -2e^u V, \qquad p_y \equiv \gamma^{-1} z, \qquad (4.22)$$

which form two canonical pairs,

$$\{\rho, p_\rho\}_D = 1, \qquad \{y, p_y\}_D = 1, \qquad (4.23)$$

(other DBs are vanishing), we can put the Hamiltonian (4.21) in the following form:

$$H_1 = \frac{1}{2} \rho^{-2} \mathcal{P}_i^+ \mathcal{P}_i^+ + \gamma(\rho p_\rho + y p_y) p_y - \gamma \rho^{-4} y. \qquad (4.24)$$

Now we can present our model in a more economical formulation. Namely, we can use the following first-order Hamiltonian form of the action:

$$S_1 = \int dt[\mathcal{P}_i^+ \dot{\mathcal{X}}_i^+ + p_\rho \dot{\rho} + p_y \dot{y} - \frac{1}{2}\rho^{-2} \mathcal{P}_i^+ \mathcal{P}_i^+ - \gamma(\rho p_\rho + y p_y) p_y + \gamma \rho^{-4} y]. \qquad (4.25)$$

Eliminating momenta $\mathcal{P}_i^+$, p_ρ, and p_y by their equations of motion, we finally obtain

$$S_1 = \int dt\left[\frac{1}{2}\rho^2 \dot{\mathcal{X}}_i^+ \dot{\mathcal{X}}_i^+ + \frac{1}{\gamma \rho}\left(\dot{y}\,\dot{\rho} - \frac{y}{\rho}\dot{\rho}\,\dot{\rho}\right) + \frac{\gamma y}{\rho^4}\right]. \qquad (4.26)$$

This new action is a generalization of the conformal mechanics action (2.24). Besides invariance under the one-dimensional conformal symmetry SO(2,1) acting on ρ and y [recall the definitions (4.22)], the model (4.26) is invariant under the full GC symmetry with $d \neq 0$.

2. *Square-root action*

The second way of introducing the dynamics in the sector of vector coset parameters provides the Lagrangian as the square root of the product of vector one-forms, in a way resembling the model of free relativistic particle. We consider the action

$$S_2 = m \int \sqrt{\omega_{A,i}^+ \omega_{A,i}^+}, \qquad (4.27)$$

where m is a constant. It is a particular case of more general action

$$S_2 = m \int \sqrt{m_{ab} \omega_i^a \omega_i^b}, \qquad (4.28)$$

where ω_i^a are vector MC one-forms.

By applying variational principle to the action (4.27) we get Eqs. (4.4), (4.5), (4.6), (4.7), and (4.8) with the minor modification in Eq. (4.7),

$$\delta u\colon\ \mathcal{P}_i^+(e^u \dot{\mathcal{X}}_i^- + 2\gamma v_i) = 0, \qquad (4.29)$$

and with the additional property that in all equations the expression

$$\mathcal{P}_i^+ = m \mathcal{D}\mathcal{X}_i^+ (\mathcal{D}\mathcal{X}_k^+ \mathcal{D}\mathcal{X}_k^+)^{-1/2} \qquad (4.30)$$

should be substituted in place of the variable $\mathcal{P}_i^+$. As in the previous model, from Eqs. (4.4) and (4.5), it follows that $\mathcal{D}\mathcal{X}_i^+ = e^u \dot{\mathcal{X}}_i^+$, and as well $\mathcal{P}_i^+ = m \dot{\mathcal{X}}_i^+ (\dot{\mathcal{X}}_k^+ \dot{\mathcal{X}}_k^+)^{-(1/2)}$. From the latter expression for $\mathcal{P}_i^+$ we derive the important relation

$$\mathcal{M} \equiv \mathcal{P}_i^+ \mathcal{P}_i^+ - m^2 = 0. \qquad (4.31)$$

Despite the difference between Eqs. (4.29) and (4.7), after elimination of v_i Eqs. (4.29) and (4.8) yield the same equation (4.11) for $X = \mathcal{X}_i^- \mathcal{P}_i^+$.[6] Thus the two-parameter sector (ρ, X) is described by the same equations as in Sec. IVA 1: by the AFF equation (2.23) and the equation (4.11).

However, the dynamics in the sector of vector variable $\mathcal{X}_i^+$ is now different. Distinctly from the model considered in Sec. IVA 1, the quantities (4.30), which become the conjugate momenta for $\mathcal{X}_k^+$ in the Hamiltonian formalism, are constrained by Eq. (4.31). As a consequence of this constraint, and taking into account Eqs. (4.4) and (4.5), Eq. (4.6) proves to be linearly dependent

$$\dot{\mathcal{X}}_i^+ \dot{\mathcal{P}}_i^+ \equiv 0. \qquad (4.32)$$

[6] Note that adding of the "cosmological" term $(-\frac{1}{2} m^2 \int e^{-u})$ to the action (4.27) leads to the new term $(\frac{1}{2} m^2)$ in the left-hand side of (4.29) which coincides, due to (4.31), with the term $(\frac{1}{2} \mathcal{P}_i^+ \mathcal{P}_i^+)$ in (4.7).

The condition (4.31) becomes transparent in the Hamiltonian language, where it appears as the additional first class constraint.

The definitions of the momenta yield the same set of primary constraints (4.13) and (4.14), and also the additional constraint $\mathcal{M} \approx 0$ (4.31). The last constraint indicates that momentum vector $\mathcal{P}_i^+$ parametrizes the sphere S^{d-1}. The constant m plays the role of its radius (in the case of the Brink-Schwarz superparticle described by a similar square-root action, an analogous constant is identified with a mass of the relativistic particle). The canonical Hamiltonian of the square-root system (4.27) is given by the second term in (4.16):

$$H_2 = -[2z\mathcal{X}_i^- + \gamma^{-1}v_i(e^u z^2 - \gamma^2 e^{-u})]\mathcal{P}_i^+. \tag{4.33}$$

The constraint (4.31) has the vanishing Poisson brackets with all other constraints (4.13) and (4.14). Thus, the set of constraints is the same as in the model (4.2). The only difference between these two systems is the presence of additional first class "mass" constraint (4.31) in the second system. Similarly to the previous case, after gauge-fixing for the first class constraints presented in (4.13) and (4.14), eliminating the auxiliary phase space variables, and introducing new variables (4.22) which are canonical with respect to DBs, we find that the "square-root" model (4.27) is described by the following Hamiltonian [compare with (4.24)]:

$$H_2 = \lambda(\mathcal{P}_i^+\mathcal{P}_i^+ - m^2) + \gamma(\rho p_\rho + y p_y)p_y - \gamma\rho^{-4}y. \tag{4.34}$$

Here, the first term reflects the presence of the first class constraint (4.31), with $\lambda(t)$ being a Lagrange multiplier.

To summarize, in Sec. IV A we illustrated the method of nonlinear realizations on the simplest particular cases of the general actions (4.2) and (4.27). The study of these general actions, and, perhaps, of their further extensions, with added actions for the scalar Goldstone fields u, z, deserves further studies. It is important to note that the additional vector variables $\mathcal{X}_i^+$ and $\mathcal{P}_i^+$, which are the characteristic feature of the new GCA invariant models, can be presumably treated as a kind of "angular" variables, keeping in mind that, in multidimensional mechanical models with SO(2,1) invariance, the standard conformal mechanics describes a radial variable.

B. Exotic $D = 2 + 1$ case

In this case, the presence of the MC one-form associated with the central charge makes it possible to consider GC mechanics described by the action

$$\tilde{S} = \tilde{S}_{\rm conf} + \tilde{S}_\theta = -\gamma\int\omega_R^+ + \theta\int\omega_\Theta. \tag{4.35}$$

The first term defines the standard conformal mechanics sector [11,15], whereas the sector of the vector coset fields is represented by the Wess-Zumino (WZ) term.

To obtain a minimal formulation we use the inverse Higgs conditions

$$\omega_{P,i} = 0, \qquad \omega_{B,i} = 0 \tag{4.36}$$

in the action (4.35).

First we consider the case with the stability subalgebra $[o(d)\oplus K\oplus D]$ [case (i)]. The solutions of the constraints (4.36) are given in (3.43). Inserting these solutions in the action (4.35), we obtain, modulo a total derivative under the integral, the action

$$\tilde{S}_\theta = \frac{1}{2}\theta\int dt\epsilon_{ij}\dot{x}_i\ddot{x}_j. \tag{4.37}$$

It is the action of the "massless" particle with the higher-order Chern-Simons-type term [32], which is the $m \to 0$ limit of the action (2.14). Other choices (ii), (iii) of the stability subalgebra, i.e., $[o(d)\oplus K]$ or $[o(d)\oplus D]$, lead to the same result: the dynamics is again described by the action (4.37) [in case (ii)—after redefining the variable x_i].

In the canonical case, when the stability subalgebra does not contain the $o(2,1)$ generators, the first term in (4.35) produces dynamics in the $o(2,1)$ sector. Inserting in the action (4.35) the expressions

$$v_i = e^u[\dot{x}_i + (\dot{u} - 2z)x_i],$$
$$f_i = e^u\Big[\frac{1}{2}\dot{v}_i + e^u(\dot{z} + z^2)x_i\Big], \tag{4.38}$$

which follow from the inverse Higgs constraints (4.36), as well as the expressions for the one-forms (2.18), we obtain

$$\omega_\Theta = \Big[\frac{1}{2}\epsilon_{ij}\dot{y}_i\ddot{y}_j + \frac{d}{dt}(\phi - z\epsilon_{ij}y_i\dot{y}_j)\Big]dt. \tag{4.39}$$

Then, eliminating the field z by its algebraic equation of motion, we obtain (modulo a total derivative) the action

$$\tilde{S} = \int dt\Big(\dot{\rho}^2 - \frac{\gamma^2}{\rho^2} + \frac{\theta}{2}\epsilon_{ij}\dot{y}_i\ddot{y}_j\Big), \tag{4.40}$$

where $y_i \equiv e^u x_i$.

Thus we ended up with a decoupled pair of the GC-invariant $D = 2 + 1$ models. One of them is the AFF conformal mechanics with the action (2.24) [11], and the other one is described by the WZ action (4.37), first proposed in [32].

V. CONCLUSIONS

We have investigated nonlinear realizations of the Galilean conformal group in arbitrary space-time dimensions D, including the exotic $D = 2 + 1$ case with the additional central charge. The analysis of the MC one-forms with the appropriate inverse Higgs and dynamical covariant constraints in many cases is sufficient to reveal the underlying dynamics of the new mechanical systems with Galilean conformal symmetry. Alternatively, one can use the MC one-forms for the construction of the invariant actions and obtaining some further examples of the Galilean conformal mechanics. Following [28,29,31], in the $D = 2 + 1$ case we use the central charge MC one-form to describe the WZ term in the action.

We recall that, until recently, the only known GC-invariant classical mechanics Lagrangian model was the one related to the exotic dimension $D = 2 + 1$, that is, the nonrelativistic particle model described by the action with the higher-order Chern-Simons-type term [30,32].[7] The powerful techniques of the nonlinear realizations allowed us to obtain the whole family of new models exhibiting Galilean conformal symmetry for any $D = d + 1$. Moreover, using the covariant MC one-forms, it is possible to construct many such models—we considered only some simple examples.

The difficulty with finding dynamical realizations of GCA with $d \neq 0$ was one of the undesirable features of this symmetry. In this paper, by using the nonlinear realizations approach, we obtained new dynamical realizations, including those for an abitrary space dimension d. We proposed three GC-invariant models of classical mechanics which contain, besides the scalar coset coordinate ρ, also nonrelativistic vector coordinates. In Sec. IV we obtained the known model of AFF conformal mechanics (in the first-order formalism, with the original degrees of freedom $u = \frac{1}{2}\ln\rho$ and z), accompanied by couplings to the additional nonrelativistic vector variables $\mathcal{X}_i^{\pm}$. It is interesting that in this case one can define two actions with different Lagrangians which lead to similar dynamical equations, such that in both cases the sector of the AFF conformal mechanics is decoupled. One more model (Sec. II A) is specific for the $D = 2 + 1$ case. It involves the GC-covariant coupling between the degrees of freedom u, z and the nonrelativistic vector coordinate x_i through the WZ term defined by the MC one-form associated with the exotic central charge. It turns out that the conformal mechanics degrees of freedom (u, z) and the vectorial ones $y_i = e^u x_i$ decouple again in this model. We would like to add that the model (4.40) in $D = 2 + 1$ contains a higher-(third-)order time derivative, while the field equations (4.4), (4.5), (4.6), (4.7), and (4.8) of the first two models are of the first and second orders only.

In this paper we studied the Galilean conformally invariant models at the classical level, based on the geometric properties of the Galilean conformal symmetry. The next step will consist in analyzing quantum properties of the new mechanical systems constructed in our paper. In particular, the quantized version of simple model (2.14) in $D = 2 + 1$ space-time was studied earlier [30,32]. As was shown in [30], despite the presence of the states with negative norm in this model due to higher-order time derivatives in the action, it is possible to remove such states by imposing the appropriate constraints and maintain unitarity in the physical subspace of states. As a mechanical model on the noncommutative two-dimensional plane, the model (2.14) reveals also direct links to the description of anyons, to the quantum Hall effect and related issues of the condensed-matter physics (see, e.g., [38,39] and references therein). In the subsequent studies, we plan to elaborate in similar contexts on the quantum properties of the new models presented here and to consider as well their field-theoretical extensions.

Finally we add that we did not address in this paper supersymmetric generalizations of the Galilean conformal algebra [40,41] which should yield extensions of the $D = 0 + 1$ superconformal mechanics models. Currently, such extensions are under consideration. Also, it would be interesting to perform the quantization, to find the quantum spectrum of the new GC-invariant models, and to clarify the role of the additional vector variables in physical considerations.

[7]For completeness, it should be mentioned that the field equations of magnetic-like Galilean electrodynamics [36] are also covariant under GCA [26]. Furthermore, one can construct the systems of nonlinear partial differential equations invariant under GCA (as was shown in [37] for any d and in [8,37] for the planar $d = 2$ case, with GCA enlarged by exotic central charge).

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