

D-dimensional log gravity

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We study Einstein gravity in dimensions $D \geq 4$ modified by curvature squared at critical point where the theory contains only massless gravitons. We show that at the critical point a new mode appears leading to a logarithmic gravity in the theory. The corresponding logarithmic solution may provide a gravity description of logarithmic conformal field theories (CFT) in higher dimensions. We note also that for special values of the parameters of the theory, the model admits solutions with nonrelativistic isometries.

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I. INTRODUCTION

Four-dimensional Einstein gravity modified by curvature squared terms may provide a toy model to study quantum gravity in four dimensions. The corresponding action of the theory is given by [1,2]

$$I = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left(R + \frac{6}{\ell^2} + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu} \right). \quad (1.1)$$

When the cosmological constant is zero, $\Lambda = 0$, the theory is renormalizable and contains massless gravitons, massive spin 2, and a massive scalar field [1,2]. Nevertheless, the theory has ghosts due to negative energy excitations of the massive tensor. We note, however, that at the special values of the parameters of the theory either the massive tensor or the scalar would be absent. We note also that when $\beta = 0$ the model is unitary but nonrenormalizable, while when the curvature squared term is given by the Weyl tensor squared the model is neither unitary nor renormalizable.

With nonzero cosmological constant, the model exhibits new features. Although in this case the theory still contains massless gravitons, massive spin 2, and a massive scalar field, it is possible to tune the parameters such that only massless gravitons remain in the spectrum.

More precisely, it has been shown [3] that for the special values of α and β (critical point) given by

$$\beta = -3\alpha = -\frac{\ell^2}{2}, \quad (1.2)$$

the spin 2 field becomes massless and at the same time the massive scalar is removed from the spectrum. As a result we are left with four-dimensional gravity with only massless gravitons.

In this paper, we would like to further study the model at the critical point. In particular, we study AdS wave solutions in the mode and we will show that at the critical point it admits logarithmic solutions. These solutions may provide gravity descriptions for logarithmic conformal field theories (CFT's) in higher dimensions.

The paper is organized as follows. In the Sec. II, we will consider anti-de Sitter (AdS) wave solutions in four-dimensional theory at critical point. Generalization to higher dimensions is presented in Sec. III. Section IV is devoted to conclusions.

II. ADS WAVE SOLUTION AND LOG GRAVITY

In this section, we will study AdS wave solutions in the model given by the action (1.1). To proceed, we start with the equations of motion of the action (1.1)

$$\mathcal{G}_{\mu\nu} + E_{\mu\nu} = 0, \quad (2.1)$$

where [3]

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{3}{\ell^2} g_{\mu\nu}$$

$$\begin{aligned} E_{\mu\nu} = & 2\beta \left(R_{\mu\rho} R_{\nu}{}^{\rho} - \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) + 2\alpha R \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \\ & + \beta \left(\square R_{\mu\nu} + \frac{1}{2} \square R g_{\mu\nu} - 2 \nabla_{\rho} \nabla_{(\mu} R_{\nu)}{}^{\rho} \right) \\ & + 2\alpha (g_{\mu\nu} \square R - \nabla_{\mu} \nabla_{\nu} R). \end{aligned}$$

Since $E_{\mu\nu}$ vanishes for any Einstein space in four dimensions, with a negative cosmological constant (as we choose here) the model admits an AdS₄ vacuum solution, whose radius is given by ℓ^2 . This solution has been studied in [3].

Having an AdS vacuum solution, it is interesting to study AdS wave solutions in the model.¹ To proceed we consider an ansatz for AdS wave solutions as follows

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + F k_{\mu} k_{\nu}, \quad (2.2)$$

where k_{μ} is a null vector field with respect to the metric $\bar{g}_{\mu\nu}$, with $\bar{g}_{\mu\nu}$ being the AdS₄ metric parametrized as

¹AdS wave solutions for topologically massive gravity and new massive gravity models have been studied in [4–6], where it was shown that at the critical value of the parameters the solution develops logarithmic behaviors. The same situation has also been observed in bi-gravity and Born-Infeld gravity [7,8].

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$$ds^2 = \frac{\ell^2}{r^2}(-2dx^+ dx^- + dy^2 + dr^2). \quad (2.3)$$

Note that F is an arbitrary function which is independent of the integral parameter along k_μ . In other words, the ansatz may be given by

$$ds^2 = \frac{\ell^2}{r^2}(-F(x^+, y, r)dx^{+2} - 2dx^+ dx^- + dy^2 + dr^2). \quad (2.4)$$

Plugging this ansatz into the equations of motion (2.1), one finds

$$\left[\frac{\beta r^3}{\ell^2} \left(\frac{\partial^4}{\partial r^4} + 2 \frac{\partial^2}{\partial y^2} \frac{\partial^2}{\partial r^2} + \frac{\partial^4}{\partial y^4} \right) + \left(1 - \frac{8(3\alpha + \beta)}{\ell^2} \right) \right. \\ \left. \times \left(r \frac{\partial^2}{\partial r^2} - 2 \frac{\partial}{\partial r} + r \frac{\partial^2}{\partial y^2} \right) \right] F(x^+, r, y) = 0. \quad (2.5)$$

For simplicity, we assume that F is independent of y coordinate. In this case the above equation reads

$$\left[\frac{\beta r^3}{\ell^2} \frac{\partial^4}{\partial r^4} + \left(1 - \frac{8(3\alpha + \beta)}{\ell^2} \right) \left(r \frac{\partial^2}{\partial r^2} - 2 \frac{\partial}{\partial r} \right) \right] F(x^+, r) = 0. \quad (2.6)$$

It is clear that the most general solution of the above differential equation is in the form of r^x with constant x satisfying the following characteristic equation

$$x(x-3)[\beta(x-1)(x-2) - 8(3\alpha + \beta) + \ell^2] = 0. \quad (2.7)$$

Therefore a generic solution of the equations of motion is²

$$F(x^+, r) = f_4(x^+) + f_3(x^+)r^3 + f_2(x^+)r^{\Delta_+} + f_1(x^+)r^{\Delta_-}, \quad (2.8)$$

where $\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{\ell^2 - 24\alpha - 6\beta}{\beta}}$, and f_i 's are undetermined functions of x^+ .

It is natural to look for a possibility of having multiplicities in the roots of the characteristic equation. Actually, we observe that at the critical point where the parameters α and β are given by the Eq. (1.2), the characteristic equation degenerates, leading to new logarithmic solutions as follows

$$F(x^+, r) = f_4(x^+) + f_3(x^+)r^3 + [f_2(x^+) + f_1(x^+)r^3] \log(r). \quad (2.9)$$

In other words, at the critical point, the model admits a new vacuum solution which is not asymptotically locally AdS₄. Therefore, in order to accommodate the new solution, one needs to change the asymptotic behavior of the metric. More precisely, using the Fefferman-Graham coordinates for the metric

²A y dependent solution can also be obtained as $F(x^+, y, r) = (c_0(x^+) + c_1(x^+)y)F(x^+, r)$.

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x_i) dx^i dx^j, \quad (2.10)$$

the equations of motion give a possibility to have a wider class of boundary conditions for the metric as follows

$$g_{ij} = b_{(0)ij} \log(r) + g_{(0)ij} + g_{(3)ij} r^3 + b_{(3)ij} r^3 \log(r) + \dots \quad (2.11)$$

Typically when $b_{(0)ij}$ is nonzero, where the solution is not an asymptotically locally AdS₄, to maintain the variational principle well posed with the Dirichlet boundary condition one needs to modify the variational principle by imposing an additional boundary condition [9]. Indeed, from the above expression for the asymptotic behavior of the metric, one finds

$$g_{(0)ij} = \lim_{r \rightarrow 0} \left(g_{ij} - r \log(r) \frac{\partial g_{ij}}{\partial r} \right), \quad b_{(0)ij} = \lim_{r \rightarrow 0} r \frac{\partial g_{ij}}{\partial r}, \quad (2.12)$$

showing that in order to fix the boundary conditions not only one needs the value of the boundary metric but also its radial derivative.

In the context of AdS/CFT correspondence [10], both $g_{(0)ij}$ and $b_{(0)ij}$ may be treated as two sources for two operators in the boundary three-dimensional CFT. We note, however, that since in the presence of nonzero $b_{(0)ij}$ the geometry is not asymptotically locally AdS₄, the parameter $b_{(0)ij}$ should be considered as a source for an irrelevant operator in the dual CFT (see for example [9]). Nevertheless, for a sufficiently small $b_{(0)ij}$, one could still use the AdS/CFT correspondence to describe the dual theory, which is expected to be a logarithmic CFT.³

As a result, the logarithmic solution of the action (1.1) indicates that critical gravity gives a gravity description for three-dimensional logarithmic CFT's. Actually, the situation is similar to those in topologically massive gravity and new massive gravity models, where it is believed that the dual theory is logarithmic CFT [12,13] at critical point. Following [9,14], it would be interesting to find two-point functions and the corresponding new anomaly parameter in this model using the holographic renormalization method.

As a final remark, we note that away from the critical point and for a specific values of α and β , the model admits another one parameter solution. Indeed, when

$$24\alpha = 2(2n^2 - n - 4)\beta + \ell^2, \quad (2.13)$$

with n being a free parameter, for a specific choice of the intergral constants one finds the following solution

³Logarithmic CFT's in the context of AdS/CFT correspondence have also been studied in [11]. We note, however, that in this paper the authors have fixed the background to be AdS and the logarithmic behavior comes for the specific action of the fields in the bulk.

$$ds^2 = \frac{\ell^2}{r^2} \left(-\frac{dt^2}{r^{2n-2}} - 2dt d\xi + dy^2 + dr^2 \right). \quad (2.14)$$

We recognize this solution as a gravity solution whose dual theory is a nonrelativistic field theory [15].⁴ In particular, for $n = 2$, the isometry of the metric is Schrödinger group and the solution provide a gravity description for a non-relativistic CFT.

It is worth mentioning that in our model the nonrelativistic holographic solution is obtained in a pure gravitational system, though the one studied in [15] has been obtained in a model which contains a gravity coupled to a massive gauge field. Of course we have not checked whether this vacuum is stable in the sense that small fluctuations above it have non-negative mass spectrum.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu} + 2\alpha R \left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R \right) + (2\alpha + \beta)(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)R + 2\gamma[RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} + R_{\mu\sigma\rho\tau}R_{\nu}^{\sigma\rho\tau} - 2R_{\mu\sigma}R_{\nu}^{\sigma} - \frac{1}{4}g_{\mu\nu}(R_{\tau\lambda\sigma\rho}^2 - 4R_{\sigma\rho}^2 + R^2)] + \beta\square \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right) + 2\beta \left(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho} \right) R^{\sigma\rho} = 0. \quad (3.2)$$

For generic values of the parameters Λ , α , β and γ the model has two distinct vacua such that $R_{\mu\nu} = \frac{2\Lambda}{D-2}g_{\mu\nu}$, where Λ is a root of the following equation [18]

$$\Lambda_0 - \Lambda = 2\Lambda^2 \left[(D\alpha + \beta) \frac{D-4}{(D-2)^2} + \frac{(D-3)(D-4)}{(D-1)(D-2)} \gamma \right]. \quad (3.3)$$

It is always possible to tune the parameters such that at least one of the vacua will be an AdS_D geometry. In this case, the radius of the AdS geometry is given in terms of Λ as follows⁵

$$\ell^2 = -\frac{(D-1)(D-2)}{2\Lambda}. \quad (3.4)$$

It was shown in [18] that for appropriate choice of the parameters there exists a critical point where

$$\beta = -\frac{4(D-1)}{D}\alpha, \quad \frac{(D-1)(D-2)}{4(-\Lambda)} = (D-1)(D\alpha + \beta) + (D-3)(D-4)\gamma, \quad (3.5)$$

at which the model has only massless tensor gravitons.

⁴Lifshitz black holes in Einstein gravity with curvature squared terms have also been studied in [16,17].

⁵When the right hand side of the Eq. (3.3) vanishes with the assumption of negative cosmological constant, $\Lambda = \Lambda_0 < 0$, the model admits a unique AdS solution. In this case when $D \neq 4$ the parameters of the model have to obey the constraint $D\alpha + \beta + \frac{(D-2)(D-3)}{D-1}\gamma = 0$.

III. HIGHER DIMENSIONS

In this section we would like to extend our previous discussions to higher dimensions. Recently D -dimensional extended gravities have been studied in [19], where the authors have considered the following gravitational action

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g} [R - 2\Lambda_0 + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu} + \gamma(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2)]. \quad (3.1)$$

The corresponding equations of motion are [19]

Following our discussions in the previous section it is natural to look for AdS wave solutions in the model and, in particular, to see if the model supports a logarithmic solution at the critical point. To proceed, we consider an ansatz as follows

$$ds^2 = \frac{\ell^2}{r^2} (-F(x^+, r, x_i) du^2 - 2dx^+ dx^- + dr^2 + (dx_i)^2). \quad (3.6)$$

Plugging this ansatz into the equations of motion one finds

$$\left\{ \frac{\beta r^3}{D-2} \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x_i^2} \right)^2 - \frac{2(D-4)}{D-2} \beta r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x_i^2} \right) \frac{\partial}{\partial r} + \beta r(D-4) \frac{\partial^2}{\partial r^2} + \frac{r}{D-2} [\ell^2 - 2D(D-1)\alpha - 4(D-2)\beta - 2(D-3)(D-4)\gamma] \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x_i^2} \right) - [\ell^2 - 2D(D-1)\alpha - (3D-4)\beta - 2(D-4)(D-3)\gamma] \frac{\partial}{\partial r} \right\} F(x^+, r, x_i) = 0. \quad (3.7)$$

The equation may be simplified with the assumption that the function F is independent of transverse directions x_i 's. In this case, a generic solution of the resultant equation will be of the form r^x for constant x . From the above equation, the characteristic equation reads

$$\frac{\beta}{D-2} x(x-D+1) \left(x^2 - (D-1)x + \frac{A}{\beta} \right) = 0, \quad (3.8)$$

where $A = \ell^2 - 2(D\alpha + \beta)(d-1) - 2(D-3)(D-4)\gamma$.

Therefore the most general solution of the equations of motion is

$$F(x^+, r) = f_4(x^+) + f_3(x^+)r^{D-1} + f_2(x^+)r^{\Delta_+} + f_1(x^+)r^{\Delta_-}, \quad (3.9)$$

where $\Delta_{\pm} = \frac{D-1}{2} \pm \sqrt{(\frac{D-1}{2})^2 - \frac{A}{\beta}}$, and f_i 's are undetermined functions of x^+ .

We note that at the critical point (3.5) where $A = 0$, the characteristic equation degenerates, leading to new logarithmic solutions as follows

$$F(x^+, r) = f_4(x^+) + f_3(x^+)r^{D-1} + [f_2(x^+) + f_1(x^+)r^{D-1}] \times \log(r). \quad (3.10)$$

As a result, following our discussions in the previous section, the gravitational model based on the action (3.1) at the critical point may provide a gravity description for $D - 1$ dimensional logarithmic CFT's.

The model also admits nonrelativistic solutions when the parameters of the model obey the constraint $A + 2(n - 1)(2n - D - 1)\beta = 0$ for $n \neq 1$. In this case, for special choice of the integral constants, one finds

$$ds^2 = \frac{\ell^2}{r^2} \left(-\frac{dt^2}{r^{2n-2}} - 2dtd\xi + dr^2 + (dx_i)^2 \right). \quad (3.11)$$

IV. CONCLUSIONS

In this paper, we have studied AdS wave solutions in D -dimensional Einstein gravity with curvature squared modification. At the critical point where the theory contains only massless gravitons the model admits logarithmic solutions.

We have also shown that for special values of the parameters of the model, one could have nonrelativistic solutions. In particular the model admits solutions with Schrödinger isometry. Therefore these models could provide gravity descriptions for nonrelativistic and logarithmic CFT's.

It is important to note that the existence of these solutions are due to the curvature squared terms in the action. Actually restricting to four dimensions we note that any Einstein solutions are the solutions of the model with curvature squared action. In fact, black hole solutions of the Einstein kind have already been discussed in [3], where the authors have shown that the corresponding black holes have zero mass and entropy!

It would be interesting to find other solutions in the model which are not Einstein solutions. In particular, one may seek for non-Einstein black hole solutions in the model. These black holes could be of the logarithmic solution, as well.

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Note added.—After we submitted our paper to arXiv, two other papers, [20,21], appeared in arXiv where the similar logarithmic solutions have been discussed. It was also conjectured that the corresponding dual field theory could be a logarithmic CFT.

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