

Post-1-Newtonian tidal effects in the gravitational waveform from binary inspirals

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(Received 12 January 2011; published 28 April 2011)

The gravitational wave signal from an inspiralling binary neutron star system will contain detailed information about tidal coupling in the system, and thus, about the internal physics of the neutron stars. To extract this information will require highly accurate models for the gravitational waveform. We present here a calculation of the gravitational wave signal from a binary with quadrupolar tidal interactions which includes all post-1-Newtonian-order effects in both the conservative dynamics and wave generation. We consider stars with adiabatically induced quadrupoles moving in circular orbits, and work to linear order in the stars' quadrupole moments. We find that post-1-Newtonian corrections increase the tidal signal by approximately 20% at gravitational wave frequencies of 400 Hz.

DOI: [10.1103/PhysRevD.83.084051](https://doi.org/10.1103/PhysRevD.83.084051)

PACS numbers: 04.30.Db, 04.25.Nx

I. INTRODUCTION**A. Background and motivation**

Inspiralling and coalescing binary neutron stars are key sources for ground-based gravitational wave (GW) detectors [1]. An important science goal in the detection of such sources is to obtain robust information on the highly uncertain equation of state (EoS) of neutron star matter [2]. The effects of the EoS on the GW signal are largest during the late inspiral and merger stages of binary evolution, at GW frequencies $\gtrsim 500$ Hz, and the strong gravity and complex hydrodynamics involved in these regimes require the use of fully relativistic numerical simulations for their study (see, e.g., Refs. [3,4] and references therein). A small but clean EoS signature will also be present in the early inspiral waveform, at frequencies $\lesssim 500$ Hz within the most sensitive band of the Laser Interferometer Gravitational Wave Observatory (LIGO), arising from the effects of tidal coupling [5]. The relative weakness of orbital gravity in this regime makes it possible to construct good approximate waveforms using post-Newtonian-based analytic models [6].

For point-particle models of binary inspiral, analytic gravitational waveforms have been computed to 3PN accuracy [7], and spin effects have been computed to 2PN accuracy [8].¹ More recent efforts to improve the analytic description of neutron star binary GW signals by including tidal effects began with Refs. [5,9], which used a leading-order model of the tidal coupling and GW emission to demonstrate the potential feasibility of measuring EoS effects in inspiralling neutron stars in the low-frequency ($\lesssim 400$ Hz) regime with Advanced LIGO.

(See also Refs. [10,11] for earlier analyses of tidal effects in inspiralling binaries.) The tidal contribution to the GW signal computed in Ref. [5] depends on a single tidal deformability parameter λ , which characterizes the star's deformation response to a static (or adiabatically changing) tidal field and which is sensitive to the star's EoS.

The quadrupolar tidal deformability λ was defined in a fully relativistic context and calculated for a variety of EoS models in Refs. [5,9,12–14], and Refs. [13,14] extended the analysis to include higher-multipolar tidal responses of both electric- and magnetic-type. It was found in Ref. [12] that Advanced LIGO should be able to constrain the neutron stars' tidal deformability to $\lambda \lesssim (1.2 \times 10^{37} \text{ g cm}^2 \text{ s}^2)$ ($D/100$ Mpc) with 95% confidence, for a binary of two $1.4M_{\odot}$ neutron stars at a distance D from the detector, using only the portion of the signal with GW frequencies less than 400 Hz. The calculations of λ for a $1.4M_{\odot}$ neutron star in Refs. [9,12–14], using several different equations of state, give values in the range $0.03\text{--}1.0 \times 10^{37} \text{ g cm}^2 \text{ s}^2$, so nearby events may allow Advanced LIGO to place useful constraints on candidate equations of state.

To detect or constrain the tidal deformability λ will require models for the tidal contribution to the GW signal that are accurate to $\lesssim 10\%$, much less than the current uncertainty in λ . References [5,12] estimate the fractional corrections to the tidal signal at GW frequencies below 400 Hz due to several effects neglected by the model of the GW phasing used in Ref. [5], namely, nonadiabaticity ($\lesssim 1\%$), higher-multipolar tidal coupling ($\lesssim 0.7\%$), nonlinear hydrodynamic effects ($\lesssim 0.1\%$), spin effects ($\lesssim 0.3\%$), nonlinear response to the tidal field ($\lesssim 3\%$), viscous dissipation (negligible), and post-Newtonian effects ($\lesssim 10\%$). The largest expected corrections, from post-Newtonian effects in the orbital dynamics and GW emission, are thus essential for an accurate analysis of the tidal signal. These corrections will depend on the neutron

¹The shorthand n PN, for post- n -Newtonian, is used to describe corrections of order c^{-2n} relative to Newtonian gravity, where c is the speed of light.

star physics only through the same tidal deformability parameter λ used in the Newtonian treatment and thus can be easily incorporated into the same data analysis methods used in the Newtonian (tidal) case.

The extension of the tidal signal calculation to 1PN order was recently discussed in Ref. [15] by Damour and Nagar (DN). Working within the framework of the effective-one-body (EOB) formalism, DN gave a complete description of the 1PN conservative dynamics of tidally interacting binaries in circular orbits, parametrized the forms of further 1PN corrections to the GW emission, and made comparisons with numerical simulations (see also Ref. [16]). The 1PN conservative dynamics has also been recently studied in Ref. [17] by Vines and Flanagan (VF). Working from the formalism for 1PN celestial mechanics developed in Refs. [18,19] and extended by Ref. [20], VF found the explicit equations of motion and action principle for generic orbits and generic evolution of the bodies' quadrupoles. Specializing to adiabatically induced quadrupoles and circular orbits, the results of VF agree with those of DN for the 1PN conservative dynamics. The construction of the 1PN metric given by VF also allows for explicit computation of the binary system's 1PN-accurate mass multipole moments.

In the present paper, we use the results of VF [17] to derive the 1PN-accurate GW signal from an inspiralling binary with quadrupolar tidal interactions. Working to linear order in the stars' quadrupole moments, and using adiabatically induced quadrupoles and circular orbits, we compute the binary's binding energy and GW energy flux and use them to determine the phase evolution of the emitted GW signal in the stationary phase approximation. The results presented here can be used to extend the validity of analytic GW signals to higher frequencies, and to provide useful information for hybrid schemes that attempt to bridge the gap in frequencies between analytic inspiral models and the start of numerical simulations, such as the EOB formalism of Ref. [15]. Our expressions for the orbital equations of motion and binding energy may also be useful for the construction of quasioequilibrium initial data for numerical simulations [21]. We note that the 1PN corrections calculated here slightly improve the prospects for detection of tidal effects in binary GW signals, as they increase the tidal signal by $\sim 20\%$ at GW frequencies of 400 Hz. The studies in Refs. [22,23] have recently made use of our results in making assessments of the measurability of tidal effects and in calibrating analytic models against numerical simulations.

B. Organization

The organization of this paper is as follows. In Sec. II, we briefly state the key results of Ref. [17] for the 1PN conservative dynamics of a binary in which one member has a mass quadrupole moment. We specialize to the adiabatic limit and circular orbits and compute the

gauge-invariant binding energy as a function of orbital frequency. In Sec. III, we consider the gravitational radiation and obtain the 1PN tidal corrections to the radiated energy flux. We then compute the resulting 1PN tidal corrections to the phase of the Fourier transform of the waveform in the stationary phase approximation and conclude in Sec. IV with a short discussion of the results.

C. Notation and conventions

We use units where Newton's constant is $G = 1$, but retain factors of the speed of light c , with $1/c^2$ serving as the formal expansion parameter for the post-Newtonian expansion. We use lowercase Latin letters a, b, i, j, \dots for indices of spatial tensors. Spatial indices are contracted with the Euclidean metric, $v^i w^i = \delta_{ij} v^i w^j$, with up or down placement of the indices having no meaning. We use angular brackets to denote the symmetric, trace-free projection of tensors, e.g., $T^{(ab)} = T^{(ab)} - \frac{1}{3} \delta^{ab} T^{cc}$. We also use multiple indices to denote tensorial powers of vectors, e.g., $v^{ij} = v^i v^j$, $n^{(ijk)} = n^{(i} n^j n^k)$, etc.

II. CONSERVATIVE DYNAMICS IN THE ADIABATIC LIMIT

In this section we briefly review the key results of VF [17] concerning the 1PN conservative dynamics of a binary system with quadrupolar tidal coupling. For simplicity, we consider a binary composed of one point-mass (body 1) and one deformable star (body 2). Since we consistently work to linear order in the quadrupole, our results can be easily generalized to the case of two deformable bodies by interchanging body labels. The binary's orbital dynamics can be formulated in terms of the separation (three-)vector $z^i = z_2^i - z_1^i$ between the bodies, the bodies' masses M_1 and M_2 , and the quadrupole moment Q_2^{ij} of body 2.

The 1PN-accurate worldlines $x^i = z_1^i(t)$ and $x^i = z_2^i(t)$ of the bodies' centers of mass-energy and their separation $z^i(t) = z_2^i(t) - z_1^i(t)$ are defined in a "global" 1PN coordinate system (t, x^i) . The global coordinates are conformally Cartesian and harmonic, and they tend to inertial coordinates in Minkowski spacetime as $|x| \rightarrow \infty$. Also, the binary system's center of mass-energy is taken to be at rest at the origin $x^i = 0$ (the system's 1PN-accurate mass dipole moment is set to zero), so that the (t, x^i) coordinates correspond to the center-of-mass-energy frame of the system. We use the following notation for the relative position, velocity, and acceleration:

$$\begin{aligned} z^i &= z_2^i - z_1^i, & r &= |z| = \sqrt{\delta_{ij} z^i z^j}, & n^i &= z^i/r, \\ v^i &= \dot{z}^i, & \dot{r} &= v^i n^i, & a^i &= \ddot{z}^i, \end{aligned}$$

with dots denoting derivatives with respect t .

We take M_1 and M_2 to be the bodies' conserved rest masses,² and we define the total mass M , mass fractions χ_1, χ_2 , reduced mass μ , and symmetric mass ratio η by

$$\begin{aligned} M &= M_1 + M_2, & \chi_1 &= M_1/M, \\ \chi_2 &= M_2/M, & \mu &= \eta M = \chi_1 \chi_2 M. \end{aligned} \quad (2.1)$$

Note that there are only two independent parameters among these quantities; we will tend to express our results in terms of the total mass M and the mass fraction χ_2 of the deformable body, unless factorizations make it more convenient to use $\chi_1 = 1 - \chi_2$ or $\eta = \chi_1 \chi_2$.

The tidal deformation of body 2 is described by its 1PN-accurate Blanchet-Damour [24] mass quadrupole moment $Q_2^{ij}(t)$. We will work in the limit where the quadrupole is adiabatically induced by the tidal field; i.e., we assume that the quadrupole responds to the instantaneous tidal field according to

$$Q_2^{ij}(t) = \lambda G_2^{ij}(t). \quad (2.2a)$$

Here, the constant λ is the tidal deformability,³ and $G_2^{ij}(t)$ is the quadrupolar gravito-electric Damour-Soffel-Xu [19] tidal moment of body 2 which encodes the leading-order ($l = 2$) tidal field felt by body 2. For the binary system under consideration, the tidal moment is given by

$$\begin{aligned} G_2^{ij} &= \frac{3\chi_1 M}{r^3} n^{(ij)} + \frac{1}{c^2} \frac{3\chi_1 M}{r^3} \left[\left(2v^2 - \frac{5\chi_2^2}{2} \dot{r}^2 \right. \right. \\ &\quad \left. \left. - \frac{6 - \chi_2}{2} \frac{M}{r} \right) n^{(ij)} + v^{(ij)} - (3 - \chi_2^2) \dot{r} n^{(i} v^{j)} \right] \\ &\quad + O(c^{-4}) + O(\lambda). \end{aligned} \quad (2.2b)$$

With the quadrupole given by Eqs. (2.2) in the adiabatic limit, the only independent degree of freedom is the binary's relative position $z^i(t)$. It was shown by VF [17] that the evolution of $z^i(t)$ is governed by the Lagrangian

²Note that the mass M_2 used here is not the 1PN-accurate Blanchet-Damour [24] mass monopole moment (which was called M_2 in VF [17]); rather, the M_2 used here is the conserved part of the BD mass monopole (called ${}^m M_2$ in VF [17]). The full 1PN-accurate monopole also receives contributions from the body's internal elastic energy (and from the tidal gravitational potential energy), which, for a deformable body, will vary as tidal forces do work on the body. The effects of these time-dependent contributions to the monopole have been separately accounted for in the Lagrangian (2.3), and the mass M_2 appearing there is constant.

³The tidal deformability is related to the dimensionless order-unity Love number k_2 [11] and the star's areal radius R by $\lambda = 2k_2 R^5/3$.

$$\begin{aligned} \mathcal{L}[z^i] &= \frac{\mu v^2}{2} + \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) \\ &\quad + \frac{\mu}{c^2} \left\{ \theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) \right. \right. \\ &\quad \left. \left. + \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\} \\ &\quad + O(c^{-4}) + O(\lambda^2), \end{aligned} \quad (2.3)$$

with $\Lambda = (3\chi_1/2\chi_2)\lambda$, and with the dimensionless coefficients

$$\begin{aligned} \theta_0 &= (1 - 3\eta)/8, & \theta_1 &= (3 + \eta)/2, & \theta_2 &= \eta/2, \\ \theta_3 &= -1/2, & \xi_1 &= (\chi_1/2)(5 + \chi_2), \\ \xi_2 &= -3(1 - 6\chi_2 + \chi_2^2), & \xi_3 &= -7 + 5\chi_2. \end{aligned} \quad (2.4)$$

The orbital equation of motion resulting from this Lagrangian, via $(d/dt)(\partial \mathcal{L}/\partial v^i) = \partial \mathcal{L}/\partial z^i$, is given by

$$\begin{aligned} a^i &= -\frac{Mn^i}{r} \left(1 + \frac{6\Lambda}{r^5} \right) + \frac{M}{c^2 r^2} \left[v^2 n^i \left(\phi_1 + \zeta_1 \frac{\Lambda}{r^5} \right) \right. \\ &\quad \left. + \dot{r}^2 n^i \left(\phi_2 + \zeta_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} n^i \left(\phi_3 + \zeta_3 \frac{\Lambda}{r^5} \right) \right. \\ &\quad \left. + \dot{r} v^i \left(\phi_4 + \zeta_4 \frac{\Lambda}{r^5} \right) \right] + O(c^{-4}) + O(\lambda^2), \end{aligned} \quad (2.5)$$

with coefficients

$$\begin{aligned} \phi_1 &= -1 - 3\eta, \\ \phi_2 &= 3\eta/2, \\ \phi_3 &= 2(2 + \eta), \\ \phi_4 &= 2(2 - \eta), \\ \zeta_1 &= -3(2 - \chi_2)(1 + 6\chi_2), \\ \zeta_2 &= 24(1 - 6\chi_2 + \chi_2^2), \\ \zeta_3 &= 66 + 9\chi_2 - 19\chi_2^2, \\ \zeta_4 &= 6(2 - \chi_2)(3 - 2\chi_2). \end{aligned} \quad (2.6)$$

The conserved energy constructed from the Lagrangian (2.3) is

$$\begin{aligned} E &= v^i \partial \mathcal{L}/\partial v^i - \mathcal{L} \\ &= \frac{\mu v^2}{2} - \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) \\ &\quad + \frac{\mu}{c^2} \left\{ 3\theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) \right. \right. \\ &\quad \left. \left. + \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) - \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\} \\ &\quad + O(c^{-4}) + O(\lambda^2), \end{aligned} \quad (2.7)$$

which is a constant of motion of the equation of motion (2.5).

The orbital equation of motion (2.5) admits solutions of the form

$$z^i(t) = rn^i(t) = r(\cos(\omega t), \sin(\omega t), 0), \quad (2.8a)$$

with $\dot{r} = 0$, $v^2 = r^2\omega^2$, and $a^i = -r\omega^2 n^i$, corresponding to circular orbits in the x - y plane with frequency ω . For later convenience, we introduce the unit vector ϕ^i in the direction of the velocity v^i , which satisfies

$$\dot{z}^i = v^i = r\omega\phi^i, \quad \dot{n}^i = \omega\phi^i, \quad \dot{\phi}^i = -\omega n^i, \quad n^i\phi^i = 0, \quad (2.8b)$$

for circular orbits. Working to linear order both in the post-Newtonian parameter c^{-2} and in the tidal deformability parameter λ , Eqs. (2.5) and (2.8) yield the radius-frequency relationship

$$r(\omega) = \frac{M^{1/3}}{\omega^{2/3}} \left[1 + \frac{3\chi_1}{\chi_2} \hat{\lambda} + \frac{\eta - 3}{3} x - \frac{\chi_1}{2\chi_2} (6 - 26\chi_2 + \chi_2^2)x\hat{\lambda} \right] + O(c^{-4}) + O(\lambda^2). \quad (2.9)$$

Here, we have introduced the ω -dependent dimensionless quantities

$$\hat{\lambda} \equiv \frac{\lambda\omega^{10/3}}{M^{5/3}}, \quad x \equiv \frac{(M\omega)^{2/3}}{c^2}, \quad (2.10)$$

which characterize the fractional corrections due to tidal effects and to post-Newtonian effects. We note that the parameter $\hat{\lambda}$ takes values in the range $\sim 1-8 \times 10^{-5}$ for two $1.4M_\odot$ stars at orbital frequencies of 200 Hz, given the range of λ values from Ref. [12], while $x \sim 0.07$. Using Eqs. (2.7), (2.8), and (2.9), we can also find the gauge-invariant energy-frequency relationship for circular orbits:

$$E(\omega) = \mu(M\omega)^{2/3} \left[-\frac{1}{2} + \frac{9\chi_1}{2\chi_2} \hat{\lambda} + \frac{9 + \eta}{24} x + \frac{11\chi_1}{4\chi_2} (3 + 2\chi_2 + 3\chi_2^2)x\hat{\lambda} \right] + O(c^{-4}) + O(\lambda^2). \quad (2.11)$$

This expression for the binding energy can be directly compared with Eqs. (37, 38, 50–57) of DN [15], and indicates that their parameter $\bar{\alpha}'_1$ giving the 1PN tidal contribution to the binding energy should have the value $\bar{\alpha}'_1 = (11/18)(3 + 2\chi_2 + 3\chi_2^2)$ instead of $55\chi_2/18$ (note that the quantity denoted here by χ_2 is denoted by X_A in DN [15]). For the case of equal masses ($\chi_1 = \chi_2 = 1/2$, $\eta = 1/4$), the binding energy (2.11) simplifies to

$$E(\omega) = -\frac{M^{5/3}\omega^{2/3}}{8} \left[1 - \frac{37}{48}x - 18\hat{\lambda} \left(1 + \frac{209}{72}x \right) \right].$$

For orbital frequencies of 200 Hz (GW frequencies of 400 Hz) and total mass $M = 2.8M_\odot$, the 1PN fractional

correction to the Newtonian tidal term in the binding energy is $(209/72)x \approx 19\%$.

III. GRAVITATIONAL RADIATION

The energy flux from the binary due to gravitational radiation is determined by the time variation of the binary system's multipole moments [6]. The flux \dot{E} to 3.5PN-order (or to 1PN-order relative to the leading 2.5PN flux) is given in terms of the total system's mass quadrupole moment $Q_{\text{sys}}^{ij}(t)$, current quadrupole moment $S_{\text{sys}}^{ij}(t)$, and mass octupole moment $Q_{\text{sys}}^{ijk}(t)$ by

$$\dot{E} = -\frac{1}{5c^5} (\partial_t^3 Q_{\text{sys}}^{ij})^2 - \frac{1}{189c^7} (\partial_t^4 Q_{\text{sys}}^{ijk})^2 + \frac{16}{45c^7} (\partial_t^3 S_{\text{sys}}^{ij})^2 + O(c^{-8}), \quad (3.1)$$

cf., Eq. (223) of Ref. [6].

The binary system's multipole moments can be computed from the asymptotic form of the global metric, as in Sec. IV of Ref. [17]. The mass quadrupole Q_{sys}^{ij} , which is needed to 1PN accuracy in the flux formula (3.1), can be found from Eqs. (4.5, 4.6, B4, B5, 6.1) of VF [17] with $S^i = 0$; the result is

$$Q_{\text{sys}}^{ij} = Q_2^{ij} + \mu r^2 n^{(ij)} + \frac{\mu r^2}{c^2} \left\{ n^{(ij)} \left[v^2 \left(\tau_1 + \sigma_1 \frac{\lambda}{r^5} \right) + \dot{r}^2 \left(\tau_2 + \sigma_2 \frac{\lambda}{r^5} \right) + \frac{M}{r} \left(\tau_3 + \sigma_3 \frac{\lambda}{r^5} \right) \right] + v^{(ij)} \left(\tau_4 + \sigma_4 \frac{\lambda}{r^5} \right) + \dot{r} n^{(i} v^{j)} \left(\tau_5 + \sigma_5 \frac{\lambda}{r^5} \right) \right\} + O(c^{-4}) + O(\lambda^2), \quad (3.2a)$$

where the 1PN-accurate body quadrupole Q_2^{ij} is given by Eqs. (2.2) above and the dimensionless τ and σ coefficients are given by

$$\begin{aligned} \tau_1 &= (29/42)(1 - 3\eta), \\ \tau_2 &= 0, \\ \tau_3 &= (1/7)(8\eta - 5), \\ \tau_4 &= (11/21)(1 - 3\eta), \\ \tau_5 &= (4/7)(3\eta - 1), \\ \sigma_1 &= 13\chi_1^2/7\chi_2, \\ \sigma_2 &= 185\chi_1^2/14\chi_2, \\ \sigma_3 &= -(3\chi_1/14\chi_2)(8 + 23\chi_2 + 13\chi_2^2), \\ \sigma_4 &= 38\chi_1^2/7\chi_2, \\ \sigma_5 &= -151\chi_1^2/7\chi_2. \end{aligned} \quad (3.2b)$$

This result holds for generic orbits (in a binary where body 2 has an adiabatically induced quadrupole). Using Eqs. (2.2) for the body quadrupole, Eqs. (2.8) to specialize

to circular orbits, and the radius-frequency relationship (2.9), the system quadrupole simplifies to

$$Q_{\text{sys}}^{ij} = \frac{\eta M^{5/3}}{\omega^{4/3}} [n^{(ij)}(1 + \sigma_0 \hat{\lambda} + \tau_6 x + \sigma_6 x \hat{\lambda}) + \phi^{(ij)}(\tau_4 x + \sigma_7 x \hat{\lambda})] + O(c^{-4}) + O(\lambda^2), \quad (3.3a)$$

with τ_4 as in Eq. (3.2b), and with

$$\begin{aligned} \sigma_0 &= (3/\chi_2)(3 - 2\chi_2), & \tau_6 &= -(85 + 11\eta)/42, \\ \sigma_6 &= (1/14\chi_2)(4 + 56\chi_2 + 264\chi_2^2 - 219\chi_2^3), \\ \sigma_7 &= (1/7\chi_2)(103 - 252\chi_2 + 302\chi_2^2 - 132\chi_2^3). \end{aligned} \quad (3.3b)$$

The expression (3.3a) for the total quadrupole determines the unknown 1PN correction coefficient introduced in Eq. (71) of DN [15].⁴

Similarly, the system's mass octupole and current quadrupole, which are needed only to Newtonian order, are given by

$$\begin{aligned} Q_{\text{sys}}^{ijk} &= \mu r^3 n^{(ijk)} \left[(\chi_1 - \chi_2) + \frac{9\chi_1}{\chi_2} \frac{\lambda}{r^5} \right] \\ &= \frac{\eta M^2}{\omega^2} n^{(ijk)} \left[(\chi_1 - \chi_2) + \frac{18\chi_1^2}{\chi_2} \hat{\lambda} \right], \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} S_{\text{sys}}^{ij} &= \mu r^2 \epsilon^{kl(i} n^{j)k} v^l \left[(\chi_1 - \chi_2) + \frac{9\chi_1}{2\chi_2} \frac{\lambda}{r^5} \right] \\ &= \frac{\eta M^2}{\omega} \epsilon^{kl(i} n^{j)k} \phi^l \left[(\chi_1 - \chi_2) + \frac{9\chi_1(3 - 4\chi_2)}{2\chi_2} \hat{\lambda} \right]. \end{aligned} \quad (3.5)$$

These expressions can be found from Eqs. (4.5, 4.6, B4, B5, 6.1) of VF [17] with $S^i = 0$. Here, the first equalities hold for generic orbits, and the second equalities hold for circular orbits; also, both Q_{sys}^{ijk} and S_{sys}^{ij} are subject to $O(c^{-2}) + O(\lambda^2)$ corrections.

Having gathered the expressions (3.3a), (3.4), and (3.5) for the system multipole moments, we can insert them into the flux formula (3.1). Using also Eqs. (2.8) for the time derivatives of n^i and ϕ^i (which are the only time-dependent quantities in the final expressions for the multipoles), and working out the STF projections and contractions (e.g. $n^{(ijk)} n^{(ijk)} = 2/5$) using the STF identities from (e.g.) Ref. [20], we find the GW energy flux from the binary to be given by

⁴The parametrization of the tidal contribution to the system quadrupole given in Eqs. (68–71) of DN [15] does not quite match the form given in Eq. (3.3a) here, as no $\phi^{(ij)}$ term is included. Also, their parametrization leaves some dependence on the radius r , while ours eliminates r in favor of the gauge-invariant quantity ω . Still, as the coefficients of $x\hat{\lambda}n^{(ij)}$ and $x\hat{\lambda}\phi^{(ij)}$ in our Eq. (3.3a) end up additively combined in the final contribution to the energy flux, one could in principle determine an effective value for the coefficient β_1 in Eq. (71) of DN [15] that would lead to the correct flux \dot{E} .

$$\begin{aligned} \dot{E}(\omega) &= -\frac{32}{5} \eta^2 x^{5/2} \left[1 - \left(\frac{1247}{336} + \frac{35\eta}{12} \right) x \right. \\ &\quad + \frac{6(3 - 2\chi_2)}{\chi_2} \hat{\lambda} + \frac{1}{28\chi_2} (-704 - 1803\chi_2 \\ &\quad \left. + 4501\chi_2^2 - 2170\chi_2^3) x \hat{\lambda} + O(c^{-3}) + O(\lambda^2) \right]. \end{aligned} \quad (3.6)$$

The coefficients for the 1PN point-mass (second) and Newtonian tidal (third) terms match those given in Refs. [5,6].

Using energy balance and the stationary phase approximation [1,25], the Fourier transform of the gravitational waveform can be written as $h = \mathcal{A} e^{i\psi}$, with the phase $\psi(\omega)$ determined from the binding energy $E(\omega)$ and flux $\dot{E}(\omega)$ as functions of the orbital frequency ω by the relation

$$\frac{d^2 \psi}{d\omega^2} = \frac{2}{\dot{E}} \frac{dE}{d\omega}. \quad (3.7)$$

Taking \dot{E} from Eq. (3.6), finding $dE/d\omega$ from a derivative of Eq. (2.11), and integrating twice (dropping unimportant integration constants) yields the phase:

$$\begin{aligned} \psi(\omega) &= \frac{3}{128\eta x^{5/2}} [1 + \psi_{0,1} \hat{\lambda} + \psi_{1,0} x + \psi_{1,1} x \hat{\lambda} \\ &\quad + O(c^{-3}) + O(\lambda^2)] \\ &= \frac{3c^5}{128\eta(M\omega)^{5/3}} \left[1 + \psi_{0,1} \frac{\lambda\omega^{10/3}}{M^{5/3}} + \psi_{1,0} \frac{(M\omega)^{2/3}}{c^2} \right. \\ &\quad \left. + \psi_{1,1} \frac{\lambda\omega^4}{Mc^2} + O(c^{-3}) + O(\lambda^2) \right], \end{aligned} \quad (3.8a)$$

with coefficients

$$\begin{aligned} \psi_{0,1} &= -\frac{24}{\chi_2} (1 + 11\chi_1), \\ \psi_{1,0} &= \frac{20}{9} \left(\frac{743}{336} + \frac{11\eta}{4} \right), \\ \psi_{1,1} &= -\frac{5}{28\chi_2} (3179 - 919\chi_2 - 2286\chi_2^2 + 260\chi_2^3). \end{aligned} \quad (3.8b)$$

The above results concern a binary where only one body (body 2) develops a tidally induced quadrupole, with quadrupolar tidal deformability $\lambda_2 = \lambda$. For the case of two deformable bodies, the contribution to the tidal signal from the other body (body 1) can simply be added to the phase by interchanging body labels ($1 \leftrightarrow 2$) in the tidal terms. For the case of equal masses and identical equations of state, $M_1 = M_2 = M/2$ and $\lambda_1 = \lambda_2 = \lambda$, the phase correction is

$$\psi(\omega) = \frac{3}{32x^{5/2}} \left[1 - 624\hat{\lambda} + \frac{2435}{378} x - \frac{3115}{2} x \hat{\lambda} \right].$$

The 1PN correction increases the tidal signal by $\approx 17\%$ at gravitational wave frequencies of 400 Hz for $M = 2.8M_\odot$.

From the expressions (3.6) and (2.11) for the gravitational wave luminosity $\dot{E}(\omega)$ and the binding energy $E(\omega)$, it is straightforward to construct the phase $\varphi(t)$ of the time-domain gravitational waveform based on the various PN Taylor approximants used in several approaches to interfacing analytical and numerical relativity [26]. We provide here the explicit expressions for the Taylor T4 approximant, in which the function $\mathcal{F} \equiv \dot{E}/(dE/dx)$ is expanded in a Taylor series and the differential equations

$$\frac{dx}{dt} = \mathcal{F}, \quad \frac{d\varphi}{dt} = 2x^{3/2}/M, \quad (3.9)$$

are integrated numerically [with x as in (2.10)]. The tidal contribution to the function \mathcal{F}^{T4} adds linearly to the 3.5PN point-mass terms and is given to 1PN order by

$$\mathcal{F}_{\text{tidal}}^{\text{T4}} = \frac{32\chi_1\lambda_2}{5M^6} \left[12(1 + 11\chi_1)x^{10} + \left(\frac{4421}{28} - \frac{12\,263}{28}\chi_2 + \frac{1893}{2}\chi_2^2 - 661\chi_2^3 \right) x^{11} \right] + (1 \leftrightarrow 2). \quad (3.10)$$

We note that the Newtonian and 1PN tidal terms in the expressions (3.8) and (3.10) for the waveform phasing are formally of 5PN and 6PN orders, having the same frequency dependencies as x^5 and x^6 . However, the coefficients of the tidal terms are not of order unity, but rather scale as $(R/M)^5 \sim 10^5$, where R and M are the stellar mass and radius (see footnote 3). As discussed in more detail in Refs. [11,15,22,23], this makes the magnitudes of the tidal terms comparable to those of 3PN-order and 3.5PN-order point-mass terms, which have also been computed analytically and are now routinely included in waveform templates [7]. We also note that including the 1PN tidal terms presented here in waveform templates will (slightly) change the constraints on λ achievable by Advanced

LIGO quoted in the Introduction. We refer the reader to Refs. [22,23] for analyses of the measurability of tidal effects in inspiral waveforms including 1PN tidal terms.

IV. DISCUSSION AND CONCLUSIONS

We have provided the 1PN accurate description of quasi-circular binary inspiral with quadrupolar tidal coupling and obtained the 1PN tidal contributions to the phasing of the emitted gravitational radiation in the low-frequency, adiabatic limit. Our results show that 1PN effects increase the tidal corrections by approximately 20% at gravitational wave frequencies of 400 Hz in the case of two $1.4M_\odot$ stars. These results should be of use in constructing GW measurement templates and can be easily be incorporated into the EOB formalism as discussed by DN [15]; the unknown coefficients introduced by DN pertaining to 1PN quadrupolar tidal effects have been determined here. Our results for the 1PN tidal corrections to GW generation have been used in Refs. [22,23] in making comparisons between numerical simulations and analytic models of the inspiral waveform and in assessing the measurability of tidal effects.

While we have restricted attention here to the case of circular orbits, the results necessary to compute the GW signal for generic orbits can all be found in this paper. This work could also be extended to consider 1PN tidal coupling at higher-multipolar orders; the necessary machinery (and the template of the quadrupolar case) is fully contained in VF [17].

ACKNOWLEDGMENTS

This research was supported at Cornell by NSF Grant No. PHY-0757735, and at Caltech by the Sherman Fairchild Foundation.

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