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Matter in inhomogeneous loop quantum cosmology: The Gowdy T^3 model

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We apply a hybrid approach which combines loop and Fock quantizations to fully quantize the linearly polarized Gowdy T^3 model in the presence of a massless scalar field with the same symmetries as the metric. Like in the absence of matter content, the application of loop techniques leads to a quantum resolution of the classical cosmological singularity. Most importantly, thanks to the inclusion of matter, the homogeneous sector of the model contains flat Friedmann-Robertson-Walker solutions, which are not allowed *in vacuo*. Therefore, this model provides a simple setting to study at the quantum level interesting physical phenomena such as the effect of the anisotropies and inhomogeneities on flat Friedmann-Robertson-Walker cosmologies.

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I. INTRODUCTION

Loop quantum cosmology (LQC) [1–4] is a quantization of cosmological models inspired in loop quantum gravity ideas and methods [5,6], in which the geometry has a discrete quantum nature. The first model successfully quantized to completion in LQC was the flat Friedmann-Robertson-Walker (FRW) model minimally coupled to a massless scalar field, whose dynamical analysis shows that a quantum bounce replaces the initial singularity [7]. The resolution of the cosmological singularity is a robust property of the theory [8,9], owing to the polymeric representation adopted for the geometry, and it is also achieved in the rest of homogeneous models quantized so far in LQC (see for instance [10–18] and references therein).

In order to allow for the presence of inhomogeneities within the framework of LQC, recently a hybrid approach to the quantization has been developed in the example of the simpler case: the Gowdy T^3 model with linear polarization [19–22]. This is a midisuperspace with three-torus spatial topology that contains inhomogeneities varying in a single direction [23].

The introduced hybrid approach combines the techniques of LQC with those of the Fock quantization for reduced models in which only global constraints remain to be imposed at the quantum level. The phase space is split in homogeneous and inhomogeneous sectors. The former is described by the degrees of freedom that parametrize the subset of homogeneous solutions, and the second one is formed by the rest of degrees of freedom. In the quantum theory, the inhomogeneous sector is represented *à la* Fock, in order to deal with the field complexity, while the homogeneous sector is represented following LQC, with the aim at obtaining a quantum model with no analog of the classical cosmological singularity. The approach assumes a hierarchy of quantum phenomena, so that the most relevant effects of the loop quantum geometry are those that affect the homogeneous degrees of freedom. In the case of the quantized Gowdy model, the homogeneous sector coincides with the phase space of the Bianchi I model, which has been extensively studied in LQC [14–16,24]. Concerning the inhomogeneous sector, the requirement that the conventional description for the inhomogeneities should be recovered when the quantum geometry effects of the homogeneous sector are negligible and that this description respect unitarity selects the Fock quantization of Refs. [25–27] without ambiguity. In fact, with the commented requirement, it has been shown that this is the unique satisfactory Fock quantization that the totally deparametrized Gowdy T^3 model admits [28,29].

Our aim is to further analyze inhomogeneous cosmologies in LQC by means of this hybrid quantization, now allowing for the presence of matter. In order to do this, we will include in the Gowdy T^3 model a minimally coupled massless scalar field with the same symmetries of the geometry. Choosing suitable field parametrizations for the inhomogeneities of both gravitational waves and matter, the corresponding field contributions appear in the constraints in the same way [30,31]. As a consequence, the uniqueness results of Refs. [28,29] for the Fock quantization apply to the nonvacuum case as well, and hence we have at our disposal a preferred Fock description also for the inhomogeneities of the matter field.

The interest of this work lies not only in the fact that it provides a complete quantization of a cosmological model with an inhomogeneous matter field in the framework of LQC, but also in that it means a further step towards the quantum analysis of physical inhomogeneities in cosmology, in the sense that these inhomogeneities propagate on a geometry not very different from that of our universe. Indeed, thanks to the inclusion of matter, now the homogeneous sector of the model (nonvacuum Bianchi I) admits

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flat FRW cosmologies as a subset of solutions, namely, the isotropic ones, and it is widely known that the observed universe can be approximated at large scales by a spacetime of this type. Therefore, it is natural to compare the dynamics of our inhomogeneous model with that of the flat isotropic model, and analyze the quantum effects that anisotropies and inhomogeneities produce over a hypothetical FRW-like background. In particular, this setting would allow us to investigate questions like the robustness of the quantum bounce scenario of LQC when inhomogeneities are included, or modifications to the evolution of the matter inhomogeneities when quantum geometry effects are taken into account.

Let us mention that, owing to the isometries of the Gowdy model, this family of spacetimes presents a particular subset of solutions with local rotational symmetry (LRS), in which the two scale factors of the directions of homogeneity coincide. Therefore, in order to simplify the analysis, it is convenient to focus on this kind of solution, which we will call LRS-Gowdy cosmologies in the following. We will carry out this LRS reduction at the quantum level using an adaptation of the (so-called) *projection* procedure introduced in Ref. [16] to pass from the loop quantized Bianchi I model to the loop quantized FRW model.

This work is intended as a first contribution to the analysis of the Gowdy system with matter. Specifically, here we quantize to completion the model, putting special attention to the new features that the consideration of the matter field introduces in comparison with the vacuum case. We also present a general discussion of the lines of attack that can be pursued to extract the physics from our quantum model. We leave for a future work a more rigorous and deeper study of this underlying physics and its consequences. The structure of this paper is as follows. The classical model is described in Sec. II. In Sec. III, we carry out its quantization, promoting the constraints to operators and characterizing the physical Hilbert space. We also show how to reduce the quantum model to the corresponding LRS-Gowdy counterpart. Finally, in the concluding Sec. IV, we point out the possibilities that this model provides to analyze quantum phenomena in cosmology and to reach physical predictions.

II. CLASSICAL MODEL

The linearly polarized Gowdy T^3 cosmologies are globally hyperbolic spacetimes with three-torus spatial topology and two axial and hypersurface orthogonal Killing vector fields [23]. We provide this system with a minimally coupled massless scalar field, Φ, with the same symmetries. We use global coordinates $\{t, \theta, \sigma, \delta\}$, where θ, σ, $\delta \in S^1$, such that the Killing fields are ∂_σ, ∂_δ. Then, all the fields (metric and matter) only depend on the coordinates t and θ. We reduce the system by performing a partial gauge fixing, as in Refs. [20,21,26]. As a result, the gravitational sector of the phase space turns out to be described by two pairs of canonically conjugate point-particle variables (they do not depend on θ) and by one field, together with its canonical momentum. We expand the fieldlike variables in Fourier series in the coordinate θ and split the phase space into two sectors: one formed by all the homogeneous degrees of freedom (the two point-particle gravitational variables and the zero modes of both matter and gravitational fields, together with their momenta) and the other formed by the nonzero modes of the two fields of the system and of their conjugate momenta. We call them homogeneous and inhomogeneous sectors, respectively.

In the totally deparametrized model, there is a particular field parametrization of the metric in which the gravitational wave content is described by a field which behaves exactly as the matter field Φ, namely, as a massless scalar field propagating in $2 + 1$ gravity [32]. Nonetheless, this description does not admit any Fock quantization with unitary dynamics [29,33,34]. For that, it is necessary to apply a time dependent canonical transformation on both fields [26,29]. The resulting gravitational and matter fields, which we call ξ and φ respectively, follow the equation of motion of a free scalar field with a time dependent mass in a static spacetime of $1 + 1$ dimensions. Consistent with our restriction to the inhomogeneous sector, we consider both fields already devoid of zero modes. Now, introducing for these fields creation and annihilationlike variables defined like one would naturally do in the case of free massless scalar fields, one reaches a Fock quantization whose evolution is indeed unitary [25,26] and such that the vacuum is invariant under S^1 translations, which is the gauge group of the reduced system. Moreover, it has been shown that these two natural properties of unitary dynamics and vacuum invariance in fact pick up this Fock quantization as the unique acceptable one, up to unitary equivalence [28,29]. So, taking into account this result for the totally deparametrized model, we adopt the suitable field parametrization of Refs. [25,26] both for the gravitational and matter inhomogeneities of our current model, and describe them in terms of the creation and annihilationlike variables mentioned above, in order to eventually carry out the corresponding Fock quantization of the inhomogeneous sector. We will call these variables $(a_m^{(\alpha)*}, a_m^{(\alpha)})$, with $m \in \mathbb{Z} - \{0\}$ and $\alpha = \xi, \varphi$.

On the other hand, the homogeneous sector describes Bianchi I cosmologies with spatial three-torus topology and with a minimally coupled homogeneous massless scalar field, given by the zero mode of Φ. From now on, we call ϕ this *homogeneous* matter field and P_ϕ its momentum. Since the homogeneous sector is to be quantized using LQC methods, we describe the gravitational variables of this sector in the Ashtekar-Barbero formalism. Using a diagonal gauge, the nontrivial components of the densitized triad are $p_j/4\pi^2$, with $j = \theta, \sigma, \delta$, whereas those of the $su(2)$ connection are $c_j/2\pi$ (see e.g. [15]).

These variables satisfy $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$, where γ is the Immirzi parameter and G is the Newton constant (throughout the text, we set the speed of light equal to the unity).

Two global constraints still remain on this reduced system: the spatial average of the densitized Hamiltonian constraint, $\mathcal{C}$, and the generator of S^1 translations, $\mathcal{C}_\theta$. On the one hand, $\mathcal{C}$ can be split into two terms, $\mathcal{C} = \mathcal{C}_{\text{hom}} + \mathcal{C}_{\text{inh}}$, the former involving only the homogeneous sector. Then, $\mathcal{C}_{\text{inh}}$ couples the homogeneous gravitational sector with both the gravitational and matter inhomogeneous sectors through two identical terms, one per field, $\mathcal{C}_{\text{inh}} = \mathcal{C}^\xi_{\text{inh}} + \mathcal{C}^\varphi_{\text{inh}}$, where $\mathcal{C}^\xi_{\text{inh}}$ denotes the corresponding coupling term for the vacuum case [20]. The homogeneous term $\mathcal{C}_{\text{hom}}$ is the densitized Hamiltonian constraint of the Bianchi I model minimally coupled to a homogeneous massless scalar field [15,16]. Thanks to the presence of matter, the classical Bianchi I model admits solutions of the FRW type. On the other hand, as it happens to be the case with the inhomogeneous term $\mathcal{C}_{\text{inh}}$, $\mathcal{C}_\theta$ is the sum of two identical contributions, $\mathcal{C}_\theta = \mathcal{C}^\xi_\theta + \mathcal{C}^\varphi_\theta$, where $\mathcal{C}^\xi_\theta$ denotes the analog constraint *in vacuo* [20]. We see that, with our choice of variables, matter and gravitational inhomogeneities contribute in the same way to the constraints, and then it is straightforward to promote $\mathcal{C}_\theta$ and $\mathcal{C}$ to operators following the hybrid quantization developed for the vacuum case.

III. QUANTUM MODEL

A. Kinematics and constraint operators

The quantization of the system starts with the introduction of a kinematical Hilbert space where the basic variables are represented as operators and where the constraints are imposed quantum mechanically. For this kinematical Hilbert space, $\mathcal{H}_{\text{kin}}$, a natural selection is the tensor product of the kinematical Hilbert space of the gravitational sector $\mathcal{H}^{\text{grav}}_{\text{kin}}$ times the kinematical Hilbert space of the matter sector $\mathcal{H}^{\text{matt}}_{\text{kin}}$. Both of these spaces are in turn the tensor product of two spaces corresponding to homogeneous and inhomogeneous sectors, respectively. Physically, the nontriviality of the system comes from the couplings introduced at the moment of imposing the quantum constraints.

For the gravitational sector, then, we carry out the hybrid quantization of Refs. [21,22], namely $\mathcal{H}^{\text{grav}}_{\text{kin}}$ is the tensor product of the kinematical Hilbert space of the Bianchi I model in LQC [16,22], $\mathcal{H}^{\text{BI}}_{\text{kin}}$, times the standard Fock space for the inhomogeneities, $\mathcal{F}^\xi$, defined in terms of the annihilation and creation variables previously described for the field ξ. The homogeneous matter sector, on the other hand, is formed by the zero modes of the massless scalar field and its momentum, determined by ϕ and P_ϕ. In analogy with the nonvacuum cases analyzed in homogeneous LQC (in particular the Bianchi I model minimally coupled to a massless scalar [16]), we take the standard representation for these variables, choosing $L^2(\mathbb{R}, d\phi)$ as the Hilbert space. Finally, since matter and gravitational inhomogeneities have identical behavior, the kinematical Hilbert space accounting for the matter inhomogeneities, $\mathcal{F}^\varphi$, is totally analogous to $\mathcal{F}^\xi$. Summarizing,

$$\mathcal{H}_{\text{kin}} = \mathcal{H}^{\text{BI}}_{\text{kin}} \otimes L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}^\xi \otimes \mathcal{F}^\varphi. \tag{3.1}$$

For the inhomogeneous sector, the chosen representation is obtained by promoting the classical variables $a^{(\alpha)*}_m$ and $a^{(\alpha)}_m$, with $m \in \mathbb{Z} - \{0\}$ and $\alpha \in \{\xi, \varphi\}$, to creation and annihilation operators, $\hat{a}^{(\alpha)\dagger}_m$ and $\hat{a}^{(\alpha)}_m$, respectively. With them, it is straightforward to construct the quantum counterpart of the constraint $\mathcal{C}_\theta$, for which we choose normal ordering. The result is [20,21]

$$\hat{\mathcal{C}}_\theta = \sum_{m=1}^{\infty} m\hat{X}^\xi_m + \sum_{m=1}^{\infty} m\hat{X}^\varphi_m, \qquad \hat{X}^\alpha_m = \hat{a}^{(\alpha)\dagger}_m \hat{a}^{(\alpha)}_m - \hat{a}^{(\alpha)\dagger}_{-m}\hat{a}^{(\alpha)}_{-m}. \tag{3.2}$$

The same strategy is adopted when representing the inhomogeneous contributions to the coupling terms $\mathcal{C}^\alpha_{\text{inh}}$. It turns out that the inhomogeneities of the field α (ξ or φ) appear in $\mathcal{C}^\alpha_{\text{inh}}$ only via two different quadratic combinations, H^α_0 and H^α_{int}, whose normal ordered quantum counterparts are [20,21]

$$\hat{H}^\alpha_0 = \sum_{m=1}^{\infty} m\hat{N}^\alpha_m, \qquad \hat{H}^\alpha_{\text{int}} = \sum_{m=1}^{\infty} \frac{1}{m}(\hat{N}^\alpha_m + \hat{a}^{(\alpha)\dagger}_m \hat{a}^{(\alpha)\dagger}_{-m} + \hat{a}^{(\alpha)}_m \hat{a}^{(\alpha)}_{-m}), \tag{3.3}$$

with $\hat{N}^\alpha_m = \hat{a}^{(\alpha)\dagger}_m \hat{a}^{(\alpha)}_m + \hat{a}^{(\alpha)\dagger}_{-m}\hat{a}^{(\alpha)}_{-m}$. The above operators $\hat{X}^\alpha_m$, $\hat{H}^\alpha_0$, and $\hat{H}^\alpha_{\text{int}}$ act nontrivially on $\mathcal{F}^\alpha$ and have as a common dense domain the space of n-particle states. We call n^α_m the number of particles of the field α in the mode m.

On the other hand, for the homogeneous sector, the basic matter variables are represented by the operators $\hat{\phi}$, which acts by multiplication, and $\hat{P}_\phi = -i\hbar\partial_\phi$, while for the gravitational part we adopt the operator representation discussed in detail in Ref. [21] (see also Ref. [22]), adhering to the improved dynamics scheme put forward by Ashtekar and Wilson-Ewing [16] (and which was called "case B" in Ref. [21]). Let us briefly review this quantization scheme. First we recall that, on $\mathcal{H}^{\text{BI}}_{\text{kin}}$, the operators $\hat{p}_i$ ($i = \theta, \sigma, \delta$), which represent the nontrivial coefficients of the densitized triad of the Bianchi I model, have a discrete spectrum equal to the real line. The corresponding eigenstates, $|p_\theta, p_\sigma, p_\delta\rangle$, form an orthonormal basis (in the discrete norm) of $\mathcal{H}^{\text{BI}}_{\text{kin}}$. Owing to this discreteness, there is no well-defined operator representing the connection, but rather its holonomies. The representation of the matrix elements of these holonomies incorporates the so-called improved dynamics prescription, which states that there exists a dynamical (state dependent) minimum length $\bar{\mu}_i$ for the straight edges in the ith-direction along which the holonomies are computed. We use the specific improved dynamics prescription put forward in Ref. [16]. Then, the elementary operators which represent the matrix elements of the holonomies, called $\hat{\mathcal{N}}_{\bar{\mu}_i}$, produce all a constant shift in the physical Bianchi I volume [16,21]. The resulting action of $\hat{\mathcal{N}}_{\bar{\mu}_i}$ on the states $|p_\theta, p_\sigma, p_\delta\rangle$ is quite involved. In order to simplify the analysis, it is convenient to relabel the basis states in the form $|v, \lambda_\sigma, \lambda_\delta\rangle$, where v is an affine parameter proportional to the volume of the compact spatial section, such that any of the operators $\hat{\mathcal{N}}_{\pm\bar{\mu}_i}$ ($i = \theta, \sigma, \delta$) causes a unit (positive or negative) shift on it. The parameters λ_i are all equally defined in terms of the corresponding parameters p_i, and verify that $v = 2\lambda_\theta\lambda_\sigma\lambda_\delta$ (see the explicit definitions in Ref. [16]).

Employing the basic homogeneous gravitational operators $\hat{p}_i$ and $\hat{\mathcal{N}}_{\pm\bar{\mu}_i}$, we can complete the construction of the constraint operator $\hat{\mathcal{C}} = \hat{\mathcal{C}}_{\text{hom}} + \hat{\mathcal{C}}_{\text{inh}}$ exactly in the same way as in the vacuum case [21]. This densitized Hamiltonian constraint operator is formed by

$$\hat{\mathcal{C}}_{\text{hom}} = -\sum_{i\neq j}\sum_{j}\frac{\hat{\Theta}_i\hat{\Theta}_j}{16\pi G\gamma^2} - \frac{\hbar^2}{2}\left[\frac{\partial}{\partial\phi}\right]^2, \tag{3.4}$$

$$\hat{\mathcal{C}}_{\text{inh}} = 2\pi\hbar\widehat{|p_\theta|}(\hat{H}_0^\xi + \hat{H}_0^\varphi) + \hbar\left[\widehat{\frac{1}{|p_\theta|^{1/4}}}\right]^2\frac{(\hat{\Theta}_\delta + \hat{\Theta}_\sigma)^2}{16\pi\gamma^2}\left[\widehat{\frac{1}{|p_\theta|^{1/4}}}\right]^2(\hat{H}_{\text{int}}^\xi + \hat{H}_{\text{int}}^\varphi), \tag{3.5}$$

with $i, j \in \{\theta, \delta, \sigma\}$. Here, $[1/\widehat{|p_\theta|}^{1/4}]$ is a regularized triad operator which has a diagonal action on the considered basis of states. On the other hand, the operator $\hat{\Theta}_i$ is the quantum counterpart of the classical quantity c_ip_i and its action on the basis states is highly nontrivial. In particular, $\hat{\Theta}_i$ and $\hat{\Theta}_j$ do not commute for $i \neq j$. We will not give here the explicit action of these operators on our basis states (which can be found in Ref. [21]). Instead, in the following section, we will write down explicitly the general equation that must be satisfied by the solutions of the quantum densitized Hamiltonian constraint.

The above constraint operator leaves invariant certain subspaces of $\mathcal{H}_{\text{kin}}$, which provide superselection sectors [4,21]. When symmetrizing $\hat{\mathcal{C}}$, we have chosen a specific factor ordering which leads to superselection sectors which are particularly simple and with most convenient properties. More precisely, instead of considering $\mathcal{H}_{\text{kin}}^{\text{BI}}$, we can restrict the homogenous gravitational sector to be the completion with respect to the discrete norm of the space spanned by the states $|v, \lambda_\sigma, \lambda_\delta\rangle$ such that v, λ_σ, and λ_δ belong to an octant, for instance $v, \lambda_\sigma, \lambda_\delta > 0$ (the case on which we will focus our attention from now on), and with v belonging then to any semilattice $\mathcal{L}_\epsilon$ of step four included in $\mathbb{R}^+$:

$$\mathcal{L}_\epsilon = \{\epsilon + 4k; k \in \mathbb{N}\}. \tag{3.6}$$

In this expression, ϵ is any number in the interval $(0, 4]$, and provides the minimum value that v takes. In addition, given ϵ, the labels λ_a ($a = \sigma$ or δ) are restricted to sectors of the form $\lambda_a = \lambda_a^\star\omega_\epsilon$, where the $\lambda_a^\star$'s are any two fixed positive numbers and ω_ϵ runs over the following numerable and dense subset of $\mathbb{R}^+$:

$$\left\{\left(\frac{\epsilon-2}{\epsilon}\right)^z\prod_k\left(\frac{\epsilon+2m_k}{\epsilon+2n_k}\right)^{p_k}\right\}. \tag{3.7}$$

Here $m_k, n_k, p_k \in \mathbb{N}$, and $z \in \mathbb{Z}$ when $\epsilon > 2$, while $z = 0$ otherwise [21].

Once we restrict the study to any of the above superselection sectors, the null eigenspace of the homogeneous densitized triad operator (which is a proper subspace of $\mathcal{H}_{\text{kin}}^{\text{BI}}$) ceases to be included in our theory. As a consequence, there is no analog of the classical cosmological singularity in the quantum model anymore. In this sense, it is ensured that the singularity is resolved, already at the kinematical level.

B. Physical Hilbert space

Once we have constructed the constraint operators, we can proceed to determine the physical states, which must be annihilated by these constraints. Notice that the two constraint operators commute and can hence be imposed consistently.

Let us consider first, e.g., the S^1 symmetry generated by $\hat{\mathcal{C}}_\theta$, which amounts to the following condition

$$\sum_{m=1}^{\infty} m(X_m^\xi + X_m^\varphi) = 0, \qquad X_m^\alpha = n_m^\alpha - n_{-m}^\alpha, \qquad \alpha = \xi, \varphi. \tag{3.8}$$

The states that satisfy this condition form a proper subspace of $\mathcal{F}^\xi \otimes \mathcal{F}^\varphi$, which we call $\mathcal{F}_{\text{p}}$.

The Hamiltonian constraint operator imposes a more complicated condition, mainly because of the nontrivial actions of both $\hat{\Theta}_i$ on the homogeneous gravitational sector and $\hat{H}_{\text{int}}^\alpha$ on the inhomogeneous sector [21,22]. For our purposes here, it suffices to make explicit the action of the Hamiltonian constraint operator on just the homogeneous sector. With this aim, it proves convenient to introduce an alternate labeling of the basis states of $\mathcal{H}_{\text{kin}}^{\text{BI}}$. The new labeling is given by $|v, \Lambda, \Upsilon\rangle$, where $\Lambda = \ln(\lambda_\sigma\lambda_\delta)$ and $\Upsilon = \ln(\lambda_\delta/\lambda_\sigma)$. Next, we expand a general state $|\Psi\rangle$ in this basis:

$$|\Psi\rangle = \sum_{\bar{v},\bar{\Lambda},\bar{\Upsilon}} |\Psi(\bar{v}, \bar{\Lambda}, \bar{\Upsilon})\rangle \otimes |\bar{v}, \bar{\Lambda}, \bar{\Upsilon}\rangle. \tag{3.9}$$

Here, $\bar{v}$, $\bar{\Lambda}$, and $\bar{\Upsilon}$ take values in the corresponding superselection sectors. Let us clarify that the kets $|\Psi(\bar{v}, \bar{\Lambda}, \bar{\Upsilon})\rangle$

are actually not wave function coefficients, but rather *states* inasmuch as we have not expanded $|\Psi\rangle$ in a basis of the whole kinematical Hilbert space, but only of the homogeneous gravitational sector. On the other hand, based on our experience with other similar cosmological models, the *states* $|\Psi(\bar{v}, \bar{\Lambda}, \bar{\Upsilon})\rangle$ for the solutions of the Hamiltonian constraint are not expected to be normalizable in $L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}^\xi \otimes \mathcal{F}^\varphi$, but rather to belong to a larger space from which one should construct the physical Hilbert space of the theory. Acting with $\hat{\mathcal{C}}$ on $|\Psi\rangle$ and projecting over $\langle v, \Lambda, \Upsilon|$, we obtain:

$$\begin{aligned} &-\frac{8}{\pi G}\left[\frac{\partial}{\partial \phi}\right]^2 |\Psi(v,\Lambda,\Upsilon)\rangle \\ &+ \sum_{\kappa\in\{0,4\}}\sum_{s\in\{+,-\}} x^s_\kappa(v)|\Psi^s_\kappa(s\kappa+v,\Lambda,\Upsilon)\rangle \\ &- 4\beta e^{2\Lambda} b^2(v)[\hat{H}^\xi_{\rm int} + \hat{H}^\varphi_{\rm int}] \sum_{\kappa\in\{0,4\}}\sum_{s\in\{+,-\}} b^2(s\kappa+v) \\ &\times \frac{s\kappa+v}{v} x^s_\kappa(v)|\Psi^{s\prime}_\kappa(s\kappa+v,\Lambda,\Upsilon)\rangle \\ &+ \frac{8}{\beta}\frac{v^2}{e^{2\Lambda}}[\hat{H}^\xi_0 + \hat{H}^\varphi_0]|\Psi(v,\Lambda,\Upsilon)\rangle = 0. \end{aligned} \tag{3.10}$$

Here, $\beta = [G\hbar/(16\pi^2\gamma^2\Delta)]^{1/3}$, with Δ denoting the minimum nonzero eigenvalue allowed for the area in loop quantum gravity [16,21], and we have defined

$$b(v) = |\sqrt{|v+1|} - \sqrt{|v-1|}|, \tag{3.11}$$

$$\begin{aligned} x^s_\kappa(v) = &-\frac{e^{i\pi\kappa/4}}{2}|s2+v|\sqrt{v|s\kappa+v|} \\ &\times\left\{1 + \mathrm{sgn}\left(s\left[2+\frac{\kappa}{2}\right]+v\right)\right\}. \end{aligned} \tag{3.12}$$

On the other hand, the objects $|\Psi^s_\kappa(s\kappa+v,\Lambda,\Upsilon)\rangle$, are linear combinations of six contributions in the form

$$\begin{aligned} &|\Psi^s_\kappa(s\kappa+v,\Lambda,\Upsilon)\rangle \\ &= \sum_{r\in\{1,-1\}} (|\Psi(s\kappa+v,\Lambda+w_v(s2),\Upsilon+rw_v(s2))\rangle \\ &+ |\Psi(s\kappa+v,\Lambda+w_v(s\kappa),\Upsilon+rw_v(s\kappa)-2rw_v(s2))\rangle \\ &+ |\Psi(s\kappa+v,\Lambda+w_v(s\kappa)-w_v(s2),\Upsilon+rw_v(s\kappa) \\ &- rw_v(s2))\rangle), \end{aligned} \tag{3.13}$$

where $w_v(sn) = \ln(sn+v) - \ln(v)$. The three last lines of Eq. (3.10) correspond to the action produced by $\hat{\mathcal{C}}_{\rm inh}$, where we have introduced the notation

$$\begin{aligned} &|\Psi^{s\prime}_\kappa(s\kappa+v,\Lambda,\Upsilon)\rangle \\ &= \sum_{r\in\{1,-1\}} (|\Psi(s\kappa+v,\Lambda+w_v(s\kappa),\Upsilon+rw_v(s\kappa))\rangle \\ &+ |\Psi(s\kappa+v,\Lambda+w_v(s\kappa),\Upsilon+rw_v(s\kappa)-2rw_v(s2))\rangle). \end{aligned} \tag{3.14}$$

Condition (3.10), coming from the constraint, is a difference equation in the variable v and can be seen as an evolution equation in this variable. In the vacuum model, it has been proven that, formally, a(n infinite but countable) set of initial data on the section given by the minimum value of v, $v_{\rm min} = \epsilon \in (0,4]$, completely determines a solution of the densitized Hamiltonian constraint [21,22]. Since the Hamiltonian constraint of our model and the one *in vacuo* have identical structure, the above result applies also to our case. This property allows us to identify the physical Hilbert space of the system, that we call $\mathcal{H}_{\rm phys}$, as the Hilbert space of these initial data.

The resulting physical Hilbert space, taking into account condition (3.8) as well, is given by $\mathcal{H}_{\rm phys} = \mathcal{H}^{\rm BI}_{\rm phys} \otimes L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}_{\rm p}$, where $\mathcal{H}^{\rm BI}_{\rm phys}$ is the physical Hilbert space of the Bianchi I model determined in Ref. [22]. As discussed in that reference, the inner product that provides this Hilbert space structure on the space of initial data is obtained by the requirement that the complex conjugation relations between a complete set of classical observables turn into adjoint relations between the corresponding operators.

C. *Projection* to LRS-Gowdy

The Gowdy T^3 model with linear polarization is symmetric under the interchange of the directions coordinatized by σ and δ. Owing to this, it has a subset of classical solutions with local rotational symmetry, in which the scale factors of these two directions can be identified during the entire evolution. We can then restrict the Gowdy model, both *in vacuo* and with matter, to the LRS-Gowdy model in which every solution is of this kind. The restriction can be performed classically, prior to quantization, or starting with the quantized model. We will focus our attention on the latter approach, passing from quantum Gowdy to quantum LRS-Gowdy, and leave for the interested reader the proof that the quantum model obtained in this way is indeed recovered by a direct quantization of the classical LRS-Gowdy spacetimes along the lines explained in this work.

In analogy with the discussion of Ref. [16], in which the quantum FRW model is obtained from quantum Bianchi I, we define the following map from (generalized) states associated with the Gowdy model to those of the LRS-Gowdy cosmologies, denoted by $|\psi(v,\Lambda)\rangle$:

$$|\Psi(v,\Lambda,\Upsilon)\rangle \to \sum_\Upsilon |\Psi(v,\Lambda,\Upsilon)\rangle \equiv |\psi(v,\Lambda)\rangle. \tag{3.15}$$

The sum is carried out over all values of Υ in the considered superselection sector. Applying this map in the Hamiltonian constraint (3.10), we obtain

$$- \frac{4}{\pi G}\left[\frac{\partial}{\partial \phi}\right]^2 |\psi(v,\Lambda)\rangle + \sum_{\kappa\in\{0,4\}}\sum_{s\in\{+,-\}} x_\kappa^s(v)|\psi_\kappa^s(s\kappa+v,\Lambda)\rangle - 8\beta b^2(v)e^{2\Lambda}[\hat{H}_{\rm int}^{\xi}+\hat{H}_{\rm int}^{\varphi}]\sum_{\kappa\in\{0,4\}}\sum_{s\in\{+,-\}} b^2(s\kappa+v) \times \frac{s\kappa+v}{v}x_\kappa^s(v)|\psi(s\kappa+v,\Lambda+w_v(s\kappa))\rangle + \frac{4}{\beta}\frac{v^2}{e^{2\Lambda}}[\hat{H}_0^{\xi}+\hat{H}_0^{\varphi}]|\psi(v,\Lambda)\rangle = 0, \quad (3.16)$$

where $|\psi_\kappa^s(s\kappa+v,\Lambda)\rangle$ are the combinations

$$|\psi_\kappa^s(s\kappa+v,\Lambda)\rangle = |\psi(s\kappa+v,\Lambda+w_v(s\kappa)-w_v(s2))\rangle + |\psi(s\kappa+v,\Lambda+w_v(s\kappa))\rangle + |\psi(s\kappa+v,\Lambda+w_v(s2))\rangle. \quad (3.17)$$

As we have already remarked, the result agrees with the constraint obtained by a suitable hybrid quantization of the classical LRS-Gowdy model. It is worth noting that the introduced map works because the coefficients appearing in Eq. (3.10) do not depend on the variable Υ, over which one sums to perform the *projection*. Indeed, this kind of map only makes sense if the classical model admits the imposition of an additional symmetry which allows its reduction into a dynamically stable submodel. A similar *projection* summing over Λ is not viable because the coefficients of the inhomogeneous contributions in the constraint depend explicitly on this variable, reflecting the fact that the associated kind of isotropic solutions exist just when the inhomogeneities are unplugged.

IV. DISCUSSION

We have completely quantized the Gowdy T^3 model with linearly polarized gravitational waves provided with a minimally coupled massless scalar field as matter content. The description adopted for the matter inhomogeneities is such that they can be treated in exactly the same way as the gravitational ones, the former just duplicating the contributions of the latter in the constraints. In this situation, we have been able to apply the hybrid quantization methods developed in Refs. [19–22] almost straightforwardly to this system with local physical degrees of freedom both in the matter content and in the gravitational field. To our knowledge, it is the first time that a model with these properties has been quantized to completion in the framework of LQC.

Since the structure of the constraints when the matter field is present is the same as *in vacuo*, all the results obtained in Ref. [21] for the vacuum Gowdy model apply as well to our model. Thus, in particular, we recover on physical states the standard quantum field theory description of both the matter and the gravitational inhomogeneities, living on a cosmological background quantized using LQC methods and consisting of a Bianchi I universe with a *homogeneous* massless scalar field. In addition, it is guaranteed that the states which are the analog of the classical singularity decouple naturally in the quantum model, so that, to this extent, the initial singularity is resolved at the kinematical level.

Conceptually, the hybrid quantization of the present family of inhomogeneous cosmologies has introduced no technical complication with respect to the vacuum case. Nonetheless, the situation is radically different when one considers the interest of the quantum model from a physical point of view. In fact, thanks to the inclusion of the massless scalar field, the homogeneous sector of the Gowdy T^3 model, namely, the Bianchi I model, admits now isotropic flat solutions of the FRW type, while *in vacuo* only the trivial Minkowskian solution is allowed.

On the other hand, the analysis of the classical solutions of the linearly polarized Gowdy T^3 model *in vacuo* [35] and the study of the effective dynamics obtained from the hybrid quantization of this model [36] show that small inhomogeneities do not increase arbitrarily in the evolution. Then, if we consider initial data which are sufficiently close to homogeneity, the corresponding solution would remain approximately homogeneous during the evolution. Besides, in the nonvacuum model, matter and gravitational inhomogeneities evolve in identical ways. This strongly indicates that initial data in a sufficiently small neighborhood of those with isotropy and homogeneity have to lead to approximately isotropic and homogeneous solutions. Therefore, it is natural to compare the dynamics of our Gowdy model with that of the flat FRW model (with three-torus topology) in order to see how the inclusion of anisotropies and inhomogeneities affects the evolution of a flat FRW background. Moreover, we are now in a perfect situation to carry out this comparison at the quantum level, since here we have accomplished the full quantization of the Gowdy T^3 model in the presence of the massless matter, and the loop quantization of the FRW model coupled to the homogeneous massless field is well known [7,9]. Even though the inhomogeneities in our model are not all those allowed in a universe like the one which we observe (but just a subfamily with the symmetries of the Gowdy T^3 cosmologies), their analysis should shed light on the kind of quantum effects affecting the evolution and on the consequences of the quantum geometry on the primordial fluctuations.

For these purposes, it is preferable to focus on the LRS-Gowdy model derived in Sec. III C. Indeed, the consideration of the 2 degrees of freedom of anisotropy that the homogeneous sector of the general Gowdy model possesses would only complicate the equations unnecessarily. The presence of either 2 degrees or just 1 degree of anisotropy does not seem to have any conceptual relevance for the proposed analysis.

In order to face this analysis, the idea is to add and subtract in Eq. (3.16) the term corresponding to the FRW model, which coincides with the first line of Eq. (3.16) but keeping the variable Λ unchanged. We can then rewrite Eq. (3.16) as the constraint equation of the FRW model coupled to a homogeneous massless scalar field plus a number of contributions coming from all other terms. These contributions contain the inhomogeneities and the difference between the gravitational parts of the constraints for LRS-Bianchi I and FRW, a difference which is due to the anisotropies. In this way, the resulting expression modifies the densitized Hamiltonian constraint of the FRW model by the effects of the anisotropies and inhomogeneities, so that it is no longer equal to zero. As we have commented, we are interested in comparing the FRW model with the inhomogeneous LRS-Gowdy model when these inhomogeneities and anisotropies are small. In this regime, it makes sense to apply a type of Born-Oppenheimer approximation, similar to others commonly employed in cosmology (see e.g. [37,38]), and assume that the variations of the isotropic degrees of freedom and those of the rest of degrees have considerably different typical scales, therefore giving them a different status. Then, in this approximation, it is easier to derive effectively the influence that anisotropies and inhomogeneities produce on the isotropic background. We leave for future research this detailed analysis.

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