Gravitational radiation and angular momentum flux from a slowly rotating dynamical black hole

Yu-Huei Wu*

Center for Mathematics and Theoretical Physics, National Central University, Chungli, 320, Taiwan, and Department of Physics, National Central University, Chungli, 320, Taiwan, ROC

Chih-Hung Wang†

Department of Physics, Tamkang University, Tamsui, Taipei 25137, Taiwan, and Institute of Physics, Academia Sinica, Taipei 115, Taiwan and Department of Physics, National Central University, Chungli, 320, Taiwan, ROC

(Received 5 November 2010; published 25 April 2011)

A four-dimensional asymptotic expansion scheme is used to study the next-order effects of the nonlinearity near a spinning dynamical black hole. The angular-momentum flux and energy flux formula are then obtained by constructing the reference frame in terms of the compatible constant spinors and the compatibility of the coupling leading-order Newman-Penrose equations. By using the slow rotation and small-tide approximation for a spinning black hole, the horizon cross-section we chose is spherical symmetric. It turns out the flux formula is rather simple and can be compared with the known results. Directly from the energy flux formula of the slow-rotating dynamical horizon, we find that the physically reasonable condition on requiring the positivity of the gravitational energy flux yields that the shear will monotonically decrease with time. Thus a slow-rotating dynamical horizon will asymptotically approach an isolated horizon during late time.

DOI: 10.1103/PhysRevD.83.084044 PACS numbers: 04.30.Db, 04.20.Ha, 04.30.Nk, 04.70.Bw

I. INTRODUCTION

Null infinity and the black hole horizon have similar geometrical properties. They are both three-dimensional hypersurfaces and have the gravitational flux across them. The physical properties of null infinity can be studied in the conformal space-time with a finite boundary. Thus the conformal method provides an alternative way to study Bondi-Sachs gravitational radiation near null infinity, which was first proposed by Penrose [1]. The boundary of a black hole is asymptotically nonflat and one may not be able to apply the conformal method to study the boundary problem of a dynamical black hole. Rather than using the symmetry for the whole space-time to locate the boundary of a black hole, Ashtekar *et al.* use a rather mild condition on the symmetry of the three-dimensional horizon [2,3]. This quasilocal definition for the black hole boundary makes it possible to study the gravitational radiation and the time evolution of the black hole.

In this paper, we use the Bondi-type coordinates to write the null tetrad for a spinning dynamical horizon (DH). The boundary conditions for the quasilocal horizons can be expressed in terms of Newman-Penrose (NP) coefficients from the Ashtekar's definition of DH. Unlike Ashtekar *et al*'s [2,3] three-dimensional analysis, we adopt a four-dimensional asymptotic expansion to study the neighborhoods of generic isolated horizons (IHs) and DHs. Since the asymptotic expansion has been used to study gravitational radiations near the null infinity [4,5], it offers a useful scheme to analyze gravitational radiation approaching another boundary of space-time, black hole horizons. We first set up a null frame with the proper gauge choices near quasilocal horizons and then expand Newman-Penrose (NP) coefficients, Weyl, and Ricci curvature with respect to radius. Their falloff can be determined from NP equations, Bianchi equations, and exact solutions, e.g., the Vaidya solution. This approach allows one to see the next-order contributions from the nonlinearity of the full theory for the quasilocal horizons.

We have shown that the quasilocal energy-momentum flux formula for a nonrotating DH by using asymptotic expansion yields the same result as Ashtekar-Krishman flux [6,7]. For slow-rotating DH, we have presented our results in [6], however, we use an assumption of vanishing NP coefficient λ on DH. Furthermore, the energy-momentum flux formula has a shear (NP coefficient σ) and a angular-momentum (NP coefficient π) coupling term. Since it is unclear whether the existence of this term carries any physical meaning or it may due to our assumptions, we thereby extend our previous work on IHs and DHs into a more general case.

An algebraically general structure (Petrov type I) of space-time is thought to be related with gravitational radiation for an isolated source and can tell us more about the inner structure of the gravitating source. The Weyl scalars $\Psi_k, k = 0, .., 4$ can be expanded in terms of an affine parameter r along each outgoing null geodesic based on assumption of compatification of null infinity [1,8]. Here $\Psi_k = O(r^{k-5})$, and one may find that it peels off more

*yhwu@phy.ncu.edu.tw
†chwang@phy.ncu.edu.tw; chwang1101@phys.sinica.edu.tw

and more when moving inward along a null ray. From Ashtekar's definition of an isolated horizon, it implies that Ψ_0, Ψ_1 vanishes on horizon. Therefore the space-time is algebraically special on an isolated horizon. However, space-time may not be algebraically special for an arbitrary DH. The corresponding peeling theorem for an arbitrary DH is crucial for our gravitational radiation study. Because of the difficulties of knowing the falloff of Weyl scalars, we use the Kerr-Vaidya solution to serve as our basis for choosing the falloff of Weyl scalars, which is Ψ_0, Ψ_1 vanishing on DH, in our previous work on a slow-rotating DH [6]. So space-time structure on a slow-rotating DH is still assumed to be algebraically special. However, according to the gravitational plane-wave solutions, Ψ_0 and Ψ_4 indicate the ingoing and outgoing gravitational waves, respectively. It seems physically unsatisfactory to assume Ψ_0 vanishing on DH. Moreover, the algebraically general space-time allows four roots of the equation, which correspond to the principal null-directions of Weyl scalars, and describes the gravitational radiation near the gravitating source. Therefore it would be more reasonable for one to consider an algebraically general space-time on an evolving DH. From the reduction and the decoupling of the equations governing the Weyl scalars, instead of assuming $\Psi_0, \Psi_1 = 0$ on DH, we set Ψ_1, Ψ_3 vanishing on a spinning DH. This is a similar setting with the perturbation method (See also Chandrasekhar [9]).

We present the results of asymptotic expansion for a spinning DH in Sec. III. However, it may be too general to yield some interesting physical results. By considering the small-tide and slow rotation of the DH and using the slow-rotating Kerr solution as a basis, we use two sphere conditions of the DH cross section for our later calculation. The NP coefficient λ_0 (shear for the incoming null tetrad n) on DH is no longer assumed to be vanished when calculating the flux formula. The index 0 on NP coefficients denotes their values on DH. Directly from nonradial NP equations, we find that σ_0 and π_0 coupling terms can be transformed into π_0 terms only, so the problems of our previous work [6] are resolved.

Though the exact solution for a stationary rotating black hole has been known for nearly 50 years, the space-time with rotation remains its ambiguity and difficulty for quasilocal mass expressions and boundary condition. For example, the existence of angular momentum will not change the boundary condition for the null infinity, however, it will affect the boundary condition of a black hole. Among the well-known quasilocal mass expressions named Komar, Brown-York and Dougan-Mason, only the Komar integral of the quasilocal mass for an arbitrary closed two-surface can go back to the unique Newtonian quasilocal mass [10]. Unfortunately, the Brown-York and Dougan-Mason mass can return to the unique surface integration of the Newtonian mass in the covariant Newtonian space-time only for the spherically symmetry sources. In GR, quasilocal mass expressions for the Kerr solution disagree with one another [11]. Different quasilocal expressions give different values of quasilocal mass for the Kerr black hole. At null infinity, there is no generally accepted definition for angular momentum [12]. Unfortunately, no explicit expression for Bramson's angular momentum in terms of the Kerr parameters m and a is given [13]. We use the Komar integral to calculate angular momentum since it gives exactly ma for the Kerr solution. Although different quasilocal expressions yield different results for the Kerr solution, our main motivation is to analyze and discuss the compatibility of the coupling NP equations from asymptotic expansions. We both calculate quasilocal mass and flux for a spinning DH based on two spinors (Dougan-Mason) and Komar integral. It is found that these two expressions yield the same result.

Bondi and Sachs use a no-incoming radiation condition for the gravitational wave on null infinity [8,14]. However, a no-incoming radiation condition is only true for linearized theories, e.g., electrodynamics and linearized GR, as to exclude the incoming rays. The incoming pulse waves do not destroy the asymptotic conditions for null infinity since they are admitted by formalism. Their existence may play an important role in the interpretation of the new conserved quantities (NP constants) [15,16]. The interpretation and physical meaning of these constants have been a source of debate and controversy until today. Some physical discussions and application of them can be found in [17,18]. Despite the vagueness of the physical meaning of these conserved quantities, in the full nonlinear gravitational theory, the mass and momentum are no longer absolutely conserved and can be carried away by the outgoing gravitational wave, so as to give a positive energy flux at infinity. Here we consider a space-time inner boundary, e.g., a spinning DH in this paper. With the aid of using an asymptotic constant spinor to define the spin frame as the reference frame for our observation, mass and angular-momentum flux can be calculated. According to the coupling NP equations from the asymptotic expansion analysis, such a system will gain energy and will cause the radius of the black hole to increase. From a similar argument, the outgoing waves *do not* change the boundary conditions of the quasilocal horizons (DHs or IHs) and make no contribution to flux, while an incoming wave will cross into DHs. The existence of an incoming wave indicates the difference between IHs and DHs. The mass and momentum are carried into the black hole by the incoming gravitational wave.

Since the negative mass loss is unlikely to make a dynamical horizon grow with time, the physically reasonable condition on gravitational energy flux should be positive. Directly from our four-dimensional asymptotical expansion scheme, we can observe that the physically reasonable condition on requiring the positivity of the gravitational energy flux yields that the shear σ_0 will monotonically decrease with time for a slow-rotating DH. It means that the slow-rotating DHs will gradually settle down to IHs as σ_0 approach zero. This is similar to a physical assumption saying that the mass loss cannot be infinitely large for null infinity [5]. However, rather than assuming that mass gain cannot be infinitely large we obtain this result directly from asymptotic expansion analysis for a slow-rotating DH together with the positive gravitational energy flux condition. Further from the commutation relations, we find that the horizon radius of a slow-rotating DH will not accelerate. The radius of a slow-rotating dynamical horizon increases with a constant speed. There is one more interesting point about the peeling properties for a slow-rotating DH. It is known that the peeling properties refer to different physical asymptotic boundary conditions of a slow-rotating black hole. By comparing our current work to a previous one [7], which has different peeling properties, and also due to the monotonic decrease of σ_0, we propose that the setting of Weyl scalars in this work excludes the possibility of absorbing the gravitational radiation from nearby the gravitating source.

The plan of this paper is as follows. In Sec. II, we review the definition of DH and express Ashtekar-Krishnan's three-dimensional analysis of DH in terms of NP coefficients. The gauge choices and boundary conditions of a spinning DH are applied to the asymptotic expansion in Sec. III. In Sec. IV, we first examine the gauge conditions of slow-rotating Kerr solution in Sec. IV A. Later we use the two-sphere condition for a slow-rotating DH with small-tide in Sec. IV B. The results of asymptotic expansion are largely simplified by considering the DH's cross section as a two-sphere. Angular momentum and its flux for a slow-rotating DH are calculated by using the Komar integral in Sec. V. Energy-momentum and its flux of a slow-rotating DH are obtained in Sec. VI. We first calculate mass and mass flux by using the Komar integral in Sec. VI A. Then, mass and mass flux of a slow-rotating DH are calculated by using the two-spinor method in Sec. VI B. The time evolution of shear flux and its monotonic decrease is discussed here. We find that either the Komar integral or the two-spinor method yields the same result.

In this paper, we adopt the same notation as in [2,3] for describing generic IHs and DHs. However, we choose the different convention $(+ - - -)$, which is a standard convention for the NP formalism [19]. The necessary equations, i.e., commutation relations, NP equations and Bianchi identities, for asymptotical expansion analysis can be found in pp. 45–p. 51 of [9]. We use "$\triangleq$" to represent quantities on a dynamical horizon (ignore $O(r')$) and use "$\cong$" to represent quantities on a slow-rotating horizon (ignore $O(a^2)$).

II. ASHTEKAR DYNAMICAL HORIZON

A. The dynamical horizon

The generic IHs are taken as the equilibrium state of the DHs. The DHs can be foliated by the marginally trapped surface S. Therefore, the expansion of the outgoing tetrad vanishes on DHs.

Definition

A smooth, three-dimensional, spacelike submanifold H of space-time is said to be a *dynamical horizon* (DH) if it can be foliated by a family of closed two-manifolds such that (1) on each leaf, S, the expansion $\Theta_{(\ell)}$ of one null normal ℓ^a vanishes, (2) the expansion $\Theta_{(n)}$ of the other null normal n^a is negative.

From this definition, it basically tells us that a dynamical horizon is a spacelike hypersurface, which is foliated by closed, marginally trapped two-surface. The requirement of the expansion of the incoming null normal is strictly negative since we want to study a black hole (future horizon) rather than a white hole. Also, it implies

$$\mathrm{Re}[\rho] \triangleq 0, \qquad \mathrm{Re}[\mu] < 0. \tag{1}$$

B. Dynamical horizon in terms of Newman-Penrose coefficients

If we contract the stress energy tensor with a timelike vector, then in components T_0^k represents the energy flux of the matter field. Therefore we can use a timelike vector T^a and contract it with the stress energy tensor to define the flux of the matter energy. Here we are more interested in the energy of the matter field associated with a null direction. One can thus calculate the flux of energy associated with $\xi^a = N\ell^a$. The flux of matter energy across H along the direction of ℓ is given by

$$F_{\mathrm{matter}} := \int_H T_{ab} T^a \xi^b d^3V. \tag{2}$$

The dynamical horizon is a spacelike surface, the Cauchy data $({}^{(3)}q_{ab}, K_{ab})$ on the dynamical horizon must satisfy the scalar and vector constraints

$$H_S := {}^{(3)}R + K^2 - K_{ab}K^{ab} = 16\pi T_{ab}T^aT^b, \tag{3}$$

$$H_V^a := \mathcal{D}_b(K^{ab} - K^{(3)}q^{ab}) = \mathcal{D}_b P^{ab} = 8\pi T^{bc} T_c{}^{(3)}q^a{}_b \tag{4}$$

where $P^{ab} := K^{ab} - K^{(3)}q^{ab}$.

If the dominate energy condition is satisfied, it turns out that H has to be a spacelike hypersurface [3]. The unit timelike vector that is normal to H is denoted by T^a and the unit spacelike vector that orthogonal to the two-sphere and tangent to H is denoted by R^a. In order to study them in terms of Newman-Penrose quantities, they can be defined by using the null normals ℓ^a and n^a. Therefore,

$$T^a = \frac{1}{\sqrt{2}}(\ell^a + n^a), \qquad R^a = \frac{1}{\sqrt{2}}(\ell^a - n^a) \tag{5}$$

where $T^aT_a = 1$, $R^aR_a = -1$. The four-metric has the form

$$g^{ab} = n^a \ell^b + \ell^a n^b + {}^{(2)}q^{ab} \tag{6}$$

$$= T^a T^b - R^a R^b + {}^{(2)}q^{ab}. \tag{7}$$

The three-metric ${}^{(3)}q^{ab}$ that is intrinsic to the dynamical horizon H is

$${}^{(3)}q^{ab} = g^{ab} - T^a T^b = {}^{(2)}q^{ab} - R^a R^b. \tag{8}$$

The two-metric ${}^{(2)}q^{ab}$ that is intrinsic to the cross section two-sphere S is

$${}^{(2)}q^{ab} = {}^{(3)}q^{ab} + R^a R^b = -(m^a \bar{m}^b + \bar{m}^a m^b). \tag{9}$$

The induced covariant derivative on H can be defined in terms of a four-dimensional covariant derivative ∇_a by

$$\mathcal{D}_b V_a := {}^{(3)}q_b{}^{c(3)}q_a{}^d \nabla_c V_d, \tag{10}$$

so the three-dimensional Ricci identity is then given by

$${}^{(3)}R_{abc}{}^d w_d = -[\mathcal{D}_a, \mathcal{D}_b] w_c. \tag{11}$$

The induced covariant derivative on cross section S can be defined in terms of the four-dimensional covariant derivative ∇_a by

$${}^{(2)}\mathcal{D}_b V_a := q_b{}^{c(2)}q_a{}^d \nabla_c V_d. \tag{12}$$

The extrinsic three curvature K_{ab} on H is

$$K_{ab} = {}^{(3)}q_{(a}{}^{c(3)}q_{b)}{}^d \nabla_c T_d = \nabla_a T_b - T_a {}^{(3)}a_b$$

where ${}^{(3)}a_b = T^c \nabla_c T_b$. One can also introduce the extrinsic two curvature ${}^{(2)}K_{ab}$ on S by

$${}^{(2)}K_{ab} = {}^{(2)}q_{(a}{}^{c(2)}q_{b)}{}^d \mathcal{D}_c R_d = \mathcal{D}_a R_b + R_a {}^{(2)}a_b,$$

where ${}^{(2)}a_b = R^c \mathcal{D}_c R_b = R^{(c(2)}q_b{}^{d)} \nabla_c R_d$. After a straightforward but tedious calculation, we can write the extrinsic curvature in terms of NP spin coefficients. Here we present the general extrinsic three curvature and two curvature without any assumption of gauge conditions. The extrinsic three curvature K_{ab} is

$$\begin{aligned} K_{ab} &= {}^{(3)}q_{(a}{}^c \nabla_c T_{b)} = {}^{(2)}q_{(a}{}^c \nabla_c T_{b)} - R_{(a} R^c \nabla_c T_{b)} \\ &= A^{(2)}q_{ab} + S_{ab} + 2W_{(a}R_{b)} + BR_a R_b \end{aligned} \tag{13}$$

where

$$\begin{aligned} S_{ab} &= \frac{1}{\sqrt{2}}[(\bar{\sigma} - \lambda) m_a m_b + C.C.], \\ A &= -\frac{\mathrm{Re}\rho - \mathrm{Re}\mu}{\sqrt{2}}, \\ W_a &:= -{}^{(2)}q_a{}^c K_{cb} R^b \\ &= \frac{1}{4}[\bar{\kappa} + \nu - \bar{\tau} - \pi - 2(\alpha + \bar{\beta})] m_a + C.C., \\ B &= -\sqrt{2}(\mathrm{Re}\epsilon - \mathrm{Re}\gamma), \end{aligned}$$

and $C.C.$ denotes the complex conjugate terms. The extrinsic two curvature ${}^{(2)}K_{ab}$ is

$$\begin{aligned} {}^{(2)}K_{ab} &= {}^{(2)}q_{(a}{}^c \mathcal{D}_c R_{b)} = {}^{(2)}q_{(a}{}^c q_c{}^d q_{b)}{}^e \nabla_d R_e \\ &= {}^{(2)}q_a{}^d q_b{}^e \nabla_d R_e = \frac{1}{2}{}^{(2)}K^{(2)}q_{ab} + {}^{(2)}S_{ab} \end{aligned}$$

where

$${}^{(2)}K = -\sqrt{2}(\mathrm{Re}\rho + \mathrm{Re}\mu), \tag{14}$$

$${}^{(2)}S_{ab} = \frac{1}{\sqrt{2}}(\bar{\sigma} + \lambda) m_a m_b + C.C. \tag{15}$$

The calculation of two acceleration ${}^{(2)}a_a$ yields

$${}^{(2)}a_a = R^b \mathcal{D}_b R_a = R^{(c(2)}q_a{}^{d)} \nabla_c R_d = C m_a + \bar{C} \bar{m}_a,$$

where

$$C = -\frac{1}{4}(\bar{\kappa} - \nu + \pi - \bar{\tau}), \tag{16}$$

so the two acceleration is tangent to S.

We now perform $2+1$ decomposition to study the various quantities on H. The curvature tensor intrinsic to S is given by

$$\begin{aligned} -{}^{(2)}R_{abc}{}^d = &-{}^{(2)}q_a{}^{f(2)}q_b{}^{g(2)}q_c{}^{k(2)}q_j{}^{d(3)}R_{fgk}{}^j \\ &- {}^{(2)}K_{ac}{}^{(2)}K_b{}^d + {}^{(2)}K_{bc}{}^{(2)}K_a{}^d, \end{aligned}$$

which is the Gauss-Codacci equation. This leads to the relation between the scalar three curvature ${}^{(3)}R$ and the scalar two curvature ${}^{(2)}R$

$$-{}^{(3)}R = -{}^{(2)}R - {}^{(2)}K^2 + {}^{(2)}K_{ab}{}^{(2)}K^{ab} - 2\mathcal{D}_a \alpha^a \tag{17}$$

where $\alpha^a := R^b \mathcal{D}_b R^a - R^a \mathcal{D}_b R^b = {}^{(2)}a^a - R^{a(2)}K$.

From (17), we obtain the Einstein tensor on H

$$-2^{(3)}G_{ab} R^a R^b = -{}^{(2)}R + {}^{(2)}K^2 - {}^{(2)}K_{ab}{}^{(2)}K^{ab}. \tag{18}$$

The expansion of the outgoing tetrad ℓ^a can be calculated to yield

$$\Theta_{(\ell)} := -\frac{1}{2}(\rho + \bar{\rho}) = \frac{1}{2\sqrt{2}}[K + {}^{(2)}K + B]. \tag{19}$$

Now we use the following relations (20)–(23) to calculate $H_S + 2R_a H_V^a$, where H_S and H_V^a are the scalar and vector constraints defined in (3) and (4).

$$K = 2A - B, \tag{20}$$

$${}^{(2)}K = -K - B + 2\sqrt{2}\Theta_{(\ell)} = -2A + 2\sqrt{2}\Theta_{(\ell)}, \tag{21}$$

$$K_{ab}K^{ab} = 2A^2 + S_{ab}S^{ab} - 2W_a W^a + B^2, \tag{22}$$

$${}^{(2)}K_{ab}{}^{(2)}K^{ab} = \frac{1}{2}{}^{(2)}K^2 + {}^{(2)}S_{ab}{}^{(2)}S^{ab}. \tag{23}$$

From the momentum constraint (4) and use integration by parts, we get

$$R_b \mathcal{D}_a P^{ab} = \mathcal{D}_a \beta^a - P^{ab} \mathcal{D}_a R_b \tag{24}$$

where $\beta^a := K^{ab}R_b - KR^a$. Thus,

$$\gamma^a := \alpha^a + \beta^a = R^b \mathcal{D}_b R^a - W^a - 2\sqrt{2}\Theta_{(l)} R^a. \quad (25)$$

For a general space-time, the matter energy flux can be calculated as following

$$\begin{aligned} H_s + 2R_a H_V^a &= {}^{(3)}R + K^2 - K_{ab}K^{ab} + 2R_a \mathcal{D}_b P^{ab} \\ &= {}^{(2)}R + {}^{(2)}K^2 - {}^{(2)}K_{ab}{}^{(2)}K^{ab} + K^2 \\ &\quad - K_{ab}K^{ab} - 2P^{ab}\mathcal{D}_a R_b + 2\mathcal{D}_a \gamma^a \,(\text{Use 17})) \\ &= {}^{(2)}R - \sigma_{ab}\bar{\sigma}^{ab} + 2W_a W^a - 2W^{a(2)}a_a \\ &\quad + 4\Theta_{(\ell)}(\Theta_{(\ell)} - \sqrt{2}B) + 2\mathcal{D}_a \gamma^a. \end{aligned}$$

By applying the $2+1$ decomposition on the covariant derivative $\mathcal{D}$ and using integration by parts, we have

$$\begin{aligned} 2\mathcal{D}_a \gamma^a &= 2\mathcal{D}_a({}^{(2)}a^a - W^a - 2\sqrt{2}\Theta_{(\ell)} R^a) \\ &= 2({}^{(2)}a^{a(2)}a_a - W^{a(2)}a_a - \mathcal{D}_a \Theta_{(\ell)} R^a + \frac{1}{2}{}^{(2)}K\Theta_{(\ell)}), \end{aligned}$$

where the term $\mathcal{D}_a({}^{(2)}a^a - W^a)$ has been discarded since it will vanish due to the integration over a compact two-surface S, and then

$$\begin{aligned} &2(W_a W^a - W_a{}^{(2)}a^a + \mathcal{D}_a \gamma^a) \\ &= 2(W_a - {}^{(2)}a_a)(W^a - {}^{(2)}a^a) - 2\mathcal{D}_a \Theta_{(\ell)} R^a - {}^{(2)}K\Theta_{(\ell)}. \end{aligned}$$

Here we can define

$$\begin{aligned} \zeta_a &:= W_a - {}^{(2)}a_a = -\sqrt{2} q_a^{(d} R^{c)} \nabla_c \ell_d \\ &= \frac{1}{2}[\bar{\kappa} - \bar{\tau} - (\alpha + \bar{\beta})] m_a + C.C. \end{aligned} \quad (26)$$

Finally, we get

$$\begin{aligned} H_s + 2R_a H_V^a &= {}^{(2)}R - \sigma_{ab}\sigma^{ab} + 2\zeta_a \zeta^a - 2\mathcal{D}_a \Theta_{(\ell)} R^a \\ &\quad + \Theta_{(\ell)}(-{}^{(2)}K + 4\Theta_{(\ell)} - 4\sqrt{2}B) \end{aligned}$$

where

$$\sigma_{ab} = \frac{1}{\sqrt{2}}(S_{ab} + {}^{(2)}S_{ab}) = \sqrt{2}\bar{\sigma} m_a m_b + C.C.$$

is the shear of null normal ℓ^a. This equation is completely general. On the dynamical horizon, the outgoing expansion $\Theta_{(\ell)}$ vanishes. It then becomes[1]

$$F_{\text{matter}} = \frac{1}{16\pi}\int_{\Delta H} N({}^{(2)}R - \sigma_{ab}\sigma^{ab} + 2\zeta_a \zeta^a) d^3V. \quad (27)$$

If the gauge condition

$$\kappa \hat{=} \pi - \bar{\tau} \hat{=} \pi - (\alpha + \bar{\beta}) \hat{=} 0 \quad (28)$$

is satisfying, where $\hat{=}$ denotes the equating on DH, then $\zeta_a = -\pi m_a + C.C.$ and $\zeta_a \zeta^a = -2\pi\bar{\pi}$. So the flux formula in terms of NP in this gauge is

$$F_{\text{matter}} = \frac{1}{16\pi}\int_{\Delta H} N({}^{(2)}R - 4\sigma\bar{\sigma} - 4\pi\bar{\pi}) d^3V. \quad (29)$$

C. Angular-momentum flux and energy fluxes

By contracting the vector constraint H_V^a with the rotational vector field ψ^a, which is tangential to S, we can obtain angular momentum of a black hole. Then we integrate the resulting equation over the region of ΔH and use the integration by parts together with the identity $\mathcal{L}_\psi {}^{(3)}q_{ab} = 2\mathcal{D}_{(a}\psi_{b)}$. It leads to

$$\begin{aligned} -dJ &= J_{S_1} - J_{S_2} \\ &= \frac{1}{8\pi}\oint_{S_2} K_{ab}\psi^a R^b dS - \frac{1}{8\pi}\oint_{S_1} K_{ab}\psi^a R^b dS \\ &= \int_{\Delta H}\left(T_{ab}T^a\psi^b + \frac{1}{16\pi}P^{ab}\mathcal{L}_\psi {}^{(3)}q_{ab}\right) d^3V. \end{aligned} \quad (30)$$

The angular momentum associated with cross section S is

$$J_S^\psi = -\frac{1}{8\pi}\oint_S K_{ab}\psi^a R^b dS \quad (31)$$

where ψ^a need not be an axial Killing field. The *flux of angular momentum* due to matter fields F_{matter}^ψ and gravitational waves F_{grav}^ψ are

$$F_{\text{matter}}^\psi = -\int_{\Delta H} T_{ab}T^a\psi^b d^3V, \quad (32)$$

$$F_{\text{grav}}^\psi = -\frac{1}{16\pi}\int_{\Delta H} P^{ab}\mathcal{L}_\psi^{(3)} q_{ab} d^3V, \quad (33)$$

and the balance equation $J_{S_2}^\psi - J_{S_1}^\psi = F_{\text{matter}}^\psi + F_{\text{grav}}^\psi$, which describes the difference of angular momentum between two cross sections, is due to the matter radiation and gravitational radiation.

Each time evolution vector t^a defines a horizon energy E_Δ^t. From Eq. (27), we find the total energy flux is the combination of the matter flux and gravitational flux

$$F_{\text{matter}} + F_{\text{grav}} = \frac{1}{16\pi}\int_{\Delta H} N^{(2)}R d^3V \quad (34)$$

where the matter flux is Eq. (2) and the gravitational flux is

$$F_{\text{grav}} = \frac{1}{16\pi}\int_{\Delta H} N(\sigma_{ab}\sigma^{ab} - 2\zeta_a\zeta^a) d^3V. \quad (35)$$

If we use the gauge conditions in (29), we then have

$$F_{\text{grav}} = \frac{1}{4\pi}\int_{\Delta H} N(|\sigma|^2 + |\pi|^2) d^3V. \quad (36)$$

The matter flux expression (2) of Vaidya solution would be

$$F_{\text{matter}} := \int_H T_{ab}T^a\ell^b N d^3V = \frac{1}{4\pi}\int \Phi_{00} N d^3V \quad (37)$$

where we use $4\pi T_{ab}\ell^a\ell^b = \Phi_{00}^0$. The total flux of Ashtekar-Krishnan then becomes

[1] Our expression has some minus sign different from Ashtekar's expression because of convention.

$$F_{\text{total}} = \frac{1}{4\pi}\int[|\sigma|^2 + |\pi|^2 + \Phi_{00}]Nd^3V. \tag{38}$$

Further, the integral of $N^{(2)}R$ can be written as

$$\int_{\Delta H} N^{(2)}Rd^3V = \int_{R_1}^{R_2} dR \oint {}^{(2)}Rd^2V = 8\pi(R_2 - R_1)$$

where R_1 and R_2 are the radii of the horizon at the boundary cross sections. For a *rotating nonspherical symmetric dynamical horizon*, we find the relation of the change in the horizon area in the dynamical processes can be written as

$$\begin{aligned}\int\frac{dR}{2} &= \frac{(R_2 - R_1)}{2} = \frac{1}{16\pi}\int_{\Delta H} N^{(2)}Rd^3V \\ &= F_{\text{matter}} + F_{\text{grav}} \\ &= \int_{\Delta H} T_{ab}T^a\xi^b d^3V + \frac{1}{16\pi}\int_{\Delta H} N(\sigma_{ab}\sigma^{ab} - 2\zeta_a\zeta^a)d^3V.\end{aligned} \tag{39}$$

Hence, from this equation we can relate the black hole area change with energy and angular-momentum change. This gives a more general black hole first law in a dynamical space-time. If we now define the *effective surface gravity* [3] as

$$\kappa_R := \frac{1}{2R}, \tag{40}$$

then the area of horizon is $A = 4\pi R^2$ and the differential of the area is $dA = 8\pi RdR$, therefore

$$\frac{\kappa_R}{8\pi}dA = \frac{1}{2}dR. \tag{41}$$

For the time evolution vector $t^a = N\ell^a - \Omega\psi^a = \xi^a - \Omega\psi^a$, the difference of the horizon energy E^t_S can be expressed as [3]

$$\begin{aligned}dE^t &= E^t_{S_2} - E^t_{S_1} \\ &= \int T_{ab}T^a t^b d^3V + \frac{1}{16\pi}\int N(\sigma_{ab}\sigma^{ab} - 2\zeta_a\zeta^a)d^3V \\ &\quad - \frac{1}{16\pi}\int_{\Delta H} \Omega P^{ab}\mathcal{L}_\psi q_{ab}d^3V.\end{aligned} \tag{42}$$

By using (30) together with the linear combination of

$$\begin{aligned}\int\frac{dR}{2} &= \int_{\Delta H} T_{ab}T^a\xi^b d^3V \\ &\quad + \frac{1}{16\pi}\int_{\Delta H} N(\sigma_{ab}\sigma^{ab} - 2\zeta_a\zeta^a)d^3V,\end{aligned} \tag{43}$$

we can obtain a *generalized black hole first law for dynamical horizon*

$$\frac{\kappa_R}{8\pi}dA + \Omega dJ = dE^t. \tag{44}$$

III. ASYMPTOTIC EXPANSION FOR A SPINNING DYNAMICAL HORIZON

A. Frame setting and gauge choice

We choose the incoming null tetrad $n_a = \nabla_a v$ to be the gradient of the null hypersurface $v = \text{const}$. We then have $g^{ab}v_{,a}v_{,a} = 0$. It gives us the gauge conditions $\nu = \mu - \bar{\mu} = \gamma + \bar{\gamma} = \bar{\alpha} + \beta - \bar{\pi} = 0$. Then we further choose the n^a flag plane parallel; it implies $\gamma = 0$. For the setting of outgoing null tetrad ℓ, we first choose ℓ to be a geodesic and use null rotation type III to make $\epsilon - \bar{\epsilon} = 0$. We choose m, $\bar{m}$ tangent to the cross section S, and thus $\rho \hat{=} \bar{\rho}$, $\pi \hat{=} \bar{\tau}$.[2] From the boundary conditions of a spinning DH (see Eq. (1)), recall

$$\rho\hat{=}0, \qquad \pi\hat{\neq}0, \qquad \sigma\hat{\neq}0 \tag{45}$$

on DH. We summarize our gauge choices and boundary conditions

$$\begin{gathered}\kappa = \epsilon - \bar{\epsilon} = \nu = \mu - \bar{\mu} = \gamma = \pi - \alpha - \bar{\beta} = 0, \\ \rho\hat{=}0, \qquad \pi\hat{=}\bar{\tau}.\end{gathered} \tag{46}$$

In order to preserve orthonormal relations, we can choose the tetrad as

$$\ell^a = (1, U, X^2, X^3), \quad n^a = (0, -1, 0, 0), \quad m^a = (0, 0, \xi^2, \xi^3)$$

in the Bondi coordinate (v, r, x^2, x^3).

Now we make a coordinate transformation to a new comoving coordinate (v, r', x^2, x^3) where $r' = r - R_\Delta(v)$ and $R_\Delta(v)$ is radius of a spinning DH. Here

$$\ell^a = (1, U - \dot{R}_\Delta, X^2, X^3), \qquad n^a = (0, -1, 0, 0),$$
$$m^a = (0, 0, \xi^2, \xi^3),$$

where $\dot{R}_\Delta(v)$ is the rate of changing effective radius of DH. From this coordinate, we may see that the dynamical horizon is a spacelike or null hypersurface. Here, the tangent vector of the DH is $R^a = \ell^a - \dot{R}_\Delta n^a \hat{=} \frac{\partial}{\partial v}$ where $\dot{R}_\Delta \geq 0$. Therefore, it implies $R^aR_a \leq 0$ and U, $X^k = O(r')$.

B. The peeling properties and falloff of the Weyl scalars

Since we use $\kappa = \nu = 0$, $\sigma \neq 0$, $\lambda \neq 0$, then we have

$$(\bar{\delta} - 4\alpha + \pi)\Psi_0 - (D - 2\epsilon)\Psi_1\hat{=}0, \tag{47}$$

$$(\Delta + \mu)\Psi_0 - (\delta - 4\bar{\pi} - 2\beta)\Psi_1\hat{=}3\sigma\Psi_2, \tag{48}$$

$$(D + 2\epsilon)\sigma\hat{=}\Psi_0, \tag{49}$$

$$(D + 4\epsilon)\Psi_4 - (\bar{\delta} + 4\pi + 2\alpha)\Psi_3\hat{=} - 3\lambda\Psi_2, \tag{50}$$

[2]This also implies $\omega\hat{=}0$. The Kerr solution preferred gauge on the horizon is $\mu\hat{=}\bar{\mu}$, $\pi\hat{=}\alpha + \bar{\beta}\hat{=}\bar{\tau}$.

$$(\delta + 4\beta - \tau)\Psi_4 - (\Delta + 4\mu)\Psi_3 \hat{=} 0, \tag{51}$$

$$(\Delta + 2\mu)\lambda \hat{=} -\Psi_4 \tag{52}$$

in vacuum. Therefore, one can set $\Psi_1 \hat{=} \Psi_3 \hat{=} 0$ as peeling properties for a spinning DH. This is similar to the perturbation method and one may refer to p. 175 and p. 180 in [9].

The falloff of the Weyl scalars is algebraically general (this is a more general setting than [6,7]) on the DH where

$$\Psi_1 = \Psi_3 = O(r'), \qquad \Psi_0 = \Psi_2 = \Psi_4 = O(1). \tag{53}$$

By considering the Vaidya solution as our compared basis for the matter field part, the falloff of the Ricci spinor components are

$$\begin{aligned} &\Phi_{00} = O(1),\\ &\Phi_{22} = \Phi_{11} = \Phi_{02} = \Phi_{01} = \Phi_{21} = O(r'). \end{aligned} \tag{54}$$

C. From the radial equations

$$\begin{aligned}
\mu &= \mu_0 + (\mu_0^2 + \lambda_0\bar{\lambda}_0)r' + O(r'^2),\\
\lambda &= \lambda_0 + (2\mu_0\lambda_0 + \Psi_4^0)r' + O(r'^2),\\
\alpha &= \alpha_0 + [\lambda_0(\bar{\pi}_0 + \beta_0) + \alpha_0\mu_0]r' + O(r'^2),\\
\beta &= \beta_0 + [\mu_0\bar{\pi}_0 + \beta_0\mu_0 + \alpha_0\bar{\lambda}_0]r' + O(r'^2),\\
\rho &= [\Psi_2^0 - \bar{\eth}_0\bar{\pi}_0 + \pi_0\bar{\pi}_0 + \sigma_0\lambda_0]r' + O(r'^2),\\
\sigma &= \sigma_0 + (-\eth_0\bar{\pi}_0 - \bar{\pi}_0^2 + \mu_0\sigma_0)r' + O(r'^2),\\
\pi &= \pi_0 + [2\mu_0\pi_0 + 2\lambda_0\bar{\pi}_0]r' + O(r'^2),\\
\epsilon &= \epsilon_0 + [2\alpha_0\bar{\pi}_0 + 2\beta_0\pi_0 + \bar{\pi}_0\pi_0]r' + O(r'^2),\\
\xi^k &= \xi^{k0} + [\bar{\lambda}_0\bar{\xi}^{k0} + \mu_0\xi^{k0}]r' + O(r'^2),\\
U &= 2\epsilon_0 r' + O(r'^2),\\
X^k &= 2(\pi_0\xi^{k0} + \bar{\pi}_0\bar{\xi}^{k0})r' + O(r'^2).\\
\Psi_0 &= \Psi_0^0 + (\mu_0\Psi_0^0 - 3\sigma_0\Psi_2^0 + \bar{\lambda}_0\Phi_{00}^0)r' + O(r'^2),\\
\Psi_1 &= (-\delta_0\Psi_2^0 + 3\bar{\pi}_0\Psi_2^0)r' + O(r'^2),\\
\Psi_2 &= \Psi_2^0 + (3\mu_0\Psi_2^0 - \sigma_0\Psi_4^0)r' + O(r'^2),\\
\Psi_3 &= [-\delta_0\Psi_4^0 + (\bar{\pi}_0 - 4\beta_0)\Psi_4^0]r' + O(r'^2).\\
\Phi_{00} &= \Phi_{00}^0 + 2\mu_0\Phi_{00}^0 r' + O(r'^2),\\
\Phi_{11} &= \frac{1}{2}\dot{R}_\Delta\Phi_{22}^0 r'^2 + O(r'^3),\\
\Phi_{01} &= -\dot{R}_\Delta\Phi_{12}^0 r'^2 + O(r'^3),\\
\Phi_{22} &= \Phi_{22}^0 r'^2 + O(r'^3),\\
\Phi_{12} &= \Phi_{12}^0 r'^2 + O(r'^3),\\
\Phi_{02} &= \Phi_{02}^0 r'^2 + O(r'^2).
\end{aligned}$$

D. From the nonradial equations

The following equations refer to the equation numbers from pp. 45–p. 47 in [9]. We relabel (304)–(306) as (NC1), (NC2), (NC3).

$$\begin{aligned}
&(\mathbf{NC1})\ \delta_0\epsilon_0 = 0,\\
&(\mathbf{NC2})\ \dot{P} = \dot{R}_\Delta[\mu_0 P + \bar{\lambda}_0\bar{P}] + \sigma_0\bar{P},\\
&(\mathbf{NC3})\ \bar{P}\overset{\bar{c}}{\nabla}\ln P = \alpha_0 - \bar{\beta}_0,
\end{aligned} \tag{55}$$

where

$$P(v, x^k) := \xi^{20} = -i\xi^{30}, \quad P\overset{c}{\nabla} := \delta_0, \quad \overset{c}{\nabla} := \frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}.$$

We relabel (a), (b), (c), (g), (d), (e), (h), (k), (m), (l) as (NR1), (NR2), (NR3), (NR4), (NR5), (NR6), (NR7), (NR8), (NR9), (NR10).

$$\begin{aligned}
&(\mathbf{NR1}) - \dot{R}_\Delta(\Psi_2^0 - \bar{\eth}_0\bar{\pi}_0 + \pi_0\bar{\pi}_0 + \sigma_0\lambda_0)\\
&\qquad = \sigma_0\bar{\sigma}_0 + \Phi_{00}^0,\\
&(\mathbf{NR2})\ \dot{\sigma}_0 = \dot{R}_\Delta[-\eth_0\bar{\pi}_0 - \bar{\pi}_0^2 + \sigma_0\mu_0] + 2\epsilon_0\sigma_0 + \Psi_0^0,\\
&(\mathbf{NR3})\ \dot{\pi}_0 = \dot{R}_\Delta[2\mu_0\pi_0 + 2\lambda_0\bar{\pi}_0] + 2\bar{\sigma}_0\bar{\pi}_0,\\
&(\mathbf{NR4})\ \dot{\lambda}_0 = \dot{R}_\Delta[2\mu_0\lambda_0 + \Psi_4^0] + \bar{\eth}_0\pi_0 + \pi_0^2 - 2\lambda_0\epsilon_0\\
&\qquad + \mu_0\bar{\sigma}_0,\\
&(\mathbf{NR5})\ \dot{\alpha}_0 = \dot{R}_\Delta[\alpha_0\mu_0 + \lambda_0(\bar{\pi}_0 + \beta_0)] + \bar{\sigma}_0\beta_0,\\
&(\mathbf{NR6})\ \dot{\beta}_0 = \dot{R}_\Delta[\alpha_0\bar{\lambda}_0 + \mu_0(\bar{\pi}_0 + \beta_0)] + \sigma_0(\alpha_0 + \pi_0),\\
&(\mathbf{NR7})\ \dot{\mu}_0 = \dot{R}_\Delta(\mu_0^2 + \lambda_0\bar{\lambda}_0) + \eth_0\pi_0 + \pi_0\bar{\pi}_0 + \sigma_0\lambda_0\\
&\qquad - 2\mu_0\epsilon_0 + \Psi_2^0,\\
&(\mathbf{NR8})\ \bar{\eth}_0\sigma_0 = 0,\\
&(\mathbf{NR9})\ \delta_0\lambda_0 - \bar{\delta}_0\mu_0 = \mu_0\pi_0 + \lambda_0(\bar{\alpha}_0 - 3\beta_0),\\
&(\mathbf{NR10})\ \eth_0\pi_0 - \bar{\eth}_0\bar{\pi}_0 = 2\,\mathrm{Im}\,\eth_0\pi_0 = -2\,\mathrm{Im}(\lambda_0\sigma_0) - 2\,\mathrm{Im}\Psi_2^0,\\
&(\mathbf{NR10})\ \mathrm{Re}\Psi_2^0 = -\mathrm{Re}[\delta_0(\alpha_0 - \bar{\beta}_0)] - \mathrm{Re}(\lambda_0\sigma_0)\\
&\qquad + (\bar{\alpha}_0 - \beta_0)(\alpha_0 - \bar{\beta}_0).
\end{aligned}$$

The following equations refer to the equation numbers (a), (b), (c), (d) on p. 49 of [9]. Here, we relabel (a), (b), (c), (d) as (NB1), (NB2), (NB3), (NB4).

$$\begin{aligned}
&(\mathbf{NB1}) - \bar{\delta}_0\Psi_0^0 + (4\alpha_0 - \pi_0)\Psi_0^0 - \dot{R}_\Delta[-\delta_0\Psi_2^0 + 3\bar{\pi}_0\Psi_2^0]\\
&\qquad = -\delta_0\Phi_{00}^0 + \bar{\pi}_0\Phi_{00}^0,\\
&(\mathbf{NB2})\ \dot{\Psi}_2^0 = \dot{R}_\Delta(3\mu_0\Psi_2^0 - \sigma_0\Psi_4^0) - \lambda_0\Psi_0^0 + \mu_0\Phi_{00}^0,\\
&(\mathbf{NB3}) - \bar{\delta}_0\Psi_2^0 - 3\pi_0\Psi_2^0 = \dot{R}_\Delta(-\delta_0\Psi_4^0 + (\bar{\pi}_0 - 4\beta_0)\Psi_4^0),\\
&(\mathbf{NB4})\ \dot{\Psi}_4^0 = \dot{R}_\Delta\Psi_4^1 - 3\lambda_0\Psi_2^0 - 4\epsilon_0\Psi_4^0.
\end{aligned}$$

E. Compatible constant spinor conditions for a rotating dynamical horizon

In this section, we adopt a similar idea of Bramson's asymptotic frame alignment for null infinity [13] and apply it to set up spinor frames on the quasilocal horizons. We define the spinor frames as

$$Z_A{}^{\underline{A}} = (\lambda_A, \mu_A) \tag{56}$$

where $\lambda_A = \lambda_1 o_A - \lambda_0 \iota_A$, $\mu_A = \mu_1 o_A - \mu_0 \iota_A$. We expand λ_1, λ_0 as

$$\lambda_1 = \lambda_1^0(v, \theta, \phi) + \lambda_1^1(v, \theta, \phi) r' + O(r'^2), \tag{57}$$

$$\lambda_0 = \lambda_0^0(v, \theta, \phi) + \lambda_0^1(v, \theta, \phi) r' + O(r'^2). \tag{58}$$

Here λ_1 is type $(-1, 0)$ and λ_0 is type (1,0).

First, we require the frame to be parallelly transported along the outgoing null normal ℓ^a.

$$\lim_{r'\to 0} D Z_A{}^{\underline{A}} = 0. \Rightarrow \ell^a \nabla_a (\lambda_1 o_A - \lambda_0 \iota_A) = 0. \tag{59}$$

Then it gives the condition $\text{þ}\lambda_0^0 = 00$ on DH. The compatible conditions are

$$\text{þ}\lambda_0^0 = 0, \qquad \Rightarrow \dot{\lambda}_0^0 - \epsilon_0 \lambda_0^0 = 00 \tag{60}$$

$$\eth_0 \lambda_0^0 + \sigma_0 \lambda_1^0 = 0, \tag{61}$$

$$\eth_0 \lambda_1^0 - \mu_0 \lambda_0^0 = 0, \tag{62}$$

$$\text{þ}\lambda_1^0 = -\bar{\eth}_0 \lambda_0^0 .0 \tag{63}$$

IV. SLOW-ROTATING BLACK HOLE AND SETTINGS ON A TWO-SPHERE

A. Slow-rotating Kerr horizon in the Bondi coordinate

The Kerr metric in the Eddington-Finkelstein coordinate (v, r, θ, χ) is

$$\begin{aligned} ds^2 = {} & \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dv^2 - 2 dv dr \\ & + \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} dv d\chi + 2a \sin^2\theta d\chi dr \\ & - \Sigma d\theta^2 - \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta d\chi^2. \end{aligned} \tag{64}$$

By changing the coordinate from (v, θ, χ) to (v, θ, χ')

$$d\chi = \Omega_\Delta dv + d\chi' \tag{65}$$

where $\Omega_\Delta = \frac{a}{r_\Delta^2 + a^2}$ is the angular velocity on the horizon and r_Δ is the horizon radius of the Kerr solution, we can make the term $g_{v\chi} dv d\chi$ vanish in the 3-D metric. The 3-D metric in the new coordinate (v, θ, χ') will be

$$\begin{aligned} ds^2 \hat{=} {} & -\frac{a^2 \sin^2\theta}{\Sigma_\Delta} dv^2 + 2 \frac{a^2 \sin^2\theta}{\Sigma_\Delta} \Omega_\Delta^{-1} dv d\chi - \Sigma_\Delta d\theta^2 \\ & - \Omega_\Delta^{-2} \frac{a^2 \sin^2\theta}{\Sigma_\Delta} d\chi^2 \end{aligned} \tag{66}$$

$$\hat{=} 0 \cdot dv^2 - \Sigma_\Delta d\theta^2 - \Omega_\Delta^{-2} \frac{a^2 \sin^2\theta}{\Sigma_\Delta} d\chi'^2 \tag{67}$$

$$\cong 0 \cdot dv^2 - r_\Delta^2 (d\theta^2 + \sin\theta^2 d\chi'^2). \tag{68}$$

Here the surface area of slow-rotating Kerr is $A_{Kerr} \cong 16\pi r_\Delta^2$.

Now, we consider the case of slow rotation so that a is small and we ignore the a^2 terms. Thus the tetrad components in the Bondi coordinate $(\tilde{v}, \tilde{r}', \tilde{\theta}, \tilde{\phi})$ are

$$\ell^a = \left(1, U r', 0, \frac{a}{r_\Delta^2} + D r'\right) \tag{69}$$

$$n^a = (0, -1, 0, 0) \tag{70}$$

$$m^a = \frac{1}{\sqrt{2}\eta_\Delta} \left(0, 0, 1 - \frac{r'}{\eta_\Delta}, \frac{-i}{\sin\theta} \left(1 - \frac{r'}{\eta_\Delta}\right)\right) \tag{71}$$

where $U := \frac{r_\Delta - M}{r_\Delta^2}$ and $D := \frac{a(2r_\Delta - M)}{r_\Delta^4}$. The NP coefficients and Weyl tensors are

$$\kappa = \sigma = \lambda = \nu = 0, \tag{72}$$

$$\rho = \frac{U(-r_\Delta + r') r'}{(\eta_\Delta - r')(\bar{\eta}_\Delta - r')} \hat{=} 0, \tag{73}$$

$$\mu = \frac{-r_\Delta + r'}{(\eta_\Delta - r')(\bar{\eta}_\Delta - r')} \hat{=} -\frac{r_\Delta}{\Sigma_\Delta} \cong -\frac{1}{r_\Delta}, \tag{74}$$

$$\pi = \bar{\tau} = \frac{i\sqrt{2} D \eta_\Delta^2 \sin\theta}{4(\eta_\Delta - r')} \hat{=} \frac{i\sqrt{2} D \eta_\Delta \sin\theta}{4} \cong \frac{i3\sqrt{2} a}{2 r_\Delta^2}, \tag{75}$$

$$\epsilon = \frac{U[(r' - r_\Delta)^2 + a^2 \cos^2\theta + i a \cos\theta r']}{2[(r' - r_\Delta)^2 + a^2 \cos^2\theta]} \hat{=} \frac{U}{2} \cong \frac{1}{4 r_\Delta}, \tag{76}$$

$$\gamma \hat{=} -\frac{i a \cos\theta}{2\Sigma_\Delta}, \qquad \gamma + \bar{\gamma} \hat{=} 0, \tag{77}$$

$$\pi = \alpha + \bar{\beta}, \tag{78}$$

$$\Psi_0 = 0, \qquad \Psi_1 = O(r'), \tag{79}$$

$$\operatorname{Im} \Psi_2 \hat{=} -\frac{i D \cos\theta}{\Sigma_\Delta} (r_\Delta^2 + a^2 \cos^2\theta - a^2 \sin^2\theta), \tag{80}$$

$$\Psi_3 \hat{=} \frac{i\sqrt{2} \sin\theta r_\Delta \eta_\Delta}{4\Sigma_\Delta^3} [D \Sigma_\Delta^2 + 2 i a^2 \cos^2\theta], \tag{81}$$

$$\Psi_4 = 0. \tag{82}$$

Remark. In this approximate Kerr tetrad in the Bondi coordinate, the NP coefficients satisfy

$$\begin{aligned} &\nu = \mu - \bar{\mu} \hat{=} \pi - \alpha - \bar{\beta} \hat{=} \gamma + \bar{\gamma} \hat{=} \epsilon - \bar{\epsilon} \hat{=} 0, \\ &\pi = \bar{\tau}, \qquad \rho \hat{=} \bar{\rho}, \qquad \mu < 0. \end{aligned} \tag{83}$$

By examining the approximate Kerr tetrad in the Bondi coordinates, we found it is compatible with our frame setting for the asymptotic expansions.

B. Setting on a two-sphere: on horizon cross section

Solving the coupling equations from nonradial NP equations would be rather complicated and may be too general to yield some interesting physical results. By considering the small-tide and slow rotate of the DH and considering the slow rotate Kerr solution as a basis from the previous subsection, we use two-sphere conditions of the DH cross section for our later calculation. On a sphere with horizon radius $R_\Delta(v)$, one can set

$$\mu_0 = -\frac{1}{R_\Delta}. \tag{84}$$

Let P, μ_0 on a sphere with radius R_Δ, then $P \propto \frac{1}{R_\Delta}$. From (NC2), $\dot{\xi}^{k0} = \dot{R}_\Delta(\mu_0 \xi^{k0} + \bar{\lambda}_0 \bar{\xi}^{k0}) + \sigma_0 \bar{\xi}^{k0}$ which depends on the next-order nonlinear effect offhorizon, we obtain

$$\lambda_0 = -\frac{\bar{\sigma}_0}{\dot{R}_\Delta}, \tag{85}$$

and

$$\dot{P} = \dot{R}_\Delta \mu_0 P = -\frac{\dot{R}_\Delta P}{R_\Delta}. \tag{86}$$

Moreover, the effective surface gravity is $\tilde{\kappa} = 2\epsilon_0 = \frac{1}{2R_\Delta}$, and then $\mu_0 = -4\epsilon_0$ (recall Eq. (40)).

Check the commutation relation $[\delta_0, D_0]\lambda_0$ and $[\delta_0, D_0]\sigma_0$, it implies

$$\ddot{R}_\Delta = 0. \tag{87}$$

This means that the horizon radius will not accelerate (no inflation). The dynamical horizon will increase with a constant speed. We note here that if the two-sphere condition does not hold, then this result is no longer true.

After applying these conditions, we list the main equations that will be used in the later Secs. V and VI:

$$\begin{aligned}
&(\mathbf{NR1'})\ \dot{R}_\Delta\Big[-\frac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0) + \frac{1}{2}(\eth_0 \pi_0 + \bar{\eth}_0 \bar{\pi}_0) - \pi_0 \bar{\pi}_0\Big] \\
&\qquad = \Phi_{00}^0, \\
&(\mathbf{NR2'})\ \dot{\sigma}_0 = \dot{R}_\Delta[-\eth_0 \bar{\pi}_0 - \bar{\pi}_0^2 + \sigma_0 \mu_0] + 2\epsilon_0 \sigma_0 + \Psi_0^0, \\
&(\mathbf{NR3'})\ \dot{\pi}_0 = 2\dot{R}_\Delta \mu_0 \pi_0, \\
&(\mathbf{NR4'})\ \frac{\bar{\sigma}_0 \ddot{R}_0 - \dot{R}_\Delta \dot{\bar{\sigma}}_0}{(\dot{R}_\Delta)^2} = \dot{R}_\Delta \Psi_4^0 + \bar{\eth}_0 \pi_0 + \pi_0^2 \\
&\qquad + 2\frac{\bar{\sigma}_0 \epsilon_0}{\dot{R}_\Delta} - \mu_0 \bar{\sigma}_0, \\
&(\mathbf{NR5'})\ \dot{\alpha}_0 = \dot{R}_\Delta \alpha_0 \mu_0 - \bar{\sigma}_0 \bar{\pi}_0, \\
&(\mathbf{NR6'})\ \dot{\beta}_0 = \dot{R}_\Delta \mu_0(\bar{\pi}_0 + \beta_0) + \sigma_0 \pi_0, \\
&(\mathbf{NR7'})\ \mathrm{Re}\Psi_2^0 = 2\mu_0 \epsilon_0 - \pi_0 \bar{\pi}_0 - \mathrm{Re}\eth_0 \pi_0, \\
&\qquad \mathrm{Im}\Psi_2^0 = -\mathrm{Im}\eth_0 \pi_0, \\
&(\mathbf{NR8'})\ \bar{\eth}_0 \sigma_0 = 0, \\
&(\mathbf{NR9'})\ -2\bar{\sigma}_0 \bar{\pi}_0 = \dot{R}_\Delta \mu_0 \pi_0, \\
&\qquad \sigma_0 \bar{\eth}_0 \pi_0 = -\frac{1}{2}\dot{R}_\Delta \mu_0 \bar{\eth}_0 \bar{\pi}_0, \\
&(\mathbf{NB2'})\ \dot{\Psi}_2^0 = \dot{R}_\Delta[3\mu_0 - \sigma_0 \Psi_4^0] + \bar{\sigma}_0 \Psi_0^0 + \mu_0 \Phi_{00}^0, \\
&(\mathbf{NR1'}) + (\mathbf{NR7'})\ 2\,\mathrm{Re}\eth_0 \pi_0 = \frac{\Phi_{00}^0}{\dot{R}_\Delta} + 2\mu_0 \epsilon_0.
\end{aligned}$$

V. ANGULAR MOMENTUM AND ANGULAR-MOMENTUM FLUX OF A SLOW-ROTATING DH

Here we use an asymptotically rotating Killing vector ϕ^a for a spinning DH where $\phi^a \neq 0$. It coincides with a rotating vector $\phi^\alpha \hat{=} \psi^a$ on a DH and is divergent free. It implies $\Delta_a \phi^a := S^a_{a0} S^{b0}_a \nabla_{b0} \phi^{a0} = 0$. Therefore,

$$\bar{m}_a \delta \phi^a = -m_a \bar{\delta} \phi^a. \tag{88}$$

Let $\phi^a = A m^a + B \bar{m}^a$; we get $A = -B$. Therefore, there exists a function f such that

$$\phi^a = \bar{\delta} f m^a - \delta f \bar{m}^a, \tag{89}$$

which is type (0, 0). Since f is type (0, 0), therefore $\delta f = \eth f$.

By using the Komar integral, the quasilocal angular momentum on a slow-rotating DH is

$$\begin{aligned}
J(R_\Delta) &= \frac{1}{8\pi}\Big[\oint_S \nabla^a \phi^b dS_{ab}\Big]\Big|_\Delta \\
&= \frac{1}{8\pi}\oint_S \mathrm{Im}(\bar{\pi}_0 \bar{\eth}_0 f) dS_\Delta \ \text{(use integration by part)} \\
&= -\frac{1}{4\pi}\oint_S f\,\mathrm{Im}\eth_0 \pi_0 dS_\Delta \\
&= -\frac{1}{4\pi}\oint_S f\,\mathrm{Im}\Psi_2^0 dS_\Delta.
\end{aligned} \tag{90}$$

From (NB2'), we get $\mathrm{Im}\dot{\Psi}_2^0 = 3\frac{\dot{R}_\Delta}{R_\Delta}\mathrm{Im}\eth_0\pi_0 = -3\frac{\dot{R}_\Delta}{R_\Delta}\mathrm{Im}\Psi_2^0$. Together with $\frac{\partial}{\partial v}dS_\Delta = 2\frac{\dot{R}_\Delta}{R_\Delta}dS_\Delta$, the angular-momentum flux for a slow-rotating DH is

$$\dot{J}(R_\Delta) = -\frac{1}{4\pi}\oint_S\left(\dot{f} - \frac{\dot{R}_\Delta}{R_\Delta}f\right)\mathrm{Im}\Psi_2^0 dS_\Delta = \frac{1}{4\pi}\oint_S \mathrm{Im}\left[\left(\dot{f} - \frac{\dot{R}_\Delta}{R_\Delta}f\right)\eth_0\pi_0\right]dS_\Delta. \quad (91)$$

We note that from $\frac{d}{dv}$ (NR7'), it yields the same result. Here if $\pi_0 \neq 0$ and $f(v,\theta,\phi) = G(\theta,\phi)R_\Delta(v)$, then $\dot{J}(R_\Delta) = 0$. It then returns to the stationary case. If $\pi_0 = 0$, i.e., $\mathrm{Im}\Psi_2^0 = 0$, then J and $\dot{J} = 0$. It then returns to the nonrotating black hole.

Now, we compare with Ashtekar-Krishnan's results in Sec. II C and try to give some physical interpretation for ζ_a. From our gauge conditions (46) of a spinning DH, we have $W_a \hat{=} -\pi m_a + C.C.$, $(2)a_0 \hat{=} 0$ and $\zeta_a \hat{=} W_a$. Then, one can calculate the angular momentum (31) (recall $J = -\frac{1}{8/\pi}\oint_S K_{ab}\phi^a R^b dS$) from Ashtekar-Krishnan's construction where

$$K_{ab}\phi^a R^b = 2W_{(a}R_{b)}\phi^a R^b = -W_a\phi^a = -\zeta_a\phi^a = -\pi m_a\phi^a + C.C. = -\pi\eth f + \bar{\pi}\bar{\eth}f. \quad (92)$$

We make the following notes:

(1) Angular-momentum of Ashtekar-Krishnan's construction (31) yields the same result with our angular-momentum computation from the Komar integral (90) on the DH cross section.

(2) Under our gauge condition (46), if $\zeta_a \hat{=} 0$ then $\pi \hat{=} 0$, i.e., $J \hat{=} 0$, $dJ \hat{=} 0$ angular momentum vanishes, and vice versa.

(3) For the IH, we first rewrite

$$\oint K_{ab}\phi^a R^b = \oint \mathcal{D}_a n_b \ell^b \phi^a \quad (93)$$

then take $\mathcal{D}_a$ to be the induced derivative on the degenerate metric on the IH. It goes back to the angular-momentum expression of the IH. Here we have

$$\oint \omega_a\phi^a = \oint f\,\mathrm{Im}\eth\pi = \oint f\,\mathrm{Im}\Psi_2 \quad (94)$$

where ω_a is the connection one-form from Ashtekar's isolated horizon construction and we use integration by parts in [19]. More on angular momentum of the IH can be found in [20–22].

VI. THE QUASI-LOCAL ENERGY-MOMENTUM AND FLUX OF A SLOW-ROTATING DH

A. Mass and mass flux from the Komar integral

The asymptotic time Killing vector on a DH can be expressed as $t_0^a = \frac{\partial}{\partial v} = [\ell^a + (U - \dot{R}_\Delta)n^a]|_\Delta$ in a corotating coordinate. The Komar mass on a DH is then

$$M_\Delta = \frac{1}{8\pi}\oint_S \nabla^a t_0^b N dS_{ab} = \frac{1}{4\pi}\oint \epsilon_0 N dS_\Delta = \frac{1}{4\pi}\oint \frac{1}{4R_\Delta}N dS_\Delta, \quad (95)$$

where $\epsilon_0 = -\mu_0/4$. This yields the same with Eq. (i) in Sec. VI B from the two-spinor calculation when one chooses $N = \lambda_0^0\bar{\lambda}_{0'}^0$.

We then obtain that the mass flux on a DH from the Komar integral is

$$\dot{M}_\Delta = \frac{1}{4\pi}\oint \frac{\dot{R}_\Delta}{R_\Delta^2}N dS_\Delta, \quad (96)$$

and later we shall see that it agrees with Eq. (97) from two-spinor method.

B. Mass and mass flux from the two-spinor method

By using the compatible constant spinor conditions for a spinning dynamical horizon (61) and (62) and the results of the asymptotic expansion, we get the quasilocal energy-momentum integral on a slow-rotating dynamical

$$I(R_\Delta) = -\frac{1}{4\pi}\oint \mu_0\lambda_0^0\bar{\lambda}_{0'}^0 dS_\Delta \quad \text{(i)}$$
$$= -\frac{1}{4\pi}\oint \frac{\mathrm{Re}}{2\epsilon_0}[\Psi_2^0 + \delta_0\pi_0 + 2\beta_0\pi_0]\lambda_0^0\bar{\lambda}_{0'}^0 dS_\Delta. \quad \text{(ii)}$$

In order to calculate flux we need the time-related condition (60) of the constant spinor of the dynamical horizon in Sec. III E and rescale it. Then $\dot{\lambda}_0^0 = 0$. It is tedious but straightforward to calculate the flux expression. It largely depends on the nonradial NP equations and the second-order NP coefficients. By using Sec. IV B, we substitute them back into the energy-momentum flux formula to simplify our expression.

From (i) Apply the time derivative to (i), and then we obtain the *quasilocal energy-momentum flux for the dynamical horizon*

$$\dot{I}(R_\Delta) = \frac{1}{4\pi}\oint \dot{\mu}_0\lambda_0^0\bar{\lambda}_{0'}^0 dS_\Delta, \quad (97)$$

where it is always *positive*. Here $\dot{\mu}_0$ is the *news function of DH* that always has mass gain.

From the choice of $\mu_0 = -\frac{1}{R_\Delta}$, we have $\dot{\mu}_0 = \frac{\dot{R}_\Delta}{R_\Delta^2} = \frac{\dot{R}_\Delta}{2}{}^{(2)}R$ where the two-scalar curvature is ${}^{(2)}R = \frac{2}{R_\Delta^2}$. (The metric of a two-sphere with radius R_Δ is $d\tilde{s}^2 = -R_\Delta^2(d\theta^2 + \sin^2\theta d\phi^2)$.) Integrate the above equation with respect to v and use $\dot{\mu}_0 = \dot{R}_\Delta{}^{(2)}R/2$; we then have [6]

$$dI(R_\Delta) = \frac{1}{8\pi}\int {}^{(2)}R\lambda_{0'}^0\bar{\lambda}_0^0 dS_\Delta dR_\Delta. \quad (98)$$

From (ii) We first apply $\partial/\partial v$ on (NR7) to get

$$\dot{\mu}_0 = \frac{\dot{\Psi}_2^0 + \dot{\delta}_0\pi_0 + \delta_0\dot{\pi}_0 + 2\dot{\beta}_0\pi_0 + 2\beta_0\dot{\pi}_0}{2\epsilon_0} - \frac{\mu_0\dot{\epsilon}_0}{\epsilon_0}. \tag{99}$$

Now, we apply the time derivative on (ii) and use Sec. IV B, which yields

$$\dot{I}(R_\Delta) = \frac{1}{4\pi}\oint \frac{1}{2\epsilon_0}\left\{\frac{1}{R_\Delta}\Phi_{00}^0 - 2\frac{\sigma_0\bar{\sigma}_0}{\dot{R}_\Delta}\left(\frac{\partial}{\partial v}\ln(R_\Delta^2\sigma_0\bar{\sigma}_0)\right) + 3\frac{\dot{R}_\Delta\pi_0\bar{\pi}_0}{R_\Delta}\right\}\lambda_0^0\bar{\lambda}_{0'}^0 dS_\Delta \tag{100}$$

where the total energy-momentum flux F_{total} is the left-hand side of (100) and is equal to the matter flux plus gravitational flux $F_{\text{total}} = F_{\text{matter}} + F_{\text{grav}}$. We can write the gravitational flux equal to the shear flux plus angular-momentum flux.

$$F_{\text{grav}} = F_\sigma + F_J \tag{101}$$

where the shear flux F_σ is the second term of the right-hand side of (100) and the angular-momentum flux F_J is the third term of the right-hand side of (100). The coupling of the shear σ_0 and π_0 can be transformed into π_0 terms by using (NR9'). We then integrate the above equation with respect to v, and we have

$$dI(R_\Delta) = \frac{1}{8\pi}\int \frac{R_\Delta}{\dot{R}_\Delta}\left\{\frac{1}{R_\Delta}\Phi_{00}^0 - 2\frac{\sigma_0\bar{\sigma}_0}{\dot{R}_\Delta}\left(\frac{\partial}{\partial v}\ln(R_\Delta^2\sigma_0\bar{\sigma}_0)\right) + 3\frac{\dot{R}_\Delta\pi_0\bar{\pi}_0}{R_\Delta}\right\}\lambda_0^0\bar{\lambda}_{0'}^0 dS_\Delta dR_\Delta, \tag{102}$$

where $dv = \frac{dR_\Delta}{\dot{R}_\Delta}$. Here we note that if one wants to observe a positive shear flux $-\frac{\partial}{\partial v}\ln(R_\Delta^2\sigma_0\bar{\sigma}_0) \geq 0$, it implies that

$$\dot{\sigma}_0 \leq 0, \tag{103}$$

where $\dot{R}_\Delta, R_\Delta > 0$ have been considered. So the shear on a spinning DH is monotonically decreasing with respect to v.

By recalling the total flux of Ashtekar-Krishnan (38), we compare our expression with Ashtekar's expression. If we choose $N = \lambda_0^0\bar{\lambda}_{0'}^0$, then (102) together with (98) gives

$$\begin{aligned} dI(R_\Delta) &= \frac{1}{8\pi}\int {}^{(2)}RNdS_\Delta dR_\Delta \\ &= \frac{1}{8\pi}\int \frac{R_\Delta}{\dot{R}_\Delta}\left\{\frac{1}{R_\Delta}\Phi_{00}^0 + 2k\frac{\sigma_0\bar{\sigma}_0}{\dot{R}_\Delta} + 3\frac{\dot{R}_\Delta\pi_0\bar{\pi}_0}{R_\Delta}\right\}NdS_\Delta dR_\Delta \end{aligned} \tag{104}$$

where we define $\frac{\partial}{\partial v}\ln(R_\Delta^2\sigma_0\bar{\sigma}_0) := -k$ for convenience. This is the relation between the change in DH area (recall (39)) and the total flux including the matter flux and gravitational flux.

Shear flux: In the special case $\frac{\partial}{\partial v}\ln(R_\Delta^2\sigma_0\bar{\sigma}_0) := -k$ where k is a constant, we then have

$$R_\Delta^2\sigma_0\bar{\sigma}_0 = Ae^{-kv}. \tag{105}$$

If $k > 0$, $\sigma_0 \searrow$. If $k < 0$, $\sigma_0 \nearrow$. Therefore, if we want to get a positive gravitational flux, the shear σ_0 must decrease with time v and $k > 0$. On the contrary, the negative gravitational flux implies the shear must grow with time. That the negative mass loss from the shear flux will make the dynamical horizon grow with time is physically unreasonable. Therefore, the second term of the right-hand side in Eq. (100) should be positive. This says that the shear on a spinning DH will decay to zero when time v goes to infinity and the amount of shear flux F_σ is finite.

$$\sigma_0 \to 0, \qquad |v| \to \infty, \tag{106}$$

$$F_\sigma < \infty. \tag{107}$$

Hence a slow-rotating dynamical horizon will settle down to an equilibrium state, i.e., isolated horizon at late time.

1. Discussion

(1) If $\pi_0 = 0$ and the shear does not vanish $\sigma_0 \neq 0$ we have

$$\dot{I}(R_\Delta) = \frac{1}{4\pi}\oint \frac{2R_\Delta}{\dot{R}_\Delta}[\Phi_{00}^0 + 2k\sigma_0\bar{\sigma}_0]NdS_\Delta.$$

This goes back to the result of the flux of the nonrotating dynamical horizon. When $k = \frac{1}{2}$, it goes back to the result of the nonrotating DH in [7].

(2) If both shear and π_0 vanish, we have

$$\dot{I}(R_\Delta) = \frac{1}{4\pi}\oint \frac{R_\Delta\Phi_{00}^0}{\dot{R}_\Delta}NdS_\Delta.$$

This result can be compared with the dynamical horizon of the Vaidya solution.

(3) We chose the cross section of the DH to be two-sphere, however, it still implies that the shear term cannot make it to zero. This is because the contribution of the shear comes from the next-order nonlinear effect of the equations.

2. Laws of black hole dynamics

The left-hand side of Eq. (104) can be written as

$$\frac{dI(R_\Delta)}{2} = \frac{\tilde{\kappa}}{8\pi}dA = \frac{dR_\Delta}{2} \tag{108}$$

where $A = 4\pi R_\Delta^2$. For a time evolution vector $t^a = N\ell^a - \Omega\phi^a$, the difference of horizon energy dE^t can be calculated as follows:

$$dE^t = \frac{1}{16\pi}\int \frac{R_\Delta}{\dot{R}_\Delta}\left\{\frac{1}{R_\Delta}\Phi_{00}^0 + 2k\frac{\sigma_0\bar{\sigma}_0}{\dot{R}_\Delta}\right\}N + 3N\pi_0\bar{\pi}_0 - 4\frac{\Omega}{\dot{R}_\Delta}\operatorname{Im}\left[\left(\eth_0\dot{f} - \frac{\dot{R}_\Delta}{R_\Delta}\eth_0 f\right)\pi_0\right]dV$$

and the generalized black hole first law for a slow-rotating dynamical horizon is

$$\frac{\tilde{\kappa}}{8\pi} dA + \Omega dJ = dE^t. \tag{109}$$

VII. CONCLUSIONS

Since Ψ_0, Ψ_4 are gauge-invariant quantities in a linear perturbation theory, it allows us to choose a gauge, in which Ψ_1, Ψ_3 vanish on DH. This choice of gauge is crucial for the coupling of the NP equations and the consequence of physical interpretation. In this paper, we use a different peeling property from our earlier work [6,7]. This leads to a physical picture that captures a collapsing slow-rotating star and formation of a dynamical horizon that finally settles down to an isolated horizon at late time. Further from the peeling property, if the shear flux is positive, it excludes the possibility for a slow-rotating DH to absorb the gravitational radiation from nearby gravitational sources. The mass and momentum are carried in by the incoming gravitational wave and cross into the dynamical horizon. We shall see that though an outgoing wave may exist on the horizon, it will not change the boundary condition or make the contribution to the energy flux. A dynamical horizon forms inside the star and eats up all of the incoming wave when it reaches the equilibrium state, i.e., isolated horizon.

The NP equations are simplified by using two-sphere conditions for a slow-rotating DH with small-tide. By using the compatibility of the coupling NP equations and the asymptotic constant spinors, the energy flux that crosses into a slow-rotating DH should be positive. The mass gain of a slow-rotating DH can be quantitatively written as matter flux, shear flux and angular-momentum flux. Further, the result that the shear flux must be positive implies that the shear must monotonically decay with respect to time. This is physically reasonable since the black hole cannot eat an infinite amount of gravitational energy when there is no other gravitational source near a slow-rotating DH. We further found that the mass and mass flux based on the Komar integral can yield the same result. Therefore, our results are unlikely expression dependent. For other quasilocal expressions remains the open question for future study. It would be interesting if one can free the two-sphere conditions, then obtain the metric distorted by the gravitational wave.

ACKNOWLEDGMENTS

Y. H. W. and C. H. W. would like to thank Professor James M. Nester for helpful discussion and the referee for the comments and criticism. Y. H. W. would like to thank *Top University Project of NCU* supported by the Ministry of Education, Taiwan, ROC. Y. H. W. was supported by the Center for Mathematics and Theoretical Physics, National Central University, Taiwan, ROC. C. H. W. was supported by the National Science Council of Taiwan under the grants NSC 96-2112-M-032-006-MY3 and 98-2811-M-032-014.

[1] R. Penrose and W. Rindler, *Cambridge Monographs on Mathematical Physics* (Cambridge University Press, Cambridge, New York, 1986).
[2] A. Ashtekar, C. Beetle, and S. Fairhurst, Classical Quantum Gravity **17**, 253 (2000).
[3] A. Ashtekar and B. Krishnan, Phys. Rev. D **68**, 104030 (2003).
[4] A. R. Exton, E. T. Newman, and R. Penrose, J. Math. Phys. (N.Y.) **10**, 1566 (1969).
[5] E. T. Newman and T. W. J. Unti,J. Math. Phys. (N.Y.) **3**, 891 (1962).
[6] Y. H. Wu, Ph.D. thesis, University of Southampton, 2007.
[7] Y. H. Wu and C. H. Wang, Phys. Rev. D **80**, 063002 (2009).
[8] R. K. Sachs, Proc. R. Soc. A **264**, 309 (1961).
[9] S. Chandrasekhar, *The Mathematical Theory of Black Hole* (Oxford University Press, Oxford, 1983).
[10] Y. H. Wu and C. H. Wang, Classical Quantum Gravity **25**, 135007 (2008).
[11] G. Bergqvist, Classical Quantum Gravity **9**, 1753 (1992).
[12] L. B. Szabados, Living Rev. Relativity (2004) (http://relativity.livingreviews.org/Articles/lrr-2004-4).
[13] B. D. Bramson, Proc. R. Soc. A **341**, 451 (1975).
[14] H. Bondi, M. G. J. Van der Burg, and A. W. K. Metzner, Proc. R. Soc. A **269**, 21 (1962).
[15] E. T. Newman and R. Penrose, Phys. Rev. Lett. **15**, 231 (1965).
[16] E. T. Newman and R. Penrose, Proc. R. Soc. A **305**, 175 (1968).
[17] S. Dain and J. A. Valiente Kroon, Classical Quantum Gravity **19**, 811 (2002).
[18] J. L. Jaramillo, J. A. Valiente Kroon, and E. Gourgoulhon, Classical Quantum Gravity **25**, 093001 (2008).
[19] R. Penrose and W. Rindler, *Cambridge Monographs on Mathematical Physics* (Cambridge University Press, Cambridge, New York, 1984).
[20] G. Date, Classical Quantum Gravity **17**, 5025 (2000).
[21] E. Gourgoulhon and J. L. Jaramillo, Physics Reports No. 423 159-294, 2006.
[22] B. Krishnan, Ph.D. thesis, Pennsylvania State University, 2002.