Collision of two general geodesic particles around a Kerr black hole

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We obtain an explicit expression for the center-of-mass (CM) energy of two colliding general geodesic massive and massless particles at any spacetime point around a Kerr black hole. Applying this, we show that the CM energy can be arbitrarily high only in the limit to the horizon and then derive a formula for the CM energy of two general geodesic particles colliding near the horizon in terms of the conserved quantities of each particle and the polar angle. We present the necessary and sufficient condition for the CM energy to be arbitrarily high in terms of the conserved quantities of each particle. To have an arbitrarily high CM energy, the angular momentum of either of the two particles must be fine-tuned to the critical value $L_i = \Omega_H^{-1} E_i$, where Ω_H is the angular velocity of the horizon and E_i and L_i are the energy and angular momentum of particle $i(= 1, 2)$, respectively. We show that, in the direct collision scenario, the collision with an arbitrarily high CM energy can occur near the horizon of maximally rotating black holes not only at the equator but also on a belt centered at the equator. This belt lies between latitudes $\pm \mathrm{acos}(\sqrt{3} - 1) \simeq \pm 42.94°$. This is also true in the scenario through the collision of a last stable orbit particle.

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I. INTRODUCTION

Bañados, Silk and West [1] discovered that the center-of-mass (CM) energy can be arbitrarily high if two particles which begin at rest at infinity collide near the horizon of a maximally rotating Kerr black hole [2] and if the angular momentum of either particle is fine-tuned to the critical value. They argue this scenario in the context of the collision of dark matter particles around intermediate-mass black holes. This scenario is generalized to charged black holes [3], the Kerr-Newman family of black holes [4] and general rotating black holes [5]. A general explanation for the arbitrarily high CM energy is presented in terms of the Killing vectors and Killing horizon by Zaslavskii [6].

The scenario by Bañados, Silk and West [1] was subsequently criticized by several authors [7,8]. One of the most important points is the limitations of the test particle approximation upon which their calculation relies. The validity of the test particle approximation is now under investigation. However, as we can see for the exact analysis of the analogous system [9], it is quite reasonable that the physical CM energy outside the horizon is bounded from above due to the violation of the test particle approximation. On the other hand, it is also reasonable that the upper limit on the CM energy is still considerably high in the situation where the test particle approximation is good.

To circumvent the fine-tuning problem of the angular momentum, Harada and Kimura [10] proposed a scenario, where the fine-tuning is naturally realized by the innermost stable circular orbit (ISCO) around a Kerr black hole [11]. They discovered that the CM energy for the collision between an ISCO particle and another generic particle becomes arbitrarily high in the limit of the maximal rotation of the black hole. Even for the nonmaximally rotating black holes, Grib and Pavlov [12,13] proposed a different scenario to obtain the arbitrarily high CM energy of two colliding particles. In this case, the particle with a near-critical angular momentum cannot reach the horizon from well outside through the geodesic motion because of the potential barrier. In their scenario, the angular momentum of the particle must be fine-tuned to the critical value through the preceding scattering near the horizon.

The geometry of a vacuum, stationary and asymptotically flat black hole is uniquely given by the Kerr metric [2]. In the background of the Kerr spacetime, the expressions for the CM energy and its near-horizon limit are given for two colliding geodesic particles of the same rest mass, different energies and angular momenta in [10] and of different masses, energies and angular momenta in [13], although both are restricted to the motion on the equatorial plane. It is quite important to extend the analysis to general geodesic particles not only because the analysis applies to realistic collisions in astrophysics but also because we can get a deeper physical insight into the phenomenon itself. The general geodesic motion of massive and massless particles in the Kerr spacetime was analyzed by Carter [14]. See also [15,16]. The last stable orbit (LSO) is the counterpart of the ISCO for the nonequatorial motion and defined by Sundararajan [17].

Based on Carter's formalism, we generalize the analysis of the CM energy of two colliding particles to general geodesic massive and massless particles. In this paper, we adopt the test particle approximation and hence neglect

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the effects of self-gravity and back reaction. We then obtain an explicit expression for the CM energy of two colliding general geodesic particles at any spacetime point in the Kerr spacetime and derive a formula for the CM energy of two general geodesic particles colliding near the horizon of a Kerr black hole in terms of the conserved quantities of each particle and the polar angle. We show that the collision with an arbitrarily high CM energy is possible only in the limit to the horizon. We present the necessary and sufficient condition to obtain an arbitrarily high CM energy and find that this condition is met only through the three scenarios, the direct collision scenario proposed by Bañados, Silk and West [1], the LSO (ISCO) collision scenario by Harada and Kimura [10] and the multiple scattering scenario by Grib and Pavlov [12,13]. We find that the collision with an arbitrarily high CM energy is possible near the horizon of maximally rotating black holes not only at the equator but also at the latitude up to $\mathrm{acos}(\sqrt{3}-1) \simeq 42.94°$ even if we do not admit the multiple scattering scenario.

This paper is organized as follows. In Sec. II, we briefly review general geodesic particles in the Kerr spacetime. In Sec. III, we obtain an expression for the CM energy of two general geodesic particles at any spacetime point and then by taking the near-horizon limit obtain a general formula for the near-horizon collision. In Sec. IV, we classify critical particles, determine the region of the collision with an arbitrarily high CM energy with and without multiple scattering. Section V is devoted to conclusion and discussion. We use the units in which $c = G = 1$ and the abstract index notation of Wald [18].

II. GENERAL GEODESIC MOTION IN THE KERR SPACETIME

A. The Kerr metric in the Boyer-Lindquist coordinates

The line element in the Kerr spacetime in the Boyer-Lindquist coordinates is given by [2,16,18]

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \quad (2.1)$$

where a and M are the spin and mass parameters, respectively, $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. If $0 \leq a^2 \leq M^2$, Δ vanishes at $r = r_\pm = M \pm \sqrt{M^2 - a^2}$, where $r = r_+$ and $r = r_-$ correspond to an event horizon and Cauchy horizon, respectively. Here, we denote $r_+ = r_H$. In this coordinate system, the time translational and axial Killing vectors are given by $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial\phi)^a$, respectively. The surface gravity of the Kerr black hole is given by $\kappa = \sqrt{M^2 - a^2}/(r_H^2 + a^2)$. Thus, the black hole has a vanishing surface gravity and hence is extremal for the maximal rotation $a^2 = M^2$, while it is subextremal for the nonmaximal rotation $a^2 < M^2$. The angular velocity of the horizon is given by

$$\Omega_H = \frac{a}{r_H^2 + a^2}. \quad (2.2)$$

The Killing vector $\chi^a = \xi^a + \Omega_H\psi^a$ is a null generator of the event horizon. We can assume $a \geq 0$ without loss of generality.

B. The Hamilton-Jacobi equation and the Carter constant

We here briefly review general geodesic particles in the Kerr spacetime based on [15,16]. Let $S = S(\lambda, x^\alpha)$ be the action as a function of the parameter λ and coordinates x^α, or the Hamilton-Jacobi function. The conjugate momentum is given by $p_\alpha = \partial S/\partial x^\alpha$. Since the Hamiltonian for a geodesic particle is given by $\mathcal{H} = (1/2)\sum_{\mu,\nu} g^{\mu\nu}p_\mu p_\nu$, we can explicitly write down the Hamilton-Jacobi equation with the metric (2.1) in the following form:

$$-\frac{\partial S}{\partial\lambda} = \frac{1}{2\rho^2}\left\{-\frac{1}{\Delta}\left[(r^2 + a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial\phi}\right]^2 + \frac{1}{\sin^2\theta}\left[\frac{\partial S}{\partial\phi} + a\sin^2\theta\frac{\partial S}{\partial t}\right]^2 + \Delta\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{\partial S}{\partial\theta}\right)^2\right\}. \quad (2.3)$$

Since λ, t and ϕ are cyclic coordinates, S is written through the separation of variables as

$$S = \frac{1}{2}m^2\lambda - Et + L\phi + S_r(r) + S_\theta(\theta), \quad (2.4)$$

where m, E and L are constants which correspond to the rest mass, conserved energy and angular momentum through $m^2 = -p_a p^a$, $E = -p_t = -\xi^a p_a$, and $L = p_\phi = \psi^a p_a$, respectively. Note that the proper time τ along the world line is given by $d\tau = m d\lambda$ and the four velocity u^a is given by $p^a = m u^a$ for a massive particle.

Substituting Eq. (2.4) into Eq. (2.3), we obtain

$$-\Delta\left(\frac{dS_r}{dr}\right)^2 - m^2r^2 + \frac{[(r^2 + a^2)E - aL]^2}{\Delta} = \left(\frac{dS_\theta}{d\theta}\right)^2 + m^2a^2\cos^2\theta + \frac{1}{\sin^2\theta}[L - aE\sin^2\theta]^2. \quad (2.5)$$

It follows that both sides must be the same constant, which we denote with $\mathcal{K}$. That is to say,

$$\mathcal{K} = -\Delta\left(\frac{dS_r}{dr}\right)^2 - m^2r^2 + \frac{[(r^2 + a^2)E - aL]^2}{\Delta}, \quad (2.6)$$

$$\mathcal{K} = \left(\frac{dS_\theta}{d\theta}\right)^2 + m^2a^2\cos^2\theta + \frac{1}{\sin^2\theta}[L - aE\sin^2\theta]^2. \quad (2.7)$$

Clearly, $\mathcal{K} \geq 0$ follows from Eq. (2.7). The Carter constant $\mathcal{Q}$ is a conserved quantity defined by $\mathcal{Q} \equiv \mathcal{K} - (L - aE)^2$ or

$$\mathcal{Q} = \left(\frac{dS_\theta}{d\theta}\right)^2 + \cos^2\theta\left[a^2(m^2 - E^2) + \frac{L^2}{\sin^2\theta}\right]. \quad (2.8)$$

Note that $\mathcal{Q}$ can be negative but $\mathcal{Q} + (L - aE)^2 \geq 0$ must be satisfied. On the other hand, we find $\mathcal{Q} \geq 0$ if $m^2 \geq E^2$ from Eq. (2.8).

We integrate Eqs. (2.6) and (2.7) to give

$$S_\theta = \sigma_\theta \int^\theta d\theta \sqrt{\Theta}, \qquad S_r = \sigma_r \int^r dr \frac{\sqrt{R}}{\Delta},$$

where the choices of the two signs $\sigma_\theta = \pm 1$ and $\sigma_r = \pm 1$ are independent and

$$\Theta = \Theta(\theta) = \mathcal{Q} - \cos^2\theta\left[a^2(m^2 - E^2) + \frac{L^2}{\sin^2\theta}\right], \quad (2.9)$$

$$R = R(r) = P(r)^2 - \Delta(r)[m^2r^2 + (L - aE)^2 + \mathcal{Q}], \quad (2.10)$$

$$P = P(r) = (r^2 + a^2)E - aL. \quad (2.11)$$

Thus, we obtain the Hamilton-Jacobi function. Note that for the allowed motion both $\Theta \geq 0$ and $R \geq 0$ must be satisfied.

Using $dx^\alpha/d\lambda = p^\alpha = \sum_\beta g^{\alpha\beta}p_\beta$, we obtain

$$\rho^2 \frac{dt}{d\lambda} = -a(aE\sin^2\theta - L) + \frac{(r^2 + a^2)P}{\Delta}, \quad (2.12)$$

$$\rho^2 \frac{dr}{d\lambda} = \sigma_r \sqrt{R}, \quad (2.13)$$

$$\rho^2 \frac{d\theta}{d\lambda} = \sigma_\theta \sqrt{\Theta}, \quad (2.14)$$

$$\rho^2 \frac{d\phi}{d\lambda} = -\left(aE - \frac{L}{\sin^2\theta}\right) + \frac{aP}{\Delta}. \quad (2.15)$$

C. Properties of geodesic particles in the Kerr spacetime

From Eqs. (2.9) and (2.14), we can see $\mathcal{Q} = 0$ must be satisfied for a particle moving on the equatorial plane $\theta = \pi/2$. As we can see in Eqs. (2.9) and (2.14), if $L \neq 0$, the particle oscillates with respect to θ and never reaches the rotational axis $\theta = 0$ or π. A special treatment is needed for a particle which crosses the rotational axis $\theta = 0$ or π, which is a coordinate singularity. To have a regular limit to the axis in Eq. (2.5), we impose $L = 0$ to such a particle. Only particles with $L = 0$ can cross the rotational axis.

Equation (2.13) imply

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{r^4}{\rho^4} V_{\text{eff}}(r) = 0, \quad (2.16)$$

where

$$V_{\text{eff}}(r) \equiv -\frac{R(r)}{2r^4}. \quad (2.17)$$

Since r^4/ρ^4 is nonzero and finite outside the horizon, V_{eff} plays a role similar to the effective potential for the motion on the equatorial plane, although there is a coupling with θ in Eq. (2.16). The allowed and prohibited regions are given by $V_{\text{eff}}(r) \leq 0$ and $V_{\text{eff}}(r) > 0$, respectively. Since $V_{\text{eff}}(r) \to (m^2 - E^2)/2$ as $r \to \infty$, the sign of $(m^2 - E^2)$ governs the particle motion far away from the black hole. A particle is bound, marginally bound and unbound if $m^2 > E^2$, $m^2 = E^2$ and $m^2 < E^2$, respectively.

Let us consider special null geodesics with $\mathcal{K} = 0$. Then, $L = aE\sin^2\theta$, $\mathcal{Q} = -(L - aE)^2 = -(aE\cos^2\theta)^2$ and hence $\Theta = 0$. Thus, $\theta = \text{const}$, $P = E\rho^2$ and $R = E^2\rho^4$. Then, we obtain simple geodesics:

$$\frac{dt}{d\lambda} = \frac{E(r^2 + a^2)}{\Delta}, \qquad \frac{dr}{d\lambda} = \sigma_r E, \qquad \frac{d\theta}{d\lambda} = 0, \qquad \frac{d\phi}{d\lambda} = \frac{aE}{\Delta}.$$

This means that for any value of θ, there are always ingoing and outgoing null geodesics along which $\theta = \text{const}$. These geodesics are called outgoing (ingoing) principal null geodesics for $\sigma_r = 1\ (-1)$.

Since we are considering causal geodesics parametrized from the past to the future, we need to impose $dt/d\lambda \geq 0$ along the geodesic. This is called the "forward-in-time" condition. In particular, as seen from Eq. (2.12), this condition reduces to

$$E - \Omega_H L \geq 0, \quad (2.18)$$

in the near-horizon limit, where we have used Eq. (2.2). Shortly, the angular momentum must be smaller than the critical value $L_c \equiv \Omega_H^{-1}E$. This condition is identical to the forward-in-time condition near the horizon for particles restricted on the equatorial plane. We refer to particles with the angular momentum $L = L_c$, $L < L_c$ and $L > L_c$ as critical, subcritical and supercritical particles, respectively. We can easily see that $L \leq L_c$ is equivalent to the condition

$$-\chi^a p_a \geq 0,$$

for the horizon-generating Killing vector χ^a and the four-momentum p_a of the particle. This must clearly hold near the horizon for the subextremal black hole because χ^a is future-pointing timelike there and p^a is future-pointing timelike or null.

III. CM ENERGY OF TWO COLLIDING GENERAL GEODESIC PARTICLES

A. CM energy of two colliding particles of different rest masses

Let particles 1 and 2 of rest masses m_1 and m_2 have four-momenta p_1^a and p_2^a, respectively. The sum of the two momenta is given by

$$p^a_{\rm tot} = p^a_1 + p^a_2.$$

The CM energy $E_{\rm cm}$ of the two particles is then given by

$$E^2_{\rm cm} = -p^a_{\rm tot} p_{{\rm tot}a} = m^2_1 + m^2_2 - 2g^{ab} p_{1a} p_{2b}. \qquad (3.1)$$

Clearly, this applies for both massive and massless particles. Since $E_{\rm cm}$ is a scalar, it does not depend on the coordinate choice in which we evaluate it. This is the reason why we can safely determine the CM energy in the Boyer-Lindquist coordinates in spite of the coordinate singularity on the horizon.

B. CM energy of two colliding particles in the Kerr spacetime

As seen in Sec. III A, the CM energy of two particles is determined by calculating $-g^{ab} p_{1a} p_{2b}$. Using Eq. (2.1), the CM energy is then calculated to give

$$E^2_{\rm cm} = m^2_1 + m^2_2 + \frac{2}{\rho^2}\left[\frac{P_1 P_2 - \sigma_{1r}\sqrt{R_1}\sigma_{2r}\sqrt{R_2}}{\Delta} - \frac{(L_1 - a\sin^2\theta E_1)(L_2 - a\sin^2\theta E_2)}{\sin^2\theta} - \sigma_{1\theta}\sqrt{\Theta_1}\sigma_{2\theta}\sqrt{\Theta_2}\right], \qquad (3.2)$$

where and hereafter E_i, L_i, $\mathcal{Q}_i$, $\mathcal{K}_i$, $P_i = P_i(r)$, $R_i = R_i(r)$ and $\Theta_i = \Theta_i(\theta)$ are E, L, $\mathcal{Q}$, $\mathcal{K}$, $P = P(r)$, $R = R(r)$ and $\Theta = \Theta(\theta)$ for particle i, respectively. This is surprisingly simple in spite of the generality of this expression. This is due to the separability of the Hamilton-Jacobi equation in the Kerr spacetime. From Eqs. (2.7) and (2.9) with $\Theta \geq 0$ and $\rho^2 = r^2 + a^2\cos^2\theta$, it follows that

$$\left|\frac{L - a\sin^2\theta E}{\sin\theta}\right| \leq \sqrt{\mathcal{K}},$$

$$\sqrt{\Theta} \leq \sqrt{|\mathcal{Q}| + a^2|m^2 - E^2|}, \qquad r^2_H \leq \rho^2 \leq r^2 + a^2$$

outside the horizon. Moreover, in the limit $r \to \infty$, we obtain

$$E^2_{\rm cm} \to m^2_1 + m^2_2 + 2\left(E_1 E_2 - \sigma_{1r}\sqrt{E^2_1 - m^2_1}\sigma_{2r}\sqrt{E^2_2 - m^2_2}\right).$$

Therefore, Eq. (3.2) assures that if all conserved quantities m_i, E_i, L_i, $\mathcal{K}_i$ are bounded from above, $E_{\rm cm}$ is also bounded from above except in the limit to the horizon where $\Delta = 0$. In other words, only if the collision occurs near the horizon, the CM energy can be unboundedly high.

C. CM energy of two particles colliding near the horizon

If σ_{1r} and σ_{2r} have different signs near the horizon, the CM energy for two colliding particles necessarily diverges in the near-horizon limit $\Delta \to 0$ as

$$E^2_{\rm cm} \approx 4\frac{(r^2_H + a^2)^2}{r^2_H + a^2\cos^2\theta}\frac{(E_1 - \Omega_H L_1)(E_2 - \Omega_H L_2)}{\Delta},$$

where both particles are assumed to be subcritical. However, σ_{1r} and σ_{2r} must not have different signs right on the black hole horizon.

Then, we assume that σ_{1r} and σ_{2r} have the same sign. In the near-horizon limit $r \to r_H$, we can see that $(P_1 P_2 - \sqrt{R_1}\sqrt{R_2})$ vanishes. In fact, it is easy to show

$$\lim_{r\to r_H}\frac{P_1 P_2 - \sqrt{R_1}\sqrt{R_2}}{\Delta} = \frac{m^2_1 r^2_H + \mathcal{K}_1}{2}\frac{(r^2_H + a^2)E_2 - aL_2}{(r^2_H + a^2)E_1 - aL_1} + \frac{m^2_2 r^2_H + \mathcal{K}_2}{2}\frac{(r^2_H + a^2)E_1 - aL_1}{(r^2_H + a^2)E_2 - aL_2},$$

where we have assumed subcritical particles. Therefore, the CM energy of two general geodesic particles in the near-horizon limit is written as

$$E^2_{\rm cm} = m^2_1 + m^2_2 + \frac{1}{r^2_H + a^2\cos^2\theta}\times\left[(m^2_1 r^2_H + \mathcal{K}_1)\frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1} + (m^2_2 r^2_H + \mathcal{K}_2)\times\frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2} - \frac{2(L_1 - a\sin^2\theta E_1)(L_2 - a\sin^2\theta E_2)}{\sin^2\theta} - 2\sigma_{1\theta}\sqrt{\Theta_1}\sigma_{2\theta}\sqrt{\Theta_2}\right], \qquad (3.3)$$

where we have used Eq. (2.2). We can now find that the necessary and sufficient condition to obtain an arbitrarily high CM energy is that

$$(m^2_1 r^2_H + \mathcal{K}_1)\frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1} + (m^2_2 r^2_H + \mathcal{K}_2)\frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2}$$

is arbitrarily large. It is also clear that the necessary condition for the CM energy to be unboundedly high is that $(E - \Omega_H L)$ is arbitrarily close to zero for either of the two particles. That is to say, either of the two particles must be arbitrarily near-critical.

Furthermore, we can show $(m^2 r^2_H + \mathcal{K})$ is bounded from below by a positive value for critical particles with $E \neq 0$. This is trivial for massive particles. For the massless case, from Eq. (2.7), we find for the critical particle

$$\mathcal{K} \geq \frac{[\Omega^{-1}_H E - aE\sin^2\theta]^2}{\sin^2\theta} = \left(\frac{r^2_H + a^2\cos^2\theta}{a\sin\theta}\right)^2 E^2 \geq \left(\frac{r^2_H}{a}\right)^2 E^2 > 0,$$

where we have used Eq. (2.2). Therefore, $(m^2 r^2_H + \mathcal{K})$ is bounded from below by a positive value except for the case where $m = E = L = 0$. Although this exceptional case might be physically meaningful, we do not need to deal with it for the present purpose. Note that since a null geodesic is principal null if and only if $\mathcal{K}=0$, no principal null geodesic can be critical as a contraposition. In fact, any principal null geodesic turns out to be subcritical because $L = aE\sin^2\theta \le aE < \Omega_H^{-1}E = L_c$.

Unless the critical particle is massless with vanishing energy, the necessary and sufficient condition to obtain an arbitrarily high CM energy reduces so that the ratio

$$\frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2}$$

is arbitrarily large or arbitrarily close to zero. If this ratio is arbitrarily close to zero, Eq. (3.3) is approximated as

$$E_{\rm cm}^2 \approx \frac{m_1^2 r_H^2 + \mathcal{K}_1}{r_H^2 + a^2\cos^2\theta}\frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1}.$$

For the particles moving on the equatorial plane, we set $\theta = \pi/2$ and $\mathcal{Q} = 0$. Then, Eq. (3.3) reduces to

$$E_{\rm cm}^2 = m_1^2 + m_2^2 + \frac{1}{r_H^2}\Big\{[m_1^2 r_H^2 + (L_1 - aE_1)^2] \times \frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1} + [m_2^2 r_H^2 + (L_2 - aE_2)^2] \times \frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2} - 2(L_1 - aE_1)(L_2 - aE_2)\Big\}. \quad (3.4)$$

If we further assume that the colliding particles have the same nonzero rest mass m_0, it is easy to explicitly confirm that Eq. (3.4) coincides with the formula (3.5) of Harada and Kimura [10] or

$$\frac{E_{\rm cm}}{2m_0} = \sqrt{1 + \frac{4M^2 m_0^2[(E_1 - \Omega_H L_1) - (E_2 - \Omega_H L_2)]^2 + (E_1 L_2 - E_2 L_1)^2}{16M^2 m_0^2 (E_1 - \Omega_H L_1)(E_2 - \Omega_H L_2)}}$$

in the present notation.

IV. COLLISION WITH AN ARBITRARILY HIGH CM ENERGY

A. Classification of critical particles

Since either of the two colliding particles must be arbitrarily near-critical to obtain an arbitrarily high CM energy, we here study critical particles, i.e. particles with the critical angular momentum $L = L_c \equiv \Omega_H^{-1}E$. Although the critical particle may be prohibited to reach the horizon or it can do so only after an infinite proper time, the critical particle still characterizes near-critical particles as a *limit critical particle*.

From Eq. (2.10), we find

$$R(r_H) = (r_H^2 + a^2)^2 (E - \Omega_H L)^2.$$

Therefore, $R(r_H) \ge 0$. In particular, only for critical particles, i.e. $E - \Omega_H L = 0$, $R(r_H) = 0$ holds. For the first derivative, from Eq. (2.10), we find

$$R'(r_H) = 4r_H(r_H^2 + a^2)E(E - \Omega_H L) - 2(r_H - M)(m^2 r_H^2 + \mathcal{K}).$$

As we have seen in Sec. III C, the factor $(m^2 r_H^2 + \mathcal{K})$ is positive for critical particles. Therefore, we conclude $R'(r_H) \le 0$ for the critical particle because $r_H \ge M$.

If $R'(r_H) = 0$ for the critical particle, the Kerr black hole is necessarily extremal. In this case, R for the critical particle becomes

$$R = (r - M)^2[(E^2 - m^2)r^2 + 2ME^2 r - \mathcal{Q}] \quad (4.1)$$

and hence

$$R''(r_H) = 2[(3E^2 - m^2)M^2 - \mathcal{Q}]. \quad (4.2)$$

Although one might expect a circular orbit of massive particles on the horizon for $R = R' = 0$ there, this is fake as is proven in [10].

Suppose $R'(r_H) = 0$ and $R''(r_H) > 0$, i.e. $(3E^2 - m^2)M^2 > \mathcal{Q}$. Then, $R(r) > 0$ at least in the vicinity of the horizon for the critical particle. This class includes what Bañados, Silk and West [1] originally assume and we refer to this class as class I. A critical particle of class I can reach the horizon along a geodesic from outside after an infinite proper time.

The condition $R'(r_H) = 0$ and $R''(r_H) = 0$, i.e. $(3E^2 - m^2)M^2 = \mathcal{Q}$, corresponds to the marginal case and this is exactly the situation studied in Harada and Kimura [10] for the equatorial case. This is of particular physical interest because the sequence of the prograde ISCO particle converges to this limit, where the fine-tuning of the angular momentum is naturally realized in the astrophysical context. Since the function R takes an inflection point at the ISCO radius and hence $R = R' = R'' = 0$ there, the potential of the limit critical particle should satisfy $R(r_H) = R'(r_H) = R''(r_H) = 0$. Hence, we treat this class as a separate case and refer to this class as class II. For $\mathcal{Q} \ne 0$, the critical particle of this class corresponds to the limit critical particle of the sequence of particles orbiting the inclined LSO in the limit $a \to 1$ according to the definition $R = R' = R'' = 0$ given by Sundararajan [17]. This means that the scenario of the high-velocity collision of an ISCO particle generalizes to the nonequatorial case, as the high-velocity collision of an LSO particle.

We can also consider the case where $R'(r_H) = 0$ and $R''(r_H) < 0$, i.e. $(3E^2 - m^2)M^2 < \mathcal{Q}$. Although this case has not been mentioned so far in the literature in the present

TABLE I. Classification of critical particles and the collision scenarios.

Class	$R(r)$ at $r = r_H$	Scenario	Reference	Parameter region
I	$R = R' = 0,\ R'' > 0$	direct collision	[1]	$a^2 = M^2,\ 3E^2 > m^2,\ \mathcal{Q} < (3E^2 - m^2)M^2$
II	$R = R' = R'' = 0$	LSO collision	[10]	$a^2 = M^2,\ 3E^2 \geq m^2,\ \mathcal{Q} = (3E^2 - m^2)M^2$
III	$R = R' = 0,\ R'' < 0$	multiple scattering	-	$a^2 = M^2,\ \mathcal{Q} > (3E^2 - m^2)M^2$
IV	$R = 0,\ R' < 0$	multiple scattering	[12,13]	$0 < a^2 < M^2$

context, we refer to this class as class III. The behavior of the critical particles of this class is similar to that of the critical particles of class IV described below.

The possibility $R'(r_H) < 0$ for the limit critical particle was first raised by Grib and Pavlov [12,13]. This is possible only for the subextremal black hole. We refer to this class as class IV. In the sequence approaching the critical particle of this class, near-critical particles with an angular momentum $L = L_c - \delta$ for sufficiently small $\delta(>0)$ can approach the horizon along a geodesic only from the vicinity of the horizon. Such near-critical particles are possible only through multiple scattering because they must be inside the potential barrier before the relevant collision. All the critical particles in a subextremal black hole belong to this class.

In principle, one might expect that there is a critical particle with $R'(r_H) > 0$. Such particles should have similar characteristics to those of class I. However, as we have seen, such a critical particle does not exist in the Kerr spacetime.

The conditions for $R(r_H)$, $R'(r_H)$ and $R''(r_h)$ are easily converted to those for $V_{\rm eff}(r_H)$, $V'_{\rm eff}(r_H)$ and $V''_{\rm eff}(r_H)$ in terms of the effective potential $V_{\rm eff}(r)$ defined by Eq. (2.17). Table I summarizes the four classes of critical particles and the three scenarios of the collision with an arbitrarily high CM energy. Note that classes III and IV belong to the same scenario so that we have four classes in spite of three scenarios. Figure 1 shows the examples of the effective potentials for the critical particles of these four classes. Although the classification in this subsection only concerns the signs of the function R and its derivatives at the horizon, it turns out that critical particles of class I with $E^2 \geq m^2$ correspond to the direct collision scenario from infinity, as we will see in Sec. IV C.

B. The high-velocity collision belts on the extremal Kerr black hole

It is not necessarily clear how the fine-tuning of the angular momentum is realized near the horizon through multiple scattering processes. Hence, hereafter we concentrate on the direct collision scenario and the LSO collision scenario. Then, critical particles of classes I and II are relevant, which are possible only if the black hole is extremal and

$$(3E^2 - m^2)M^2 \geq \mathcal{Q} \qquad (4.3)$$

is satisfied for the critical particle, as we have seen in the previous section. Together with Eq. (2.8), $\mathcal{Q}$ must satisfy the following condition:

$$\cos^2\theta\left[M^2(m^2 - E^2) + \frac{4M^2E^2}{\sin^2\theta}\right] \leq \mathcal{Q} \leq (3E^2 - m^2)M^2, \qquad (4.4)$$

where $a^2 = M^2$ and $L = L_c = 2ME$ have been used. We will see here whether this condition restricts the polar angle. From Eq. (4.4), the following condition must be satisfied:

$$(m^2 - E^2)\sin^4\theta + 2(4E^2 - m^2)\sin^2\theta - 4E^2 \geq 0. \qquad (4.5)$$

Conversely, if Eq. (4.5) holds, we can always find $\mathcal{Q}$ which satisfies Eq. (4.4).

For the marginally bound orbit $m^2 = E^2$, we can easily find from Eq. (4.5)

$$\sin\theta \geq \sqrt{\frac{2}{3}}.$$

This means that critical particles can occur only on the belt between latitudes $(\pi/2 - \theta) = \pm \mathrm{acos}\sqrt{2/3} \simeq \pm 35.26°$.

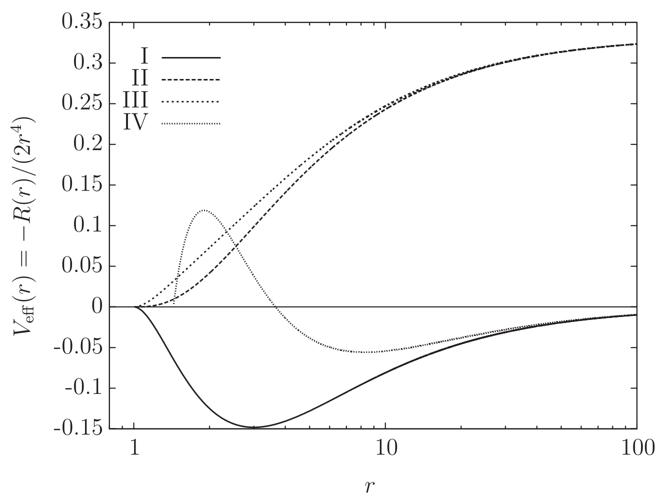

FIG. 1. The examples of the effective potential $V_{\rm eff}(r) = -R(r)/(2r^4)$ for the critical particles. The solid, long-dashed, dashed and short-dashed curves show the potentials for the particles of classes I ($M = a = 1$, $m = E = 1$, $L = 2$, $\mathcal{Q} = 0$), II ($M = a = 1$, $m = 1$, $E = 1/\sqrt{3}$, $L = 2/\sqrt{3}$, $\mathcal{Q} = 0$), III ($M = a = 1$, $E = 1/\sqrt{3}$, $L = 2/\sqrt{3}$, $\mathcal{Q} = 1$), and IV ($M = 1$, $a = 0.9$, $m = E = 1$, $L = 2$, $\mathcal{Q} = 0$), respectively.

It is also easy to generalize this bound to nonmarginally bound particles because the left-hand side of inequality (4.5) is only quadratic with respect to $\sin^2\theta$. The result is that E^2 must satisfy $3E^2 \geq m^2$ and then θ must satisfy the following condition:

$$\sin\theta \geq \sqrt{\frac{-(4E^2 - m^2) + \sqrt{12E^4 - 4E^2m^2 + m^4}}{m^2 - E^2}}. \quad (4.6)$$

Therefore, the absolute value of the latitude must be lower than the angle $\alpha(E, m)$, where

$$\alpha(E, m) = \mathrm{acos}\left(\sqrt{\frac{-(4E^2 - m^2) + \sqrt{12E^4 - 4E^2m^2 + m^4}}{m^2 - E^2}}\right).$$

The above applies to both bound ($m^2 > E^2$) and unbound ($m^2 < E^2$) particles.

For $3E^2 = m^2$, Eqs. (4.4) and (4.5) imply $\theta = \pi/2$ and $\mathcal{Q} = 0$ so that the critical particle belongs to class II and it is on the equatorial plane. This is an ISCO particle for the maximal black hole spin. In the limit $E^2 \to m^2$, the right-hand side of Eq. (4.6) approaches $\sqrt{2/3}$ and hence reproduces the result for the marginally bound particles. It is quite intriguing to see the limit $E^2 \to \infty$. In this limit, the right-hand side of Eq. (4.6) approaches $\sqrt{3} - 1$ and hence

$$\sin\theta \geq \sqrt{3} - 1.$$

Noting that the right-hand side of Eq. (4.6) is monotonically decreasing as a function of E^2, the belt where critical particles can occur becomes larger as the energy of the particle is greater. However, the latitude limit of the belt does not reach the poles but approaches $\pm\mathrm{acos}(\sqrt{3} - 1) \simeq \pm 42.94°$ as the energy of the particle is increased to infinity. In other words, no critical particle occurs with the latitude higher than this angle. The highest absolute value of the latitude is shown in Fig. 2 as a function of the specific energy of the particle.

For a massless particle, i.e. $m = 0$, Eq. (4.6) simply reduces to

$$\sin\theta \geq \sqrt{3} - 1,$$

irrespective of the energy of the particle. Thus, the highest absolute value of the latitude is $\mathrm{acos}(\sqrt{3} - 1) \simeq 42.94°$ if the near-critical particle is massless.

The result is schematically shown in Fig. 3. This figure shows the regions of high-velocity collision on the extremal Kerr black hole. The red (solid thick) line shows the equator. The collisions with an arbitrarily high CM energy occur on the belt colored with blue and cyan (shaded darkly and lightly) if we allow all the critical particles. On the other hand, such collisions occur on the belt colored with blue (shaded darkly) if we only allow bound and marginally bound massive critical particles. On the uncolored (unshaded) region, the collision with an arbitrarily high CM energy is prohibited.

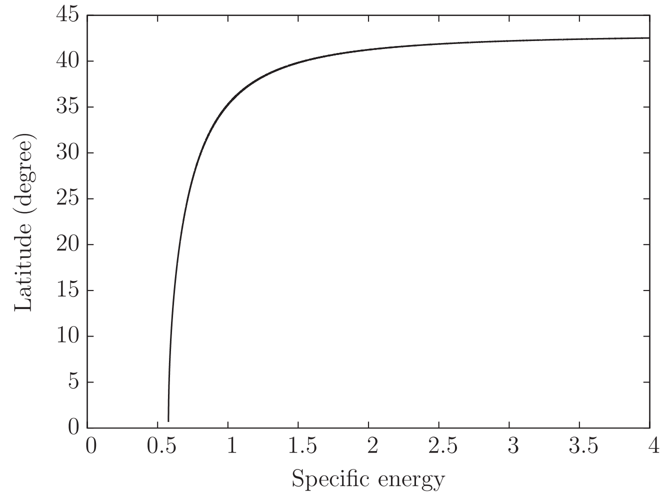

FIG. 2. The highest absolute value of the latitude for the critical particles of classes I and II to occur on the extremal Kerr black hole as a function of the specific energy.

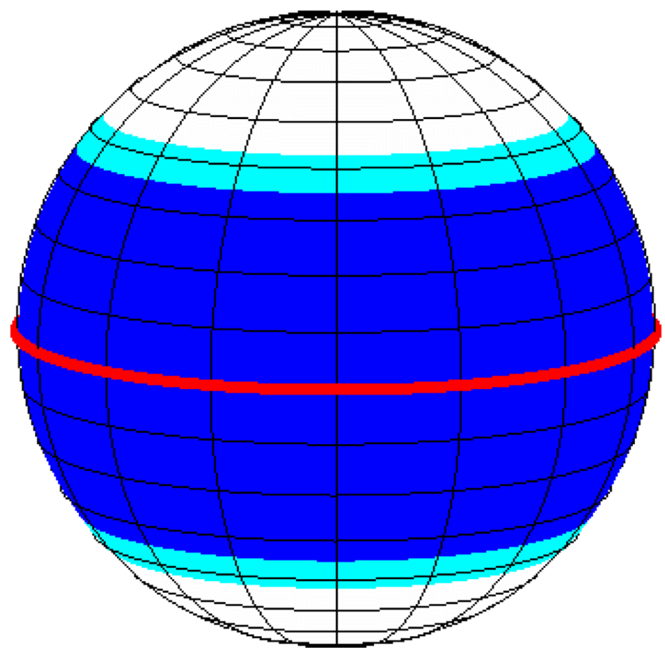

FIG. 3 (color online). The belts of high-velocity collision on the extremal Kerr black hole. The red (solid thick) line shows the equator. The collisions with an arbitrarily high CM energy occur on the belt colored with blue and cyan (shaded darkly and lightly) between latitudes $\pm\mathrm{acos}(\sqrt{3} - 1) \simeq \pm 42.94°$ if we allow all the critical particles. On the other hand, such collisions occur on the belt colored with blue (shaded darkly) between latitudes $\pm\mathrm{acos}\sqrt{2/3} \simeq \pm 35.26°$ if we only allow bound and marginally bound massive critical particles. On the uncolored (unshaded) region, the collision with an arbitrarily high CM energy is prohibited.

C. Direct collision from infinity with nonequatorial geodesics

Bañados, Silk and West [1] originally proposed a scenario where a massive particle which is at rest at infinity, i.e. $E^2 = m^2$, with a near-critical angular momentum $L \approx L_c = 2Mm$ falls towards an extremal Kerr black hole on the equatorial plane and collides with another particle near the horizon with an arbitrarily high CM energy in the limit $L \to L_c$.

First, we only relax the restriction of the equatorial motion in their scenario and see whether the CM energy can still be arbitrarily high. In the original scenario by Bañados, Silk and West [1], it is important that the geodesic motion from infinity to the horizon is allowed. This means that the function $R(r)$ must be positive for $r_H < r < \infty$, with which we have not been concerned in Secs. IV A and IV B. As seen in Eq. (4.1) with $E^2 = m^2$, this is the case if and only if $\mathcal{Q} \le 2m^2M^2$. Then, marginally bound particles with a near-critical angular momentum $L = L_c - \delta$ for a sufficiently small $\delta(>0)$ can approach the horizon from infinity and collide with another particle near the horizon. Actually, the condition $\mathcal{Q} \le 2m^2M^2$ is identical to that for the marginally bound critical particle $E^2 = m^2$ obtained in Sec. IV B and hence we obtain

$$\sin\theta \ge \sqrt{\frac{2}{3}}.$$

Thus, we can extend the original scenario by Bañados, Silk and West [1] from the equator up to the latitude $\pm \mathrm{acos}\sqrt{2/3} \simeq \pm 35.26°$.

Moreover, we can also extend the analysis to include both marginally bound and unbound particles. Also in this case, as seen in Eq. (4.1), the geodesic motion of the critical particle from infinity to the horizon is allowed if and only if $E^2 \ge m^2$ and $(3E^2 - m^2)M^2 \ge \mathcal{Q}$. In other words, the condition obtained in Sec. IV B also applies to the direct collision from infinity for both marginally bound and unbound particles. So the upper limit on the latitude for an arbitrarily high CM energy rises up to $\pm \mathrm{acos}(\sqrt{3} - 1) \simeq \pm 42.94°$ as the energy of the particle is increased to infinity. Therefore, Fig. 3 still applies if the original scenario by Bañados, Silk and West [1] is generalized to nonequatorial motion.

We have proven that the consideration of the global behavior does not change the condition for an arbitrarily high CM energy for the marginally bound and unbound critical particles in the Kerr black hole. However, it will not necessarily be true in more general black hole spacetimes.

V. CONCLUSION AND DISCUSSION

We have presented an expression for the CM energy of two general geodesic particles around a Kerr black hole. This is the generalization of the formula obtained in the previous paper [10] of the present authors, where the analysis was restricted to two massive geodesic particles of the same rest mass moving on the equatorial plane. Applying this general expression, we have shown that an unboundedly high CM energy can be realized only in the limit to the horizon and derived a formula for the CM energy for the near-horizon collision of two general geodesic particles. Then, we have written down the necessary and sufficient condition for an unboundedly high CM energy explicitly in terms of the conserved quantities of each particle and found that this reduces to that the ratio $(E_1 - \Omega_H L_1)/(E_2 - \Omega_H L_2)$ is infinitely large or infinitely close to zero for the energy E_i and angular momentum L_i of particle i ($i = 1, 2$). Such a collision is possible at any latitude for any Kerr black hole with $0 < a \le M$ if the angular momentum is fine-tuned through multiple scattering in the vicinity of the horizon.

However, if we concentrate on the direct collision scenario and the LSO collision scenario, the black hole in the limiting case must be maximally rotating to obtain an unboundedly high CM energy. Then, we find that the collision with an unboundedly high CM energy can occur only on the belt between latitudes $\pm 35.26°$ if we only allow the bound and marginally bound critical massive particles and $\pm 42.94°$ if we allow all the possible critical particles. This also applies to the original scenario proposed by Bañados, Silk and West [1]. It is suggested that the collision with a very high CM energy might have observational consequences in the contexts of the annihilation of dark matter particles [1,19,20], the high-energy hadron collision at the inner edge of the accretion disks and the high-velocity collision of the compact objects around supermassive black holes [10]. The present result strongly suggests that if signals due to high-energy collision are to be observed, such signals can be produced primarily on the high-velocity collision belt centered at the equator of a (nearly) maximally rotating black hole but not from the polar regions.

We briefly discuss the possible limitations of our result under the test particle approximation. Because of this approximation, we have neglected the self-gravity and back reaction effects. In fact, these effects on particles orbiting a Kerr black hole have not been fully understood yet. These effects are negligible and the inspiral will be always adiabatic if the mass ratio $\eta \equiv m/M$ is sufficiently small because these effects first appear at $O(\eta)$. On the other hand, when η is small but nonzero finite, the back reaction effects due to a single high-velocity collision would considerably reduce the spin of the black hole [7]. It is also discussed that infinite collision energy is attained at the horizon after an infinite proper time and radiative effects cannot be neglected for such near-critical particles [7,8]. The effects of radiation reaction and conservative self-gravity on the ISCO and LSO of a Kerr black hole are studied [17,21–25]. Based on those studies, these effects are argued on the near-critical particles around a near-maximally rotating black hole in a different context [26]. We speculate that these effects should be responsible for bounding the CM energy of the near-horizon collision. This is supported by the fully exact analysis of a system of charged spherical shells surrounding an extremal Reissner-Nordstöm black hole [9]. It is clearly important to evaluate the upper bound in terms of the mass ratio η for the collision of particles on the equatorial plane. It will be the next step to study these effects on the collision of general particles in the present context.

Here, we discuss the possible extension of the present analysis. Since $E - \Omega_H L = -\chi^a p_a$ for the horizon-generating Killing vector $\chi^a = \xi^a + \Omega_H \psi^a$ in the Kerr spacetime, we might extend the present analysis for the Kerr spacetime to more general stationary and axisymmetric spacetimes which admit a Killing vector χ^a and a Killing horizon $\mathcal{H}$, which is defined as a null hypersurface on which the Killing vector χ^a is also null. It is clear that the present analysis applies in a straightforward manner if the analysis is restricted on the equatorial plane (e.g. [3–5]). For the general geodesic orbits, the present analysis is still applicable only if the spacetime possesses three constants of motion and the geodesic equations can be written in the first-order form. Note, however, that there is no analogue of the Carter constant for more general stationary and axisymmetric spacetimes (e.g. [27]) and the present analysis will not immediately apply to such general stationary and axisymmetric black holes.

Moreover, we may speculate that an arbitrarily high CM energy can be attained for the near-horizon collision even in the spacetime which is not stationary and axisymmetric but admits a Killing horizon $\mathcal{H}$ associated with a Killing vector χ^a. For a general geodesic particle, the quantity $\mathcal{A} = -\chi^a p_a$ is conserved. $\mathcal{A}$ must be positive in the vicinity of the horizon if χ^a is future-pointing timelike there. This is the case for the nonmaximally rotating Kerr black holes. In such a case, it is clear that the critical particle, which has $\mathcal{A} = 0$, cannot approach the horizon from outside. On the other hand, for the maximally rotating Kerr black hole, this may not apply and this is exactly what Bañados, Silk and West [1] exploit. Now, we conjecture that if and only if particles 1 and 2 collide near the Killing horizon and the ratio $\mathcal{A}_1/\mathcal{A}_2$ is infinitely large or infinitely close to zero, the CM energy of the two particles is unboundedly high possibly under some genericity condition.

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