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Constraint structure of the three dimensional massive gravity

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Constraint analysis of the three-dimensional massive gravity, the so-called new massive gravity, is studied in the Palatini formalism. We show that amongst 6 components of the metric, 2 are dynamical, which is compatible with the existence of one vector massive graviton in the linearized theory (Fierz-Pauli theory).

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I. INTRODUCTION

It is well known that theories with general covariance are constrained systems [1–4]. In other words, their equations of motion in Lagrangian formalism lead to acceleration-free relations. On the other hand, constructing the Hamiltonian formulation for such theories needs care in order to consider the constraints. The *primary* constraints emerge in the phase space whenever the momenta are not independent functions of velocities. The *secondary* constraints come out as the result of the consistency of primary constraints.

The most difficulty in Hamiltonian treatment of general covariant theories is that the action depends on the second derivatives of the metric, as well. In this situation a well-behaved Hamiltonian system is not recognized, or at least is not agreed upon, even when a system is not constrained. However, for Einstein-Hilbert gravity [5] or, for instance, Hořava gravity [6] one may use the Arnowitt-Deser-Misner variables [7] whose advantage is that the Lagrangian does not contain accelerations when written in terms of these variables. This may not happen for an arbitrary general covariant Lagrangian.

The other possibility is using the so-called Palatini formalism in which the Christoffel symbols are considered as independent variable. For Einstein-Hilbert gravity, this approach does work well. The reason is the relation between the Christoffel symbols and the derivatives of the metric results naturally from the equation of motion of Christoffel symbols. It was shown recently [8] that for a gravitation theory of the Lovelock-type, the Palatini formalism is fine. However, for an arbitrary model the equations of motion give no guaranty about the relations of Christoffel symbols and derivatives of the metric. Therefore, one needs to add them to the Lagrangian using Lagrange multipliers, which should be considered as additional variables in the Lagrangian formalism.

The three-dimensional gravity has attracted intensive interest in recent years. One reason is that it is possible to construct nontrivial renormalizable models in three dimensions. Among so many attractive features, investigating the Hamiltonian structure of the models is noticeable. The topological massive gravity (TMG) [9], which is generally covariant on a closed manifold, is one of the most important ones. The Hamiltonian structure of TMG is discussed in some papers [10].

Recently, Begshoeff, Hohm, and Townsend proposed a model [11] for three-dimensional massive gravity (the so-called new massive gravity) which preserves parity and possesses general covariance on an arbitrary manifold. Linearization around the flat metric of this model leads to a Pauli-Fierz action describing massive graviton. Then Deser [12] showed that this model is finite and ghost-free. Oda and Nakasone [13] showed afterward that the model is unitary and renormalizable. Clemènt also gave some black hole solutions of the model [14].

The Lagrangian of new massive gravity (NMG), at one hand, includes accelerations (i.e. second order derivatives of the metric), which make it necessary to use Christoffel symbols (or combinations of them) as auxiliary fields. On the other hand, the model is not of the Lovelock-type. These peculiarities make the Hamiltonian treatment and constraint structure of NMG much more difficult. Moreover, the existence of quadratic terms with respect to $R_{\mu\nu}$ makes it difficult to use the Arnowitt-Deser-Misner variables. However, in spite of complicated calculations, the Hamiltonian treatment of the theory can be followed carefully. This is what we have done in this paper.

Our main task in this work is counting the physical degrees of freedom of this model. Since the Einstein gravity in three dimensions has zero degrees of freedom, one expects roughly that the NMG action, which contains two more derivatives, should have 2 degrees of freedom; in the same way as TMG with one more derivative than Einstein-Hilbert action possesses 1 degree of freedom. This result is in agreement with the number of massive gravitons in the linearized model. However, from a theoretical point of view, it is better to check the validity of such rough arguments by a careful Hamiltonian analysis.

In Sec. II we introduce the model in the Hamiltonian formalism and find the primary constraints. In Sec. III we follow the consistency conditions of the constraints and

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find the secondary constraints of system. Section IV is devoted to our conclusions.

In our work we use Greek indices for space-time components and Latin indices for space components.

II. LAGRANGIAN AND HAMILTONIAN

The action of NMG is given as

$$S=\frac{1}{16\pi G}\int d^3x\sqrt{g}\Big[\mathrm{R}-2\Lambda+\frac{1}{m^2}\Big(\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu}-\frac{3}{8}\mathrm{R}^2\Big)\Big], \tag{2.1}$$

where g is the metric determinant, $\mathrm{R}_{\mu\nu}$ is the Ricci tensor, and R is the Ricci scalar. We assume that 3d space-time is torsion-free and the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are symmetric with respect to μ and ν. This allows us to introduce new variables $\xi^\lambda_{\mu\nu}$ via

$$\Gamma^\lambda_{\mu\nu}=\xi^\lambda_{\mu\nu}-\tfrac{1}{2}(\delta^\lambda_\mu\xi^\sigma_{\nu\sigma}+\delta^\lambda_\nu\xi^\sigma_{\mu\sigma}), \tag{2.2}$$

as in Ref. [15]. The Ricci tensor in terms of ξ variables contains derivatives in the form of total 3-divergence as

$$\mathrm{R}_{\mu\nu}=\xi^\lambda_{\mu\nu,\lambda}-\xi^\lambda_{\mu\sigma}\xi^\sigma_{\nu\lambda}+\tfrac{1}{2}\xi^\sigma_{\mu\sigma}\xi^\lambda_{\nu\lambda}. \tag{2.3}$$

This is, in fact, the advantage of using ξ's in comparison with Γ's. In this way, $\xi^0_{\mu\nu}$ are the only variables whose velocities are present in the Lagrangian. As is well known, in 3 dimensions an action containing higher order derivatives can not be of the Lovelock-type. So the Palatini approach cannot be used without imposing explicitly the relation between the metric $g_{\mu\nu}$ and auxiliary variables $\xi^\lambda_{\mu\nu}$. Using the Eq. (2.2) and the definition of Christoffel symbols as

$$\Gamma^\lambda_{\mu\nu}=\tfrac{1}{2}g^{\lambda\rho}(g_{\mu\rho,\nu}+g_{\nu\rho,\mu}-g_{\mu\nu,\rho}), \tag{2.4}$$

we have

$$\begin{aligned}\Psi_{\alpha\lambda\beta}\equiv{}& g_{\lambda\rho}(\xi^\rho_{\alpha\beta}-\tfrac{1}{2}(\delta^\rho_\alpha\xi^\sigma_{\beta\sigma}+\delta^\rho_\beta\xi^\sigma_{\alpha\sigma}))\\ &-\tfrac{1}{2}(g_{\lambda\beta,\alpha}+g_{\lambda\alpha,\beta}-g_{\alpha\beta,\lambda})=0.\end{aligned} \tag{2.5}$$

Expressions $\Psi_{\alpha\lambda\beta}$ should be considered as external Lagrangian constraints which should be put by hand in the Lagrangian with Lagrange multipliers. These Lagrange multipliers then should be taken into account as new variables in addition to $g_{\mu\nu}$ and $\xi^\lambda_{\mu\nu}$. In this way, the following action should be considered instead of the original one (2.1):

$$\begin{aligned}S={}&\int d^3x\mathcal{L}\\ ={}&\int d^3x\sqrt{g}\Big[\mathrm{R}-2\Lambda+\frac{1}{m^2}\Big(\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu}-\frac{3}{8}\mathrm{R}^2\Big)\Big]\\ &+\int d^3xA^{\alpha\lambda\beta}\Big(g_{\lambda\rho}\Big(\xi^\rho_{\alpha\beta}-\frac{1}{2}(\delta^\rho_\alpha\xi^\sigma_{\beta\sigma}+\delta^\rho_\beta\xi^\sigma_{\alpha\sigma})\Big)\\ &-\frac{1}{2}(g_{\lambda\beta,\alpha}+g_{\lambda\alpha,\beta}-g_{\alpha\beta,\lambda})\Big),\end{aligned} \tag{2.6}$$

where $A^{\alpha\lambda\beta}$ is the Lagrange multiplier. Let us enumerate different degrees of freedom before considering the details of the dynamics of the system. We have 6 degrees of freedom $g_{\mu\nu}$ and $3\times 6=18$ auxiliary variables $\xi^\lambda_{\mu\nu}$, taking into account the $\mu\leftrightarrow\nu$ symmetry in both cases. In other words, by adding the auxiliary variables, the number of Lagrangian degrees of freedom are multiplied by 4 to avoid higher order derivatives (greater than 2) in the equations of motion. We have also introduced 18 more degrees of freedom $A^{\alpha\lambda\beta}$, since Eq. (2.5) is symmetric with respect to indices α and β and we have $A^{\alpha\lambda\beta}=A^{\beta\lambda\alpha}$. Putting all these points together we have a priory $18+18+6=42$ Lagrangian variables which is equivalent to 84 phase space variables. Now we proceed to the Hamiltonian formalism. The canonical momenta conjugate to $g_{\mu\nu}$, $\xi^\lambda_{\mu\nu}$ and $A^{\alpha\lambda\beta}$ are defined, respectively, as

$$\Pi^{\mu\nu}_\gamma=\frac{\partial\mathcal{L}}{\partial\dot{\xi}^\gamma_{\mu\nu}}=\sqrt{g}\delta^0_\gamma\Big(g^{\mu\nu}+\frac{2}{m^2}G^{\mu\alpha\nu\beta}\mathrm{R}_{\alpha\beta}\Big), \tag{2.7}$$

$$\pi^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\dot{g}_{\mu\nu}}=-\frac{1}{2}(A^{0\mu\nu}+A^{0\nu\mu}-A^{\mu0\nu}), \tag{2.8}$$

$$P_{\alpha\lambda\beta}=\frac{\partial\mathcal{L}}{\partial\dot{A}^{\alpha\lambda\beta}}=0. \tag{2.9}$$

where $G^{\mu\alpha\nu\beta}$ is the generalized metric defined by

$$G^{\mu\alpha\nu\beta}=g^{\mu\alpha}g^{\nu\beta}-\tfrac{3}{8}g^{\mu\nu}g^{\alpha\beta}. \tag{2.10}$$

From Eq. (2.7), with $\gamma=0$, we have

$$\mathrm{R}_{\mu\nu}=\tfrac{1}{2}m^2\mathcal{G}_{\mu\alpha\nu\beta}(g^{-1/2}\Pi^{\alpha\beta}_0-g^{\alpha\beta}), \tag{2.11}$$

where $\mathcal{G}_{\mu\alpha\nu\beta}$ is the inverse of the generalized metric $G^{\mu\alpha\nu\beta}$ such that $\mathcal{G}_{\mu\alpha\nu\beta}G^{\mu\sigma\nu\gamma}=\delta^\sigma_\alpha\delta^\gamma_\beta$. Hence, using (2.3) we find

$$\begin{aligned}\dot{\xi}^0_{\mu\nu}={}&\tfrac{1}{2}m^2g^{-1/2}\mathcal{G}_{\mu\alpha\nu\beta}\Pi^{\alpha\beta}_0+4m^2g_{\mu\nu}-\xi^i_{\mu\nu,i}\\ &+\xi^\lambda_{\mu\sigma}\xi^\sigma_{\nu\lambda}-\tfrac{1}{2}\xi^\sigma_{\mu\sigma}\xi^\lambda_{\nu\lambda}.\end{aligned} \tag{2.12}$$

Using Eqs. (2.11) and (2.12) the canonical Hamiltonian density can be derived in the usual way as $H_C=\int d^2x\mathcal{H}_C$, where

$$\begin{aligned}\mathcal{H}_C={}&\pi^{\mu\nu}\dot{g}_{\mu\nu}+\Pi^{\mu\nu}_\gamma\dot{\xi}^\gamma_{\mu\nu}+P_{\alpha\lambda\beta}\dot{A}^{\alpha\lambda\beta}-\mathcal{L}\\ ={}&\tfrac{1}{4}m^2g^{-1/2}\mathcal{G}_{\mu\alpha\nu\beta}\Pi^{\alpha\beta}_0\Pi^{\mu\nu}_0+2\sqrt{g}(\Lambda-3m^2)\\ &+\tfrac{1}{2}(A^{i\mu\nu}+A^{i\nu\mu}-A^{\mu i\nu})g_{\mu\nu,i}\\ &-\Pi^{\mu\nu}_0[\xi^i_{\mu\nu,i}-\xi^\lambda_{\mu\sigma}\xi^\sigma_{\nu\lambda}+\tfrac{1}{2}\xi^\sigma_{\mu\sigma}\xi^\lambda_{\nu\lambda}]\\ &-A^{\alpha\lambda\beta}g_{\lambda\rho}(\xi^\rho_{\alpha\beta}-\tfrac{1}{2}(\delta^\rho_\alpha\xi^\sigma_{\beta\sigma}+\delta^\rho_\beta\xi^\sigma_{\alpha\sigma})).\end{aligned} \tag{2.13}$$

In deriving the canonical Hamiltonian the following primary constraints resulted from Eqs. (2.7), (2.8), and (2.9) are imposed:

$$\phi^{\mu\nu} := \pi^{\mu\nu} + \tfrac{1}{2}(A^{0\mu\nu} + A^{0\nu\mu} - A^{\mu 0\nu}) \approx 0,$$
$$\Phi_i^{\mu\nu} := \Pi_i^{\mu\nu} \approx 0, \qquad \Omega_{\mu\lambda\nu} := P_{\mu\lambda\nu} \approx 0, \tag{2.14}$$

where the symbol "≈" means weak equality, i.e., equality on the constraint surface. We recall that primary constraints are identities amongst coordinates and momenta which follow directly from the definition of canonical momenta. The total Hamiltonian reads

$$H_T = \int d^3x \mathcal{H}_T,$$
$$\mathcal{H}_T = \mathcal{H}_C + U_{\mu\nu}\phi^{\mu\nu} + \lambda^i_{\mu\nu}\Phi_i^{\mu\nu} + V^{\mu\alpha\nu}\Omega_{\mu\alpha\nu}, \tag{2.15}$$

where $U_{\mu\nu}$, $\lambda^i_{\mu\nu}$, and $V^{\mu\alpha\nu}$ are Lagrange multipliers (in the context of Hamiltonian constrained systems) corresponding to the primary constraints (2.14), respectively. The fundamental Poisson brackets of field variables are

$$\{g_{\mu\nu}(x), \pi^{\alpha\beta}(y)\} = \Delta^{\alpha\beta}_{\mu\nu}\delta^{(3)}(x-y),$$
$$\{\xi^\lambda_{\mu\nu}(x), \Pi^{\alpha\beta}_\gamma(y)\} = \delta^\lambda_\gamma\Delta^{\alpha\beta}_{\mu\nu}\delta^{(3)}(x-y), \tag{2.16}$$
$$\{A^{\alpha\lambda\beta}(x), P_{\mu\gamma\nu}(y)\} = \delta^\lambda_\gamma\Delta^{\alpha\beta}_{\mu\nu}\delta^{(3)}(x-y),$$

where $\Delta^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu)$.

III. CONSTRAINT DYNAMICS AND COUNTING PHYSICAL DEGREES OF FREEDOM

The number of primary constraints as well as their corresponding Lagrange multipliers are as follows:

$$\#\phi^{\mu\nu} \text{ corresponding to } U_{\mu\nu} = 6,$$
$$\#\Phi_i^{\mu\nu} \text{ corresponding to } \lambda^i_{\mu\nu} = 2\times 6 = 12,$$
$$\#\Omega_{\mu\lambda\nu} \text{ corresponding to } V^{\mu\lambda\nu} = 3\times 6 = 18,$$
$$\# \text{ total primary constraints } = 36.$$

As in all constrained systems, the primary constraints should be valid in the course of time. This means that their Poisson brackets with the total Hamiltonian, which is responsible for the dynamics of the system, should vanish. This process is the so-called "consistency of the constraints." It is important to remind the reader that at each step of consistency, two main things may happen. If a given constraint has a nonvanishing Poisson bracket with some primary constraints, the corresponding Lagrange multiplier would be determined in terms of phase space variables. This is the case when the related constraint is second class.[1] The other possibility is that a new constraint emerges as the consistency of the given constraint and corresponding Lagrange multiplier is not determined. These constraints at different levels of consistency are called second level, third level, and so forth, which altogether are remembered as secondary constraints. The process of consistency will continue up to the last level in which either a Lagrange multiplier is determined (when we have a chain of second class constraints) or the consistency is established identically (when the constraints in the corresponding chain are first class). Now, we follow the consistency procedure for our problem.

Consistency of $\Phi_i^{\mu\nu}$'s causes the following expressions to vanish:

$$\chi_i^{\mu\nu} \equiv \{\Phi_i^{\mu\nu}, H_T\}$$
$$= -\tfrac{1}{2}(\partial_i\Pi_0^{\mu\nu} - \tfrac{1}{2}\Pi_0^{\lambda\mu}\xi^\sigma_{\sigma\lambda}\delta^\nu_i - \tfrac{1}{2}(g_{\lambda i}A^{\mu\lambda\nu}$$
$$- g_{\lambda\sigma}A^{\sigma\lambda\mu}\delta^\nu_i) + 2\Pi_0^{\lambda\mu}\xi^\nu_{\lambda i}) + \mu\longleftrightarrow\nu. \tag{3.1}$$

Since $\Phi_i^{\mu\nu}$ have vanishing Poisson brackets with all primary constraints, no term containing Lagrange multipliers has appeared in Eq. (3.1). Therefore, consistency of 12 primary constraints $\Phi_i^{\mu\nu}$ gives 12 second level constraints $\chi_i^{\mu\nu}$. The consistency of $\chi_i^{\mu\nu}$ will be investigated afterward.

For $\Omega_{\mu\lambda\nu}$ we have

$$\{\Omega_{\mu\lambda\nu}, H_T\} = g_{\lambda\rho}(\xi^\rho_{\mu\nu} - \tfrac{1}{2}(\xi^\sigma_{\mu\sigma}\delta^\rho_\nu + \xi^\sigma_{\nu\sigma}\delta^\rho_\mu))$$
$$- \tfrac{1}{2}(g_{\lambda\mu,i}\delta^i_\nu + g_{\lambda\nu,i}\delta^i_\mu - g_{\mu\nu,i}\delta^i_\lambda)$$
$$- \tfrac{1}{2}(U_{\lambda\mu}\delta^0_\nu + U_{\lambda\nu}\delta^0_\mu - U_{\mu\nu}\delta^0_\lambda). \tag{3.2}$$

The last term above includes Lagrange multipliers $U_{\mu\nu}$. For $\mu = i$, $\nu = j$, and $\lambda = k$ this term vanishes and the consistency of constraints Ω_{ikj} lead to the following second level constraints:

$$\Theta_{ikj} = g_{k\rho}(\xi^\rho_{ij} - \tfrac{1}{2}(\xi^\sigma_{i\sigma}\delta^\rho_j + \xi^\sigma_{j\sigma}\delta^\rho_i)) - \tfrac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k}). \tag{3.3}$$

The constraints Θ_{ikj} are in fact the same as Ψ_{ikj} given in Eq. (2.5). We should investigate the consistency of Θ_{ikj}'s in the next level of consistency. Let us come back to Eq. (3.2). The Lagrange multipliers $U_{\alpha\beta}$ have appeared in the last term due to Poisson brackets $\{\Omega_{\mu\lambda\nu}, \phi^{\alpha\beta}\}$ with one of indices μ or λ or ν considered as zero. This is a (12×6) rectangular matrix of rank 6. So it is possible to divide $\Omega_{\mu\lambda\nu}$'s with at least one zero index into two 6-member sets, as follows:

[1]Remember that a set of constraints are second class if the matrix of their mutual Poisson brackets is nonsingular. On the other hand, first class constraints have vanishing Poisson brackets with all of the constraints (at least on the constraint surface).

$$\Omega^{(1)} = \begin{cases} B_1 \equiv \Omega_{001} \\ B_2 \equiv \Omega_{002} \\ B_{11} \equiv \frac{1}{2}\Omega_{011} + \Omega_{101} \\ B_{22} \equiv \frac{1}{2}\Omega_{022} + \Omega_{202} \\ B_{12} \equiv \frac{1}{2}(\Omega_{012} + \Omega_{021}) + \Omega_{102} \\ B'_{12} = \Omega_{012} - \Omega_{021} \end{cases}, \qquad \Omega^{(2)} = \begin{cases} C_0 \equiv \Omega_{000} \\ C_1 \equiv \Omega_{010} \\ C_2 \equiv \Omega_{020} \\ C_{11} \equiv \Omega_{011} - \frac{1}{2}\Omega_{101} \\ C_{22} \equiv \Omega_{022} - \frac{1}{2}\Omega_{202} \\ C_{12} \equiv \frac{1}{2}(\Omega_{021} - \Omega_{012} - 2\Omega_{102}) \end{cases}. \tag{3.4}$$

The constraints of the set $\Omega^{(1)}$ commute (i.e. has vanishing Poisson brackets) with $\phi_{\alpha\beta}$'s and the other set, $\Omega^{(2)}$, constitutes a second class system with $\phi^{\alpha\beta}$'s, so that the 6×6 matrix $\{\Omega^{(2)}, \phi\}$ is nonsingular. We can redefine the Lagrange multipliers $V_{\mu\lambda\nu}$ corresponding to the division of the constraints $\Omega^{\mu\lambda\nu}$ into the sets Ω^{ikj}, $\Omega^{(1)}$. and $\Omega^{(2)}$, such that

$$\sum V_{\mu\lambda\nu}\Omega^{\mu\lambda\nu} = \sum V_{ikj}\Omega^{ikj} + \sum V_{(1)}\Omega^{(1)} + \sum V_{(2)}\Omega^{(2)}. \tag{3.5}$$

This gives

$$V_{(1)} = \begin{cases} V^1_{(1)} \equiv V^{001} \\ V^2_{(1)} \equiv V^{002} \\ V^{11}_{(1)} \equiv \frac{2}{5}(2V^{101} + V^{011}) \\ V^{22}_{(1)} \equiv \frac{2}{5}(2V^{202} + V^{022}) \\ V^{12}_{(1)} \equiv V^{012} + V^{021} \\ V'^{12}_{(1)} = V^{012} - \frac{1}{2}V^{102} \end{cases}, \qquad V_{(2)} = \begin{cases} V^0_{(2)} \equiv V^{000} \\ V^1_{(2)} \equiv V^{010} \\ V^2_{(2)} = V^{020} \\ V^{11}_{(2)} \equiv \frac{2}{5}(2V^{011} - V^{101}) \\ V^{22}_{(2)} \equiv \frac{2}{5}(2V^{022} - V^{202}) \\ V^{12}_{(2)} \equiv V^{012} + V^{021} - V^{102} \end{cases}. \tag{3.6}$$

If we consider at this point the consistency of $\phi^{\mu\nu}$'s, we get

$$\{\phi^{\mu\nu}, H_T\} = -\frac{1}{4}m^2 g_{\alpha\beta} g^{-1/2}(\Pi_0^{\alpha\mu}\Pi_0^{\beta\nu} - 3\Pi_0^{\alpha\beta}\Pi_0^{\mu\nu}) + \frac{1}{16}m^2 g^{-1/2} g^{\mu\nu} \mathcal{G}_{\alpha\sigma\beta\lambda}\Pi_0^{\alpha\beta}\Pi_0^{\sigma\lambda} - 2m^2\Pi_0^{\mu\nu} - \sqrt{g}g^{\mu\nu}(\Lambda - 3m^2) + \frac{1}{2}A^{\alpha\mu\beta}(\xi^{\nu}_{\alpha\beta} - \frac{1}{2}(\xi^{\sigma}_{\alpha\sigma}\delta^{\nu}_{\beta} + \xi^{\sigma}_{\beta\sigma}\delta^{\nu}_{\alpha})) - \frac{1}{2}(A^{i\mu\nu} - \frac{1}{2}A^{\mu i\nu})_{,i} + \frac{1}{2}(V^{0\mu\nu} - \frac{1}{2}V^{\mu 0\nu}) + \mu \longleftrightarrow \nu. \tag{3.7}$$

It can be seen easily that the last term in Eq. (3.7) contains Lagrange multipliers $V_{(2)}$ corresponding to the constraints in the set $\Omega^{(2)}$ which have nonvanishing Poisson brackets with $\phi^{\mu\nu}$'s. In this way the set of constraints

$$\phi \leftrightarrow \Omega^{(2)} \tag{3.8}$$

constitute a one-level 12-member family of second class constraints which is constructed of two cross-conjugate chains that determine 12 Lagrange multipliers $U_{\alpha\beta}$ and $V_{(2)}$. In this description, we have used the language of Ref. [16] in classifying the families of constraints. We just recall here that a family of constraints are determined as a set of constraints which results from the consistency of a limited subset of primary constraints and make a close algebra of Poisson brackets among themselves, and with the canonical Hamiltonian. Consistency of the set $\Omega^{(1)}$ gives 6 other constraints of the next level as follows:

$$\Psi^{(1)} = \begin{cases} D_{11} \equiv \frac{1}{2}\Psi_{011} + \Psi_{101} \\ D_{22} \equiv \frac{1}{2}\Psi_{022} + \Psi_{202} \\ D_{12} \equiv \frac{1}{2}(\Psi_{012} + \Psi_{021}) + \Psi_{102} \end{cases}, \tag{3.9}$$

$$\Psi^{(2)} = \begin{cases} D_j \equiv \Psi_{0j0} j = 1, 2 \quad j = 1, 2 \\ D'_{12} = \Psi_{012} - \Psi_{021} \end{cases},$$

where $\Psi_{\alpha\lambda\beta}$ are Lagrangian constraints given in Eq. (2.5). We should continue to investigate consistency of the above constraints in the next level. This will make the meaning of the classification given in Eq. (3.9) more clear. As can be seen, second level constraints Θ_{ikj}, $\Psi^{(1)}$, and $\Psi^{(2)}$ are 12 out of 18 Lagrangian constraints (2.5). The remaining 6 Lagrangian constraints correspond to expressions (2.5) including Lagrange multipliers $U_{\mu\nu}$. In fact, the equations of motion of $g_{\mu\nu}$ give $\dot{g}_{\mu\nu} = U_{\mu\nu}$. Putting this into Eq. (3.2) gives the corresponding Lagrangian constraints (2.5) for the cases which include time derivatives of the metric.

Now we proceed to the next level by considering the consistency of $\chi_i^{\mu\nu}$'s, Θ_{ikj}'s, and the sets $\Psi^{(1)}$ and $\Psi^{(2)}$. For Θ_{ikj}'s we find

$$\begin{aligned}\{\Theta_{ikj},H_T\}={}&\frac{m^2}{4}[g_{k0}(\mathcal{G}_{\alpha i\beta j}+\mathcal{G}_{\alpha j\beta i})-\frac{1}{2}(g_{ki}(\mathcal{G}_{\alpha j\beta 0}+\mathcal{G}_{\alpha 0\beta j})+g_{kj}(\mathcal{G}_{\alpha i\beta 0}+\mathcal{G}_{\alpha 0\beta i}))]\Pi_0^{\alpha\beta}-g_{k0}\Big(\xi^l_{ij,l}-\xi^\lambda_{i\sigma}\xi^\sigma_{j\lambda}+\frac{1}{2}\xi^\lambda_{i\lambda}\xi^\sigma_{j\sigma}\Big)\\&+\frac{1}{2}\Big(g_{ki}\Big(\xi^l_{j0,l}-\xi^\lambda_{j\sigma}\xi^\sigma_{0\lambda}+\frac{1}{2}\xi^\lambda_{j\lambda}\xi^\sigma_{0\sigma}\Big)+g_{kj}\Big(\xi^l_{i0,l}-\xi^\lambda_{i\sigma}\xi^\sigma_{0\lambda}+\frac{1}{2}\xi^\lambda_{i\lambda}\xi^\sigma_{0\sigma}\Big)\Big)+4m^2\Big(g_{k0}g_{ij}-\frac{1}{2}(g_{ki}g_{0j}+g_{kj}g_{0i})\Big)\\&+\Big(\xi^\lambda_{ij}-\frac{1}{2}(\xi^\sigma_{j\sigma}\delta^\lambda_i+\xi^\sigma_{i\sigma}\delta^\lambda_j)\Big)U_{k\lambda}-\frac{1}{2}(U_{kj,i}+U_{ki,j}-U_{ij,k})+g_{kl}\lambda^l_{ij}-\frac{1}{2}(g_{ki}\lambda^l_{jl}+g_{kj}\lambda^l_{il}).\end{aligned}\tag{3.10}$$

Since $U_{\mu\nu}$ are determined previously, the Eq. (3.10) should be considered as equations to find λ^l_{mn}. It is easy to check that the matrix of coefficients of λ^l_{mn}'s, i.e., $\{\Theta_{ikj},\Phi_l^{mn}\}$, is nonsingular and these Lagrange multipliers can be determined completely. Consistency of constraints in the set $\Psi^{(2)}$ gives

$$\begin{aligned}\{D_j,H_T\}={}&\tfrac{1}{8}m^2g^{-1/2}(g_{00}(\mathcal{G}_{0\mu j\nu}+\mathcal{G}_{j\mu 0\nu})-2g_{0j}\mathcal{G}_{0\mu 0\nu})\Pi_0^{\mu\nu}-\tfrac{1}{2}(g_{00}(\xi^i_{0j,i}-\xi^\lambda_{0\sigma}\xi^\sigma_{j\lambda}+\tfrac{1}{2}\xi^\sigma_{0\sigma}\xi^\lambda_{j\lambda})\\&-2g_{0j}(\xi^i_{00,i}-\xi^\lambda_{0\sigma}\xi^\sigma_{0\lambda}+\tfrac{1}{2}\xi^\sigma_{0\sigma}\xi^\lambda_{0\lambda}))-\tfrac{1}{2}(g_{00}\lambda^i_{ji}+g_{0j}\lambda^i_{0i}-2g_{0i}\lambda^i_{0j}),\quad j=1,2,\end{aligned}\tag{3.11}$$

$$\begin{aligned}\{D'_{12},H_T\}={}&\tfrac{1}{8}m^2g^{-1/2}g_{10}(\mathcal{G}_{0\mu 2\nu}+\mathcal{G}_{2\mu 0\nu})\Pi_0^{\mu\nu}-\tfrac{1}{2}g_{10}(\xi^i_{02,i}-\xi^\lambda_{0\sigma}\xi^\sigma_{2\lambda}+\tfrac{1}{2}\xi^\sigma_{0\sigma}\xi^\lambda_{2\lambda})\\&-\tfrac{1}{2}(g_{10}\lambda^i_{2i}+g_{20}\lambda^i_{1i}-2g_{1i}\lambda^i_{02})-1\longleftrightarrow 2.\end{aligned}\tag{3.12}$$

Since Lagrange multipliers λ^i_{jk} are already determined, Eqs. (3.11) and (3.12) are three equations for seven unknowns $V^1_{(1)}$, $V^2_{(1)}$, $V'^{12}_{(1)}$, and λ^i_{j0}'s. Therefore, we should keep these equations in mind and wait to find four other equations which should be solved together with (3.11) and (3.12) to find the above unknowns. Anyhow, the consistency of D_1, D_2, and D'_{12} does not go further. Hence, we have a 6 member family of second class constraints as

$$\begin{array}{ccc}B_1 & B_2 & B'_{12}\\ \downarrow & \downarrow & \downarrow\\ D_1 & D_2 & D'_{12}\end{array}.\tag{3.13}$$

Consistency of $\chi_i^{\mu\nu}$'s leads the following expression to vanish:

$$\begin{aligned}\{\chi_i^{\mu\nu},H_T\}={}&(\Pi_0^{\lambda\mu}(\xi^\nu_{\lambda 0}-\xi^\sigma_{\lambda\sigma}\delta^\nu_0))_{,i}-\tfrac{1}{2}(g_{\lambda 0}A^{\mu\lambda\nu}-g_{\lambda\sigma}A^{\sigma\lambda\mu}\delta^\nu_0)_{,i}-\tfrac{1}{4}m^2g^{-1/2}((\mathcal{G}_{\alpha\lambda\beta i}+\mathcal{G}_{\alpha i\beta\lambda})\delta^\nu_0\\&-\tfrac{1}{2}(\mathcal{G}_{\alpha\lambda\beta 0}+\mathcal{G}_{\alpha 0\beta\lambda})\delta^\nu_i)\Pi_0^{\lambda\mu}\Pi_0^{\alpha\beta}-4m^2(g_{\lambda i}\delta^\nu_0-\tfrac{1}{2}g_{\lambda 0}\delta^\nu_i)\Pi_0^{\lambda\mu}+(\Pi_0^{\sigma\lambda}\xi^\nu_{\sigma 0}+\Pi_0^{\sigma\nu}\xi^\lambda_{\sigma 0})\xi^\mu_{\lambda i}\\&-\tfrac{1}{2}(\Pi_0^{\alpha\beta}\xi^\nu_{\beta i}\delta^\mu_0+\Pi_0^{\alpha\mu}\xi^\nu_{0i})\xi^\rho_{\alpha\rho}-\tfrac{1}{2}(\Pi_0^{\alpha\beta}\xi^\mu_{\alpha 0}+\Pi_0^{\alpha\mu}\xi^\beta_{\alpha 0})\xi^\rho_{\beta\rho}\delta^\nu_i+(\xi^j_{\alpha i,j}-\xi^\rho_{\alpha\sigma}\xi^\sigma_{i\rho}+\tfrac{1}{2}\xi^\rho_{\alpha\rho}\xi^\sigma_{i\sigma})\Pi_0^{\alpha\mu}\delta^\nu_0\\&-\tfrac{1}{4}(\Pi_0^{\sigma\alpha}\xi^\lambda_{\alpha\lambda}\delta^\mu_0+\Pi_0^{\sigma\mu}\xi^\lambda_{0\lambda})\xi^\rho_{\sigma\rho}\delta^\nu_i-\tfrac{1}{2}(\xi^j_{\lambda 0,j}-\xi^\rho_{\lambda\sigma}\xi^\sigma_{0\rho}+\tfrac{1}{2}\xi^\rho_{\lambda\rho}\xi^\sigma_{0\sigma})\Pi_0^{\lambda\mu}\delta^\nu_i-g_{\rho 0}(\xi^\nu_{\lambda i}-\tfrac{1}{2}\xi^\sigma_{\lambda\sigma}\delta^\nu_i)A^{\lambda\rho\mu}\\&+g_{\rho\alpha}(A^{\alpha\rho\lambda}\xi^\nu_{\lambda i}\delta^\mu_0+A^{\alpha\rho\mu}\xi^\nu_{0i}-\tfrac{1}{2}(A^{\alpha\rho\lambda}\xi^\sigma_{\lambda\sigma}\delta^\mu_0+A^{\alpha\rho\mu}\xi^\sigma_{0\sigma})\delta^\nu_i)-\Pi_0^{\sigma\mu}(\lambda^j_{\sigma i}\delta^\nu_j-\tfrac{1}{2}\lambda^j_{\sigma j}\delta^\nu_i)\\&+\tfrac{1}{2}(A^{\mu\sigma\nu}U_{i\sigma}-A^{\rho\sigma\mu}U_{\sigma\rho}\delta^\nu_i)+\tfrac{1}{2}(g_{\sigma i}V^{\mu\sigma\nu}-g_{\sigma\rho}V^{\rho\sigma\mu}\delta^\nu_i)+\mu\leftrightarrow\nu.\end{aligned}\tag{3.14}$$

Let us first consider 6 equations concerning the cases $\mu=j$ and $\nu=k$ in Eq. (3.14). It can be seen that the 6×6 matrix of the coefficients of Lagrange multipliers V^{ikj} is nonsingular. Moreover, the corresponding equations do not include the yet undetermined Lagrange multipliers λ^i_{0j} and λ^i_{00}. They include, however, the Lagrange multipliers $U_{\mu\nu}$, $V_{(2)}$, and λ^i_{lm} which are determined previously. Therefore, the Eq. (3.14) for the cases considered can be used to determine 6 Lagrange multiplier V^{ikj}. Nonsingularity of the (6×6) submatrix $\{\chi_i^{jk},\Omega_{lmn}\}$ has an interesting meaning in the terminology of ref. [16] on classifying the constraint families. To this end, the set of constraints

$$\begin{array}{ccc}\Omega_{ikj} & \nwarrow\!\nearrow & \Phi_i^{jk}\\ \Theta_{ikj} & \nearrow\!\searrow & \chi_i^{jk},\end{array}\tag{3.15}$$

constitute a family of a 24-member, 2-level and cross-conjugate second class system in which the consistency of constraints of the first row gives the constraints of the second row, while the constraints at the end of any chain have nonvanishing Poisson brackets with the constraints at the top of the other chain. We can check that $\{\Omega_{ikj}, \Theta_{mln}\}$ as well as $\{\Phi_i^{jk}, \chi_l^{mn}\}$ vanish.

Let us come back to Eq. (3.14) and consider the case $(\mu = i, \nu = 0)$ or $(\mu = 0, \nu = i)$. We have four equations in this case again for seven unknowns $V^1_{(1)}$, $V^2_{(1)}$, $V'^{12}_{(1)}$, and λ^i_{j0}'s. These equations are in fact, 4 equations which we were expecting, after Eq. (3.12). Hence, we have 7 independent equations for 7 unknowns. In this way the constraints χ_i^{0j} and their parents Φ_i^{0j} constitute an 8-member, 2-level family of second class constraints shown as

$$\begin{array}{c} \Phi_i^{0j} \\ \downarrow \\ \chi_i^{0j} \end{array} . \tag{3.16}$$

The only remaining case in Eq. (3.14) is $\mu = \nu = 0$. No term containing λ^i_{00} appears in Eq. (3.14). This corresponds to two constraints χ_i^{00} for which the term including Lagrange multipliers λ_i^{00} vanishes. Consistency of χ_i^{00} leads to third level constraints Σ_i^{00} as

$$\begin{aligned} \Sigma_i^{00} = {} & -2(\Pi_0^{\lambda 0}\xi^j_{\lambda j})_{,i} - ((g_{\lambda 0}A^{0\lambda 0} - g_{\lambda\sigma}A^{\sigma\lambda 0}))_{,i} - 8m^2 g_{\lambda i}\Pi_0^{\lambda 0} + 2(\Pi_0^{\sigma\lambda}\xi^0_{\sigma 0} + \Pi_0^{\sigma 0}\xi^\lambda_{\sigma 0})\xi^0_{\lambda i} \\ & - \frac{1}{2}m^2 g^{-1/2}(\mathcal{G}_{\alpha\lambda\beta i} + \mathcal{G}_{\alpha i\beta\lambda})\Pi_0^{\lambda 0}\Pi_0^{\alpha\beta} + 2(\xi^j_{\alpha i,j} - \xi^\rho_{\alpha\sigma}\xi^\sigma_{i\rho} + \tfrac{1}{2}\xi^\rho_{\alpha\rho}\xi^\sigma_{i\sigma})\Pi_0^{\alpha 0} - (\Pi_0^{\alpha\beta}\xi^0_{\beta i} + \Pi_0^{\alpha 0}\xi^0_{0i})\xi^\rho_{\alpha\rho} \\ & - 2(g_{\rho 0}\xi^0_{\lambda i}A^{\lambda\rho 0} - g_{\rho\alpha}(A^{\alpha\rho\lambda}\xi^0_{\lambda i} + A^{\alpha\rho 0}\xi^0_{0i})) + A^{0\lambda 0}U_{i\lambda} + g_{\sigma i}V^{0\sigma 0}. \end{aligned} \tag{3.17}$$

Direct calculation shows that $\{\Sigma_i^{00}, \Phi_j^{00}\}$ is a nonsingular matrix. Therefore, consistency of Σ_i^{00} determines two Lagrange multipliers λ_i^{00} and shows that the constraints Σ_i^{00} as well as their parents in the corresponding chain are second class. In this way we have derived a 6-member family of second class constraints gathered in three-level chains as

$$\begin{array}{c} \Phi_i^{00} \\ \downarrow \\ \chi_i^{00} \\ \downarrow \\ \Sigma_i^{00} \end{array} . \tag{3.18}$$

Hence, 12 second class constraints in families (3.8) determine 12 Lagrange multipliers $U_{\alpha\beta}$ and $V_{(2)}$ at first level of consistency; 38 second class constraints in families (3.15), (3.16), and (3.18), determine 19 Lagrange multipliers λ_i^{jk}, λ_i^{0j}, V^{ikj}, $V^1_{(1)}$, $V^2_{(1)}$, and $V'^{12}_{(1)}$ at second level of consistency; and 6 second class constraints in family (3.18) determine 2 Lagrange multipliers λ_i^{00} at the third level of consistency. We have a total of $S = 12 + 38 + 6 = 56$ second class constraints which determine $12 + 19 + 2 = 33$ Lagrange multipliers. There remain $36 - 33 = 3$ undetermined Lagrange multipliers which are $V^{11}_{(1)}$, $V^{22}_{(1)}$, and $V^{12}_{(1)}$ corresponding to primary constraints B_{11}, B_{22}, and B_{12} given in (3.4).

Let us recall that consistency of primary constraints B_{11}, B_{22}, and B_{12} gave us the second level constraints D_{11}, D_{22}, and D_{12} as in (3.9). Straightforward calculations show that the Poisson brackets of D_{11}, D_{22}, and D_{12} with the total Hamiltonian vanishes. Therefore, we collect the following constraints:

$$\begin{array}{ccc} B_{11} & B_{22} & B_{12} \\ \downarrow & \downarrow & \downarrow \\ D_{11} & D_{22} & D_{12} \end{array} \tag{3.19}$$

as a 6-member, 2-level and first class family of constraints. So the number of first class constraints in the form of family (3.19) is $F = 6$.

For the number of dynamical variables, using the famous formula [17]

$$D = N - S - 2F, \tag{3.20}$$

where N is the number of initially introduced variables in phase space, we have $D = 84 - 56 - 2 \times 6 = 16$. This is in Hamiltonian formalism. In Lagrangian formalism, we have half of this number as dynamical variables. By dynamical variables, we mean those variables which obey differential equations containing accelerations. Taking a look on the complete action (2.6) shows that the auxiliary variables $A^{\alpha\lambda\beta}$ are not within these variables. Therefore, Eq. (3.20) says that after eliminating the redundant variables by using the constraints and gauge fixing conditions, we have 8 dynamical equations for $g_{\mu\nu}$'s and $\xi^\lambda_{\mu\nu}$'s. As we mentioned before, the total number of variables $g_{\mu\nu}$ and $\xi^\lambda_{\mu\nu}$ is 4 times greater than the number of the principle variable $g_{\mu\nu}$. Therefore, we conclude that the number of dynamical variables is $8/4 = 2$ out of six $g_{\mu\nu}$.

Notice that throughout our calculations concerning the constraint structure of the NMG model we have kept the cosmological constant term up to end. Hence, it is easy to find the constraint structure of the model without the cosmological model just by putting $\Lambda = 0$. It should be noted that although the main structure of the constraints and the number of degrees of freedom is the same, the cosmological constant has a serious effect on the form of constraints [see Eq. (3.7)] as well as the final Hamiltonian, after imposing the derived forms of Lagrange multipliers in the total Hamiltonian (2.13). Therefore, it can be expected that the dynamics of the remaining physical degrees of freedom in the reduced phase space is affected deeply by the presence of the cosmological constant. Specially, particular solutions of the equations of motion appear when $\Lambda \neq 0$ which are forbidden in the absence of cosmological constant. However, our purpose in this paper is not investigating the properties of particular solutions, which stand beyond studying the constraint structure of the system, although it could be interesting in turn.

IV. CONCLUDING REMARKS

In this paper we studied the Hamiltonian structure of the new massive gravity model. This is a complicated model in 3d gravity that contains higher order derivatives as well as higher than quadratic terms. We used some form of the Palatini formalism in which combinations of Christoffel symbols, i.e., the variables $\xi^{\lambda}_{\mu\nu}$, are used as independent variables while their relation with derivatives of the metric is imposed as additional conditions in the Lagrangian using the auxiliary variables $A^{\alpha\lambda\beta}$. As is expected, the system is highly constrained with 36 primary and 26 secondary constraints, where 56 of them are second class and 6 are first class. This classification makes the system suitable to be studied more carefully in the context of constrained systems. For example, one may be interested in finding the generator of gauge transformations in terms of first class constraints and studying more carefully the gauge symmetries of the system. Moreover, if one decides to fix the gauges, one needs to know carefully which gauge fixing conditions should be imposed. As is well known [16,18], for this proposes it is necessary to know the constraint structure of the system. We showed, finally, in phase space that there remain 16 physical variables. This means that there are 8 dynamical Lagrangian variables composed of the metric and Christoffel symbols. If one eliminates the Christoffel symbols, there remain 2 dynamical degrees of freedom out of 6 components of the metric. This conclusion is in agreement with the result that the NMG model constitute 2 gravitons under linearization of the equations of motion.

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Note added.—When completing our paper we observed Ref. [19] in which the total number of 2 degrees of freedom is derived in another approach.

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