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Type *D* solutions of 3D new massive gravity

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In a recent reformulation of three-dimensional new massive gravity, the field equations of the theory consist of a massive (tensorial) Klein-Gordon type equation with a curvature-squared source term and a constraint equation. Using this framework, we present *all algebraic type D solutions* of new massive gravity with constant and nonconstant scalar curvatures. For constant scalar curvature, they include homogeneous anisotropic solutions which encompass both solutions originating from topologically massive gravity, Bianchi types *II*, *VIII*, *IX*, and those of non-topologically massive gravity origin, Bianchi types VI_0 and VII_0. For a special relation between the cosmological and mass parameters, $\lambda = m^2$, they also include conformally flat solutions, and, in particular, those being locally isometric to the previously-known Kaluza-Klein type $\mathrm{AdS}_2 \times S^1$ or $\mathrm{dS}_2 \times S^1$ solutions. For nonconstant scalar curvature, all the solutions are conformally flat and exist only for $\lambda = m^2$. We find two general metrics which possess at least one Killing vector and comprise all such solutions. We also discuss some properties of these solutions, delineating among them black hole type solutions.

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I. INTRODUCTION

General relativity in three dimensions suffers from a number of undesirable properties: The usual Newtonian limit does not exist, there are no local dynamical degrees of freedom and no black hole solutions without a negative cosmological constant [1,2] (see also [3] and references therein). With a negative cosmological constant, the theory admits black holes with an asymptotically anti-de Sitter (AdS_3) behavior, which are known as Bañados-Teitelboim-Zanelli (BTZ) black holes [4]. The BTZ black holes have played a profound role in understanding the classical and quantum structures of the "simple" general relativity, albeit its final quantum status continues to remain unclear [5]. Onpar with this, intriguing developments have also been towards giving the theory dynamical degrees of freedom by adding higher-derivative terms to the Einstein-Hilbert (EH) action. Such a procedure generically introduces a topological mass parameter in the theory and entails the appearance of propagating modes with negative energy, known as *ghosts*. However, there are some fortunate exceptions in which cases the ghosts can be avoided. Among such exceptions, topologically massive gravity (TMG) occupies a central place [6,7]. This theory is obtained by adding to the EH action a gravitational Chern-Simons term which is third-order in a derivative expansion. Nevertheless, the resulting wave equation represents a single massive spin 2 mode of helicity either $+2$ or -2, depending on the sign of the topological mass parameter. This mode has positive energy only for the opposite sign (with respect to the conventional one in four dimensions) of the EH term in the total action. Thus, TMG is a unitary theory only for the "wrong" sign of the EH term and it is a parity-violating theory by its very nature [6].

Recently, it was shown that there is a special case of TMG, determined by a critical value of the topological mass parameter, for which one can still allow the usual "right" sign of the EH term [8]. In this case, the massive gravitons disappear and the theory admits BTZ black holes with positive mass. This in turn resolves the unitarity conflict between the bulk and boundary theories in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. That is, the theory of TMG in its special case becomes dual to a conformal field theory (CFT_2) on the boundary (see Refs. [8,9] for details). This result has also renewed the interest in massive gravity theories in three dimensions on their own right and, as another fortunate exception, a new ghost-free theory of massive gravity appeared [10]. This theory is known as *new massive gravity* (NMG). It is a parity-preserving theory and is obtained by adding to the EH action a particular quadratic-curvature term. Though such an action gives rise to the field equations of fourth-order in derivatives, its canonical structure involves two propagating massive modes of helicities ± 2. Moreover, the equivalence of this action, when expanding about the Minkowski vacuum, to the Fierz-Pauli action confirms unitary property of the theory. Subsequently, this property has also been confirmed in [11–13].

Further developments have been towards searching for exact solutions to the theory of NMG and studying its holographic properties. Shortly after the advent of NMG, it was found that the theory admits regular warped AdS_3 black holes which are counterparts of those appearing in TMG [14]. Continuing the line of analogy with TMG solutions [15–17], the other constant scalar curvature solutions of NMG, such as AdS_3-waves [18] and Bianchi type IX homogeneous space solution [19], were also found. A new class of solutions with nonconstant scalar curvature, which includes the black hole, gravitational soliton and kink type solutions, was discussed in [20,21]. Other studies of the issue of exact solutions to NMG can be found in [22–26]. The holographic properties of NMG exhibit some similarities with those of TMG: The bulk/boundary unitarity conflict still exists and its sensible resolution at some "chiral" points turns out to depend crucially on the nature of asymptotic boundary conditions [20,27–29]. Therefore, finding pertinent stable vacua of the theory is of great importance and this motivates us to search for further exact solutions.

In striving to undertake an exhaustive investigation of the exact solutions, the theory of NMG was recently reformulated in terms of a first-order differential operator (appearing in TMG and resembling a Dirac type operator) and the traceless Ricci tensor [30]. With this novel proposal, the field equations of NMG consist of a massive (tensorial) Klein-Gordon type equation with an "effective" curvature-squared source term and a constraint equation. This approach has a striking consequence for finding algebraic types D and N (in the Petrov-Segre classification) exact solutions to NMG. Remarkably, for these types of spacetimes, possessing constant scalar curvature, one can interpret TMG as the "square root" of NMG. This allows us to establish a simple framework for mapping of all types D and N solutions of TMG into NMG [30]. In other words, there is no need to solve in each case the field equations of NMG for a given metric ansatz as it becomes possible to generate solutions to NMG from the known solutions of TMG. Moreover, the novel proposal turns out to be a powerful tool for finding new types D and N solutions to NMG which do not have their counterparts in TMG. Some intriguing and simple examples of these "non-TMG origin" solutions were given [30]. In the meantime, a universal metric for all type N solutions of NMG was found in [31].

The purpose of this paper is to explore algebraic type D solutions of NMG with both constant and nonconstant curvatures. In Sec. II we begin by describing a theoretical framework which includes reformulation of NMG in terms of a first-order differential operator, appearing TMG and acting on the traceless Ricci tensor. In Sec. III we introduce an orthonormal basis of three real vectors (one timelike and two spacelike vectors) and describe the properties of type D spacetimes in NMG, in terms of these vectors and their covariant derivatives. Here we also prove two defining mathematical statements for these spacetimes. In Sec. IV we present type D solutions with constant scalar curvature, which include homogeneous anisotropic space solutions of both TMG and non-TMG origins. Next, we present conformally flat solutions with constant scalar curvature, for which the cosmological and mass parameters are related by $\lambda = m^2$. In Sec. V we discuss the solutions with nonconstant scalar curvature and possessing at least one Killing vector. They all are conformally flat and require $\lambda = m^2$ by their very existence. We find two general metrics which comprise all such solutions. We also discuss some properties of these solutions, among them are black hole type solutions.

II. THEORETICAL FRAMEWORK

For our purposes in the following it is convenient to begin with TMG. The theory is described by the action [6,7]

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R - 2\Lambda + \frac{1}{\mu}\mathcal{L}_{\rm CS}\right), \qquad (1)$$

where R is the three-dimensional Ricci scalar, Λ is the cosmological constant, μ is the mass parameter and the Chern-Simons term is given by

$$\mathcal{L}_{\rm CS} = \frac{1}{2}\epsilon^{\mu\nu\rho}\Gamma^{\alpha}_{\mu\beta}\left(\partial_\nu\Gamma^{\beta}_{\rho\alpha} + \frac{2}{3}\Gamma^{\beta}_{\nu\gamma}\Gamma^{\gamma}_{\rho\alpha}\right). \qquad (2)$$

Here $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor and the quantities $\Gamma^{\mu}_{\alpha\beta}$ are the usual Christoffel symbols.

The field equations of TMG, obtained from this action, are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \qquad (3)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and $C_{\mu\nu}$ is the Cotton tensor given by

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\left(R_{\nu\beta} - \frac{1}{4}g_{\nu\beta}R\right)_{;\alpha} \qquad (4)$$

and the semicolon stands for covariant differentiation. We note that this tensor is a symmetric, traceless and covariantly conserved quantity.

It is not difficult to see that with the traceless Ricci tensor

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R, \qquad (5)$$

Eq. (3) can be put in the form

$$S_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0. \qquad (6)$$

Next, we introduce a first-order differential operator $\not{D}$ which is defined as

$$\not{D}\Phi_{\mu\nu} = \frac{1}{2}(\epsilon_{\mu}{}^{\alpha\beta}\Phi_{\nu\alpha;\beta} + \epsilon_{\nu}{}^{\alpha\beta}\Phi_{\mu\alpha;\beta}), \qquad (7)$$

where $\Phi_{\mu\nu}$ is a symmetric tensor. It is straightforward to show that with the subsidiary relation

$$\Phi_{\mu\nu}{}^{;\nu} = (\Phi_{\nu}{}^{\nu})_{;\mu}, \qquad (8)$$

Eq. (7) takes the most simple form

$$\not{D}\Phi_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\Phi_{\nu\alpha;\beta}. \qquad (9)$$

Since condition (8) is fulfilled for the Schouten tensor

$$\Phi_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R, \tag{10}$$

it is easy to verify that

$$C_{\mu\nu} = -\not{D}\Phi_{\mu\nu} = -\not{D}S_{\mu\nu}. \tag{11}$$

With these relations the field equations of TMG, as seen from Eq. (6), acquire the form

$$\not{D}S_{\mu\nu} = \mu S_{\mu\nu}. \tag{12}$$

We note that this equation resembles the Dirac equation $\not{D}\Psi_A = \gamma_A^{\mu B}\nabla_\mu \Psi_B = \mu\Psi_A$ for a massive spinor field. In the case under consideration, the operator $\not{D}$ acts on the traceless Ricci tensor, $\not{D}S_{\mu\nu} = D_{(\mu\nu)}{}^{(\alpha\beta)\rho}\nabla_\rho S_{\alpha\beta}$, and therefore it looks like the "cousin" of the Dirac operator (see also Refs. [30,31]).

Using Eq. (12), it easy to show that the secondary action of the operator $\not{D}$ yields the Klein-Gordon type equation

$$(\not{D}^2 - \mu^2)S_{\mu\nu} = 0. \tag{13}$$

It is also important to note that with the differential operator $\not{D}$ we have the following identity

$$(\not{D}^2 S_{\mu\nu})^{;\nu} = \epsilon_\mu{}^{\rho\sigma} S_{\rho\nu} C^\nu{}_\sigma. \tag{14}$$

This identity can be verified by using the relation

$$\not{D}^2 S_{\mu\nu} = \Phi_{\mu\nu;\sigma}{}^{;\sigma} - \Phi_{\mu\rho;\nu}{}^{;\rho} \tag{15}$$

along with the fact that in three dimensions the Riemann tensor is given as

$$R_{\mu\nu\alpha\beta} = 2\left(R_{\mu[\alpha} g_{\beta]\nu} - R_{\nu[\alpha} g_{\beta]\mu} - \frac{R}{2} g_{\mu[\alpha} g_{\beta]\nu}\right), \tag{16}$$

where the square brackets stand for antisymmetrization over the indices enclosed.

There also exists another important identity

$$(\not{D}^2 S_{\mu\nu})^{;\mu;\nu} = C_{\mu\nu}C^{\mu\nu} - S^{\mu\nu}\not{D}^2 S_{\mu\nu}, \tag{17}$$

that is obtained by a straightforward calculation of the quantity $S^{\mu\nu}\not{D}^2 S_{\mu\nu}$, using the definition in (7), with Eqs. (9) and (11) in mind, as well as the identity in (14).

We now proceed to NMG whose action has the form [10]

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R - 2\lambda - \frac{1}{m^2}K\right), \tag{18}$$

where λ is the cosmological parameter, m is the mass parameter and the scalar K is given by

$$K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2. \tag{19}$$

In a recent work [30], it was shown that the field equations, which follow from this action, can be written in terms of the square of the operator $\not{D}$, resulting in the Klein-Gordon type equation

$$(\not{D}^2 - m^2)S_{\mu\nu} = T_{\mu\nu}, \tag{20}$$

where $T_{\mu\nu}$ can be thought of as the effective source tensor. We have

$$T_{\mu\nu} = S_{\mu\rho}S^\rho{}_\nu - \frac{R}{12}S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S_{\alpha\beta}S^{\alpha\beta}. \tag{21}$$

Equation (20) is also accompanied by the subsidiary (constraint) equation

$$S_{\mu\nu}S^{\mu\nu} + m^2 R - \frac{R^2}{24} = 6m^2\lambda. \tag{22}$$

It is remarkable that such a description of NMG greatly simplifies the search for exact solutions to the theory. It is instructive to demonstrate this using the case of maximally symmetric solutions to TMG, for which the Cotton tensor in (6) vanishes identically and we have $S_{\mu\nu} = 0$. Then Eq. (20) is trivially satisfied, whereas Eq. (22) yields

$$\Lambda = 2m^2\left(1 \pm \sqrt{1 - \lambda/m^2}\right), \tag{23}$$

where we have used the fact that $R = 6\Lambda$. Thus, with the cosmological constant adjusted according to this relation, every maximally symmetric solution of TMG (including its identification with a BTZ black hole [4]) can be mapped into two inequivalent solutions of NMG [10,14].

The most striking feature of this description emerges for algebraic types D and N spacetimes. It is straightforward to verify that for these spacetimes the relation

$$T_{\mu\nu} = \kappa S_{\mu\nu} \tag{24}$$

holds, where κ is a function of the scalar curvature. With this relation, Eq. (20) reduces to the form

$$(\not{D}^2 - \mu^2)S_{\mu\nu} = 0, \tag{25}$$

where

$$\mu^2 = m^2 + \kappa. \tag{26}$$

We see that for type N spacetimes and type D spacetimes with constant scalar curvature this equation is equivalent to that given (13). Thus, comparing Eqs. (12), (13), and (25) we conclude that for these types of spacetimes TMG can be interpreted as the square root of NMG. This is a remarkable fact and it paves the way for mapping all types D and N exact solutions of TMG into NMG by means of an algebraic procedure that adjusts the physical parameters of the corresponding solutions in both theories [30]. On the other hand, it is clear that Eq. (25) admits types D and N exact solutions of non-TMG origin as well. (The square root does not always exist!) Some simple examples of such solutions were given in [30]. Meanwhile, in [31] we managed to find a universal metric which comprises all type N solutions of NMG. Here we wish to continue the line of works [30,31], giving an exhaustive investigation of type D solutions to NMG.

III. TYPE D SPACETIMES

Let us introduce an orthonormal basis of three real vectors $\{t_\mu, s_\mu, \upsilon_\mu\}$, satisfying the relations

$$t_\mu t^\mu = -1, \qquad s_\mu s^\mu = 1, \qquad \upsilon_\mu \upsilon^\mu = 1, \tag{27}$$

with all other products vanishing, such that the spacetime metric can be written in the form

$$g_{\mu\nu} = -t_\mu t_\nu + s_\mu s_\nu + \upsilon_\mu \upsilon_\nu. \tag{28}$$

In three dimensions, depending on whether the one-dimensional eigenspace of $S^\mu{}_\nu$ is timelike or spacelike, one can distinguish types D_t and D_s spacetimes, respectively [32] (see also [33,34]). For type D_t spacetime, the canonical form of the traceless Ricci tensor is given by

$$S_{\mu\nu} = p(g_{\mu\nu} + 3t_\mu t_\nu), \tag{29}$$

whereas, for type D_s spacetime we have

$$S_{\mu\nu} = p(g_{\mu\nu} - 3s_\mu s_\nu), \tag{30}$$

where p is a scalar function. In what follows, we shall perform a detailed analysis for type D_t spacetimes. The same analysis can be performed for type D_s spacetimes as well, either directly or simply by an appropriate analytical continuation of the type D_t results.

Clearly, using expression (29) in Eq. (22) one can express the function p in terms of the scalar curvature, the mass and cosmological parameters, m and λ. Thus, we have

$$6p^2 = 6m^2\lambda - m^2 R + \frac{R^2}{24}. \tag{31}$$

On the other hand, taking into account expression (29) in Eq. (24) and using the result in Eq. (26), we find that

$$\mu^2 = m^2 - p - \frac{R}{12}. \tag{32}$$

We turn now to the Cotton tensor in (4). It is straightforward to verify that the most general form of this tensor, written in terms of the orthonormal basis vectors, must have the following representation

$$C_{\mu\nu} = aS_{\mu\nu} + b(s_\mu s_\nu - \upsilon_\mu \upsilon_\nu) + 2c\upsilon_{(\mu}s_{\nu)} + 2ht_{(\mu}s_{\nu)}, \tag{33}$$

where the round brackets denote symmetrization of the indices enclosed. The coefficients a, b, c and h are scalar functions and we have gauged out the term proportional to $t_{(\mu}\upsilon_{\nu)}$ using the invariance of Eqs. (28) and (29) with respect to rotations in the (s_μ, υ_μ)-plane. Using this representation in Eq. (14), we find that

$$(\not{D}^2 S_{\mu\nu})^{;\nu} = 3ph\upsilon_\mu. \tag{34}$$

Next, taking the divergence of Eq. (25) and comparing the result with Eq. (34), we arrive at the relation

$$S_\alpha{}^\beta(\mu^2)_{;\beta} + \frac{1}{6}\mu^2 R_{;\alpha} = 3ph\upsilon_\alpha, \tag{35}$$

where we have used the contracted Bianchi identity

$$S_{\mu\nu}{}^{;\nu} = \frac{1}{6}R_{;\mu}. \tag{36}$$

Further it is convenient to consider in (32) the cases of $\mu^2 \neq 0$ and $\mu^2 = 0$ separately: First, we begin with the case $\mu^2 \neq 0$. Contracting both sides of Eq. (35) with s^μ, we find that

$$s^\mu R_{;\mu} = 0. \tag{37}$$

That is, the scalar curvature is constant along the basis vector s^μ. Meanwhile, the contraction of Eq. (35) with υ^μ yields

$$h = \frac{\mu^2}{12p}\upsilon^\mu R_{;\mu}, \tag{38}$$

where we have used Eqs. (31) and (32). We note that the same contraction procedure with t^μ gives a trivial result. Next, we need to find the representation for the covariant derivative of the vector t^μ. For this purpose, we substitute expression (29) into Eq. (11) and taking into account the relations

$$t_\mu = \epsilon_{\mu\nu\rho}s^\nu \upsilon^\rho, \quad s_\mu = \epsilon_{\mu\nu\rho}t^\nu\upsilon^\rho, \quad \upsilon_\mu = \epsilon_{\mu\nu\rho}s^\nu t^\rho \tag{39}$$

together with the Bianchi identity in (36), we compare the result with the representation of the Cotton tensor given in (33). After straightforward calculations, we find that

$$t_{\mu;\nu} = \frac{a}{3}\epsilon_{\mu\nu}{}^\rho t_\rho - \frac{2b}{3p}s_{(\mu}\upsilon_{\nu)} + \frac{c}{3p}(s_\mu s_\nu - \upsilon_\mu\upsilon_\nu) + \frac{g}{3p}(s_\mu s_\nu + \upsilon_\mu \upsilon_\nu) + \frac{f}{3p}\upsilon_\mu t_\nu, \tag{40}$$

where

$$g = \frac{3p}{2}t^\mu{}_{;\mu} = -\left(p' + \frac{1}{12}\right)t^\mu R_{;\mu}, \tag{41}$$

$$f = \left(p' - \frac{1}{6}\right)\upsilon^\mu R_{;\mu}, \tag{42}$$

the prime stands for the derivative with respect to the scalar curvature. In obtaining (41) we have also used expression (29) in Eq. (36) with the subsequent contraction of the result with t^μ.

We are now ready to prove that the following statement: *Suppose that the scalar function $h = 0$. Then, for $\mu^2 \neq 0$, all type D solutions of NMG must possess constant scalar curvature.*

The proof will be given by contradiction, assuming that $R \neq$ const. First, we note that for $h = 0$ Eq. (38) yields

$$\upsilon^\mu R_{;\mu} = 0. \tag{43}$$

Moreover, comparing Eqs. (14) and (34) we obtain that

$$C_{\rho[\mu}S_{\nu]}{}^{\rho} = 0. \qquad (44)$$

This relation is trivially satisfied in the TMG case, whereas it is a governing geometrical relation for the NMG solutions. It is important to note that with $h = 0$ in (33), one can always gauge out either b or c due to the rotational symmetry of the spacetime metric (28) in the (s_μ, v_μ)-plane. Therefore, in the following we can set c equal to zero, without loss of generality.

We also note that with relations (37) and (43), one can always write down the simple decomposition

$$R_{;\mu} = \chi t_\mu, \qquad (45)$$

where χ is some function and $\chi \neq 0$ as $R \neq$ const by our assumption. Using the fact that $R_{[;\mu;\nu]} = 0$, we take the covariant derivative of this equation. As a consequence, we have

$$\chi t_{[\mu;\nu]} + t_{[\mu}\chi_{;\nu]} = 0. \qquad (46)$$

Substituting now (40) into this equation and contracting the result with $v^\mu s^\nu$, we find that $a = 0$, in addition to $c = 0$. Thus, Eq. (40) reduces to the form

$$t_{\mu;\nu} = -\frac{2b}{3p}s_{(\mu}v_{\nu)} + \frac{g}{3p}(s_\mu s_\nu + v_\mu v_\nu). \qquad (47)$$

Meanwhile, the Cotton tensor in (33) becomes as

$$C_{\mu\nu} = b(s_\mu s_\nu - v_\mu v_\nu). \qquad (48)$$

The value of b in these equations can be fixed by using (17) along with (25) and (34). We find that

$$b^2 = 3p^2\mu^2. \qquad (49)$$

Next, we calculate the divergence of expression (48). Taking into account the property $C_{\mu\nu}{}^{;\nu} = 0$, and contracting the result with the vectors s^μ and v^μ, we have

$$s_\mu{}^{;\mu} = 0, \qquad v_\mu{}^{;\mu} = 0, \qquad (50)$$

where we have also used relations (37) and (43).

With Eqs. (47) and (50), as well as using the properties of the orthonormal basis vectors given in (27), it is straightforward to write down the explicit expressions for the covariant derivatives of the basis vectors s_μ and v_μ. They are given by

$$s_{\mu;\nu} = -\frac{b}{3p}t_\mu v_\nu + \frac{g}{3p}t_\mu s_\nu + \zeta v_\mu t_\nu, \qquad (51)$$

$$v_{\mu;\nu} = -\frac{b}{3p}t_\mu s_\nu + \frac{g}{3p}t_\mu v_\nu - \zeta s_\mu t_\nu, \qquad (52)$$

where ζ is a scalar function.

Finally, the use of these expressions in the Ricci identity

$$x_{\mu;\nu}{}^{;\mu} - (x_\mu{}^{;\mu})_{;\nu} = \left(S_{\mu\nu} + \frac{R}{3}g_{\mu\nu}\right)x^\mu, \qquad (53)$$

written for the vectors s_μ and v_μ, gives us $\zeta = 0$ as well as the relation

$$(\mu^2)'\left(1 + \frac{4\mu^2}{3p}\right)t^\mu R_{;\mu} = 0. \qquad (54)$$

We recall that $\mu^2 \neq 0$. Furthermore, using Eqs. (31) and (32), it is easy to show that the expression in the round brackets does not vanish identically. Thus, from Eq. (54) (see also (37) and (43)) it follows that the scalar curvature R is constant. This contradicts to our initial assumption that $R \neq$ const and completes the proof of the statement made above.

A similar analysis shows that this statement holds for type D_s spacetimes as well. However, in both cases an interesting question is that *what happens if one drops the condition* $h = 0$. It turns out that in this case one can still prove the same statement, provided that the spacetime admits a hypersurface orthogonal Killing vector. However, the general case with $h \neq 0$ becomes very involved and unclear. In our opinion, even in this case all the solutions with at least one Killing vector possess constant scalar curvature.

We turn now to the case $\mu^2 = 0$, which is equivalent to

$$p = m^2 - \frac{R}{12}, \qquad (55)$$

and taking this into account in Eq. (31), we find that

$$\lambda = m^2. \qquad (56)$$

Meanwhile, from the field equation (25) it follows that $\not{D}^2 S_{\mu\nu} = 0$. Using this in Eq. (34), we see that $h = 0$. With these results, from Eq. (17) it follows that

$$C_{\mu\nu}C^{\mu\nu} = 0. \qquad (57)$$

Substituting (33) into this equation, we find the relation

$$3p^2a^2 + b^2 = 0, \qquad (58)$$

which implies that both $a = 0$ and $b = 0$. Therefore,

$$C_{\mu\nu} = 0, \qquad (59)$$

as can be seen from Eq. (33). (We recall that c is gauged out as $h = 0$). Thus, *with* $\mu^2 = 0$, *or equivalently* $\lambda = m^2$, all type D *solutions of NMG must be conformally flat*.

IV. SOLUTIONS WITH CONSTANT SCALAR CURVATURE

We shall now discuss all type D solutions with constant scalar curvature, focusing first on the case $\mu^2 \neq 0$. These are homogeneous anisotropic solutions which can be formally divided into two categories: Solutions of TMG origin and solutions of non-TMG origin i.e. those which do not have their counterparts in TMG.

We begin by noting that for $R =$ const, the functions h, g and f, as seen from Eqs. (38), (41), and (42), vanish. Then, Eq. (40) can be written in the form

$$t_{\mu;\nu} = \frac{a}{3}\epsilon_{\mu\nu}{}^{\rho}t_{\rho} - \frac{2b}{3p}s_{(\mu}v_{\nu)}. \qquad (60)$$

Similarly, Eq. (33) reduces to the form

$$C_{\mu\nu} = aS_{\mu\nu} + b(s_{\mu}s_{\nu} - v_{\mu}v_{\nu}). \qquad (61)$$

Contracting now Eq. (25) with $t^{\mu}t^{\nu}$ and using Eqs. (9) and (11), after some algebraic manipulations, we obtain that

$$\epsilon^{\mu\rho\sigma}t_{\mu}(2pat_{\rho;\sigma} + C_{\rho}{}^{\nu}t_{\nu;\sigma}) = 2p\mu^2. \qquad (62)$$

Substitution expressions (60) and (61) into this equation yields the relation

$$a^2 + \frac{b^2}{3p^2} = \mu^2. \qquad (63)$$

On the other hand, writing down the Ricci identity (53) for the vector t_{μ} and contracting the result with t^{ν}, we find the relation

$$a^2 - \frac{b^2}{p^2} = 9p - \frac{3}{2}R. \qquad (64)$$

Since p is constant, as can be seen from (31), comparing Eqs. (63) and (64) it is easy to see that a and b are constants as well. Furthermore, with Eq. (61) it is not difficult to see that for $b \neq 0$, we again arrive at the relations given in (50).

A. Solutions of TMG origin

When $b = 0$, the Cotton tensor in (61) becomes as

$$C_{\mu\nu} = aS_{\mu\nu}, \qquad (65)$$

with

$$a = \pm\mu. \qquad (66)$$

Comparing this equation with those given in (11) and (12), and taking into account (13) and (25), we deduce that all type D solutions of TMG can be mapped into NMG. The resulting solutions, as can be seen from Eqs. (32) and (64), are characterized by

$$p = \frac{m^2}{10} + \frac{17}{120}R. \qquad (67)$$

It is straightforward to show that the use of this value of p in (31) and (32) results in the adjusting relations, earlier obtained in [30], for mapping of the TMG solutions into NMG. For these solutions Eq. (60) takes the form

$$t_{\mu;\nu} = \frac{a}{3}\epsilon_{\mu\nu}{}^{\rho}t_{\rho} = \frac{a}{3}(v_{\mu}s_{\nu} - s_{\mu}v_{\nu}), \qquad (68)$$

where in the last step we have used the relations in (39). That is, the vector t_{μ} is the Killing vector (see also [17,32]).

We now want to give an elegant classification of these solutions by a dimensional reduction on this Killing vector, where the sign of the scalar curvature of the two-dimensional subspace (the factor space) plays a crucial role. We define the projection operator onto the factor space perpendicular to t_{μ}. It is given by

$$h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + t^{\mu}t_{\nu}, \qquad (69)$$

and $h^{\mu}{}_{\nu}t^{\nu} = 0$. We also define the derivative operator D_{μ} with respect to the two-dimensional "spatial" metric $h_{\mu\nu}$, and the associated Riemann tensor $r^{\mu}{}_{\nu\lambda\tau}$. We have

$$D_{\nu}V_{\mu} = h^{\lambda}{}_{\mu}h^{\sigma}{}_{\nu}V_{\lambda;\sigma}, \quad r^{\mu}{}_{\nu\lambda\tau}V_{\mu} = 2D_{[\tau}D_{\lambda]}V_{\nu}. \qquad (70)$$

With these definitions, the relation between the spacetime Riemann tensor and the Riemann tensor of the factor space is derived in a usual way (see for instance, Ref. [35]). After straightforward calculations, we have

$$r_{\mu\nu\alpha\beta} = h^{\lambda}{}_{\mu}h^{\tau}{}_{\nu}h^{\rho}{}_{\alpha}h^{\sigma}{}_{\beta}(R_{\lambda\tau\rho\sigma} + 2t_{\tau;\lambda}t_{\rho;\sigma} + t_{\rho;\lambda}t_{\tau;\sigma} - t_{\rho;\tau}t_{\lambda;\sigma}). \qquad (71)$$

Taking the projection of Eqs. (16) and (68) onto the factor space and using the result in (71), we find that

$$r_{\mu\nu\alpha\beta} = \left(2p + \frac{R}{6} - \frac{\mu^2}{3}\right)(h_{\mu\alpha}h_{\nu\beta} - h_{\mu\beta}h_{\nu\alpha}). \qquad (72)$$

From this expression it follows that the scalar curvature of the factor space is given by

$$r = \frac{21}{20}R - \frac{m^2}{5}, \qquad (73)$$

where we have also used Eqs. (32) and (67).

Next, we need to find the covariant derivatives of the basis vectors s_{μ} and v_{μ}. Using Eq. (68), together with the properties of the orthonormal basis in (27), it is not difficult to show that the most general expressions for these derivatives are given by

$$s_{\mu;\nu} = -\frac{a}{3}t_{\mu}v_{\nu} + Av_{\mu}t_{\nu} + Bv_{\mu}s_{\nu} + Cv_{\mu}v_{\nu}, \qquad (74)$$

$$v_{\mu;\nu} = \frac{a}{3}t_{\mu}s_{\nu} - As_{\mu}t_{\nu} - Bs_{\mu}s_{\nu} - Cs_{\mu}v_{\nu}, \qquad (75)$$

where the functions A, B and C need to be determined. Projecting these expressions onto the factor space, by means of (69), and comparing the result, for certainty, with that obtained for the metric of two-dimensional hyperbolic space

$$ds_2^2 = \frac{1}{k^2}(d\theta^2 + \sinh^2\theta d\phi^2), \qquad k^2 = -\frac{r}{2}, \qquad (76)$$

we find that

$$B = 0, \qquad C = k\coth\theta. \qquad (77)$$

Substitution of these quantities in Eqs. (74) and (75) yields

$$s_{\mu}{}^{;\mu} = C, \qquad v_{\mu}{}^{;\mu} = 0. \qquad (78)$$

On the other hand, using the Ricci identity (53) for the vector v_μ, with Eq. (78) in mind, and contracting the result with v^ν, we find that $A = -a/3$. Thus, Eqs. (74) and (75) take their final forms given by

$$s_{\mu;\nu} = -\frac{a}{3}(t_\mu v_\nu + v_\mu t_\nu) + C v_\mu s_\nu, \qquad (79)$$

$$v_{\mu;\nu} = \frac{a}{3}(t_\mu s_\nu + s_\mu t_\nu) - C s_\mu s_\nu. \qquad (80)$$

It is not difficult to show that the associated Lie brackets are given by

$$[s,t]=0, \quad [v,t]=0, \quad [v,s]=-\frac{2a}{3}t+Cv. \qquad (81)$$

It is convenient now to choose a coordinate system in which the Killing vector is given as $t = \partial_\tau$. Then, the use of (76) and (81) enables us to specify the remaining vectors in the form

$$s = k\partial_\theta, \qquad v = \frac{k}{\sinh\theta}\left(\partial_\phi - \frac{2a}{3k^2}\cosh\theta\partial_\tau\right). \qquad (82)$$

Meanwhile, the corresponding dual one-forms result in the spacetime metric

$$ds^2 = -\left(d\tau + \frac{2a}{3k^2}\cosh\theta d\phi\right)^2 + \frac{1}{k^2}(d\theta^2 + \sinh^2\theta d\phi^2). \qquad (83)$$

We see that this metric has an apparent singularity at $k = 0$. However, by redefining the coordinates as

$$\tau \to \tau - \frac{2a}{3k^2}\phi, \qquad \theta \to k\theta, \qquad (84)$$

one can put it in the form

$$ds^2 = -\left(d\tau + \frac{4a}{3k^2}\sinh^2\frac{k\theta}{2}d\phi\right)^2 + d\theta^2 + \frac{\sinh^2 k\theta}{k^2}d\phi^2, \qquad (85)$$

which is suitable for taking the limit $k \to 0$.

It is also worth noting that in terms of the left invariant one-forms of $SU(1,1)$, parametrized by the Euler angles,

$$\begin{aligned}\sigma_1 &= -\sin\psi d\theta + \cos\psi\sinh\theta d\phi,\\ \sigma_2 &= \cos\psi d\theta + \sin\psi\sinh\theta d\phi,\\ \sigma_3 &= d\psi + \cosh\theta d\phi,\end{aligned} \qquad (86)$$

and with $\tau = (2a/3k^2)\psi$, the metric in (83) takes its most simple form given by

$$ds^2 = -\left(\frac{2a}{3k^2}\right)^2\sigma_3^2 + \frac{1}{k^2}(\sigma_1^2 + \sigma_2^2). \qquad (87)$$

In this case, instead of (81), we have the following Lie algebra

$$[s,t]=-\frac{3r}{4a}v, \quad [v,t]=\frac{3r}{4a}s, \quad [v,s]=-\frac{2a}{3}t. \qquad (88)$$

We recall that the quantities a and k^2 are as given in (66) and (76) and they are determined by Eqs. (73) and (67) together with (31) and (32). We see that, in general, depending on the sign of the scalar curvature r of the two-dimensional factor space, we have three possible spatial geometries. Namely, a sphere for $r > 0$, a hyperboloid for $r < 0$ and a flat space for $r = 0$. Accordingly, the associated spacetimes are homogeneous anisotropic solutions: (i) the solution of Bianchi type IX, or with $SU(2)$ symmetry, (ii) the solution of Bianchi type $VIII$, or with $SU(1,1)$ symmetry, (iii) the solution of Bianchi type II, or with the symmetry of the Heisenberg group.

We note that the type D_s counterpart of spacetime (87) can be obtained by taking $t \to is$ and $s \to it$ in the above expressions (see also Eqs. (29) and (30)). In this case, Bianchi type II remains unchanged, whereas Bianchi types $VIII$ and IX goes over into each other.

B. Solutions of non-TMG origin

We turn now to the case of $b \neq 0$ in Eqs. (60) and (61). This results in solutions which are only inherent in NMG. Clearly, in this case the vector t_μ is no longer the Killing vector, but it is still divergence-free. Furthermore, as we have mentioned above, the vectors s_μ and v_μ are divergence-free as well. With these in mind, using Eq. (60) and the properties of the orthonormal basis vectors given in (27), we find that

$$s_{\mu;\nu} = -\frac{1}{3}\left(a + \frac{b}{p}\right)t_\mu v_\nu + \psi v_\mu t_\nu, \qquad (89)$$

$$v_{\mu;\nu} = \frac{1}{3}\left(a - \frac{b}{p}\right)t_\mu s_\nu - \psi s_\mu t_\nu, \qquad (90)$$

where ψ is a scalar function. Using these relations in the Ricci identity (53) for the vectors s_μ and v_μ, and taking into account Eq. (50), we find that

$$\psi = 0, \qquad p = -\frac{R}{3}. \qquad (91)$$

Next, we calculate the action of the operator $\not{D}$ on the Cotton tensor in (61), by means of Eqs. (11), (89), and (90). Then, comparing the result with Eq. (25), we find that $a = 0$, whereas b is given by the same relation as in (49). (See also Eq. (63)).

On the other hand, from Eqs. (31), (32), and (64) it follows that

$$p = -\frac{4}{15}m^2, \qquad \lambda = \frac{m^2}{5}. \qquad (92)$$

Finally, using these results in Eqs. (60), (89), and (90) it is straightforward to show that we have the following Lie algebra

$$[s,t]=\sqrt{2/5}m v, \quad [v,t]=\sqrt{2/5}m s, \quad [v,s]=0. \tag{93}$$

It is convenient to choose a coordinate system in which the Lie algebra admits the following representation for the basis vectors

$$t=\partial_\tau, \qquad s=\frac{1}{\sqrt{2}}(e^{-\sqrt{2/5}m\tau}\partial_x + e^{\sqrt{2/5}m\tau}\partial_y),$$
$$v=\frac{1}{\sqrt{2}}(e^{-\sqrt{2/5}m\tau}\partial_x - e^{\sqrt{2/5}m\tau}\partial_y). \tag{94}$$

Then, the spacetime metric can easily be written down by using the associated dual basis. We have

$$ds^2 = -d\tau^2 + e^{2\sqrt{2/5}m\tau}dx^2 + e^{-2\sqrt{2/5}m\tau}dy^2. \tag{95}$$

This solution, as can be seen from the Lie algebra in (93), is of a homogeneous anisotropic spacetime of Bianchi type VI_0, or with $E(1,1)$ symmetry.

Again, the corresponding type D_s solution can be obtained by making the replacement $t \to is$ and $s \to it$. In this case, we have the Lie algebra

$$[t,s]=-\sqrt{2/5}m v, \quad [v,s]=\sqrt{2/5}m t, \quad [v,t]=0, \tag{96}$$

which enables us to specify the basis vectors in the form

$$s=\partial_x, \qquad v=\sin(\sqrt{2/5}mx)\partial_\tau + \cos(\sqrt{2/5}mx)\partial_y,$$
$$t=-\cos(\sqrt{2/5}mx)\partial_\tau + \sin(\sqrt{2/5}mx)\partial_y. \tag{97}$$

The associated dual basis results in the solution given by

$$ds^2 = \cos(2\sqrt{2/5}mx)(-dt^2+dy^2)+dx^2 + 2\sin(2\sqrt{2/5}mx)dtdy. \tag{98}$$

This is a homogeneous anisotropic spacetime of Bianchi type VII_0, or with $E(2)$ symmetry.

To complete this subsection, we consider now type D solutions with constant scalar curvature, which correspond to the case $\mu^2=0$. For these solutions, we have the special relation between the cosmological and mass parameters, as given in (56), and the Cotton tensor in (61) vanishes as both a and b are zero. Using these facts in Eq. (60), we see that

$$t_{\mu;\nu}=0. \tag{99}$$

It follows that $t=\partial_\tau$ is a hypersurface orthogonal Killing vector of constant length. With this in mind, we employ the Ricci identity (53) for the vector t_μ and obtain that

$$p=\frac{R}{6}. \tag{100}$$

Using this value of p in Eqs. (55) and (72), we find that

$$R=r=4m^2. \tag{101}$$

Clearly, type D_t spacetime metric can now be written as

$$ds^2=-d\tau^2+\frac{1}{k^2}(d\theta^2+\sinh^2\theta d\phi^2), \tag{102}$$

where

$$k^2=-2m^2=-2\lambda. \tag{103}$$

Meanwhile, type D_s solution is obtained from this metric by performing the coordinate changes $\tau \to i\phi$ and $\phi \to i\tau$. We have

$$ds^2=d\phi^2+\frac{1}{k^2}(d\theta^2-\sinh^2\theta d\tau^2). \tag{104}$$

It is straightforward to show that this metric is locally isometric to the Kaluza-Klein solution $AdS_2 \times S^1$ found earlier in [14] (see also Ref. [20]). We note that the spacetime metric (102) has also a cousin with spherical spatial section (the flat case is trivial). One can show that the type D_s counterpart of the latter is locally isometric to the $dS_2 \times S^1$ solution of [20]. We recall that similar solutions with a hypersurface orthogonal Killing vector absent in TMG due to the no-go theorem of [36].

V. SOLUTIONS WITH NONCONSTANT SCALAR CURVATURE

Now we discuss all type D solutions which possess nonconstant scalar curvature. We recall that these are conformally flat solutions for which $C_{\mu\nu}=0$ $(a=b=c=0)$. Using the traceless Ricci tensor (29) in the Bianchi identity (36) and contracting the result with the vector t^μ, we obtain that

$$t_\mu{}^{;\mu}=0, \tag{105}$$

which in turn implies that $g=0$. In obtaining these results we have also used Eqs. (41) and (55). With these in mind, from Eq. (40) we have

$$t_{\mu;\nu}=\frac{v^\rho p_{;\rho}}{p}v_\mu t_\nu. \tag{106}$$

In what follows, we will restrict ourselves to the case when the spacetime under consideration admits at least one Killing vector ξ. For further convenience, we use the invariance of metric (28) with respect to rotations in the (s_μ, v_μ)-plane and represent the Killing vector in the form

$$\xi_\mu=\alpha t_\mu+\beta s_\mu, \tag{107}$$

where α and β are scalar functions to be specified later. Then, due to the same rotations we have

$$t_{\mu;\nu}=\frac{v^\rho p_{;\rho}}{p}v_\mu t_\nu+\frac{s^\rho p_{;\rho}}{p}s_\mu t_\nu, \tag{108}$$

instead of (106). From the Bianchi identity (36) it follows that

$$\xi^\mu S_{\mu\nu}{}^{;\nu}=0. \tag{109}$$

Substituting Eq. (29) in this expression and using relation (105), we find that

$$(\alpha p)_{;\mu}t^\mu=0. \tag{110}$$

On the other hand, the use of the fact that

$$\mathcal{L}_\xi S_{\mu\nu} = 0, \tag{111}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the Killing vector, gives us the relations

$$(\alpha p)_{;\mu} s^\mu = 0, \qquad (\alpha p)_{;\mu} v^\mu = 0. \tag{112}$$

From Eqs. (110) and (112) it follows that

$$\alpha = a_0 p^{-1}, \tag{113}$$

where a_0 is a constant.

Using the properties of the orthonormal basis in (27) along with expression (108) and the Killing equation

$$\xi_{(\mu;\nu)} = 0, \tag{114}$$

we obtain the relations

$$s_{\mu;\nu} = \frac{s^\rho p_{;\rho}}{p} t_\mu t_\nu - \frac{v^\rho \beta_{;\rho}}{\beta} v_\mu s_\nu, \tag{115}$$

$$v_{\mu;\nu} = \frac{v^\rho p_{;\rho}}{p} t_\mu t_\nu + \frac{v^\rho \beta_{;\rho}}{\beta} s_\mu s_\nu, \tag{116}$$

and

$$t^\mu \beta_{;\mu} = 0, \qquad s^\mu \beta_{;\mu} = 0. \tag{117}$$

Next, we need to obtain the determining equations for unknown functions β and p. They are obtained by employing the Ricci identity (53) for the basis vectors t^μ, s^μ and v^μ, respectively, as well as using Eqs. (108), (115), and (116). After some manipulations, we arrive at the "oscillator" type equation for β,

$$v^\mu v^\nu \beta_{;\mu;\nu} - k^2 \beta = 0, \tag{118}$$

as well as at the following set of equations for p,

$$s^\mu s^\nu (p^{-1})_{;\mu;\nu} = 3 + k^2 p^{-1} - \frac{v^\rho \beta_{;\rho}}{\beta} v^\mu (p^{-1})_{;\mu}, \tag{119}$$

$$v^\mu v^\nu (p^{-1})_{;\mu;\nu} = 3 + k^2 p^{-1}, \tag{120}$$

$$v^\mu s^\nu (p^{-1})_{;\mu;\nu} = 0. \tag{121}$$

We note that k^2 is the same as that given in (103). It is straightforward to show that the Killing vector in (107) commutes with the basis vector v in the sense of their Lie bracket. That is, we have

$$[\xi, v] = 0, \tag{122}$$

which can easily be verified by means of Eqs. (108), (115), and (116). This fact allows us to choose the vectors ξ and v as

$$\xi = \partial_z \qquad v = \partial_x \tag{123}$$

in a coordinate system (x, y, z). With this in mind, we turn to Eq. (118) whose general solution is given by

$$\beta = f_1(y) \sinh(kx) + f_2(y) \cosh(kx), \tag{124}$$

where $f_1(y)$ and $f_2(y)$ are arbitrary functions. Alternatively, the use of the reparametrization invariance of the vector fields ξ and v with respect to the coordinate freedom $x \to x + g_1(y)$ enables us to reduce this solution into three different forms given by

$$\beta = \omega(y) \sinh(kx), \tag{125}$$

$$\beta = \omega(y) \cosh(kx), \tag{126}$$

$$\beta = \omega(y) e^{kx}, \tag{127}$$

where the function $\omega(y)$ will be specified below.

We turn now to Eqs. (107) and (113) and consider the cases $a_0 = 0$ and $a_0 \neq 0$ separately.

A. The case $a_0 = 0$

In this case, the Killing vector, as seen from Eq. (107), becomes proportional to the spacelike basis vector s. That is, we have

$$s = \frac{1}{\beta} \partial_z. \tag{128}$$

Next, using the relations

$$[\xi, t] = 0, \qquad [v, t] = \frac{v^\mu p_{;\mu}}{p} t, \tag{129}$$

where the first relation follows from Eq. (111) and the second one is obtained by means of Eqs. (108) and (116), we fix the timelike basis vector as

$$t = p \partial_y. \tag{130}$$

With this in mind, from the first relation of (117), we find that $\omega = \text{const}$ in Eqs. (125)–(127). Using now the dual one-forms to the basis vectors in Eqs. (123), (128), and (130) we arrive at the spacetime metric

$$ds^2 = -p^{-2} d\tau^2 + dx^2 + \beta^2 dy^2, \tag{131}$$

where we have relabeled the coordinates as $y = \tau$ and $z = y$. We see that this metric in general is characterized by two functions $\beta = \beta(x)$ and $p = p(\tau, x)$. The first function is given by solutions (125)–(127), whereas the second function is determined by the system of differential Eqs. (119)–(121), which admits the general solution

$$p^{-1} = -\frac{3}{k^2} + f(\tau) \partial_x \beta, \tag{132}$$

where $f(\tau)$ is a smooth function. Performing an analytical continuation of solution (131) by $\tau \to iy$ and $y \to i\tau$, we arrive at its type D_s counterpart. It is straightforward to show that the resulting spacetime metrics for the explicit forms of β given in (125)–(127), after appropriate coordinate changes, represent the black hole, gravitational soliton and kink type solutions with one Killing vector, respectively. Meanwhile, for $f = \text{const}$ the second Killing vector appears as well, and these solutions reduce to those with two commuting Killing vectors, found earlier in [21]. Below, we shall focus only on the black hole type solution with one Killing vector.

Let us take $\beta = \sinh(kx)$ in metric (131). Then, passing to the coordinates $3/k^2 - r = M\cosh(\nu x)$, where r is a radial coordinate and M is a constant, $\tau \to i\phi$ and $y \to ikM\tau$, we arrive at the metric

$$ds^2 = -k^2(r-r_+)(r-r_-)d\tau^2 + \frac{dr^2}{k^2(r-r_+)(r-r_-)} + [r - (r_+ + r_-)/2 + F(\phi)]^2 d\phi^2, \quad (133)$$

where the radii of outer (r_+) and inner (r_-) horizons are given by

$$r_\pm = \frac{3}{k^2} \pm M, \quad (134)$$

and $F(\phi)$ is an arbitrary function. This metric can be interpreted as describing a "generalized" black hole type solution, which is asymptotically AdS spacetime with $\Lambda = 2m^2$ ($m^2 < 0$). For $r_+ = r_- = r_0$, it corresponds to an extremal black hole with one Killing vector, which was earlier described in [30]. It is also easy to see that for $F(\phi) = \text{const}$, this solution goes over into a standard new black hole metric (with two commuting Killing vectors) of [20,21].

In light of this, it is also interesting to ask whether the solution in (131) admits another limit with two Killing vectors. It turns out that the answer is affirmative, but the Killing vectors are no longer commuting. In fact, solving the equation $\eta_{(\mu;\nu)} = 0$ for the putative Killing vector η, we find that

$$\eta = \partial_\tau - \frac{b_0}{k}\partial_x + b_0 y \partial_y, \quad (135)$$

provided that the metric functions are given by

$$f = e^{b_0\tau}, \qquad \beta = e^{kx}. \quad (136)$$

Here b_0 is a constant of integration. We see that the Killing vectors η and ξ do not commute for $b_0 \neq 0$. Clearly, similar analysis is also true for the type D_s counterpart of (131).

B. The case $a_0 \neq 0$

We begin with calculating, in addition to (122), the other Lie brackets of the vectors ξ, s and v. Using Eqs. (107) and (115) and taking into account Eq. (117) it is easy to show that

$$[\xi, s] = 0, \quad (137)$$

whereas, the use of Eqs. (115) and (116) yields

$$[s, v] = \frac{v^\mu \beta_{;\mu}}{\beta} s. \quad (138)$$

With these Lie brackets and with Eq. (123), one can show that the vector s admits the simple representation

$$s = \frac{1}{\beta}\partial_y, \quad (139)$$

and $\omega = \text{const}$ in Eqs. (125)–(127). In obtaining these results we have also used the first equation in (117) and the coordinate freedom $z \to z + g_2(y)$.

Next, using the above representation of the vectors ξ and s in Eq. (107), we see that the timelike basis vector t is given by

$$t = p(\partial_z - \partial_y), \quad (140)$$

where we have set $a_0 = 1$, for certainty. It is easy to see that the associated dual one-forms enable us to write down the spacetime metric in the form

$$ds^2 = -p^{-2}d\tau^2 + dx^2 + \rho^2(dy + \omega d\tau)^2, \quad (141)$$

where we have passed to the new coordinates $\rho \to \beta/\omega$, $\tau \to z$ and $y \to y\omega$. We recall that $\rho = \rho(x)$ is determined by the solutions in (125)–(127), and $p = p(x, y)$ is given by the solutions of Eqs. (119)–(121). Thus, we have the following three pairs of solutions

$$p^{-1} = -\frac{3}{k^2} + c_0\cosh(kx) + (c_1 e^{iky} + c_2 e^{-iky})\sinh(kx), \quad \rho = \sinh(kx), \quad (142)$$

$$p^{-1} = -\frac{3}{k^2} + c_0\sinh(kx) + (c_1 e^{ky} + c_2 e^{-ky})\cosh(kx), \quad \rho = \cosh(kx), \quad (143)$$

$$p^{-1} = -\frac{3}{k^2} + c_0 e^{kx} + c_1(k^2y^2e^{kx} + e^{-kx}) + c_2 y e^{kx}, \quad \rho = e^{kx}, \quad (144)$$

where c_0, c_1 and c_2 are constants of integration. The type D_s counterpart of solution (141) is obtained by making the coordinate changes $\tau \to iy$ and $y \to i\tau$.

It is important to note that metric (141) with one Killing vector is intrinsically different from that in (131) even for $\omega = 0$. In particular, this can be seen from the fact that for metric (131), for which $a_0 = 0$, the Killing vector in (107) is an eigenvector of (29), i.e. $S_\mu{}^\nu \xi_\nu = p\xi_\nu$. However, this is not the case for metric (141), where $a_0 \neq 0$ and we have $S_\mu{}^\nu \xi_\nu = p\xi_\nu - 3a_0 t_\nu$. On the other hand, for the vanishing constants c_1 and c_2 an additional Killing isometry appears in (141), and ∂_y becomes the second Killing vector, commuting with the first one, $\xi = \partial_\tau$. With these two commuting Killing vectors, solutions (131) and (141) become locally isometric to each other.

In analogy with the case of (131), it is not difficult to show that for the special forms of the metric functions, solution (141) still admits a second Killing vector. This vector does not commute with the Killing vector ξ. Again, solving the associated Killing equation, we find the explicit form of the Killing vector. It is given by

$$\eta = e^{k(\omega\tau+y)}[\tanh(kx)\partial_y - \partial_x], \tag{145}$$

provided that the metric functions are as in Eq. (143) with $c_0 = 0$ and $c_2 = 0$. We note that for $\omega = 0$, this vector commutes with the Killing vector ξ. Then, it is not difficult to show that the resulting metric with these commuting Killing vectors is locally isometric to that in (131) with $f = \text{const}$ and with $\beta = e^{kx}$. It is also worth to note that the Killing vector, similar to that in (145), does also exist for the solutions in (142), again with the vanishing c_0 and c_2. On the other hand, there is no such a Killing vector for the solutions in (144). Finally, it is certainly interesting to know what is the relation between metrics (131) and (141) in their special limits with the Killing vectors in (135) and (145), respectively. This issue requires further investigation.

In summary, we note that the spacetime metrics in (131) and (141), both admitting one Killing vector, represent all type D solutions of NMG with nonconstant scalar curvature. We recall once again that all these solutions are conformally flat and they do exist provided that the special relation $\lambda = m^2$ holds.

VI. CONCLUSION

This paper completes our program of the exhaustive investigation of types D and N exact solutions of NMG, which was first begun in [30] and then continued in [31]. Here we have presented an exhaustive set of solutions, which includes all type D solutions with both constant and nonconstant curvatures. As in the previous cases, this was achieved in the framework of a novel proposal that amounts to reformulation of NMG in such a way that the field equations underlying the theory acquire a remarkably simple form. Namely, they reduce to a massive (tensorial) Klein-Gordon type equation which is accompanied by a constraint equation as well. The "mass term" in this equation plays a distinguished role in many aspects of the delineation of exact solutions. In particular, for its constant value being achieved only for type N and type D (with constant scalar curvature) spacetimes, there exists an intimate relation between the theories of NMG and TMG; the latter can be understood as the square root of the former, thereby opening up the way for mapping all the associated solutions of TMG into NMG.

We have proved two defining mathematical statements which in essence classify all type D solutions with at least one Killing vector. For the nonzero value of the mass term, all type D solutions of NMG must have constant scalar curvature, whereas for the vanishing mass term all the solutions turn out to be conformally flat, possessing both constant and nonconstant scalar curvatures. Introducing an orthonormal basis of three real vectors and using the gauge freedoms provided by the associated Lorentz symmetries, we have given a detailed description of type D spacetimes in NMG, in terms of these basis vectors and their covariant derivatives.

With type D solutions of constant scalar curvature, we have shown that all the solutions fall into two classes: (i) homogeneous anisotropic solutions which consist of those having their counterparts in TMG and those of being only inherent in NMG, (ii) solutions with the vanishing Cotton tensor, i.e. conformally flat solutions. Using the dimensional reduction procedure on a Killing vector, we have given an elegant classification of the homogeneous anisotropic solutions of TMG origin in terms of the scalar curvature of two-dimensional subspace. We have obtained the most compact expression for the spacetime metric and established a *universal* Lie algebra for the associated basis vectors. Depending on the value (positive, negative or zero) of the two-dimensional scalar curvature, appearing in the Lie algebra, there exist three possible spatial geometries and the associated spacetimes are of Bianchi types IX, $VIII$ and II, respectively. We have also obtained all homogeneous anisotropic solutions of non-TMG origin, which are of Bianchi types VI_0 and VII_0 spacetimes. It is important to note that these solutions require a special relation between the cosmological and mass parameters ($\lambda = m^2/5$), unlike the Bianchi types of TMG origin. As for conformally flat solutions, they exist only for $\lambda = m^2$ and possess a hypersurface orthogonal Killing vector of constant length. We have discussed these solutions as well, emphasizing that some of them are locally isometric to the previously-known Kaluza-Klein type $\mathrm{AdS}_2 \times S^1$ or $\mathrm{dS}_2 \times S^1$ solutions.

With type D solutions of nonconstant scalar curvature, we have found two general metrics which admit at least one Killing vector and comprise the entire set of such solutions. All these solutions are conformally flat and require $\lambda = m^2$ by their very existence. We have discussed the special limits of these metrics with two commuting Killing vectors. In the latter case, the resulting metrics become locally isometric to each other, recovering all the previously-known solutions in the literature. We have shown that there also exists a special limit of both general metrics when two noncommuting Killing vectors appear. The question of whether in this case the resulting metrics are locally isometric or not is unclear and it requires further investigation. As an illustrative example, we have briefly discussed global properties of the solutions with at least one hypersurface orthogonal Killing vector, focusing on black hole type solutions.

In this paper, we have given all the solutions in their local form. It is certainly of great interest to perform the complete global analysis of all type D solutions presented here, especially in the case with nonconstant scalar curvature. It is also of great interest to extend the results of this paper to other theories of 3D massive gravity. Using our approach, this can be done for ''general massive gravity'', for an extension of NMG by adding a parity-violating Chern-Simon term, as well as for the theory of NMG with higher-order curvature invariants. We hope to return to these issues in our future works.

[1] A. Staruszkiewicz, Acta Phys. Pol. **24**, 735 (1963).
[2] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[3] S. Carlip, Classical Quantum Gravity **22**, R85 (2005).
[4] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[5] E. Witten, arXiv:0706.3359.
[6] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406 (E) (1988).
[7] S. Deser, in *Quantum Theory of Gravity*, edited by S. Christensen (A. Hilger Ltd, London, 1984).
[8] W. Li, W. Song, and A. Strominger, J. High Energy Phys. 04 (2008) 082.
[9] A. Maloney, W. Song, and A. Strominger, Phys. Rev. D **81**, 064007 (2010).
[10] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
[11] M. Nakasone and I. Oda, Prog. Theor. Phys. **121**, 1389 (2009).
[12] S. Deser, Phys. Rev. Lett. **103**, 101302 (2009).
[13] İ. Güllü and B. Tekin, Phys. Rev. D **80**, 064033 (2009).
[14] G. Clément, Classical Quantum Gravity **26**, 105015 (2009).
[15] Y. Nutku and P. Baekler, Ann. Phys. (N.Y.) **195**, 16 (1989).
[16] Y. Nutku, Classical Quantum Gravity **10**, 2657 (1993).
[17] M. Gürses, Classical Quantum Gravity **11**, 2585 (1994).
[18] E. Ayón-Beato, G. Giribet, and M. Hassaïne, J. High Energy Phys. 05 (2009) 029.
[19] I. Bakas and C. Sourdis, Classical Quantum Gravity **28**, 015012 (2011).
[20] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. D **79**, 124042 (2009).
[21] J. Oliva, D. Tempo, and R. Troncoso, J. High Energy Phys. 07 (2009) 011.
[22] M. Gürses, Classical Quantum Gravity **27**, 205018 (2010).
[23] A. Ghodsi and M. Moghadassi, Phys. Lett. B **695**, 359 (2011).
[24] S. Nam, J. D. Park, and S. H. Yi, Phys. Rev. D **82**, 124049 (2010).
[25] E. Ayón-Beato, A. Garbarz, G. Giribet, and M. Hassaïne, Phys. Rev. D **80**, 104029 (2009).
[26] U. Camara dS and G. M. Sotkov, Phys. Lett. B **686**, 268 (2010); U. Camara dS and G. M. Sotkov, Phys. Lett. B **694**, 94 (2010).
[27] Y. Liu and Y. W. Sun, J. High Energy Phys. 04 (2009) 106.
[28] Y. Liu and Y. W. Sun, J. High Energy Phys. 05 (2009) 039.
[29] A. Sinha, J. High Energy Phys. 06 (2010) 061.
[30] H. Ahmedov and A. N. Aliev, Phys. Rev. Lett. **106**, 021301 (2011).
[31] H. Ahmedov and A. N. Aliev, Phys. Lett. B **694**, 143 (2010).
[32] D. K. Chow, C. N. Pope, and E. Sezgin, Classical Quantum Gravity **27**, 105001 (2010).
[33] G. S. Hall, T. Morgan, and Z. Perjes, Gen. Relativ. Gravit. **19**, 1137 (1987).
[34] A. García, F. W. Hehl, C. Heinicke, and A. Macías, Classical Quantum Gravity **21**, 1099 (2004).
[35] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[36] A. N. Aliev and Y. Nutku, Classical Quantum Gravity **13**, L29 (1996).