

Thermodynamic geometry of charged rotating BTZ black holesM. Akbar,¹ H. Quevedo,^{2,3} K. Saifullah,⁴ A. Sánchez,⁵ and S. Taj^{1,3}¹*Center for Advanced Mathematics and Physics, National University of Sciences and Technology, H-12, Islamabad, Pakistan*²*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, México, DF 04510, Mexico*³*ICRANet, Dipartimento di Fisica, Università di Roma La Sapienza, I-00185 Roma, Italy*⁴*Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan*⁵*Departamento de Posgrado, CIIDET, AP 752, Querétaro, QRO 76000, Mexico*

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We study the thermodynamics and the thermodynamic geometries of charged rotating Bañados-Teitelboim-Zanelli black holes in $(2 + 1)$ -gravity. We investigate the thermodynamics of these systems within the context of the Weinhold and Ruppeiner thermodynamic geometries and the recently developed formalism of geometrothermodynamics. Considering the behavior of the heat capacity and the Hawking temperature, we show that Weinhold and Ruppeiner geometries cannot describe completely the thermodynamics of these black holes and of their limiting case of vanishing electric charge. In contrast, the Legendre invariance imposed on the metric in geometrothermodynamics allows one to describe the charged rotating Bañados-Teitelboim-Zanelli black holes and their limiting cases in a consistent and invariant manner.

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I. INTRODUCTION

An important characteristic of a black hole consists of its thermodynamic features: a Hawking temperature proportional to its surface gravity on the horizon and an entropy proportional to its horizon area [1,2], satisfying the first law of black hole thermodynamics [3]. However, it is still a challenging problem to find the statistical origin of black hole thermodynamics. As one knows, the divergence of the heat capacity is an indication of a second-order phase transition in ordinary thermodynamic systems. Using this fact, Davies [4] argued that phase transitions appear in black hole thermodynamics and the phase transition point is the one where the heat capacity diverges [5].

On the other hand, several authors have proposed to include geometric concepts in order to investigate the properties of the equilibrium space of thermodynamic systems. For instance, Weinhold [6] introduced in the equilibrium space a Riemannian metric defined in terms of the second derivatives of the internal energy U with respect to entropy and other extensive variables of a thermodynamic system. The Weinhold metric is given by

$$g_{ij}^W = \partial_i \partial_j U(S, N^r). \quad (1)$$

However, the geometry based on this metric seems to be meaningless in the context of pure equilibrium thermodynamics. In 1979, Ruppeiner [7] introduced a Riemannian metric structure in thermodynamic fluctuation theory, and related it to the second derivatives of the entropy. This geometric structure was used to find the significance of the distance between equilibrium states and to study the thermodynamics of equilibrium systems. It was observed by Ruppeiner [8,9] that in thermodynamic fluctuation theory

the Riemannian curvature of the Ruppeiner metric measures the complexity of the underlying statistical mechanical model. The Ruppeiner metric is defined as

$$g_{ij}^R = -\partial_i \partial_j S(M, N^r), \quad (2)$$

where S is the entropy, U denotes the energy and N^r are other extensive variables of the system. The Ruppeiner geometry is conformally related to the Weinhold geometry [10,11] as

$$ds_R^2 = \frac{1}{T} ds_W^2, \quad (3)$$

where T is the temperature of the system under consideration. Equation (3) often provides a more convenient way to compute the Ruppeiner metric.

One of the aims of the application of geometry in thermodynamics is to describe phase transitions in terms of curvature singularities and to interpret curvature as a measure of thermodynamic interaction. Since the proposal of Weinhold, many investigations have been carried out to understand the thermodynamic geometry of various thermodynamic systems. The Weinhold and Ruppeiner geometries have been analyzed in a number of black hole families to study phase space, critical behavior, and stability properties [12–22]. In some particular cases, it was found that Weinhold and Ruppeiner geometries carry information about the phase transitions structure. In fact, this is true in the case of the ideal gas, whose curvature vanishes, and the van der Waals gas for which the thermodynamic curvature becomes singular at those points where phase transitions occur. Unfortunately, the obtained results are contradictory in the case of black holes. For instance, for the Kerr black hole the Weinhold metric predicts no phase transitions at all [15] whereas the Ruppeiner metric, with

a very specific thermodynamic potential, predicts phase transitions which are compatible with the results of standard black hole thermodynamics [18]. Nevertheless, a change of the thermodynamic potential affects the Ruppeiner geometry in such a way that the resulting curvature singularity does not correspond to a phase transition. Another example is provided by the Bañados-Teitelboim-Zanelli (BTZ) black hole thermodynamics for which the curvature of the equilibrium space turns out to be flat [19–21]. This flatness is usually interpreted as a consequence of the lack of thermodynamic interaction. However, if one applies an invariant approach the resulting manifold is curved [23].

Recently, the formalism of geometrothermodynamics (GTD) was developed in order to unify in a consistent manner the geometric properties of the phase space and the space of equilibrium states [24–26]. Legendre invariance plays an important role in this formalism. It has been shown that there exist thermodynamic metrics that correctly describe the thermodynamic behavior of the ideal and the van der Waals gas. In fact, for the ideal gas the curvature vanishes whereas for the van der Waals gas the curvature is nonzero and diverges only at those points where phase transitions take place. Moreover, there exists a thermodynamic metric with nonvanishing curvature which correctly describes the thermodynamic properties of black holes [27].

In fact, the problem of using Weinhold or Ruppeiner metrics in equilibrium space is that the results can depend on the choice of thermodynamic potential, i. e., the results are not invariant with respect to Legendre transformations [28,29]. These results indicate that, in the case of black holes, geometry and thermodynamics are compatible only for a specific thermodynamic potential. However, it is well known that ordinary thermodynamics does not depend on the thermodynamic potential, i.e., it is invariant with respect to Legendre transformations. GTD incorporates Legendre invariance into the geometric structures of the phase space and equilibrium space so that the results do not depend on the choice of thermodynamic potential. The phase transition structure contained in the heat capacity of black holes [30] becomes completely integrated in the scalar curvature of the Legendre-invariant metric so that a curvature singularity corresponds to a phase transition.

In the present work we investigate the Weinhold and Ruppeiner geometries of charged rotating BTZ (CR-BTZ) black holes. We show that both geometries are curved, indicating the presence of thermodynamic interaction. However, in the limiting case of vanishing electric charge certain inconsistencies appear. We also use GTD to derive a Legendre-invariant metric for the CR-BTZ black holes. Our main result is that GTD correctly describes the thermodynamics of the CR-BTZ black holes and that no inconsistencies appear in the limiting cases of vanishing electric charge.

This paper is organized as follows. In Sec. II, we review some known facts about CR-BTZ black holes and present the main features of their thermodynamics. In Sec. III, we examine the Weinhold and Ruppeiner geometries of the CR-BTZ black hole. We review and apply 3-dimensional GTD in Sec. IV. Finally, the last Section contains the conclusions. Throughout this paper we use the units in which $c = \hbar = 8G = 1$.

II. THE CHARGED ROTATING BTZ BLACK HOLE

The charged rotating BTZ black hole solutions [31–34] in $(2 + 1)$ spacetime dimensions are particular solutions to the field equations derived from the action [35] (see also [36] for a critical review)

$$I = \frac{1}{2\pi} \int dx^3 \sqrt{-g} \left(R + 2\Lambda - \frac{\pi}{2} F_{\mu\nu} F^{\mu\nu} \right). \quad (4)$$

The Einstein field equations are given by

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \pi T_{\mu\nu}, \quad (5)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field:

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F^2. \quad (6)$$

The corresponding line element for the CR-BTZ solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\phi - \frac{J}{2r^2} dt \right)^2, \quad (7)$$

with lapse function:

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r. \quad (8)$$

Here M and J are the mass and angular momentum, respectively, and Q is the charge carried by the black hole. The horizons of the CR-BTZ metric correspond to the roots of the lapse function $f(r)$. Depending on these roots there are three cases for the CR-BTZ configuration [37,38]: Two distinct roots r_{\pm} determine the standard CR-BTZ black holes; two repeated real roots correspond to a single horizon that determines an extreme black hole; the absence of real roots implies that no horizon exists and the configuration corresponds to that of a naked singularity. We shall investigate the first case in this work.

In terms of the exterior horizon radius r_+ , the black hole mass and the angular momentum are given, respectively, by

$$M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2} - \frac{\pi}{2} Q^2 \ln(r_+), \quad (9)$$

and

$$J = 2r_+ \sqrt{M - \frac{r_+^2}{l^2} - \frac{\pi}{2} Q^2 \ln(r_+)}, \quad (10)$$

with the corresponding angular velocity on the horizon

$$\Omega = 2r_+^2 \frac{\partial M}{\partial J} \Big|_{r=r_+} = \frac{J}{2r_+^2} = \frac{1}{r_+} \sqrt{M - \frac{r_+^2}{l^2} - \frac{\pi}{2} Q \ln(r_+)}. \quad (11)$$

The Hawking temperature T at the black hole horizon is

$$T = \frac{1}{4\pi} \frac{df}{dr} = \frac{1}{4\pi} \left(\frac{2r_+}{l^2} - \frac{J^2}{2r_+^3} - \frac{\pi}{2} \frac{Q^2}{r_+} \right), \quad (12)$$

and the electric potential is given by

$$\phi = \frac{\partial M}{\partial Q} \Big|_{r=r_+} = -\pi Q \ln r_+. \quad (13)$$

Furthermore, using the fundamental postulate of black hole thermodynamics, the entropy of the CR-BTZ black hole is defined as

$$S = 4\pi r_+. \quad (14)$$

In terms of this entropy, the corresponding thermodynamic fundamental equation and the temperature for the CR-BTZ black hole are given, respectively, by

$$M = \left(\frac{S}{4\pi l} \right)^2 + \left(\frac{2\pi J}{S} \right)^2 - \frac{\pi Q^2}{2} \ln \frac{S}{4\pi}, \quad (15)$$

and

$$T = \left(\frac{\partial M}{\partial S} \right)_{J,Q} = \frac{S}{8\pi^2 l^2} - \frac{8\pi^2 J^2}{S^3} - \frac{\pi Q^2}{2S}. \quad (16)$$

The thermodynamic quantities T , S , J , Q and M obey the first law of thermodynamics

$$dM = TdS + \Omega dJ + \phi dQ. \quad (17)$$

An important quantity for the analysis of the thermodynamic properties is the heat capacity of the CR-BTZ black hole [37], $C_{J,Q} = (\partial M / \partial T)_{J,Q}$, which is given by

$$C_{J,Q} = S \frac{S^4 - 4\pi^3 l^2 Q^2 S^2 - 64\pi^4 l^2 J^2}{S^4 + 4\pi^3 l^2 Q^2 S^2 + 192\pi^4 l^2 J^2}, \quad (18)$$

or using the horizon radius r_+ as coordinate, by

$$C_{J,Q} = 4\pi r_+ \frac{4r_+^4 - \pi l^2 Q^2 r_+^2 - l^2 J^2}{4r_+^4 + \pi l^2 Q^2 r_+^2 + 3l^2 J^2}. \quad (19)$$

Furthermore, the capacitances $\tilde{C}_{S,Q} = (\partial J / \partial \Omega)_{S,Q}$ and $\tilde{C}_{S,J} = (\partial Q / \partial \phi)_{S,J}$ can also contain important information about the thermodynamic behavior of black holes [39]. From the fundamental equation (15), we obtain

$$\tilde{C}_{S,Q} = \frac{S^2}{4\pi^2}, \quad \tilde{C}_{S,J} = -\frac{1}{\pi \ln(S/4\pi)} = -\frac{1}{\pi \ln r_+}. \quad (20)$$

From expressions (12) and (14) we conclude that the condition

$$S^4 - 4\pi^3 l^2 Q^2 S^2 - 64\pi^4 l^2 J^2 > 0 \quad (21)$$

must be satisfied in order for the temperature to be positive definite, a requirement which follows from the standard laws of black hole thermodynamics. From the above condition and the expression (18), it follows that the heat capacity is always positive definite. This is an important observation which implies that a CR-BTZ black hole with a positive definite temperature must be a thermodynamically stable configuration. In fact, a change of sign of the heat capacity is usually associated with a drastic change of the stability properties of a thermodynamic system; a negative heat capacity represents a region of instability whereas the stable domain is characterized by a positive heat capacity.

It is worth mentioning that the heat capacity is a regular function for all real positive values of the exterior horizon radius. In fact, the denominator of the expression (19) is always positive and, consequently, $C_{J,Q}$ is a regular function, except in the pathological case where $S = J = Q = 0$. On the other hand, in black hole thermodynamics, divergences of the heat capacity are associated with second-order phase transitions. This implies that a CR-BTZ black hole cannot undertake a phase transition associated with a divergence of the heat capacity. However, the capacitance $\tilde{C}_{S,J}$ becomes singular for a black hole with horizon radius $r_+ = 1$, at which the electric potential ϕ vanishes. This indicates the presence of a second-order phase transition. In fact, since the heat capacity can be written as $C_{J,Q} = \left(\frac{T}{(\partial^2 M / \partial S^2)} \right)_{J,Q}$, a second-order phase transition takes place if the condition $(\partial^2 M / \partial S^2) = 0$ is satisfied. Therefore, for the capacitance $\tilde{C}_{S,J} = \left(\frac{1}{(\partial^2 M / \partial Q^2)} \right)_{S,J}$ we can consider by analogy the singularity situated at $(\partial^2 M / \partial Q^2) = 0$ as a second-order phase transition.

The above observations demonstrate that the CR-BTZ black hole is a completely stable thermodynamic system with no phase transition structure following from the heat capacity $C_{J,Q}$. The only second-order phase transition can occur for the horizon radius $r_+ = 1$ at which the capacitance $\tilde{C}_{S,J}$ diverges. In this work, we will use this fact in order to test different geometric descriptions of the thermodynamics of the CR-BTZ black hole. Figure 1 shows for selected values of J , Q and l the behavior of heat capacity and temperature for a charged black hole (solid line) and for a neutral black hole (dotted line). The comparison of both curves shows that the charge essentially increases the value of the horizon radius at which the heat capacity and the temperature vanish. As the value of the horizon radius increases the contribution of the charge decreases. Finally, for very large values of the horizon radius the heat capacities and the temperatures coincide, indicating that the contribution of the charge is negligible.

One would expect that the limiting case $T \rightarrow 0$ or, equivalently, $C \rightarrow 0$ corresponds to an extreme black hole with only one horizon of radius, say, r_* . To analyze this question it is necessary to find the domain of parameters for which the equation $f(r) = 0$ allows only one

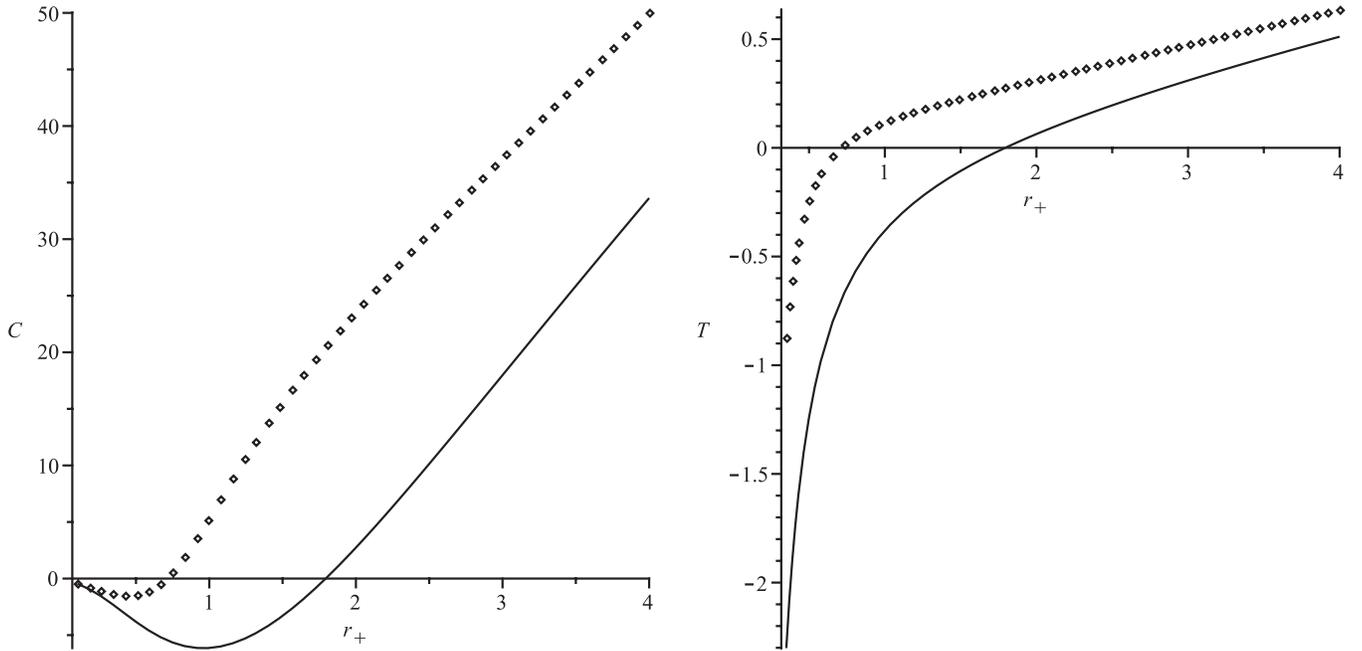


FIG. 1. Behavior of the heat capacity and temperature as functions of the event horizon radius r_+ of a CR-BTZ black hole with $Q = 2$, $J = 1$ and $l = 1$. Temperature and heat capacity vanish at $r_+ \approx 1.79$. The unstable region ($C_{J,Q} < 0$) corresponds to an unphysical region with negative temperature. To illustrate the contribution of the charge we include the plots (dotted curves) for the case of the nonstatic BTZ black hole with $Q = 0$ [23,44].

positive real root, and to calculate the value of T for this domain. However, the equation $f(r) = 0$ cannot be solved analytically because of the presence of the logarithmic term $\ln r$. An alternative procedure consists in solving the equation $T = 0$ for r^2 to obtain

$$r^2 := r_*^2 = \frac{\pi l^2 Q^2}{8} (1 + \eta), \quad \eta = \sqrt{1 + \frac{16J^2}{\pi^2 l^2 Q^4}}, \quad (22)$$

and introducing this solution into the equation $f(r) = 0$ to obtain the value of the mass at this radius, i. e.,

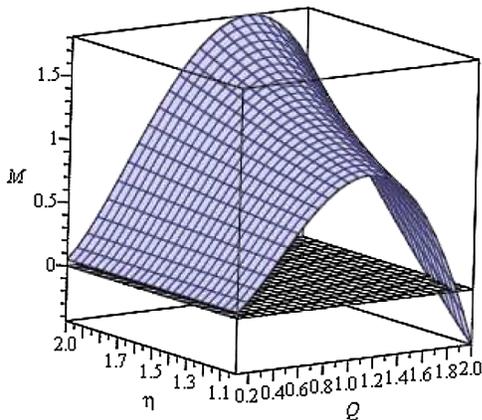


FIG. 2 (color online). The mass M of an extreme CR-BTZ black hole with horizon radius $r_* = \pi l^2 Q^2 (1 + \eta) / 8$ as a function of the charge Q and the angular parameter $\eta > 1$. Here, we chose $l = 1$ as a representative value. The plane $M = 0$ is plotted to visualize the region where $M > 0$.

$$M = \frac{\pi Q^2}{4} \left[\eta - \ln(1 + \eta) - \ln \frac{\pi l^2 Q^2}{8} \right], \quad (23)$$

where we replaced J^2 by using the definition of the auxiliary parameter η . Now the question is whether the last expression represents a physical mass, i.e. whether it is positive. A numerical analysis shows that for any value $\eta > 1$, a condition that follows from the definition of η , there always exists a combination of values for Q and l such that M is positive. Figure 2 shows an example of the behavior of the mass for a fixed value of the parameter l . We conclude that the limit $T \rightarrow 0$ indeed corresponds to an extreme black hole.

Notice that the above numerical analysis is necessary only if the solution must correspond to a black hole with positive mass. If this condition is not imposed, a simpler analysis is possible. Indeed, since the only solution of the equation $f'(r) = 0$ is $r = r_*$ and $f(r)$ is a smooth function for $r \in (0, \infty)$ with $\lim_{r \rightarrow 0, +\infty} f(r) = +\infty$, the equation $f(r) = 0$ has only one solution for an extreme black hole $r = r_{\text{ebh}}$ if and only if $f'(r_{\text{ebh}}) = 0$. On the other hand, since $T = 0$ if and only if $f'(r) = 0$, it follows that any black hole with $T = 0$ is extremal.

III. WEINHOLD AND RUPPEINER GEOMETRIES

Now we construct the thermodynamic geometry of the CR-BTZ black hole by using the Weinhold metric (1). In this case the extensive variables are $N^r = \{J, Q\}$ so that the general Weinhold metric becomes

$$ds_W^2 = \left(\frac{\partial^2 M}{\partial S^2}\right)dS^2 + \left(\frac{\partial^2 M}{\partial J^2}\right)dJ^2 + \left(\frac{\partial^2 M}{\partial Q^2}\right)dQ^2 + 2\left(\frac{\partial^2 M}{\partial S \partial J}\right)dSdJ + 2\left(\frac{\partial^2 M}{\partial J \partial Q}\right)dJdQ + 2\left(\frac{\partial^2 M}{\partial Q \partial S}\right)dQdS, \quad (24)$$

and in the special case of the CR-BTZ black hole we have

$$ds_W^2 = \left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 J^2}{S^4} + \frac{\pi Q^2}{2S^2}\right)dS^2 + \frac{8\pi^2}{S^2}dJ^2 - \pi \ln\left(\frac{S}{4\pi}\right)dQ^2 - \frac{32\pi^2 J}{S^3}dSdJ - 2\frac{\pi Q}{S}dSdQ. \quad (25)$$

The corresponding scalar curvature is given by

$$R_W = \frac{l^2 r_+^2 [-4r_+^4 (1 + 2 \ln r_+ + 4 \ln r_+^2) + \pi l^2 Q^2 (9 + 2 \ln r_+) r_+^2 + J^2 l^2 (1 + 2 \ln r_+)]}{[-4r_+^4 \ln r_+ - \pi Q^2 l^2 (\ln r_+ + 2) r_+^2 + J^2 l^2 \ln r_+]^2}. \quad (26)$$

The general behavior of the scalar curvature for the Weinhold geometry is illustrated in Fig. 3. We see that the thermodynamic curvature is regular for all positive values of the horizon radius. At the value $r_+ \approx 1.79$ (with $l = 1$, $J = 1$, $Q = 2$), at which the temperature vanishes, the scalar curvature is $R_W \approx 0.0527$. Moreover, it is positive and regular in the interval $0.185 < r_+ < 1.79$, a region where the temperature is negative, and in the interval $1.79 \leq r_+ < 2.4$, a region where the temperature is positive. This means that the Weinhold thermodynamic curvature cannot differentiate between a CR-BTZ black hole with positive temperature and a similar configuration with negative temperature.

We now investigate the limiting case of a vanishing charge. The additional extensive variable in this case is $N^r = \{J\}$ so that the Weinhold metric reduces to

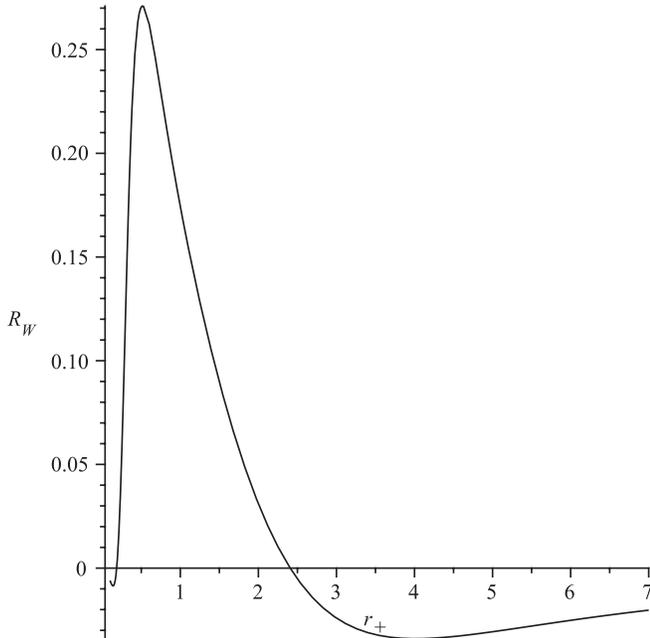


FIG. 3. Thermodynamic curvature for the Weinhold (R_W) geometry as a function of the event horizon radius, r_+ , of the CR-BTZ black hole. Here, the free parameters are chosen as $Q = 2$, $J = 1$ and $l = 1$. The curvature is completely regular in the entire domain of r_+ .

$$ds_W^2 = \left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 J^2}{S^4}\right)dS^2 - \frac{32\pi^2 J}{S^3}dSdJ + \frac{8\pi^2}{S^2}dJ^2, \quad (27)$$

and the corresponding scalar curvature becomes

$$R_W = 16 \frac{\pi^2 l^2 S^6}{(S^4 - 64\pi^4 J^2 l^2)^2}. \quad (28)$$

We see that there exists a true curvature singularity at the value $S^4 = 64\pi^4 J^2 l^2$ that, according to Eq. (21) with $Q = 0$, corresponds to the limit of vanishing temperature or, equivalently, to the extreme black hole limit. This result shows that the Weinhold thermodynamic curvature in this case correctly describes the transition from a region with positive and well-defined temperature to a region with an unphysical negative temperature. This is in contrast to what we obtained in the case of a charged black hole in which the Weinhold thermodynamic curvature is not able to recognize the transition to an extreme black hole with zero temperature.

Let us now consider the Ruppeiner geometry. A direct computation of the Ruppeiner metric (2) cannot be performed because it is not possible to rewrite explicitly the fundamental equation (15) in the form $S = S(M, J, Q)$. Nevertheless, indirect methods of computation are available. Indeed, if we assume the invariance of the line element under a change of thermodynamic potential, the relationship (3) can be used to derive the Ruppeiner metric from the Weinhold metric. Then, we obtain

$$ds_R^2 = \frac{dS^2}{S} + \frac{\pi}{T} \left[\frac{8\pi}{S^2} \left(\frac{2J}{S} dS - dJ \right)^2 + \left(\frac{Q}{S} dS - dQ \right)^2 - \left(1 + \ln \frac{S}{4\pi} \right) dQ^2 \right]. \quad (29)$$

A second indirect and equivalent method to compute the components of the Ruppeiner metric consists in applying the chain rule for partial derivatives. For instance, from the definition of the temperature given in Eq. (16) one obtains that $g_{MM}^R = -\left(\frac{\partial^2 S}{\partial M^2}\right)_{J,Q} = -\left(\frac{\partial(1/T)}{\partial M}\right)_{J,Q} = \frac{1}{T^3} \left(\frac{\partial T}{\partial S}\right)_{J,Q}$. This means that $g_{MM}^R = \frac{1}{T^3} g_{SS}^W$ or, using the appropriate

coordinates in the corresponding line elements, $g_{MM}^R = \frac{1}{l} g_{MM}^W$ as stated in Eq. (3). The remaining metric components can be calculated in a similar manner. For instance, $g_{JJ}^R = -(\frac{\partial^2 S}{\partial J^2})_{M,Q} = -\frac{\partial}{\partial J} (\frac{1}{\partial J / \partial S})_{M,Q} = (\frac{\partial J}{\partial S})^{-3} \times (\frac{\partial(\partial J / \partial S)}{\partial S})_{M,Q}$, and $g_{QQ}^R = -(\frac{\partial^2 S}{\partial Q^2})_{M,J} = (\frac{\partial Q}{\partial S})^{-3} (\frac{\partial(\partial Q / \partial S)}{\partial S})_{M,J}$, etc. None of these computations requires the explicit use of the equation $S = S(M, J, Q)$.

The thermodynamic curvature scalar R_R of the metric (29) turns out to be nonzero, i.e., the space of its thermodynamic equilibrium states is nonflat. The explicit form of R_R cannot be written in a compact form. Therefore, we perform a numerical analysis of its behavior and the result is illustrated in Fig. 4.

The singularity located at $r_+ \approx 1.79$ represents the limit for which the heat capacity vanishes and the temperature becomes negative. This shows that the Ruppeiner thermodynamic curvature describes correctly the behavior of the CR-BTZ black hole.

In the limiting case of a vanishing charge, it is possible to rewrite the fundamental equation (9) as $S = S(M, J)$ in the following manner

$$S = 4\pi r_+ = \pi \sqrt{8Ml^2 \left(1 + \sqrt{1 - \frac{J^2}{M^2 l^2}}\right)}, \quad (30)$$

so that the Ruppeiner metric can be computed by using the definition (2). Then

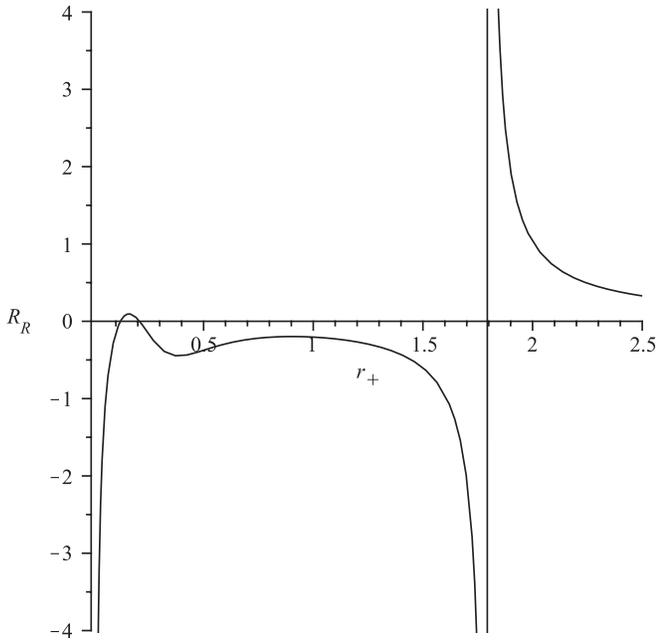


FIG. 4. Thermodynamic curvature, R_R , of the Ruppeiner geometry as a function of the event horizon, r_+ , for a CR-BTZ black hole with $J = l = 1$, $Q = 2$. The only singularity is located at $r_+ \approx 1.79$.

$$ds_R^2 = -\frac{\pi l^2}{(r_+^2 - r_-^2)^3} [r_+(r_+^2 + 3r_-^2)(l^2 dM^2 + dJ^2) - 2lr_-(3r_+^2 + r_-^2)dM dJ], \quad (31)$$

where

$$r_{\pm}^2 = \frac{l^2 M}{2} \left(1 \pm \sqrt{1 - \frac{J^2}{l^2 M^2}}\right). \quad (32)$$

A straightforward calculation shows that the curvature of this metric vanishes identically, indicating the absence of thermodynamic interaction, i.e., the thermodynamic variables M and J do not generate thermodynamic interaction. This is a peculiar result because, as we have seen above, the Ruppeiner geometry correctly describes the thermodynamic behavior of the CR-BTZ black hole. This implies that only the charge Q acts as a source of thermodynamic interaction in the Ruppeiner geometry. It seems that there is no specific reason for the existence of this difference between thermodynamic variables of this particular black hole configuration.

IV. GEOMETROTHERMODYNAMICS OF THE CR-BTZ BLACK HOLE

In order to describe a thermodynamic system with n degrees of freedom, we consider in GTD *the thermodynamic phase space* which is defined mathematically as a Riemannian contact manifold (\mathcal{T}, Θ, G) , where \mathcal{T} is a $(2n + 1)$ -dimensional manifold, Θ defines a contact structure on \mathcal{T} and G is a Legendre-invariant metric on \mathcal{T} . The *space of equilibrium states* is an n -dimensional Riemannian manifold (\mathcal{E}, g) , where $\mathcal{E} \subset \mathcal{T}$ is defined by a smooth mapping $\varphi: \mathcal{E} \rightarrow \mathcal{T}$ such that the pullback $\varphi^*(\Theta) = 0$ and a Riemannian structure g is induced naturally in \mathcal{E} by means of $g = \varphi^*(G)$. It is then expected in GTD that the physical properties of a thermodynamic system in a state of equilibrium can be described in terms of the geometric properties of the corresponding space of equilibrium states \mathcal{E} [24].

To be more specific we introduce in the phase space \mathcal{T} the set of independent coordinates $Z^A = (\Phi, E^a, I^a)$ with $A = 0, \dots, 2n$ and $a = 1, \dots, n$, where Φ represents the thermodynamic potential, and E^a and I^a are the extensive and intensive thermodynamic variables, respectively. Consider the Legendre-invariant Gibbs 1-form

$$\Theta_G = d\Phi - \delta_{ab} I^a dE^b, \quad \delta_{ab} = \text{diag}(1, \dots, 1). \quad (33)$$

The pair (\mathcal{T}, Θ) is called a contact manifold [40] if \mathcal{T} is differentiable and Θ satisfies the condition $\Theta \wedge (d\Theta)^n \neq 0$. Legendre invariance [41] guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

The smooth mapping $\varphi: \mathcal{E} \rightarrow \mathcal{T}$ is given in terms of coordinates as $\varphi: \{E^a\} \mapsto \{Z^A\} = \{\Phi(E^a), E^a, I^a(E^a)\}$.

Consequently, the condition $\varphi^*(\Theta) = 0$ can be written as the expressions

$$\frac{\partial\Phi}{\partial E^a} = \delta_{ab}I^b, \quad d\Phi = \delta_{ab}I^a dE^b, \quad (34)$$

which in ordinary thermodynamics correspond to the first law of thermodynamics and the conditions for thermodynamic equilibrium, respectively. In thermodynamics $\phi(E^a)$ is known as the fundamental equation from which all the equations of state of the system can be derived [42]. The second law of thermodynamics is implemented in GTD as the convexity condition $\partial^2\Phi/\partial E^a\partial E^b \geq 0$. Furthermore, the Gibbs-Duhem identity can be expressed as $\Phi = \delta_{ab}I^a E^b$ and $\delta_{ab}E^a dI^b = 0$.

For the geometric description of the thermodynamics of the CR-BTZ black hole in GTD, we first introduce the seven-dimensional phase space \mathcal{T} with coordinates M, S, J, Q, T, Ω and ϕ , a contact 1-form

$$\Theta = dM - TdS - \Omega dJ - \phi dQ, \quad (35)$$

which satisfies the condition $\Theta \wedge (d\Theta)^3 \neq 0$, and a Legendre-invariant metric

$$\begin{aligned} G &= ds_{\mathcal{T}}^2 \\ &= (dM - TdS - \Omega dJ - \phi dQ)^2 \\ &\quad + TS(-dTdS + d\Omega dJ + d\phi dQ). \end{aligned} \quad (36)$$

This particular metric is a special case of a metric used in [23] to describe the region of positive temperature of the BTZ black hole.

Let \mathcal{E} be a three-dimensional subspace of \mathcal{T} with coordinates $E^a = (S, Q, J)$, $a = 1, 2, 3$, defined by means of a smooth mapping $\varphi: \mathcal{E} \rightarrow \mathcal{T}$. The subspace \mathcal{E} is called the space of equilibrium states if $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ . Furthermore, a metric structure g is naturally induced on \mathcal{E} by applying the pullback on the metric G of \mathcal{T} , i.e., $g = \varphi^*(G)$. It is clear that the condition $\varphi^*(\Theta) = 0$ leads immediately to the first law of thermodynamics of black holes as given in Eq. (17). It also implies the existence of the fundamental equation $M = M(S, Q, J)$ and the conditions of thermodynamic equilibrium, Eqs. (11)–(13). Moreover, the induced metric

$$\begin{aligned} g &= ds_{\mathcal{E}}^2 = \varphi^*(G) \\ &= S \frac{\partial M}{\partial S} \left(-\frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial J^2} dJ^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2 \right) \end{aligned} \quad (37)$$

determines all the geometric properties of the equilibrium space \mathcal{E} . In the above expression we used the Euler identity to simplify the form of the conformal factor. In order to obtain the explicit form of the metric it is only necessary to specify the thermodynamic potential M as a function of S, J and Q as given in Eq. (15). Another advantage of the use of GTD is that it allows us to easily implement different thermodynamic representations of the fundamental

equation, given as $M = M(S, Q, J)$, $S = S(M, Q, J)$, $Q = Q(S, M, J)$ or $J = J(S, M, Q)$ and redefine the coordinates in \mathcal{T} and the smooth mapping ϕ in such a way that the condition $\varphi^*(\Theta) = 0$ generates on \mathcal{E} the corresponding fundamental equation in the S -, Q -, or the J -representation, respectively. The results obtained with different representations of the same fundamental equation are completely equivalent. To simplify the notation, in Eqs. (36) and (37) and in the rest of this work, we use G and g to denote the line elements $ds_{\mathcal{T}}^2$ and $ds_{\mathcal{E}}^2$, respectively, as it is usual in standard textbooks on mathematical physics (see, for instance, [43], p. 285).

For the CR-BTZ black hole, using the fundamental equation $M = M(S, J, Q)$ given in Eq. (15), the thermodynamic metric can be written as

$$\begin{aligned} g &= \frac{S^4 - 64\pi^4 J^2 l^2 - 4\pi^3 l^2 Q^2 S^2}{8\pi^2 l^2 S^2} \left[-\left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 l^2}{S^4} \right. \right. \\ &\quad \left. \left. + \frac{\pi Q^2}{2S^2} \right) dS^2 + \frac{8\pi^2}{S^2} dJ^2 - \pi \ln \frac{S}{4\pi} dQ^2 \right]. \end{aligned} \quad (38)$$

The corresponding thermodynamic curvature turns out to be nonzero and is given by

$$R_{\text{GTD}} = \frac{2l^4 r_+^4}{D_1 D_2} \left[\frac{1}{\ln^2 r_+} + \frac{1}{D_1 D_2} \left(\frac{N_1}{\ln r_+} + N_0 \right) \right] \quad (39)$$

where

$$\begin{aligned} D_1 &= 4r_+^4 + \pi l^2 Q^2 r_+^2 + 3l^2 J^2, \\ D_2 &= 4r_+^4 - l^2 J^2 - \pi l^2 Q^2 r_+^2, \end{aligned} \quad (40)$$

$$\begin{aligned} N_0 &= 4[6J^6 l^6 + 23\pi J^4 l^6 Q^2 r_+^2 + l^4 J^2 (15\pi^2 l^2 Q^4 + 8J^2) r_+^4 \\ &\quad + 4\pi l^4 Q^2 (14J^2 + \pi^2 Q^4 l^2) r_+^6 + 4l^2 (\pi^2 Q^4 l^2 - 40J^2) r_+^8 \\ &\quad - 16\pi Q^2 l^2 r_+^{10} + 128r_+^{12}], \end{aligned} \quad (41)$$

$$\begin{aligned} N_1 &= 2[-15J^6 l^6 - 6\pi J^4 l^6 Q^2 r_+^2 - 3l^4 J^2 (\pi^2 l^2 Q^4 - 20J^2) r_+^4 \\ &\quad + 128\pi J^2 l^4 Q^2 r_+^6 - 4l^2 (4J^2 - 5\pi^2 l^2 Q^4) r_+^8 \\ &\quad + 96\pi Q^2 l^2 r_+^{10} + 64r_+^{12}]. \end{aligned} \quad (42)$$

We see from the expression for the scalar curvature that the curvature singularities can be situated at those values of the parameters where $D_1 = 0$, $D_2 = 0$ or $\ln r_+ = 0$. For real values of the parameters the condition $D_1 = 4r_+^4 + \pi l^2 Q^2 r_+^2 + 3l^2 J^2 = 0$ cannot be satisfied. In fact, this term appears in the denominator of the heat capacity (19) and determines the absence of phase transitions of the CR-BTZ black hole that could be associated with divergencies of the heat capacity. Singularities determined by the roots of the equation $D_2 = 4r_+^4 - l^2 J^2 - \pi l^2 Q^2 r_+^2 = 0$ coincide with the points where $T = 0$ or, equivalently, where the heat capacity (19) vanishes. This implies that the non-physical region of negative temperatures is isolated from the allowed region with positive temperatures by a true

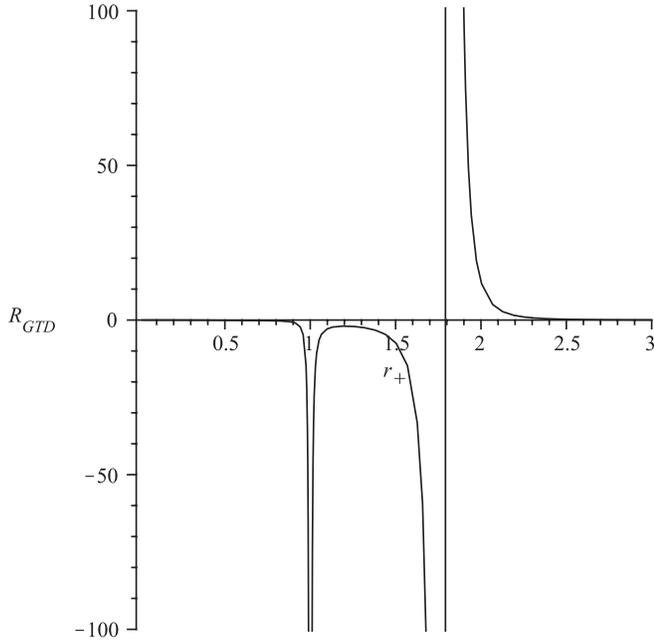


FIG. 5. Thermodynamic curvature of the CR-BTZ black hole (R_{GTD}) as a function of event horizon r_+ . A typical behavior is depicted for the specific values $Q = 2$, $J = l$ and $l = 1$.

curvature singularity. The third singularity located at $\ln r_+ = 0$ corresponds to a critical point that is not determined by the heat capacity (19). In fact, at $r_+ = 1$ capacitance $\tilde{C}_{S,J}$ given in Eq. (20) diverges, indicating the presence of a second-order phase transition. The singular behavior of the GTD scalar curvature is illustrated in Fig. 5.

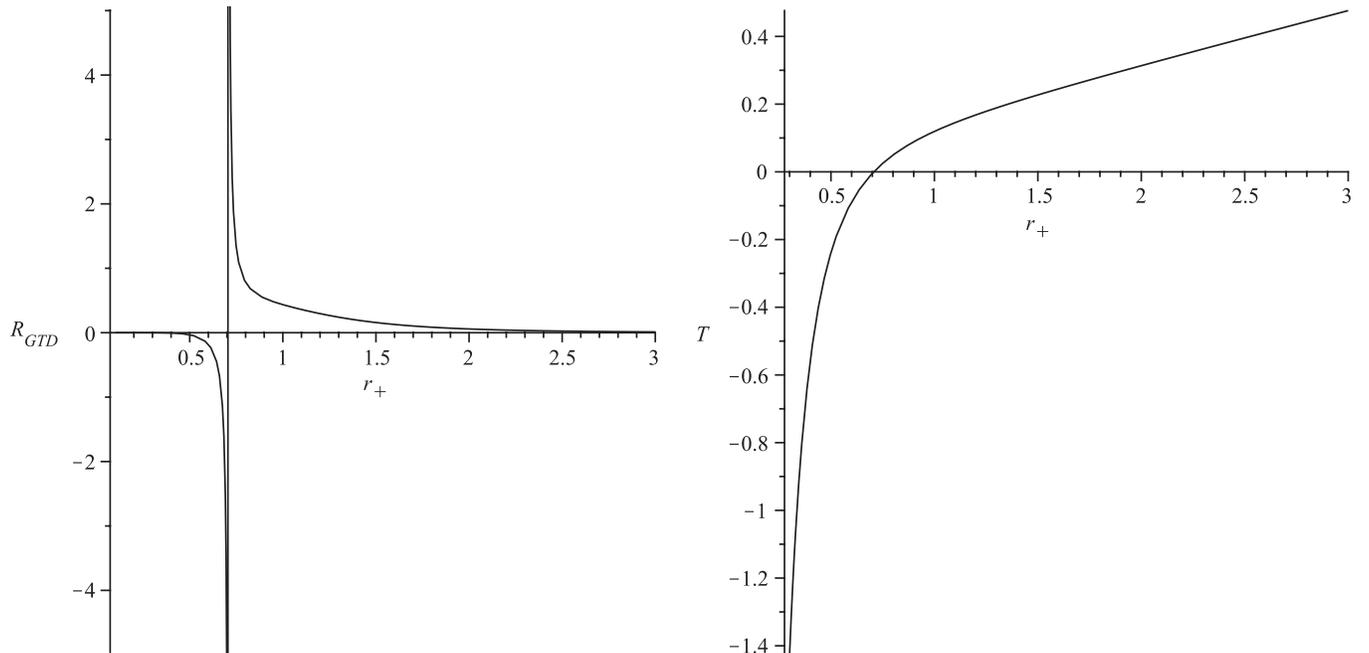


FIG. 6. Behavior of the scalar curvature in GTD and temperature as functions of the event horizon radius $r_+ = S/4\pi$ of a neutral rotating BTZ black hole with $J = 1$ and $l = 1$. The curvature singularity coincides with the point of zero temperature.

Let us now consider the limiting case of vanishing charge. The geometrothermodynamic metric reduces to

$$g = \frac{S^4 - 64\pi^4 J^2 l^2}{8\pi^2 l^2 S^2} \left[-\left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 l^2}{S^4} \right) dS^2 + \frac{8\pi^2}{S^2} dJ^2 \right], \quad (43)$$

and the corresponding scalar curvature can be written as

$$R_{GTD} = \frac{256l^4 \pi^4 S^8}{(S^4 + 192\pi^4 J^2 l^2)^2 (S^4 - 64\pi^4 J^2 l^2)}. \quad (44)$$

The behavior of this scalar and the temperature is depicted in Fig. 6. It follows that in general a curvature singularity appears when the condition $S^4 - 64\pi^4 J^2 l^2 = 0$ is satisfied which, according to Eq. (16) with $Q = 0$, corresponds to a zero temperature. We conclude that the invariant metric proposed in GTD correctly describes the limiting case of a neutral BTZ black hole.

V. CONCLUSIONS

In this work, we analyzed the thermodynamics and the thermodynamic geometry of the CR-BTZ black hole. By considering the behavior of the heat capacity and the Hawking temperature, we found that this black hole configuration is free of phase transitions and stable. In fact, the instability region is characterized by a nonphysical negative temperature. Moreover, we performed a numerical analysis which shows that in the limiting case of zero temperature the black hole becomes extreme.

We analyzed the thermodynamic geometry based on the Weinhold metric and found that the corresponding thermodynamic curvature is free of singularities in the entire equilibrium manifold. This result is not in accordance with the analysis of the behavior of the heat capacity and the Hawking temperature that indicates the presence of an unphysical region with negative temperature for $r_+ \leq 1.79$ (the additional parameters are chosen as $l = 1$, $J = 1$ and $Q = 2$). We conclude that Weinhold geometry does not describe correctly the thermodynamic geometry in this specific case. However, in the limiting case of a vanishing electric charge there exists a true curvature singularity that is located at the point where the temperature vanishes. It is not clear why the presence of an electric charge cannot be handled correctly in the context of the Weinhold thermodynamic metric.

Although it is not possible to calculate directly the Ruppeiner metric, it can be derived from the Weinhold metric by using a conformal transformation with the inverse of the temperature as the conformal factor. A numerical analysis of the Ruppeiner thermodynamic curvature shows that it is smooth and well-behaved in the region $r_+ > 1.79$, with a true curvature singularity situated at $r_+ \approx 1.79$. This is exactly the value of the horizon radius at which the Hawking temperature vanishes. We interpret this result as indicating that the Ruppeiner geometry correctly describes the thermodynamics of the CR-BTZ black hole. However, in the limiting case of vanishing electric charge the Ruppeiner metric turns out to be flat. Since a vanishing thermodynamic curvature is usually interpreted as indicating the absence of thermodynamic interaction, it is not clear why Ruppeiner geometry correctly describes the thermodynamics of the CR-BTZ black hole but fails in the limiting neutral case.

Finally, we analyzed the properties of a Legendre-invariant metric proposed in the context of GTD. In this case, the curvature can be calculated explicitly and it turns out that it possesses a true curvature singularity at those points where the Hawking temperature vanishes, and at the horizon radius $r_+ = 1$ where a second-order phase transition is located. In the entire region where the CR-BTZ black hole corresponds to a stable thermodynamic system with no phase transition structure, the thermodynamic curvature of GTD is described by a smooth function of all the thermodynamic variables. In the limiting case of vanishing electric charge, the metric proposed in GTD is also able to correctly describe the thermodynamic properties of the black hole configuration in the sense that it is finite and smooth in the region where the black hole is stable, but possesses a true curvature singularity at the point where the temperature vanishes. Since the Weinhold and Ruppeiner metrics are not invariant with respect to Legendre transformations, we conclude that the Legendre invariance imposed in the context of GTD is an important property to describe geometrically the thermodynamics of black holes without intrinsic contradictions.

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