PHYSICAL REVIEW D **83,** 084027 (2011)

Quantum modifications to gravity waves in de Sitter spacetime

Jen-Tsung Hsiang,[1,*] L. H. Ford,[2,†] Da-Shin Lee,[1,‡] and Hoi-Lai Yu[3,§]

[1]*Department of Physics, National Dong Hwa University, Hualien, Taiwan, Republic of China*
[2]*Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155, USA*
[3]*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China*
(Received 8 December 2010; published 15 April 2011)

We treat a model in which tensor perturbations of de Sitter spacetime, represented as a spatially flat model, are modified by the effects of the vacuum fluctuations of a massless conformally invariant field, such as the electromagnetic field. We use the semiclassical theory of gravity with the expectation value of the conformal field stress tensor as a source. We first study the stability of de Sitter spacetime by searching for growing, spatially homogeneous modes, and conclude that it is stable within the limits of validity of the semiclassical theory. We next examine the modification of linearized plane gravity waves by the effects of the quantum stress tensor. We find a correction term which is of the same form as the original wave, but displaced in phase by $\pi/2$, and with an amplitude which depends upon an initial time. The magnitude of this effect is proportional to the change in scale factor after this time. We discuss alternative interpretations of this time, but pay particular attention to the view that this is the beginning of inflation. So long as the energy scale of inflation and the proper frequency of the mode at the beginning of inflation are well below the Planck scale, the fractional correction is small. However, modes which are trans-Planckian at the onset of inflation can undergo a significant correction. The increase in amplitude can potentially have observable consequences through a modification of the power spectrum of tensor perturbations in inflationary cosmology. This enhancement of the power spectrum depends upon the initial time, and is greater for shorter wavelengths.

DOI: 10.1103/PhysRevD.83.084027 PACS numbers: 04.62.+v, 04.30.−w, 04.60.−m, 98.80.Cq

I. INTRODUCTION

Most versions of inflationary cosmology assume a period of exponential expansion in which the universe is approximately a portion of de Sitter spacetime. Quantum fields in de Sitter spacetime play a crucial role in creating the primordial spectrum of scalar and tensor perturbations. In addition, quantum effects can potentially modify the duration of inflation and possibly introduce instabilities. Recently, there has been work on the possible effects of quantum stress tensor fluctuations in inflation [1,2].

In the present paper, we examine some effects in the semiclassical theory, where gravity is coupled to the renormalized expectation value of a matter field stress tensor, the mean value around which stress tensor fluctuations occur. The semiclassical theory has been extensively studied and applied to scalar perturbations of de Sitter spacetime. (See, for example, Ref. [3] and references therein.) There seems to have been less attention paid to tensor perturbations, which will be the topic of this paper. A brief discussion was given by Starobinsky [4], and a more detailed derivation of the equations for tensor perturbations was given by Campos and Verdaguer [5]. We will treat a model in which the matter field is a conformal field, such as the electromagnetic field, and address two physical questions: the stability of de Sitter spacetime under tensor perturbations, and the effects of one-loop quantum matter field corrections upon the propagation of gravity waves in de Sitter spacetime.

In Sec. II, we review the aspects of the semiclassical theory needed for our analysis. Section III treats the geometric terms in the stress tensor expectation value. Here we find that these terms have no physical effect for our problems. The stability of the tensor perturbations is discussed in Sec. IV. The one-loop correction to gravity wave modes is derived in Sec. V, and the possible implications for inflationary cosmology are discussed in Sec. VI. Our results are summarized in Sec. VII.

We adopt the sign conventions of Ref. [6], and use units in which $\hbar = c = 1$.

II. WEAKLY PERTURBED DE SITTER SPACETIME

We will be concerned with the piece of global de Sitter spacetime which can be represented as a spatially flat Robertson-Walker universe with the metric

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2), \tag{1}$$

where $a(\eta) = -1/(H\eta)$ and $\eta < 0$ is the conformal time coordinate. We wish to consider tensor perturbations of this geometry, which describe gravitational waves on the de Sitter background. Let the perturbed metric be

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}, \tag{2}$$

*cosmology@gmail.com
†ford@cosmos,phy.tufts.edu
‡dslee@mail.ndhu.edu.tw
§hlyu@phys.sinica.edu.tw

where $\gamma_{\mu\nu}$ is the background metric of Eq. (1) and $h_{\mu\nu}$ is the perturbation. We will employ the transverse trace-free gauge defined by

$$h^{\mu\nu}{}_{;\nu} = 0, \qquad h = 0, \quad \text{and} \quad h^{\mu\nu}u_\nu = 0. \tag{3}$$

Here $u^\nu = \delta^\nu_t$ is the four-velocity of the comoving observers, covariant derivatives are taken with respect to the fixed de Sitter background, and indices are raised and lowered by the background metric. These conditions remove all of the gauge freedom and leave only the two physical degrees of freedom associated with the possible polarizations of a gravity wave.

It was shown long ago by Lifshitz [7] that the mixed components h^ν_μ satisfy the *scalar* wave equation

$$\Box_s h^\nu_\mu = 0, \tag{4}$$

where $\Box_s$ is the scalar wave operator. One consequence of this result is that de Sitter spacetime is classically stable against tensor perturbations, as the solutions of Eq. (4) are oscillatory functions. A second consequence is that gravitons in de Sitter spacetime behave as a pair of massless, minimally coupled quantum scalar fields [8].

It is well known that such massless scalar fields exhibit a type of quantum instability in that they do not possess a de Sitter invariant vacuum state. As a result, the mean squared field grows linearly in time [9–11] as $\langle\varphi^2\rangle \sim H^3 t/(4\pi^2)$. Similarly, the mean squared graviton field also grows linearly: $\langle h^\nu_\mu h^\mu_\nu\rangle \sim H^3 t/\pi^2$. However, this growth does not produce any physical consequences, at least in pure quantum gravity at the one-loop level. It was shown in Ref. [12] that at this level, all of the linearly growing terms cancel in the graviton effective energy momentum tensor. Whether there is an instability at higher orders is still unclear [13–15].

In this paper, we will study a model involving coupling of the tensor perturbations to a matter field. As a prelude, let us briefly recall the essential features of the renormalization of $\langle T_{\mu\nu}\rangle$, the expectation value of a matter stress tensor on a curved background [16]. This quantity is formally divergent, but under a covariant regularization, the divergent terms are of three types. The first is proportional to the metric tensor and can be absorbed in a cosmological constant renormalization. The second is proportional to the Einstein tensor and can be absorbed in a renormalization of Newton's constant. Finally, there are divergent terms proportional to two geometric tensors, $H_{\mu\nu}$ and $A_{\mu\nu}$, which arise from variation of R^2 and $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ terms in the action, respectively. Here R is the scalar curvature and $C_{\mu\nu\alpha\beta}$ is the Weyl tensor. The explicit forms of these tensors are expressible in terms of R, the Ricci tensor, $R_{\mu\nu}$, and their second derivatives as

$$H_{\mu\nu} = -2\nabla_\nu\nabla_\mu R + 2g_{\mu\nu}\nabla_\rho\nabla^\rho R - \tfrac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}, \tag{5}$$

and [17]

$$A_{\mu\nu} = -4\nabla_\alpha\nabla_\beta C_\mu{}^\alpha{}_\nu{}^\beta - 2C_\mu{}^\alpha{}_\nu{}^\beta R_{\alpha\beta}. \tag{6}$$

The derivative terms lead to a potential problem of making the Einstein equations fourth-order equations, leading to unstable solutions. This effect is analogous to the runaway solutions of the Lorentz-Dirac equation for classical charged particles. Various solutions to this problem have been suggested, including order-reduction approaches [18] and criteria for the validity of the semiclassical theory [3,17].

A well-known aspect of the quantum stress tensor is the conformal anomaly. At the classical level, the stress tensor of a conformally invariant field has a vanishing trace. This no longer holds for the renormalized stress tensor, where $\langle T^\mu_\mu\rangle \neq 0$. Furthermore, the anomalous trace for a free field is a state independent local geometric quantity which is quadratic in the Riemann tensor. In the case of a conformally invariant field in a conformally flat spacetime, the unambiguous part of the anomalous trace arises from a geometrical term in $\langle T_{\mu\nu}\rangle$ of the form $C\mathcal{B}_{\mu\nu}$, where C is a constant which depends upon the specific field, and

$$\mathcal{B}_{\mu\nu} = -2C_{\alpha\mu\beta\nu}R^{\alpha\beta} + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}R_{\mu\nu}R - R_\mu{}^\alpha R_{\nu\alpha} - \tfrac{1}{4}g_{\mu\nu}R^2, \tag{7}$$

where $C_{\alpha\mu\beta\nu}$ is the Weyl tensor. The term containing the Weyl tensor vanishes in conformally flat spacetime, but is needed to give the correct generalization to nonconformally flat spacetimes. The tensor $\mathcal{B}_{\mu\nu}$ was obtained by Davies *et al.* [19] and by Bunch [20]. The conformal anomaly is given by

$$\langle T^\mu_\mu\rangle = C\mathcal{B}^\mu_\mu = C(R_{\alpha\beta}R^{\alpha\beta} - \tfrac{1}{3}R^2). \tag{8}$$

More generally, there can be a term proportional to $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ in the anomalous trace, but this term will vanish for weakly perturbed conformally flat spacetimes, such as we consider.

The semiclassical Einstein equations for gravity with a cosmological constant Λ coupled to a quantum field can be written as

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N(\langle T_{\mu\nu}\rangle - \tfrac{1}{2}g_{\mu\nu}\langle T^\rho_\rho\rangle), \tag{9}$$

where G_N is Newton's constant. In addition to the local, geometric terms in $\langle T_{\mu\nu}\rangle$, in general, there are nonlocal terms which are difficult to compute explicitly. Fortunately, for the case of small perturbations around a conformally flat spacetime, they have been found in Refs. [4,5,21]. Here we will follow the coordinate space formulation given by Horowitz and Wald [21], which is based on earlier work by Horowitz [22] and by Horowitz and Wald [23].

To first order in the perturbation $h_{\mu\nu}$, Horowitz and Wald's result can be written as

$$\langle T_{\mu\nu}\rangle = \beta H_{\mu\nu} + C\mathcal{B}_{\mu\nu} + P_{\mu\nu} + Q_{\mu\nu}. \tag{10}$$

Here

$$P_{\mu\nu} = -16\pi\alpha a^{-2}\partial^\rho\partial^\sigma[\ln(a)\tilde{C}_{\mu\rho\nu\sigma}], \tag{11}$$

where $\tilde{C}_{\mu\rho\nu\sigma}$ is the Weyl tensor for perturbed Minkowski spacetime with the perturbation $\tilde{h}_{\mu\nu} = a^{-2}h_{\mu\nu}$, the partial derivatives are performed with respect to the Minkowski space coordinates, and α is another constant which depends upon the quantum field. The most complicated term in Eq. (10) is the nonlocal part given by

$$Q_{\mu\nu} = \alpha a^{-2}\int d^4x' H_\lambda(x-x')\tilde{A}_{\mu\nu}, \tag{12}$$

where

$$\tilde{A}_{\mu\nu} = -4\partial^\rho\partial^\sigma\tilde{C}_{\mu\rho\nu\sigma} \tag{13}$$

is the first order form of $A_{\mu\nu}$ for perturbed Minkowski spacetime with the perturbation $\tilde{h}_{\mu\nu} = a^{-2}h_{\mu\nu}$. The action of the distribution $H_\lambda(x-x')$ on a function f can be expressed in terms of radial null coordinates $u = t - r$ and $v = t + r$ and an angular integration as

$$\int d^4x' H_\lambda(x-x')f(x') = \int_{-\infty}^{0} du \int d\Omega\left[\frac{\partial f}{\partial u}\bigg|_{v=0}\ln(-u/\lambda) + \frac{1}{2}\frac{\partial f}{\partial v}\bigg|_{v=0}\right]. \tag{14}$$

This expression is an integral over the past light cone of the point x.

The result for $\langle T_{\mu\nu}\rangle$, Eq. (10), contains two constants, C and α, whose values can be determined explicitly, and are given in Table I for several fields. The remaining two constants, β and λ, are undetermined. A shift in either of these constants adds additional terms proportional to $H_{\mu\nu}$ and $A_{\mu\nu} = a^{-2}\tilde{A}_{\mu\nu}$, respectively. We could have added a term of the form $c_A A_{\mu\nu}$ to the right-hand side of Eq. (10). The result would then be invariant under changes in λ in the sense that a shift in λ would alter c_A.

III. EFFECTS OF THE LOCAL GEOMETRIC TERMS

Here we treat the local, geometric tensors $H_{\mu\nu}$ and $\mathcal{B}_{\mu\nu}$, and show that they produce no effects on the tensor perturbations other than finite shifts of the cosmological and Newton's constants. Write Eq. (9) as

$$R_{\mu\nu} - \Lambda_0 g_{\mu\nu} = 8\pi G_0(\langle T_{\mu\nu}\rangle - \tfrac{1}{2}g_{\mu\nu}\langle T^\rho_\rho\rangle), \tag{15}$$

where Λ_0 and G_0 are the cosmological and Newton's constants after all infinite renormalizations have occurred, but before these finite shifts. Here we take

$$\langle T_{\mu\nu}\rangle = \beta H_{\mu\nu} + C\mathcal{B}_{\mu\nu}. \tag{16}$$

To zeroth order, that is, on the de Sitter background, we have

$$^{(0)}\mathcal{B}_{\mu\nu} = -\tfrac{1}{3}\gamma_{\mu\nu}\Lambda^2, \qquad {}^{(0)}\mathcal{B} = -\tfrac{4}{3}\Lambda^2, \quad \text{and} \quad {}^{(0)}H_{\mu\nu} = 0. \tag{17}$$

If we insert these relations in Eq. (9), we find

$$^{(0)}R_{\mu\nu} - \Lambda\gamma_{\mu\nu} = 0, \tag{18}$$

where the shifted cosmological constant Λ is related to Λ_0 by

$$\Lambda = \Lambda_0 + \frac{8\pi}{3}G_0 C\Lambda^2. \tag{19}$$

In general, $\mathcal{B}_{\mu\nu}$ is not of the form of a cosmological constant term, but in de Sitter space, it produces an effective shift in Λ. Here we have written the second term on the right-hand side of Eq. (19) in terms of the shifted cosmological constant Λ, but to the order we are working, we could have equally well used Λ_0.

Next we need to find the explicit forms for the various tensors in Eq. (15) to first order in $h_{\mu\nu}$ in the transverse, trace-free gauge, Eq. (3). The Ricci tensor has the first order form

$$^{(1)}R_{\mu\nu} = -\tfrac{1}{2}h_{\mu\nu;\alpha}{}^{\alpha} + \tfrac{4}{3}\Lambda h_{\mu\nu}. \tag{20}$$

Thus, if $\langle T_{\mu\nu}\rangle = 0$, Eq. (9) becomes $h_{\mu\nu;\alpha}{}^{\alpha} - \frac{2}{3}\Lambda h_{\mu\nu} = 0$, which is equivalent to Eq. (4). Note that in the presence of sources, Eq. (3) is generally no longer a gauge condition, but rather a physical restriction on the perturbation. Here all the terms in the first order Einstein equations are traceless, so this condition is fulfilled. (Strictly speaking, it is $^{(1)}R^\mu_\nu$ which is a gauge invariant quantity, whereas $^{(1)}R^{\mu\nu}$ and $^{(1)}R_{\mu\nu}$ are not necessarily gauge invariant [24].) The first order form of $H_{\mu\nu}$ is

$$^{(1)}H_{\mu\nu} = 4\Lambda(h_{\mu\nu;\alpha}{}^{\alpha} - \tfrac{2}{3}\Lambda h_{\mu\nu}), \tag{21}$$

and that of $\mathcal{B}_{\mu\nu}$ is

$$^{(1)}\mathcal{B}_{\mu\nu} = -\tfrac{1}{3}\Lambda(h_{\mu\nu;\alpha}{}^{\alpha} + \tfrac{1}{3}\Lambda h_{\mu\nu}). \tag{22}$$

The net contribution of $\mathcal{B}_{\mu\nu}$ to the right-hand side of Eq. (15) is proportional to

$$^{(1)}\mathcal{B}_{\mu\nu} - \tfrac{1}{2}h_{\mu\nu}{}^{(0)}\mathcal{B} = -\tfrac{1}{3}\Lambda(h_{\mu\nu;\alpha}{}^{\alpha} - \tfrac{5}{3}\Lambda h_{\mu\nu}). \tag{23}$$

If we use Eqs. (19)–(21) and (23), then we may write Eq. (15) as

TABLE I. The coefficients C and α are listed for three different massless fields, where the spin $\frac{1}{2}$ field result refers to Weyl fermions and becomes a factor of 2 larger for 4-component Dirac fermions. This table is based on data from Refs. [16,22].

Field	C	α
Conformal scalar	$1/(2880\pi^2)$	$1/(3840\pi^3)$
Spin $\frac{1}{2}$	$11/(5760\pi^2)$	$1/(1280\pi^3)$
Photon	$31/(1440\pi^2)$	$1/(320\pi^3)$

$$\left(1+64\pi G_0\beta\Lambda-\frac{16\pi}{3}G_0C\Lambda\right)({}^{(1)}R_{\mu\nu}-\Lambda h_{\mu\nu})=0. \quad (24)$$

This implies that once we introduce additional terms in the stress tensor, the Einstein equation becomes Eq. (9), with the shifted Newton's constant given by

$$G_N=\ell_p^2=G_0\left(1+64\pi G_0\beta\Lambda-\frac{16\pi}{3}G_0C\Lambda\right)^{-1}, \quad (25)$$

where ℓ_p is the Planck length.

Now we may consider only the effects of the $P_{\mu\nu}$ and $Q_{\mu\nu}$ terms on the tensor perturbations, which satisfy the equation

$$\Box_s h_i^j=-16\pi\ell_p^2(P_i^j+Q_i^j), \quad (26)$$

in the transverse, trace-free gauge.

IV. SPATIALLY HOMOGENEOUS SOLUTIONS

In this section, we study the stability of the tensor perturbations of de Sitter spacetime in the presence of the quantum stress tensor of the conformal field. For this purpose, it is sufficient to examine spatially homogeneous solutions of Eq. (26), as these will be the most rapidly growing modes if there is an instability. Note that the tensor modes which we are considering are associated with anisotropic perturbations, even when they are spatially homogeneous. This follows from the fact that they have a nonvanishing Weyl tensor. Thus, the results of this section are distinct from, but complementary to, recent results by Pérez-Nadal *et al.* [25], who demonstrate the stability of de Sitter spacetime under isotropic perturbations at the one-loop level in semiclassical gravity.

In order to find the tensors $P_{\mu\nu}$ and $Q_{\mu\nu}$, we first need $\tilde{C}_{\mu\rho\nu\sigma}$. Here we ignore spatial derivatives, and restrict our attention to spatial components, which are the only nontrivial ones in our gauge. Then we need $\tilde{A}_{ij}=-4\tilde{C}_{i\eta j\eta,\eta\eta}$. The relevant components of the Riemann and Ricci tensors associated with the Minkowski perturbation $\tilde{h}_{ij}$ are $\tilde{R}_{i\eta j\eta}=-\frac{1}{2}\tilde{h}_{ij,\eta\eta}$ and $\tilde{R}_{ij}=\frac{1}{2}\tilde{h}_{ij,\eta\eta}$. Note that although h_{ij} is a gravity wave on de Sitter spacetime, $\tilde{h}_{ij}$ is not a source-free solution near flat spacetime. From these results, we obtain $\tilde{C}_{i\eta j\eta}=\tilde{R}_{i\eta j\eta}+\frac{1}{2}\tilde{R}_{ij}=-\frac{1}{4}\tilde{h}_{ij,\eta\eta}$ and hence

$$\tilde{A}_{ij}=\partial_\eta^4\tilde{h}_{ij}. \quad (27)$$

We may express the local tensor P_{ij} as

$$P_{ij}=4\pi\alpha a^{-2}[\ln(a)(a^{-2}h_{ij})_{,\eta\eta}]_{,\eta\eta}. \quad (28)$$

The nonlocal term involves the distribution H_λ, and an integral over the past light cone of the point x at which the stress tensor is evaluated. Take $r=0$ at this point, in which case we may write $u=\eta'-\eta-r'$ and $v=\eta'-\eta+r'$. The function on which the distribution acts depends only upon η', so $f=f(\eta')=f[\frac{1}{2}(u+v)+\eta]$. As a result, $(\partial f/\partial u)_{v=0}=(\partial f/\partial v)_{v=0}=\frac{1}{2}f'(\eta')$, and we may write Eq. (14) as

$$\int d^4x' H_\lambda(x-x')f(\eta')=4\pi\int_{-\infty}^0 d\eta'\left\{f'(\eta')\ln\left[\frac{2(\eta-\eta')}{\lambda}\right]+\frac{1}{2}f'(\eta')\right\}. \quad (29)$$

The last term in the integrand may be absorbed in a redefinition of λ, and hence will be dropped. Thus we obtain

$$Q_{ij}=4\pi\alpha a^{-2}\int_{-\infty}^0 d\eta'\partial_{\eta'}^5\tilde{h}_{ij}(\eta')\ln\left[\frac{2(\eta-\eta')}{\lambda}\right]. \quad (30)$$

We wish to look for a growing, spatially homogeneous solution of Eq. (9). In particular, let

$$\tilde{h}_{ij}=a^{-2}h_{ij}=h_i^j=e_i^j(-\eta)^{-b}, \quad (31)$$

where e_i^j is a constant tensor and b is a constant. A solution for which $b>0$ will grow as a power of conformal time as $\eta\to 0$, or exponentially in comoving time.

If we insert Eq. (31) into Eq. (28), the result is

$$P_i^j=4\pi\alpha e_i^j H^4(-\eta)^{-b}b(1+b)[2b+5-(2+b)(3+b)\times\ln(-H\eta)]. \quad (32)$$

Similarly, Eq. (30) yields

$$Q_i^j=4\pi\alpha e_i^j H^4(-\eta)^{-b}b(1+b)(2+b)(3+b)\times[\ln(-2\eta/\lambda)-\psi(b+4)-\gamma], \quad (33)$$

where γ is Euler's constant and ψ is the digamma function. Here, the scalar wave operator in de Sitter spacetime has the form

$$\Box_s h_i^j=-H^2\eta^4\frac{d}{d\eta}\left(\eta^{-2}\frac{d}{d\eta}\right)h_i^j. \quad (34)$$

Equation (26) may now be written as

$$b(3+b)=-\xi(2+b)(3+b)\{b(1+b)[\psi(b)+\gamma+\ln(H\lambda/2)]+1+2b\}, \quad (35)$$

where $\xi=64\pi^2\ell_p^2H^2\alpha$, and we have used the identity $\psi(x+1)=\psi(x)+1/x$. Thus the homogeneous solutions in the absence of the quantum stress tensor ($\xi=0$) are $b=0$ and $b=-3$, which are both stable. The only possibility for an unstable solution which is within the domain of validity of the semiclassical theory is one with a small positive value of b. If we expand Eq. (35) for $|b|\ll 1$, we find

$$b(3+b)\approx-\xi\{6[1+\ln(H\lambda/2)]b+[5+\pi^2+11\ln(H\lambda/2)]b^2+O(b^3)\}. \quad (36)$$

Thus $b=0$ is still a solution, but there are no solutions with $b>0$ so long as $\xi\ll 1$ and $\xi|\ln(H\lambda/2)|\ll 1$. These latter conditions can be considered to be criteria for the validity of the semiclassical theory. Hence we conclude that de Sitter spacetime is stable in the semiclassical theory against tensor perturbations. Here we should comment on the explicit appearance of the parameter λ in Eq. (35). Although the theory is invariant under changes in λ, so long as there is a term proportional to $A_{\mu\nu}$ in $\langle T_{\mu\nu}\rangle$, we have set the coefficient of this term to zero, which is analogous to a gauge choice. In any case, our conclusion does not depend upon the value of λ in Eq. (35), so long as $\xi|\ln(H\lambda/2)| \ll 1$. If this condition is not fulfilled, any resulting instabilities can be viewed as a breakdown of the semiclassical theory.

V. EFFECTS ON GRAVITY WAVES

A. The form of the correction

In this section, we will study the effect of the quantum stress tensor on gravity waves in de Sitter spacetime. The plane wave solutions of Eq. (4) are of the form

$$h^{\nu}_{\mu} = c_0 e^{\nu}_{\mu}(1 + ik\eta)e^{i(\mathbf{k}\cdot\mathbf{x}-k\eta)}, \tag{37}$$

where c_0 is a constant and e^{ν}_{μ} is the polarization tensor. We need to compute the quantum stress tensor in perturbed de Sitter spacetime, with this plane wave perturbation. The first step in finding the tensors $P_{\mu\nu}$ and $Q_{\mu\nu}$ is constructing $\tilde{C}_{\mu\rho\nu\sigma}$, the Weyl tensor associated with the conformally transformed perturbation of flat spacetime, $\tilde{h}_{\mu\nu}$. Note that mixed components of $\tilde{h}^{\nu}_{\mu}$ coincide with those of the original perturbation of de Sitter spacetime, h^{ν}_{μ}. However, $\tilde{h}^{\nu}_{\mu}$ is not a vacuum solution of perturbed flat space, and has a nonzero Ricci tensor

$$\tilde{R}^{\ \nu}_{\mu} = -\tfrac{1}{2}\tilde{\Box}\tilde{h}^{\nu}_{\mu}, \tag{38}$$

where $\tilde{\Box}$ is the flat space wave operator. Similarly, we find the associated Riemann tensor to satisfy

$$\partial^{\rho}\partial^{\sigma}\tilde{R}_{\mu\rho\nu\sigma} = -\tfrac{1}{2}\tilde{\Box}\,\tilde{\Box}\,\tilde{h}_{\mu\nu}. \tag{39}$$

Hence the tensor $\tilde{A}_{\mu\nu}$ and the Weyl tensor satisfy

$$\tilde{A}_{\mu\nu} = -4\partial^{\rho}\partial^{\sigma}\tilde{C}_{\mu\rho\nu\sigma} = \tilde{\Box}\,\tilde{\Box}\,\tilde{h}_{\mu\nu}. \tag{40}$$

However, when we use the perturbation given by Eq. (37), we find that $\tilde{A}_{\mu\nu} = 0$, so the nonlocal term vanishes:

$$Q_{\mu\nu} = 0. \tag{41}$$

The tensor P^{ν}_{μ} is nonzero and is given by

$$P^{\nu}_{\mu} = 8\pi i\alpha H^2 e^{\nu}_{\mu} c_0 k^3 \eta a^{-2} e^{i(\mathbf{k}\cdot\mathbf{x}-k\eta)}. \tag{42}$$

In the presence of the quantum stress tensor, the modified gravity wave may be expressed as $h^{\nu}_{\mu} + h'^{\nu}_{\mu}$, where

$$h'^{\nu}_{\mu}(x) = 16\pi\ell_p^2 \int d^4x'\sqrt{-g(x')}G_R(x,x')P^{\nu}_{\mu}, \tag{43}$$

where $G_R(x, x')$ is the scalar retarded Green's function in de Sitter space. Note that we are performing a perturbation expansion in powers of the squared Planck length ℓ_p^2, or equivalently Newton's constant G_N. Because this expansion parameter has dimensions, the effective dimensionless coupling constant is $(\ell_p/\lambda_P)^2$, where λ_P is a characteristic physical length scale associated with the perturbation.

The retarded Green's function vanishes for $\eta < \eta'$ and satisfies

$$\Box_s G_R(x, x') = -\frac{\delta(x - x')}{\sqrt{-g(x')}}. \tag{44}$$

It is convenient to take a spatial Fourier transform and write

$$G_R(x, x') = \frac{1}{a^2(\eta')(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}G(\eta, \eta'; k), \tag{45}$$

where $G(\eta, \eta'; k)$ satisfies

$$\frac{d^2G}{d\eta^2} - \frac{2}{\eta}\frac{dG}{d\eta} + k^2G = \delta(\eta - \eta'). \tag{46}$$

The explicit form for $G(\eta, \eta'; k)$ is given in Eq. (72) of Ref. [1], and may be expressed as

$$G(\eta, \eta'; k) = \frac{1}{k^3(\eta')^2}\{(1 + k^2\eta\eta')\sin[k(\eta - \eta')] - k(\eta - \eta')\cos[k(\eta - \eta')]\}, \tag{47}$$

for $\eta > \eta'$.

B. Initial conditions and explicit results

One possible initial condition is to set $h'^{\nu}_{\mu}(x) = 0$ at $\eta = \eta_0$. Now we may write the solution for $h'^{\nu}_{\mu}(x)$, which vanishes for $\eta < \eta_0$, as

$$h'^{\nu}_{\mu}(x) = 128\pi^2 i e^{\nu}_{\mu} c_0 \alpha H^2 \ell_p^2 e^{i\mathbf{k}\cdot\mathbf{x}} \times \int_{\eta_0}^{\eta} d\eta'\{(1 + k^2\eta\eta')\sin[k(\eta - \eta')] - k(\eta - \eta')\cos[k(\eta - \eta')]\}\frac{e^{-ik\eta'}}{\eta'}. \tag{48}$$

In the limit that $\eta_0 \to -\infty$, that is, $|\eta_0| \gg |\eta|$, the dominant contribution to the integral will come from terms in the integrand which are independent of η'. This leads to a result proportional to $|\eta_0|$,

$$h'^{\nu}_{\mu}(x) \sim 64\pi^2 i e^{\nu}_{\mu} c_0 \alpha H^2 k \ell_p^2 |\eta_0|(1 + ik\eta)e^{i(\mathbf{k}\cdot\mathbf{x}-k\eta)}, \tag{49}$$

which has the same functional form as does h^{ν}_{μ}, but is out of phase by $\pi/2$ due to the factor of i.

A more precise form for h'^{ν}_{μ} is obtained by replacing η_0 by $\eta_0 - \eta$ in Eq. (49). Thus the modified wave is no longer exactly a solution of the Lifshitz equation, Eq. (4). It is no longer constant when the mode has a proper wavelength larger than the horizon size, $k|\eta| < 1$. This is in contrast to the unperturbed mode, Eq. (37), whose magnitude is approximately constant when it is outside the horizon.

The most striking feature of the result, Eq. (49), is that the correction term due to the quantum stress tensor is proportional to $|\eta_0|$, and hence is larger the earlier the coupling between the quantum stress tensor and the metric perturbation is switched on. This bears some similarities to the results found in Refs. [1,2], where the effects of conformal stress tensor fluctuations in inflation were found to depend upon powers of the scale factor change during inflation. However, here we are concerned with an effect of the stress tensor expectation value, and not with fluctuations around this value.

One might think that the $|\eta_0|$ dependence is an artifact of sudden switching at $\eta = \eta_0$. However, it is possible to derive an equivalent result with a more gradual switching. For example, introduce an additional factor of $e^{p\eta'}$ in the integrand of Eq. (48) and let the lower limit of integration become $-\infty$. This introduces a gradual switch-on in which $1/p$ plays the role of $|\eta_0|$. The result is still Eq. (49), with $|\eta_0|$ replaced by $1/p$, showing that the more gradual switching has no effect.

In all cases, the dependence upon $|\eta_0|$ might appear to violate a theorem first introduced by Weinberg [26]. (See also Ref. [27].) This result states that quantum loop effects should grow no faster than logarithmically with the scale factor during inflation. However, it was argued in Ref. [2] that there is no real violation of this theorem, because the quantum effects are not so much growing as always large, and are due to very high frequency modes at $\eta = \eta_0$.

The same interpretation applies to our present result, Eq. (49). We can write the ratio of the magnitude of the correction to that of the original wave as

$$\Gamma = \left|\frac{h'^{\nu}_{\mu}}{h^{\nu}_{\mu}}\right| = 64\pi^2\alpha H^2 k\ell_p^2|\eta_0| = 64\pi^2\alpha H k_P \ell_p^2. \quad (50)$$

Here $k_P = k/a(\eta_0) = kH|\eta_0|$ is the physical wave number of the mode as measured by a comoving observer at time $\eta = \eta_0$. If we require that the curvature of the de Sitter spacetime be well below the Planck scale, then we have

$$H\ell_p \ll 1. \quad (51)$$

Similarly, if the mode in question is always below the Planck scale while it interacts with the quantum stress tensor, then

$$k_P\ell_p \ll 1. \quad (52)$$

These two conditions together imply that $|h'^{\nu}_{\mu}/h^{\nu}_{\mu}| < 1$, and hence the quantum correction to the gravity wave is smaller than the original wave.

However, if inflation lasts for a sufficiently long time, then modes which are of cosmological interest today appear to have been above the Planck scale at the onset of inflation. This is the cosmological version of the trans-Planckian problem, which also arises in black hole physics. Hawking's derivation of black hole evaporation [28] requires modes which start far above the Planck scale. It is possible to obtain black hole evaporation without the use of trans-Planckian modes [29,30], but only at the price of introducing a nonlinear dispersion relation which violates local Lorentz symmetry. An analogous choice arises in the present problem. One option is to take the trans-Planckian mode seriously, and allow their contributions. This option has the advantage of being the simplest extrapolation of known physics. It has the disadvantage of doubts about the validity of perturbation theory as an expansion in powers of $(\ell_p/\lambda_P)^2$, with λ_P the physical wavelength of the mode in question. A second option is to apply Eq. (49) only to modes which are below the Planck scale at $\eta = \eta_0$. This option avoids the possible problems with trans-Planckian modes, but seems to require a nonlocal cutoff when implemented in coordinate space. This issue was discussed in more detail in Ref. [2], where numerous references to earlier papers on the trans-Planckian issue in cosmology may be found.

In the remainder of this paper, we will explore the consequences of adopting the first option. We wish to study the possible observational effects of the modification of gravity wave modes, and their use as a possible probe of trans-Planckian physics.

VI. TENSOR PERTURBATIONS IN INFLATIONARY COSMOLOGY

One of the successes of inflationary cosmology is the prediction of a Gaussian and nearly scale invariant spectrum of primordial density fluctuations [11,32–34], which seems to be confirmed by measurements on the CMB [35]. Another prediction is a similar spectrum of tensor perturbations [36–39], which might be found in polarization measurements of the CMB, but at present, these perturbations have not been detected.

The tensor perturbations from inflation are less model dependent than are the density perturbations. The former arise from vacuum modes of the quantized graviton field in de Sitter spacetime which evolve according to the Lifshitz equation, Eq. (4), until the last scattering surface. At this time, they leave an imprint on the CMB in the form of a power spectrum of tensor perturbations given by (see, for example, Refs. [40,41])

$$\delta_h^2 \approx \frac{8}{\pi}\ell_p^2 H^2. \quad (53)$$

This is an approximately flat spectrum. If H slowly decreases as inflation progresses, then the spectrum is slightly enhanced for longer wavelengths. The numerical coefficient is fixed by the normalization of vacuum graviton modes, which leads to $c_0 = \ell_p\sqrt{16\pi/k}$ in Eq. (37).

The effect of the conformal stress tensor is to modify the amplitude of these modes by a factor of $1 - i\Gamma$, where Γ is given by Eq. (50). This in turn multiplies the power spectrum by $|1 - i\Gamma|^2 = 1 + \Gamma^2$. In order to estimate this enhancement factor, we need to make some assumptions about a model of inflation. Let E_R be the reheating energy and assume that most of the vacuum energy which drives inflation is converted into radiation at reheating. Then Einstein's equations yield

$$H^2 = \frac{8\pi}{3} \ell_p^2 E_R^4. \tag{54}$$

For this discussion, we assume that H is approximately constant throughout the inflationary era. There is expansion by a factor of about $E_R/(1\ \mathrm{eV})$ between the end of inflation and last scattering and a further expansion by a factor of 10^3 to the present. Let us choose the scale factor to be unity at the end of inflation, so its present value will be

$$a_{\mathrm{now}} = 10^3 \frac{E_R}{1\ \mathrm{eV}}. \tag{55}$$

Consider a scale which presently has a proper length of ℓ_0, and hence a physical wave number of $k_P = 2\pi/\ell_0$. At the end of inflation, its physical and comoving wave numbers coincide and are given by

$$k = \frac{2\pi a_{\mathrm{now}}}{\ell_0}. \tag{56}$$

Recall that k is constant, so this form holds throughout the cosmological expansion.

Let

$$S = H|\eta_0|, \tag{57}$$

which is the factor by which the universe expands from the initial conformal time $\eta = \eta_0$ to the end of inflation. We may combine the above relations to write

$$\Gamma^2 = \frac{8\pi}{3}(128\pi^3\alpha)^2 \ell_p^6 E_R^4 S^2 \frac{a_{\mathrm{now}}^2}{\ell_0^2}. \tag{58}$$

If we use the value of $\alpha = 1/(320\pi^3)$ corresponding to the electromagnetic field, then we may write

$$\Gamma^2 = 1.34 \times 10^{-78}\left(\frac{10^{25}\ \mathrm{cm}}{\ell_0}\right)^2 \left(\frac{E_R}{10^{15}\ \mathrm{GeV}}\right)^6 S^2. \tag{59}$$

Recall that the present horizon size is of order 10^{28} cm, so $\ell_0 \approx 10^{25}$ cm corresponds to angular scales of the order of $1°$ today.

If one has only the minimal inflation needed to solve the horizon and flatness problems, so $S \approx 10^{23}$, then the effects of the one-loop correction on the tensor perturbation spectrum are negligible. However, larger values of S have the potential to produce significant corrections. For example, $E_R \approx 10^{15}$ GeV and $S \approx 10^{39}$ would lead to an effect of order unity at $1°$ scales. One should expect the one-loop approximation to begin to break down, but this can serve as an order of magnitude estimate. In contrast to the nearly flat spectrum, Eq. (53), due to free graviton fluctuations, the one-loop effect is highly tilted toward the blue end of the spectrum.

It is of interest to compare the magnitude of this effect on the tensor perturbations with the stress tensor fluctuation effect on density perturbations which was treated in Refs. [1,2]. The latter effect becomes significant if $E_R \approx 10^{15}$ GeV and $S \approx 10^{33}$ [see Eq. (108) in Ref. [2]], and is hence somewhat larger than the effect treated in the present paper.

VII. SUMMARY

We have constructed the semiclassical Einstein equation with a conformal matter field on a weakly perturbed de Sitter background, using the coordinate space formulation of Horowitz and Wald [21–23], and examined gravity wave solutions of this equation. We found no growing, spatially homogeneous (but anisotropic) solutions in a spatially, flat universe, which implies that de Sitter spacetime is stable to tensor perturbations at the one-loop level in the presence of conformal matter.

We further examined the effects of the one-loop correction on the propagation of finite wavelength gravity waves, and found a correction term which depends upon the interval over which the interaction with the quantum matter field is switched on. One viewpoint is that this is the duration of inflation. So long as the curvature of de Sitter spacetime and the initial proper frequency of the mode are below the Planck scale, the fractional correction is small. The effect takes the form of both a phase shift and an amplitude change. If one is concerned only with the form of the gravity wave modes at late times, this effect can be absorbed in a complex amplitude shift. However, gravity wave modes are no longer exactly solutions of the Lifshitz equation, Eq. (4).

The effect is potentially observable with a sufficient amount of inflation through an increase in the amplitude of the spectrum of tensor perturbations of the cosmic microwave background. This possibility does require one to take seriously the contribution of modes which were trans-Planckian at the beginning of inflation.

ACKNOWLEDGMENTS

We would like to thank A. Higuchi, S. P. Miao, E. Mottola, A. Roura, E. Verdaguer, and R. Woodard for valuable discussions. This work is partially supported by the National Center for Theoretical Sciences, Taiwan, by Grants No. NSC 97-2112-M-001-005-MY3 and No. NSC 97-2112-M-259-007-MY3, and by the U.S. National Science Foundation under Grant No. PHY-0855360. L. H. F. would like to thank Academia Sinica and National Dong Hwa University for hospitality while this work was conducted.

[1] C.-H. Wu, K.-W. Ng, and L. H. Ford, Phys. Rev. D **75**, 103502 (2007).
[2] L. H. Ford, S. P. Miao, K.-W. Ng, R. P. Woodard, and C.-H. Wu, Phys. Rev. D **82**, 043501 (2010).
[3] P. R. Anderson, C. Molina-Paris, and E. Mottola, Phys. Rev. D **80**, 084005 (2009).
[4] A. A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 460 (1981) [JETP Lett. **34**, 438 (1981)].
[5] A. Campos and E. Verdaguer, Phys. Rev. D **49**, 1861 (1994).
[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[7] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **16**, 587 (1946) [J. Phys. USSR **10**, 116 (1946)].
[8] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).
[9] A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982).
[10] A. D. Linde, Phys. Lett. **116B**, 335 (1982).
[11] A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[12] L. H. Ford, Phys. Rev. D **31**, 710 (1985).
[13] N. C. Tsamis and R. P. Woodard, Ann. Phys. (N.Y.) **253**, 1 (1997).
[14] J. Garriga and T. Tanaka, Phys. Rev. D **77**, 024021 (2008).
[15] N. C. Tsamis and R. P. Woodard, Phys. Rev. D **78**, 028501 (2008).
[16] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982), Chaps. 6–7.
[17] P. R. Anderson, C. Molina-Paris, and E. Mottola, Phys. Rev. D **67**, 024026 (2003).
[18] L. Parker and J. Z. Simon, Phys. Rev. D **47**, 1339 (1993).
[19] P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, Ann. Phys. (N.Y.) **109**, 108 (1977).
[20] T. S. Bunch, J. Phys. A **12**, 517 (1979).
[21] G. T. Horowitz and R. M. Wald, Phys. Rev. D **25**, 3408 (1982).
[22] G. T. Horowitz, Phys. Rev. D **21**, 1445 (1980).
[23] G. T. Horowitz and R. M. Wald, Phys. Rev. D **21**, 1462 (1980).
[24] J. M. Stewart and M. Walker, Proc. R. Soc. A **341**, 49 (1974).
[25] G. Pérez-Nadal, A. Roura, and E. Verdaguer, Phys. Rev. D **77**, 124033 (2008).
[26] S. Weinberg, Phys. Rev. D **72**, 043514 (2005); **74**, 023508 (2006).
[27] K. Chaicherdsakul, Phys. Rev. D **75**, 063522 (2007).
[28] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[29] W. G. Unruh, Phys. Rev. D **51**, 2827 (1995).
[30] S. Corley and T. Jacobson, Phys. Rev. D **54**, 1568 (1996).
[31] V. Mukhanov and G. Chibisov, JETP Lett. **33**, 532 (1981).
[32] A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
[33] S. W. Hawking, Phys. Lett. **115B**, 295 (1982).
[34] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[35] E. Komatsu *et al.*, Astrophys. J. Suppl. Ser. **180**, 330 (2009); Astrophys. J. Suppl. Ser. **192**, 18 (2011).
[36] A. A. Starobinsky, JETP Lett. **30**, 682 (1979).
[37] L. F. Abbott and M. B. Wise, Nucl. Phys. **B244**, 541 (1984).
[38] A. A. Starobinsky, Sov. Astron. Lett. **11**, 133 (1985).
[39] B. Allen, Phys. Rev. D **37**, 2078 (1988).
[40] K.-W. Ng, Int. J. Mod. Phys. A **11**, 3175 (1996).
[41] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).