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Perturbative no-hair property of form fields for higher dimensional static black holes

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In this paper we examine the static perturbation of p-form field strengths around higher dimensional Schwarzschild spacetimes. As a result, we can see that the static perturbations do not exist when $p \geq 3$. This result supports the no-hair properties of p-form fields. However, this does not exclude the presence of the black objects having nonspherical topology.

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I. INTRODUCTION

Motivated by the recent progress of superstring theory, higher dimensional black holes have been actively studied so far [1]. Different from the four-dimensional cases, the conventional uniqueness theorem does not hold in stationary higher dimensional black holes. Indeed, there are several different black hole/ring spacetimes with the same mass and angular momentum [2,3]. See Ref. [4] for a new approach, "blackfolds." However, if one considers static (electro)vacuum cases, the uniqueness theorem holds [5,6] (see also Refs. [7,8]) and then the spacetimes are the Schwarzschild-Tangherlini solution [9] (the higher dimensional Reissner-Nordström solution in electrovacuum cases). However, there are open questions even in static cases. If there are other matter fields, it becomes difficult to show the uniqueness in general (see also Refs. [10,11]). For example, one might be interested in the higher form fields (say, p-form field strengths). According to the recent work [12], one can show the no-hair theorem for the cases with $(n+1)/2 \leq p \leq (n-1)$ in n-dimensional asymptotically flat spacetimes. Note that the Maxwell field ($p = 2$) is out of the condition on p and consistent with the presence of the Maxwell hair of charged black holes. However, there is a mystery about the presence of the hairs for $2 < p < (n+1)/2$. We should also note that the cases with $p \geq 3$ cannot have the conserved charge associated with $H_{(p)}$. Therefore, we intuitively guess that the monopole component of $H_{(p)}$ does not exist. In stationary cases, there is the exact solution with dipole hair [13].

In this paper, using the perturbation analysis, we will consider the possibility of the back hole spacetime with nontrivial p-form field-strength hair. Since the background spacetimes are vacuum ones, the p-form field perturbations are decoupled with the metric perturbations. Thus, this setup makes the analysis much easier than the cases of the perturbation analysis of "charged" black holes. The analysis will show us that the static perturbations of p-form field strength around the Schwarzschild-Tangherlini spacetime do not exist [14] (see the study on stationary metric perturbation for the Schwarzschild-Tangherlini spacetimes in Ref. [16]). Our result suggests that the deformed black holes with spherical topology do not exist. However, this does not exclude the presence of the black hole solution with nonspherical topology. As mentioned in the discussion in the Appendix, if the solution exists, it seems to have both the electric and magnetic hair of p-form field strengths simultaneously.

The rest of this paper is organized as follows. In Sec. II, we describe the model, boundary conditions, and hyperspherical harmonic functions (harmonics, for brevity). In Sec. III, we analyze the Maxwell fields from the pedagogical point of view. Then, in Sec. IV, we will discuss the static perturbation of general form field and show that there are no regular solutions. In Sec. V, we also have a little consideration of no-hair in asymptotically (anti–)de Sitter spacetimes. Finally, we will summarize our work and discuss future issues in Sec. VI. In the Appendix, we try to show the no-hair theorem in the cases with both electric and magnetic parts of p-form field strength. However, we fail to do so.

II. FIELD EQUATIONS, BOUNDARY CONDITIONS, AND HARMONICS

A. Model

We consider the system described by the Lagrangian

$$\mathcal{L} = R - \frac{1}{p!} H_{(p)}^2, \tag{1}$$

where R is the n-dimensional Ricci scalar and $H_{(p)}$ is the p-form field strength. $H_{(p)}$ has the $(p-1)$-form field potential as

$$H_{(p)} = dB_{(p-1)}. \tag{2}$$

The field equations are

$$R_{\mu\nu} = \frac{1}{p!}\left(pH_{\mu}{}^{\rho_1 \cdots \rho_{p-1}} H_{\nu \rho_1 \cdots \rho_{p-1}} - \frac{p-1}{n-2} g_{\mu\nu} H_{(p)}^2 \right) \tag{3}$$

and

$$\nabla_\mu H^\mu{}_{\nu_1\nu_2\cdots\nu_{p-1}} = 0, \tag{4}$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. As in Ref. [12], we can include the dilation field too. However, the effect from the dilaton does not affect our result. For simplicity, then, we will not include the dilaton fields in this study.

B. Boundary conditions

Let us consider the boundary conditions. In general, the metric of static spacetimes can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -V^2(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j, \tag{5}$$

where x^i are spatial coordinates and t is the time coordinate. Since we mainly focus on asymptotically flat spacetimes, we suppose that the asymptotic boundary conditions are given by

$$\begin{aligned} V &= 1 - \frac{m}{r^{n-3}} + O(1/r^{n-2}) \\ g_{ij} &= \left(1 + \frac{2}{n-3}\frac{m}{r^{n-3}}\right)\delta_{ij} + O(1/r^{n-2}), \end{aligned} \tag{6}$$

where m is the ADM mass. We will not use the above expressions directly. From the asymptotic flatness, $H_{(p)}$ should decay at infinity. Although we mainly discuss the asymptotically flat cases, we will address the no-hair of $H_{(p)}$ in asymptotically (anti–)de Sitter spacetimes shortly.

The boundary condition on the event horizon $V = 0$ comes from the regularity. To see this, we compute the Kretschmann invariant

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 4R_{i0j0}R^{i0j0} + R_{ijkl}R^{ijkl} \\ &= \frac{4}{V^2}D_iD_jVD^iD^jV + R_{ijkl}R^{ijkl} \\ &= \frac{4}{V^2}\left[\frac{1}{\rho^2}k_{ij}k^{ij} + \frac{1}{\rho^4}(n^i\partial_i\rho)^2 + \frac{2}{\rho^4}(\mathcal{D}\rho)^2\right] \\ &\quad + R_{ijkl}R^{ijkl} \\ &= \frac{4}{V^2}\left[\frac{1}{\rho^2}k_{ij}k^{ij} + \frac{1}{\rho^2}(k-\rho D^2V)^2 + \frac{2}{\rho^4}(\mathcal{D}\rho)^2\right] \\ &\quad + R_{ijkl}R^{ijkl}, \end{aligned} \tag{7}$$

where we used $R_{i0j0} = VD_iD_jV$ in the second line and D_i is the covariant derivative with respect to g_{ij}. In the third line, $k_{ij} = h_i^k D_k n_j$ with $n_i = \rho D_i V$. For the last line, we may be using the Einstein equation

$$\begin{aligned} R_{00} &= VD^2V \\ &= \frac{n-p-1}{(n-2)(p-1)!}H_0{}^{i_1\cdots i_{p-1}}H_{0i_1\cdots i_{p-1}} \\ &\quad + \frac{p-1}{(n-2)p!}V^2H_{i_1\cdots i_p}H^{i_1\cdots i_p}. \end{aligned} \tag{8}$$

$\mathcal{D}_i$ is the covariant derivative with respect to the induced metric $h_{ij} = g_{ij} - n_in_j$.

Thus, from Eqs. (7) and (8), the regularity implies

$$k_{ij}|_{V=0} = \mathcal{D}\rho|_{V=0} = 0, \tag{9}$$

$$H_0{}^{i_1\cdots i_{p-1}}H_{0i_1\cdots i_{p-1}}|_{V=0} = O(V^2), \tag{10}$$

and

$$H^{i_1\cdots i_p}H_{i_1\cdots i_p}|_{V=0} = O(1). \tag{11}$$

In this paper we focus on the static perturbation around vacuum and spherical symmetric solutions. That is, the background p-form field does not exist. The background metric is given by

$$ds_0^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_{n-2}^2, \tag{12}$$

where $f(r) = 1 - (r_0/r)^{n-3}$ and $d\Omega_{n-2}^2 =: \sigma_{AB}dx^Adx^B$ is the metric of the $(n-2)$-dimensional unit sphere. In this specific form, the static perturbation should satisfy

$$H_{0rA_1\cdots A_{p-2}}|_{V=0} = O(1) \tag{13}$$

$$H_{0A_1\cdots A_{p-1}}|_{V=0} = O(V) = O(\sqrt{f}) \tag{14}$$

$$H_{rA_1\cdots A_{p-1}}|_{V=0} = O(V^{-1}) = O(1/\sqrt{f}) \tag{15}$$

$$H_{A_1\cdots A_p}|_{V=0} = O(1). \tag{16}$$

C. Hyperspherical harmonic functions

Since the background spacetimes have spherical symmetry, we can decompose all quantities in terms of hyperspherical harmonics defined on the sphere S^{n-2} [17–19]. In general, there are three type of harmonics, that is, scalar, vector, and tensor types. The scalar harmonic function Y follows:

$$\mathcal{D}^2Y = -\ell(\ell + n - 3)Y. \tag{17}$$

The vector harmonic function V_A satisfies

$$\mathcal{D}^AV_A = 0 \tag{18}$$

$$\mathcal{D}^2V_A = -[\ell(\ell + n - 3) - 1]V_A. \tag{19}$$

Since quantities which we will consider are often asymmetric tensor, we consider the totally antisymmetric tensor harmonic function only:

$$\mathcal{D}^{A_1}T_{A_1\cdots A_q} = 0$$

$$\mathcal{D}^2T_{A_1\cdots A_q} = -[\ell(\ell + n - 3) - q]T_{A_1\cdots A_q}. \tag{21}$$

Note that the static perturbation of metric is decoupled with p-form fields. This is due to the nonpresence of the background field of p-form fields. We know that the possible static perturbation of the metric are $\ell = 0$, 1 modes, so if the mass is fixed, $\ell = 0$ static modes vanish. The $\ell = 1$ modes can be absorbed to the redefinition of the coordinate, that is, they correspond to the choice of the "center" of the coordinate. Therefore, we will not consider the static perturbation of the metric.

III. MAXWELL FIELDS

As a pedagogical exercise, we will first consider the Maxwell fields. As is already known, the uniqueness theorem for charged black holes (the higher dimensional Reissner-Nordström solution) holds in this system. Therefore, only the static *monopole* perturbation is permitted. We will confirm this fact in this section. See Refs. [20,21] for the perturbation analysis of Reissner-Nordström spacetimes.

Each respective component of the Maxwell equation becomes

$$\partial_r F_{rt} + \frac{n-2}{r} F_{rt} + \frac{1}{r^2 f} \mathcal{D}^A F_{At} = 0, \tag{22}$$

$$\mathcal{D}^A F_{Ar} = 0, \tag{23}$$

and

$$\partial_r F^r{}_A + \frac{n-4}{r} F^r{}_A + \frac{1}{r^2} \mathcal{D}^B F_{BA} = 0. \tag{24}$$

A. Gauge conditions

We employ the following gauge:

$$A_r = \mathcal{D}^A A_A = 0. \tag{25}$$

This can been achieved by the following standard argument. There is the gauge freedom of $A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \chi$. Then, if we choose χ as

$$\chi = -\int dr A_r(r, x^A) + \eta(x^A), \tag{26}$$

we can set

$$A_r = 0. \tag{27}$$

Equation (23) implies

$$\mathcal{D}^2 A_r - \partial_r(\mathcal{D}^A A_A) = 0, \tag{28}$$

and then

$$\partial_r(\mathcal{D}^A A_A) = 0, \tag{29}$$

that is, $\mathcal{D}^A A_A$ does not depends on the coordinate of r. Using the remaining gauge freedom of $A_A \to \tilde{A}_A = A_A + \partial_A \eta(x^B)$ satisfying

$$\mathcal{D}^2 \eta = -\mathcal{D}^A A_A, \tag{30}$$

we can set

$$\mathcal{D}^A A_A = 0. \tag{31}$$

B. Solutions

Under the gauge condition of Eq. (25), the Maxwell equation becomes

$$\partial_r^2 A_t + \frac{n-2}{r} \partial_r A_t + \frac{1}{r^2 f} \mathcal{D}^2 A_t = 0 \tag{32}$$

and

$$\partial_r^2 A_A + \left(\frac{n-4}{r} + \frac{f'}{f}\right) \partial_r A_A + \frac{\mathcal{D}^2 - (n-3)}{r^2 f} A_A = 0. \tag{33}$$

Here we expand A_t, A_A in terms of harmonics as

$$A_t = G(r) Y, \qquad A_A = H(r) V_A. \tag{34}$$

Let us first solve the equation for A_t. Introducing the new variable x defined by

$$x := \left(\frac{r_0}{r}\right)^{n-3}, \tag{35}$$

the solution can be written in the analytic form of

$$G(r) = \mathrm{Br}^{-(n+\ell-3)} F(\alpha, \beta, \gamma; x) + C r^\ell F(\alpha', \beta', \gamma'; x), \tag{36}$$

where $F(\alpha, \beta, \gamma; x)$ is the hypergeometric function, and

$$\alpha = \frac{\ell}{n-3} \tag{37}$$

$$\beta = \frac{n+\ell-3}{n-3} \tag{38}$$

$$\gamma = \frac{2(n+\ell-3)}{n-3} = \alpha + \beta + 1, \tag{39}$$

and

$$\alpha' = -\frac{\ell}{n-3} \tag{40}$$

$$\beta' = -\frac{n+\ell-3}{n-3} \tag{41}$$

$$\gamma' = -\frac{2\ell}{n-3} = \alpha' + \beta' + 1. \tag{42}$$

From the asymptotic flatness, we must set $C = 0$ and the solution becomes

$$A_t = \mathrm{Br}^{-(n+\ell-3)} F(\alpha, \beta, \gamma; x) Y. \tag{43}$$

Now, we compute the field strength:

$$F_{r0} = -(n+\ell-3)\mathrm{B}r^{-(n+\ell-2)}F(\alpha,\beta,\gamma;x)Y - (n-3)\mathrm{B}r^{-(n+\ell-3)}\frac{x}{r}\frac{d}{dx}F(\alpha,\beta,\gamma;x)Y = -(n+\ell-3)\mathrm{B}r^{-(n+\ell-2)}F(\alpha,\beta,\gamma;x)Y - (n-3)\frac{\alpha\beta}{\gamma}\mathrm{B}r^{-(n+\ell-3)}\frac{x}{r} \times F(\alpha+1,\beta+1,\gamma+1;x)Y. \tag{44}$$

Let us examine the behavior on the horizon. Since

$$F(\alpha+1,\beta+1,\gamma+1;1) = \frac{\Gamma(\gamma+1)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{45}$$

and $\Gamma(\gamma-\alpha-\beta-1) = \Gamma(0)$, it diverges except for $\ell \neq 0$. This means that the second term in the right-hand side of Eq. (44) diverges. The case of $\ell = 0$ is special. In this case, α vanishes and then the second term disappears and the solution will be regular everywhere outside of the black holes. Thus, only the monopole component ($\ell = 0$) is permitted. Of course, this is the case of the Reissner-Nordström solution.

Next, we solve the equation for A_A and then we have the analytic solution as

$$H(r) = \mathrm{B}r^{-(\ell+n-4)}F(\alpha,\beta,\gamma;x) + Cr^{\ell+1}F(\alpha',\beta',\gamma';x), \tag{46}$$

where

$$\alpha = \frac{\ell+n-4}{n-3} \tag{47}$$

$$\beta = \frac{\ell+n-2}{n-3} \tag{48}$$

$$\gamma = 2\frac{\ell+n-3}{n-3} = \alpha+\beta, \tag{49}$$

and

$$\alpha' = -\frac{\ell+1}{n-3} \tag{50}$$

$$\beta' = -\frac{\ell-1}{n-3} \tag{51}$$

$$\gamma' = -\frac{2\ell}{n-3}. \tag{52}$$

From the asymptotic flatness, we must set $C = 0$ and the solution becomes

$$A_A = \mathrm{B}r^{-(\ell+n-4)}F(\alpha,\beta,\gamma;x)V_A. \tag{53}$$

Since $\gamma = \alpha+\beta$, on the event horizon,

$$F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(0)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{54}$$

diverges. Therefore, there is no regular solution.

As a conclusion, the regular solution is the $\ell = 0$ mode only of A_t, which corresponds to the Reissner-Nordström solution. This is a well-known fact.

IV. HIGHER FORM FIELDS

In this section we examine the static perturbation of $H_{(p)}$ fields with $p \geq 3$. The field equations are

$$\mathcal{D}^A H_{AtrA_1A_2\cdots A_{p-3}} = 0, \tag{55}$$

$$\partial_r H_{rtA_1A_2\cdots A_{p-2}} + \frac{n-2(p-1)}{r}H_{rtA_1A_2\cdots A_{p-2}} + \frac{1}{r^2 f}\mathcal{D}^B H_{BtA_1\cdots A_{p-2}} = 0, \tag{56}$$

$$\mathcal{D}^A H_{ArA_1\cdots A_{p-2}} = 0, \tag{57}$$

and

$$\partial_r H_{rA_1\cdots A_{p-1}} + \left(\frac{n-2p}{r} + \frac{f'}{f}\right)H_{rA_1\cdots A_{p-1}} + \frac{1}{r^2 f}\mathcal{D}^B H_{BA_1\cdots A_{p-1}} = 0. \tag{58}$$

A. Gauge conditions

Using the gauge freedom of $B_{\mu_1\cdots\mu_{p-1}} \to \tilde{B}_{\mu_1\cdots\mu_{p-1}} = B_{\mu_1\cdots\mu_{p-1}} + \partial_{[\mu_1}C_{\mu_2\cdots\mu_{p-1}]}$, we can show that one can choose the following gauge condition:

$$\mathcal{D}^A B_{tAA_1\cdots A_{p-3}} = 0 \tag{59}$$

$$\mathcal{D}^A B_{rAA_1\cdots A_{p-3}} = 0 \tag{60}$$

$$\mathcal{D}^B B_{BA_1\cdots A_{p-2}} = 0. \tag{61}$$

With Eq. (59), the field equation shows

$$B_{trA_1\cdots A_{p-3}} = 0. \tag{62}$$

The detail can been seen by the following argument. The gauge transformation gives us

$$\mathcal{D}^A\tilde{B}_{tAA_1\cdots A_{p-3}} = \mathcal{D}^A B_{tAA_1\cdots A_{p-3}} - [\mathcal{D}^2 - (p-3)(n-p+1)]C_{tA_1\cdots A_{p-3}}, \tag{63}$$

where we already imposed $\mathcal{D}^{A_1}C_{tA_1\cdots A_{p-3}} = 0$. Then we take $C_{tA_1\cdots A_{p-3}}$, satisfying

$$[\mathcal{D}^2 - (p-3)(n-p+1)]C_{tA_1\cdots A_{p-3}} = \mathcal{D}^A B_{tAA_1\cdots A_{p-3}}. \quad (64)$$

Note that solutions exist for $C_{tA_1\cdots A_{p-3}}$. This then implies

$$\mathcal{D}^A \tilde{B}_{tAA_1\cdots A_{p-3}} = 0. \quad (65)$$

In this case, Eq. (55) becomes

$$[\mathcal{D}^2 - (p-3)(n-p-1)]B_{trA_1\cdots A_{p-3}} = 0. \quad (66)$$

In terms of harmonics, $B_{trA_1\cdots A_{p-3}}$ will be expanded as

$$B_{trA_1\cdots A_{p-3}} = J(r)T_{A_1\cdots A_{p-3}}. \quad (67)$$

Then,

$$(\ell + p - 3)(\ell + n - p)J(r) = 0. \quad (68)$$

Except for the special case with $\ell = 0$, $p = 3$, it is easy to see that

$$J(r) = 0 \quad (69)$$

holds. We can also show $J(r) = 0$ even for the $\ell = 0$, $p = 3$ case by a distinct argument. In fact, we can use the remaining gauge freedom of $\tilde{B}_{tr} = B_{tr} - \partial_r C_t(r)$. Then, taking

$$C_t(r) = \int^r dr B_{tr}(r), \quad (70)$$

we can set

$$B_{tr}(r) = 0. \quad (71)$$

Therefore, without loss of generality, we can conclude that

$$B_{trA_1\cdots A_{p-3}} = 0 \quad (72)$$

holds.

Next, we will ask if we can take the gauge condition of Eq. (60). To see this, we first look at

$$\mathcal{D}^A \tilde{B}_{rAA_1\cdots A_{p-3}} = \mathcal{D}^A B_{rAA_1\cdots A_{p-3}} + \partial_r(\mathcal{D}^A C_{AA_1\cdots A_{p-3}}) - [\mathcal{D}^2 - (p-3)(n-p+1)]C_{rA_1\cdots A_{p-3}}, \quad (73)$$

where we imposed $\mathcal{D}^A C_{rAA_1\cdots A_{p-3}} = 0$.

Using $C_{\mu_1\cdots\mu_{p-2}}$, satisfying

$$\partial_r(\mathcal{D}^A C_{AA_1\cdots A_{p-3}}) = 0 \quad (74)$$

$$[\mathcal{D}^2 - (p-3)(n-p+1)]C_{rA_1\cdots A_{p-3}} = \mathcal{D}^A B_{rAA_1\cdots A_{p-3}}, \quad (75)$$

we can set

$$\mathcal{D}^A B_{rAA_1\cdots A_{p-3}} = 0. \quad (76)$$

Finally, we consider the following transformation:

$$\mathcal{D}^A \tilde{B}_{AA_1\cdots A_{p-2}} = \mathcal{D}^A B_{AA_1\cdots A_{p-2}} + [\mathcal{D}^2 - (n-p)(p-2)]C_{A_1\cdots A_{p-2}}, \quad (77)$$

where we imposed $\mathcal{D}^A C_{AA_1\cdots A_{p-3}} = 0$. Then, taking $C_{A_1\cdots A_{p-2}}$, satisfying

$$[\mathcal{D}^2 - (n-p)(p-2)]C_{A_1\cdots A_{p-2}} = -\mathcal{D}^A B_{AA_1\cdots A_{p-2}}, \quad (78)$$

we can adopt the gauge of

$$\mathcal{D}^A B_{AA_1\cdots A_{p-2}} = 0. \quad (79)$$

B. Solutions

Now, under the current gauge conditions, Eq. (56) becomes

$$\partial_r^2 B_{tA_1\cdots A_{p-2}} + \frac{n-2(p-1)}{r}\partial_r B_{tA_1\cdots A_{p-2}} + \frac{\mathcal{D}^2 - (n-p)(p-2)}{r^2 f} B_{tA_1\cdots A_{p-2}} = 0. \quad (80)$$

Here we expand $B_{tA_1\cdots A_{p-2}}$ in terms of the harmonics as

$$B_{tA_1\cdots A_{p-2}} = K(r)T_{A_1\cdots A_{p-2}}. \quad (81)$$

Then, the above equation becomes

$$\partial_r^2 K + \frac{n-2(p-1)}{r}\partial_r K - \frac{(\ell+p-2)(\ell+n-p-1)}{r^2 f}K = 0. \quad (82)$$

The solution has been found in the analytic form of

$$K(r) = \mathrm{B}r^{-(\ell+n-p-1)}F(\alpha, \beta, \gamma; x) + Cr^{\ell+p-2}F(\alpha', \beta', \gamma'; x), \quad (83)$$

where

$$\alpha = \frac{\ell + p - 2}{n-3} \quad (84)$$

$$\beta = \frac{\ell + n - p - 1}{n-3} \quad (85)$$

$$\gamma = 2\frac{\ell + n - 3}{n-3} = \alpha + \beta + 1, \quad (86)$$

and

$$\alpha' = -\frac{\ell + p - 2}{n-3} \quad (87)$$

$$\beta' = -\frac{\ell + n - p - 1}{n - 3} \tag{88}$$

$$\gamma' = -2\frac{\ell}{n-3}. \tag{89}$$

From the asymptotic flatness, the solution will be

$$B_{tA_1\cdots A_{p-2}} = r^{-(\ell+n-p-1)}F(\alpha, \beta, \gamma; x)T_{A_1\cdots A_{p-2}}. \tag{90}$$

Now, we can compute the field strength $H_{rtA_1\cdots A_{p-2}}$:

$$\begin{aligned} H_{rtA_1\cdots A_{p-2}} = &-(\ell + n - p - 1)r^{-(\ell+n-p)} \\ &\times F(\alpha, \beta, \gamma; x)T_{A_1\cdots A_{p-2}} \\ &-(n-3)\frac{\alpha\beta}{\gamma}\frac{x}{r}r^{-(\ell+n-p-1)} \\ &\times F(\alpha+1, \beta+1, \gamma+1; x)T_{A_1\cdots A_{p-2}}. \end{aligned} \tag{91}$$

Since

$$F(\alpha+1, \beta+1, \gamma+1; 1) = \frac{\Gamma(\gamma+1)\Gamma(\gamma-\alpha-\beta-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{92}$$

and $\Gamma(\gamma - \alpha - \beta - 1) = \Gamma(0) = -\infty$, the second term in the right-hand side of Eq. (91) diverges at the horizon. Thus, there are no regular solutions.

In the current gauge, Eq. (58) becomes

$$\partial_r^2 B_{A_1\cdots A_{p-1}} + \left(\frac{n-2p}{r} + \frac{f'}{f}\right)\partial_r B_{A_1\cdots A_{p-1}} + \frac{\mathcal{D}^2 - (n-p-1)(p-1)}{r^2 f}B_{A_1\cdots A_{p-1}} = 0. \tag{93}$$

Let us expand $B_{A_1\cdots A_{p-1}}$ in terms of harmonics as

$$B_{A_1\cdots A_{p-1}} = L(r)T_{A_1\cdots A_{p-1}}. \tag{94}$$

Then, Eq. (93) then becomes

$$\partial_r^2 L + \left(\frac{n-2p}{r} + \frac{f'}{f}\right)\partial_r L - \frac{(\ell+p-1)(\ell+n-p-2)}{r^2 f}L = 0. \tag{95}$$

The solution is given by

$$L(r) = \mathrm{B}r^{-(\ell+n-p-2)}F(\alpha, \beta, \gamma; x) + Cr^{\ell+p-1}F(\alpha', \beta', \gamma'; x), \tag{96}$$

where

$$\alpha = \frac{\ell + n - p - 2}{n-3} \tag{97}$$

$$\beta = \frac{\ell + n + p - 4}{n-3} \tag{98}$$

$$\gamma = 2\frac{n+\ell-3}{n-3} = \alpha + \beta, \tag{99}$$

and

$$\alpha' = -\frac{\ell + p - 1}{n-3} \tag{100}$$

$$\beta' = -\frac{\ell - p + 1}{n-3} \tag{101}$$

$$\gamma' = -2\frac{\ell}{n-3} = \alpha' + \beta'. \tag{102}$$

The asymptotic flatness implies $C = 0$, and then we see

$$B_{A_1\cdots A_{p-1}} = \mathrm{B}r^{-(\ell+n-p-2)}F(\alpha, \beta, \gamma; x)T_{A_1\cdots A_{p-1}}. \tag{103}$$

Since

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(0)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{104}$$

we see the singular behaviors of the field strength as

$$H_{rA_1\cdots A_{p-1}} = O(1/(r - r_0)) \tag{105}$$

and

$$H_{rA_1\cdots A_{p-1}} = O(1/(r - r_0)). \tag{106}$$

As a conclusion we can show that black holes cannot have the hair of the p-form field strengths.

V. ASYMPTOTICALLY (ANTI–)DE SITTER SPACETIMES

So far we have concentrated on asymptotically flat spacetimes and could have the analytic solution for the equation of static perturbation. On the other hand, this is not the case once one turns on the cosmological constant. Without solving the equations, however, we can ask if the solution exists. Note that the equations for the static perturbations are not changed except for the expression of $f(r)$ in the metric of the background spacetimes.

A. de Sitter cases

First we consider the cases with positive cosmological constant. In this case, the background spacetime is higher dimensional Schwarzschild–de Sitter spacetime and $f(r)$ in the metric becomes

$$f(r) = 1 - (r_0/r)^{n-3} - r^2/a^2. \tag{107}$$

Under a certain case with parameters r_0 and a, there are two horizons, black hole and cosmological horizons at r_h and r_c. Note that $r_c > r_h$.

From Eq. (82), we have the following relation:

$$\int_{r_h}^{r_c} dr r^{n-2(p-1)}\left[(\partial_r K)^2 + \frac{(\ell+p-2)(\ell+n-p-1)}{r^2 f}K^2\right] = [r^{n-2(p-1)}K\partial_r K]_{r_h}^{r_c}. \tag{108}$$

Since the presence of the cosmological constant does not disturb the behavior of the horizons, the same regularity conditions are imposed on the both of the horizons. Thus, we can see that the boundary term in the right-hand side vanishes and then

$$K = 0. \tag{109}$$

In the same way, from Eq. (95), we have

$$\int_{r_h}^{r_c} dr r^{n-2p}\left[f(\partial_r L)^2 + \frac{(\ell+p-1)(\ell+n-p-2)}{r^2}L^2\right] = [r^{n-2p} f L\partial_r L]_{r_h}^{r_c}. \tag{110}$$

Since the boundary term vanishes, we can see that

$$L = 0 \tag{111}$$

holds. Therefore, there are no regular static perturbations in the region of $r_h \leq r \leq r_c$.

B. anti–de Sitter cases

Next, let us consider asymptotically anti–de Sitter cases. In this case, $f(r) = 1 - (r_0/r)^{n-3} + r^2/a^2$. Then, near infinity, K follows the equation approximately:

$$\partial_r^2 K + \frac{n-2(p-1)}{r}\partial_r K \simeq 0. \tag{112}$$

Then, the solution is approximately given by

$$K \simeq \frac{A}{r^{n-2p+1}}. \tag{113}$$

In the current case, Eq. (82) gives us

$$\int_{r_h}^{\infty} dr r^{n-2(p-1)}\left[(\partial_r K)^2 + \frac{(\ell+p-2)(\ell+n-p-1)}{r^2 f}K^2\right] = [r^{n-2(p-1)}K\partial_r K]_{r_h}^{\infty}. \tag{114}$$

Since the boundary term near infinity is roughly estimated as $\int dr 1/r^{n-2p+2}$, one has to impose $(n+1)/2 > p$ in order to make it finite. Thus, if we impose $(n+1)/2 > p$, the boundary term vanishes and then we can conclude

$$K = 0. \tag{115}$$

Similar results will be obtained for L, that is, $L = 0$.

As a consequence, we can see that there no static perturbations of p-form field strength in asymptotically anti–de Sitter spacetimes as well.

VI. SUMMARY AND DISCUSSIONS

In this paper we studied the static perturbation of p-form field strengths for the Schwarzschild-Tangherlini spacetime and were then able to show that the black holes cannot have p-form hair except for the Maxwell cases ($p = 2$, $\ell = 0$). This work is initiated by remaining issues in the no-hair theorem [12] of p-form fields in higher dimensional black hole spacetimes. That is, there is a limitation therein of p as $p \geq (n+1)/2$ in the proof of the no-hair theorem. Therefore, it was natural to ask if the no-hair properties with $p < (n+1)/2$ hold. Our current result supports no-hair properties of p-form field strength with $p \geq 3$ regardless of such limitations.

Our analysis is based on the perturbation, and then the topology of black holes is limited to sphere. If one thinks of another topology such as black ring, there is still the possibility of a solution. According to the Appendix, however, the solution may have both electric and magnetic hairs if it exists. They will be addressed in a near-future study.

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APPENDIX: NO-DIPOLE-HAIR THEOREM REVISITED

In this appendix, we revisit the no-hair theorem of p-form fields strengths in static asymptotically flat spacetimes [12]. In the theorem, one assumes only the presence of the electric part. Then, if $p \geq (n+1)/2$, we can show that the p-form hair does not exist. From this, if one assumes only the presence of the magnetic part of p-form field strengths, we expect that a similar theorem holds. In fact, its dual version is the electric part of $(n-p)$-form field strengths. Therefore, we would guess that, if $(n+1)/2 \geq (n-p)$, the magnetic parts of p-form field strengths do not exist. The above condition is rearranged as $p \geq (n-1)/2$. The above consideration indicates the breakdown of the proof of the no-hair of p-form field strengths if both parts exist. On the other hand, the argument in the main text indicates the no-hair of p-forms except for $p = 2$. Or, it may suggest the presence of the solutions which cannot be explained by the perturbation on the Schwarzschild spacetime.

Let us examine if the no-hair theorem holds in the details. Here we include the dilation to the system described by the Lagrangian

$$\mathcal{L} = R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{p!}e^{-\alpha\phi}H_{(p)}^2, \tag{A1}$$

where ϕ is the dilation field. The Einstein equation is

$$R_{\mu\nu} = \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{p!}e^{-\alpha\phi} \times\left(pH_\mu{}^{\alpha_1\cdots\alpha_{p-1}}H_{\nu\alpha_1\cdots\alpha_{p-1}} - \frac{p-1}{n-2}g_{\mu\nu}H_{(p)}^2\right). \tag{A2}$$

Since we will not perform the equations for the p-form fields and dilation, we do not write down these equations. Different from Ref. [12], we will not assume that the p-form fields have the electric components only. The metric of static spacetimes is written as Eq. (5). From the Einstein equation, then, we can see that

$$\begin{aligned} R_{00} &= VD^2V \\ &= \frac{n-p-1}{(n-2)(p-1)!}e^{-\alpha\phi}H_0{}^{i_1\cdots i_{p-1}}H_{0i_1\cdots i_{p-1}} \\ &\quad + \frac{p-1}{(n-2)p!}V^2e^{-\alpha\phi}H_{i_1\cdots i_p}H^{i_1\cdots i_p} \end{aligned} \tag{A3}$$

and

$$\begin{aligned} R_{ij} &= {}^{(n-1)}R_{ij} - \frac{1}{V}D_iD_jV \\ &= \frac{1}{2}D_i\phi D_j\phi + \frac{1}{(p-2)!}e^{-\alpha\phi}(H_i^{0k_1\cdots k_{p-2}}H_{j0k_1\cdots k_{p-2}} \\ &\quad - \frac{1}{n-2}g_{ij}H_{0k_1\cdots k_{p-1}}H^{0k_1\cdots k_{p-1}}) \\ &\quad + \frac{1}{(p-1)!}e^{-\alpha\phi}\Big(H_i{}^{k_1\cdots k_{p-1}}H_{jk_1\cdots k_{p-1}} \\ &\quad - \frac{p-1}{p(n-2)}g_{jk}H_{k_1\cdots k_{p-1}}\Big) \end{aligned} \tag{A4}$$

hold. Moreover, we can compute the Ricci scalar of g_{ij} as

$$\begin{aligned} {}^{(n-1)}R &= \frac{1}{2}(D\phi)^2 + \frac{1}{(p-1)!}\frac{e^{-\alpha\phi}}{V^2}H_0{}^{i_1\cdots i_{p-1}}H_{0i_1\cdots i_{p-1}} \\ &\quad + \frac{1}{p!}e^{-\alpha\phi}H_{i_1\cdots i_p}H^{i_1\cdots i_p}. \end{aligned} \tag{A5}$$

The outline of the proof will be as follows if it works. We first consider the conformal transformation of $t =$ constant hypersurfaces so that the Ricci scalar is non-negative and the ADM mass vanishes. Then we will apply the positive mass theorem [22,23] and then show that the conformally transformed spacetime is flat and the p-form hair does not exist. We know that the vacuum black hole spacetimes with conformally flat static slices must be spherically symmetric. Thus, the resultant spacetime is the Schwarzschild spacetime.

Let us look at the details. For the proof of no-hair, we will consider the two conformal transformations given by

$$\tilde{g}^{\pm}_{ij} = \Omega_\pm^2 g_{ij}, \tag{A6}$$

where

$$\Omega_\pm = \left(\frac{1\pm V}{2}\right)^{2/(n-3)} =: \omega_\pm^{2/(n-3)}. \tag{A7}$$

Then,

$$\begin{aligned} \Omega_\pm^2\,{}^{(n-1)}\tilde{R} &= {}^{(n-1)}R \mp \frac{2(n-2)}{n-3}\omega_\pm^{-1}D^2V \\ &= \frac{1}{2}(D\phi)^2 + \frac{1}{(p-1)!}\frac{e^{-\alpha\phi}}{V^2}\frac{\lambda_\pm}{\omega_\pm}H_0{}^{i_1\cdots i_{p-1}}H^{0i_1\cdots i_{p-1}} \\ &\quad + \frac{1}{p!}e^{-\alpha\phi}\frac{\mu_\pm}{\omega_\pm}H_{i_1\cdots i_p}H^{i_1\cdots i_p}, \end{aligned} \tag{A8}$$

where

$$\lambda_\pm = \frac{1 \mp \frac{3n-4p-1}{n-3}}{2} \tag{A9}$$

and

$$\mu_\pm = \frac{1 \pm \frac{n-4p+1}{n-3}}{2}. \tag{A10}$$

If ${}^{(n-1)}\tilde{R}_\pm \geq 0$, we can proceed with the proof. However, we cannot. The sufficient conditions for ${}^{(n-1)}\tilde{R}_\pm \geq 0$ are $\lambda_\pm \geq 0$ and $\mu_\pm \geq 0$. Each condition becomes

$$p \geq \frac{n+1}{2} \quad \text{and} \quad p \leq \frac{n-1}{2}, \tag{A11}$$

respectively. The both conditions together do not hold manifestly. Therefore, we can say nothing about the no-hair for the cases having both of electric and magnetic p-form fields. The results may suggest the presence of the p-form hairy static black object solutions.

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