Gradient expansion, curvature perturbations, and magnetized plasmas

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The properties of magnetized plasmas are always investigated under the hypothesis that the relativistic inhomogeneities stemming from the fluid sources and from the geometry itself are sufficiently small to allow for a perturbative description prior to photon decoupling. The latter assumption is hereby relaxed and predecoupling plasmas are described within a suitable expansion where the inhomogeneities are treated to a given order in the spatial gradients. It is argued that the (general relativistic) gradient expansion shares the same features of the drift approximation, customarily employed in the description of cold plasmas, so that the two schemes are physically complementary in the large-scale limit and for the low-frequency branch of the spectrum of plasma modes. The two-fluid description, as well as the magnetohydrodynamical reduction, is derived and studied in the presence of the spatial gradients of the geometry. Various solutions of the coupled system of evolution equations in the anti-Newtonian regime and in the quasi-isotropic approximation are presented. The relation of this analysis to the so-called separate universe paradigm is outlined. The evolution of the magnetized curvature perturbations in the nonlinear regime is addressed for the magnetized adiabatic mode in the plasma frame.

I. MOTIVATIONS

The analyses of the large-scale galaxy distribution [1,2], of the high-redshift type Ia supernovae [3,4], and of the CMB observables [5,6] seem to converge, these days, on a concordance model sometimes called the ΛCDM scenario, where Λ stands for the dark energy component and CDM accounts for the dark matter component. The ΛCDM scenario is just the compromise between the number of ascertainable parameters and the quality of the observational data. The quest for a concordance lore is also able to shed some light on the presence of large-scale magnetic fields in nearly all gravitationally bound systems we observe. Since we do see magnetic fields today over large distance scales, it seems natural to scrutinize their impact on the CMB observables. This is the motivation of a program aimed at bringing the unconventional study of magnetized CMB anisotropies to the same standard of the more conventional adiabatic[1] paradigm (see [7–10] and references therein). While different approaches to the problem are certainly available [11–15] (see [16] for a more complete list of earlier references), the path followed in [7,8] led to the calculation of the temperature and polarization anisotropies induced by the magnetized (adiabatic and entropic) initial conditions. The parameters of the

magnetized background have been estimated (for the first time) in [9,10] by using the TT and TE correlations[2] measured by the WMAP Collaboration. The obtained results[3] show that large-scale (comoving) magnetic fields larger than 3.5 nG are excluded to 95% C.L. and for magnetic spectral indices $n_B = 1.6_{-0.1}^{0.8}$. These determinations have been conducted in the context of the minimal $m\Lambda$CDM, where m stands for magnetized. The addition of a fluctuating dark energy background pins down systematically larger values of the magnetic field parameters (see [10] for further details).

The results obtained so far assumed the simplest setup for the inclusion of large-scale magnetic fields in the predecoupling plasma, and it is therefore mandatory to scrutinize if the main assumptions of the analysis are consistently posited. For instance, in [7–10] (as well as in nearly all other approaches) it was assumed that magnetic fields do not contribute to the electron-photon scattering. In [17] this assumption has been relaxed by explicitly including the magnetic field contribution in the electron-photon scattering matrix.

In Refs. [7–10], as well as in other perturbative approaches to the physics of magnetized CMB anisotropies, a common hypothesis is that the intensity of the

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[1]The wording "adiabatic paradigm" refers here to the situation where the unique source of inhomogeneity prior to photon decoupling is localized in the standard adiabatic mode of curvature perturbations. This is the situation contemplated by the minimal version of the ΛCDM scenario.

[2]Following the standard shorthand terminology, the TT correlations denote the temperature autocorrelations while the TE correlations denote the cross correlation between the temperature and the E-mode polarization.

[3]In [7–10] the magnetic power spectrum and the magnetic spectral index n_B are defined with the conventions employed for the scalar modes of the geometry, where the scale-invariant limit corresponds to 1.

magnetized background is sufficiently small to describe magnetic fields within the standard Einstein-Boltzmann hierarchy, where the curvature perturbations as well as the density contrasts all remain in the perturbative regime. In similar terms, perturbation theory is extremely well justified for the treatment of baryon acoustic oscillations because of the absolute smallness of the relative temperature fluctuations. As far as fully inhomogeneous magnetic fields are concerned, the perturbative assumption is a direct consequence of the closure bound applied to the magnetic field intensity. The natural question, in this context, seems to be as follows: can we go beyond the standard perturbative expansion and scrutinize its properties in a broader and firmer scheme? Can we understand which are the possible nonlinear effects in a gravitating and magnetized plasma to a given order in the spatial gradients? One of the standard tools to improve (and partially resum) the perturbative description of the relativistic fluctuations of the geometry is the gradient expansion, where the guiding criterion is not the absolute smallness of the given perturbation relative to its background value but rather the number of gradients carried by each term and defining the order of the expansion. The extension of the general relativistic gradient expansion to the case of magnetized plasmas immediately suggests a physical connection with analog approximation schemes adopted in the discussion of weakly coupled plasmas in flat space-time.

The modest aim of the present analysis will then be to combine the general relativistic gradient expansion with the drift approximation (customarily applied in plasma physics) and to derive a system of equations where both expansions can be implemented in a unified manner. We remind the reader that a known tool for computing the properties of cold plasma involves an expansion in the gradients of the magnetic and of the electric fields. The details of the scheme depend upon the problem at hand, and this richness is reflected in slightly different terminologies such as the drift approximation [18] (mainly adopted in kinetic theory) and the guiding center approximation [19,20] (often used in the physics of cold plasmas and fully ionized gases[4]).

Similarly, the general relativistic gradient expansion has been discussed in several related contexts since the papers of Lifshitz, Khalatnikov, and Belinskii (see, e.g., [22–25]) but never in combination with either the drift approximation or the guiding center approximation. More recently, various applications of the gradient expansion to inflationary models have been studied by Tomita [26,27] as well as by Deruelle and collaborators [28–30]. The neglect of the spatial curvature and of the spatial gradients to zeroth order implies that the obtainable solutions are, in a sense, opposite to the ones customarily discussed in the Newtonian

regime, and this is the reason why they are sometimes named anti-Newtonian. The latter solutions are also relevant for the so-called separate universe picture stipulating that any portion of the universe larger than the Hubble radius $r_{\rm H}$ but smaller than the physical wavelength on the perturbation will look like a separate unperturbed universe. The gradient expansion has also been applied to the problem of the evolution of non-Gaussianities (see, e.g., [31,32] and references therein) and to the problem of dark energy [33] with the purpose of demonstrating that the spatial gradients cannot be responsible for present-day acceleration (see, e.g., [34] and references therein). An instructive approach to nonlinear power spectra in Einstein gravity has been developed by Noh, Hwang, and collaborators in a series of interesting papers [35–37]. Finally, the gradient expansion can also be employed in the investigation of the so-called sudden (or quiescent) singularities which arise in the context of dark energy models with a barotropic index smaller than that of a cosmological constant (i.e. the supernegative equations of state) [38].

Even if some of the considerations developed in this paper will be rather general, it is useful to bear in mind the essentials of the predecoupling plasma which contains different components interacting both gravitationally and electromagnetically. To fix the notations, it is convenient to separate the total action of the system, i.e. $S_{\rm tot}$, in three distinct parts,

$$S_{\rm tot} = S_{\rm grav} + S_{\rm em} + S_{\rm plasma}, \qquad (1.1)$$

where $S_{\rm grav}$ and $S_{\rm em}$ denote, respectively, the gravitational and the electromagnetic contributions,

$$S_{\rm grav} + S_{\rm em} = \int d^4x \sqrt{-g} \left[-\frac{R}{2\ell_{\rm P}^2} - \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - j_\nu A^\nu \right];$$
$$(1.2)$$

note that $g = \det(g_{\mu\nu})$ and $F_{\alpha\beta}$ is the Maxwell field strength; $S_{\rm plasma}$ contains all the different components of the plasma, which can be written, in the context of the vanilla ΛCDM paradigm, as

$$S_{\rm plasma} = S_{\rm ei} + S_\gamma + S_\nu + S_{\rm cdm} + S_\Lambda; \qquad (1.3)$$

$S_{\rm ei}$ denotes the contribution of electrons and ions; S_γ and S_ν are, respectively, the contributions of the photons and of the (massless) neutrinos; and $S_{\rm cdm}$ and S_Λ account for the cold dark matter and for the dark energy.

Before photon decoupling the conditions of validity of the general relativistic gradient expansion and of the guiding center approximation are both verified, and it is therefore extremely interesting to derive and study the evolution equations describing the plasma without assuming that the geometry is *a priori* conformally flat. Various questions can be addressed in this scheme, such as, for instance, the corrections induced by the gradients of the geometry on the two-fluid plasma description, on the MHD reduction, and on all the plasma processes which are relevant for a correct

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[4]The expansion in spatial gradients is also relevant in nonlinear magnetohydrodynamics (MHD) in flat space-time [21].

description of magnetized CMB anisotropies. Needless to say, the present results are also applicable in the case of other magnetized systems in curved space-time not necessarily connected to CMB physics.

Some examples of the statements made in the previous paragraph will now be given for the benefit of the interested reader. The conventional description of the relativistic inhomogeneities of the geometry and of the fluid sources is typically performed under the tenets of the Bardeen formalism [39]. In this framework (and in the absence of large-scale magnetic fields) the initial conditions of the Einstein-Boltzmann hierarchy are set by solving the relevant evolution equations in the large-scale limit, i.e. when the spatial gradients can be neglected (see, e.g., [40]). This way of setting initial conditions is typically employed prior to decoupling when the relevant length scales are larger than the Hubble radius. The present considerations provide a self-contained formalism, where the gradients of the electromagnetic fields and the ones of the fluid sources are treated in a unified manner. It will actually be shown that the usual perturbative description of the Einstein-Boltzmann hierarchy in the presence of large-scale magnetic fields tacitly *assumes* the present form of the gradient expansion. For instance, the magnetic energy density is quadratic in the fields' strength but it affects the first-order (perturbative) inhomogeneities. This procedure can only be rigorously justified within a form of gradient expansion that treats, on the same footing, the relativistic fluctuations of the geometry and of the gauge fields. Another class of problems where the present results can be applied involves all those nonlinear phenomena arising in cold plasma in flat space which have not been studied, so far, in curved backgrounds. An example along this direction is the nonlinear Hall effect [41]. Another example involves the potential generation of a primordial vorticity associated with the fully inhomogeneous description of the geometry which could play a role in the description of magnetized collapse and of primordial vorticity. Along these directions, dedicated discussions are in progress.

The present paper is organized as follows. In Sec. II the decomposition of the inhomogeneous geometry will be introduced, paying special attention to the themes which are mostly relevant for the present discussion. In Sec. III the evolution equations of weakly coupled plasmas will be derived in the case where the metric is not assumed to be conformally flat and, in particular, without resorting to the standard separation between background geometry and relativistic fluctuations. In Sec. IV we will scrutinize the way gradients must be counted in a magnetized plasma when the background geometry is fully inhomogeneous. Section V discusses the anti-Newtonian solutions in the drift approximation. In Sec. VI the quasi-isotropic MHD solutions are presented. Section VII contains the derivation of the nonlinear magnetized adiabatic mode whose linearized counterpart is one of the elements of the simplest

version of the $m\Lambda$CDM scenario. Finally, Sec. VIII contains our concluding remarks. In Appendixes A and B the complementary results have been collected with the purpose of keeping the paper self-contained.

II. FULLY INHOMOGENEOUS GEOMETRY

The standard treatments of magnetized CMB anisotropies as well as the discussion of magnetized plasmas in the early Universe customarily assume that the geometry is separated into a homogeneous background supplemented by its relativistic fluctuations. To go beyond the latter description, the fully inhomogeneous geometry shall then be described in terms of N, N^i, and γ_{ij} denoting, respectively, the lapse function, the shift vector, and the three-dimensional metric tensor, i.e.[5]

$$g_{00} = N^2 - N_k N^k, \qquad g_{ij} = -\gamma_{ij}, \qquad g_{0i} = -N_i,$$
$$g^{00} = \frac{1}{N^2}, \qquad g^{ij} = \frac{N^i N^j}{N^2} - \gamma^{ij}, \qquad g^{0i} = -\frac{N^i}{N^2}. \tag{2.1}$$

The decomposition of Eq. (2.1) is well known and it is sometimes referred to as the ADM decomposition from Arnowitt, Deser, and Misner who applied it to the Hamiltonian formulation of the general relativistic dynamics (see, e.g., [42,43]). In the ADM variables the extrinsic curvature K_{ij} and the spatial components of the Ricci tensor r_{ij} become

$$K_{ij} = \frac{1}{2N}[-\partial_\tau \gamma_{ij} + {}^{(3)}\nabla_i N_j + {}^{(3)}\nabla_j N_i], \tag{2.2}$$

$$r_{ij} = \partial_m {}^{(3)}\Gamma^m_{ij} - \partial_j {}^{(3)}\Gamma^m_{im} + {}^{(3)}\Gamma^m_{ij}{}^{(3)}\Gamma^n_{mn} - {}^{(3)}\Gamma^m_{jn}{}^{(3)}\Gamma^n_{im}, \tag{2.3}$$

where ${}^{(3)}\nabla_i$ is the covariant derivative defined with respect to the metric γ_{ij}, ∂_τ denotes a derivation with respect to the time coordinate τ, and ${}^{(3)}\Gamma^m_{ij}$ are the Christoffel symbols computed from γ_{ij}. Note that $\Gamma^m_{ij} = {}^{(3)}\Gamma^m_{ij}$ but only in the case $N_i = 0$ (see Appendix A for details). It is now useful to introduce the basic logic of the gradient expansion by looking at the whole system of Einstein equations with generic electromagnetic and fluid sources.[6] In their contracted form the Einstein equations read

$$R^\nu_\mu = \ell_P^2 \left[\left(T^\nu_\mu - \frac{T}{2}\delta^\nu_\mu \right) + \mathcal{T}^\nu_\mu \right], \qquad T = g^{\mu\nu}T_{\mu\nu} = T^\mu_\mu, \tag{2.4}$$

where R^ν_μ is the four-dimensional Ricci tensor while T^ν_μ and $\mathcal{T}^\nu_\mu$ denote, respectively, the fluid energy-momentum

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[5]The Greek indices will run over the four space-time dimensions, while the Latin indices will denote the spatial indices.

[6]As mentioned in Sec. I the species present in the plasma prior to photon decoupling can be interacting directly with the electromagnetic fields, and this will demand a more detailed description of the sources, as will be discussed in Sec. III.

tensor [accounting for the global contribution of all the species of Eq. (1.3)] and the electromagnetic energy-momentum tensor:

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - p g_{\mu\nu},$$
$$\mathcal{T}^\nu_\mu = \frac{1}{4\pi}\left(-F_{\mu\alpha}F^{\nu\alpha} + \frac{\delta^\nu_\mu}{4}F_{\alpha\beta}F^{\alpha\beta}\right), \qquad (2.5)$$

where $g^{\mu\nu}u_\mu u_\nu = 1$ [see also Eq. (A12) of Appendix A for an explicit expression of the latter condition in the ADM metric (2.1)]. If not otherwise stated, the covariant derivatives with spatial indices listed below will always refer to the three-dimensional metric. In other words, to simplify the notation, $^{(3)}\nabla_i \equiv \nabla_i$. Details on the inhomogeneous geometry of Eq. (2.1) can be found in Appendix A and will be quoted whenever needed. Bearing in mind these conventions, the various components of Eq. (2.4) are given by

$$\partial_\tau K - N\mathrm{Tr}K^2 + \nabla^2 N = N\ell_P^2\left\{\frac{3p + \rho}{2} + (p + \rho)u^2 + \mathcal{T}^0_0\right\}, \qquad (2.6)$$

$$\nabla_i K - \nabla_k K^k_i = N\ell_P^2\left[\frac{u_i}{N}(p + \rho)\sqrt{1 + u^2} + \mathcal{T}^0_i\right], \quad (2.7)$$

$$\partial_\tau K^j_i - NKK^j_i - Nr^j_i + \nabla_i\nabla^j N$$
$$= \ell_P^2 N\left[\frac{p - \rho}{2}\delta^j_i - (p + \rho)u_i u^j + \mathcal{T}^j_i\right], \qquad (2.8)$$

where, for the sake of simplicity, the shorthand notation $u^2 = \gamma^{ij}u_i u_j$ has been adopted. Having chosen $N_i = 0$ in the general equations of Appendix A, the geometry appearing in Eqs. (2.6), (2.7), and (2.8) is described in terms of seven independent functions (i.e. γ_{ij} and N). It is useful to keep the lapse function arbitrary for practical purposes such as, for instance, the matching with the perturbative treatment of the magnetized fluctuations in the conformally Newtonian gauge (see, e.g., Appendix B). The momentum constraint [i.e. Eq. (2.7)] can be used, in the framework of the general relativistic gradient expansion, to obtain the velocity field in terms of the extrinsic curvature evaluated to the preceding order in the expansion. By combining the trace of Eq. (2.8) with Eq. (2.6), the standard form of the Hamiltonian constraint can be readily obtained,

$$K^2 - \mathrm{Tr}K^2 + r = 2\ell_P^2[\rho + \mathcal{T}^0_0 + (p + \rho)u^2]. \quad (2.9)$$

The traceless part of Eq. (2.8) can instead be written as

$$\partial_\tau \bar{K}^j_i - NK\bar{K}^j_i - N\bar{r}^j_i + \nabla_i\nabla^j N - \frac{\nabla^2 N}{3}\delta^j_i$$
$$= \ell_P^2 N\left\{-(p + \rho)\left[u_i u^j - \frac{u^2}{3}\delta^j_i\right] + \bar{\mathcal{T}}^j_i\right\}, \qquad (2.10)$$

where the barred quantities define the traceless part of the corresponding variable, i.e.

$$\bar{K}^j_i = K^j_i - \frac{K}{3}\delta^j_i, \qquad \bar{r}^j_i = r^j_i - \frac{r}{3}\delta^j_i, \qquad \bar{\mathcal{T}}^j_i = \mathcal{T}^j_i - \frac{\mathcal{T}}{3}\delta^j_i. \qquad (2.11)$$

Recalling Eq. (A3) and using Eq. (2.11) it follows that $\mathrm{Tr}K^2 = \bar{K}^j_i\bar{K}^i_j + K^2/3$. Bearing in mind the explicit form of Eqs. (2.6), (2.7), and (2.8), to zeroth order in the spatial gradients, the peculiar velocities as well as the spatial curvature are neglected. From the momentum constraint, the zeroth-order results determine the peculiar velocity which can be used as an input for the following order in the expansion. By iteration the first-order correction to the geometry can be determined. While the zeroth order of the linearized approximation is, by definition, homogeneous, the zeroth order of the gradient expansion can be fully inhomogeneous but does not contain any spatial gradient. Conversely, the first order depends upon the spatial gradients and upon the spatial curvature and so on and so forth.

If the gravitating system is a magnetized plasma (such as the one present prior to photon decoupling) the inhomogeneities of the electromagnetic fields will necessarily modify the trajectories of the charged species. In this case the nature of the physical system combines inextricably electromagnetic and gravitational gradients. The tenets of the (general relativistic) gradient expansion must be consistently combined with the expansion in spatial derivatives usually adopted in the analysis of weakly coupled plasmas in flat space-time [19,20], both in the two- and one-fluid approximations. There are different ways of introducing the guiding center approximation, and the simplest one is to think of a gradient expansion of the large-scale magnetic field; i.e. denoting with $\vec{B}$ the (flat-space) magnetic field, we can write

$$B_i(\vec{x}, \tau) \simeq B_i(\vec{x}_0, \tau) + (x^j - x^j_0)\partial_j B_i + \dots, \qquad (2.12)$$

where the ellipsis stands for the higher orders in the gradients leading to both curvature and drift corrections. A similar expansion can also be written in the case of the electric field with the caveat that, in a plasma, electric fields are anyway screened for typical length scales larger than the Debye radius:

$$E_i(\vec{x}, \tau) \simeq E_i(\vec{x}_0, \tau) + (x^j - x^j_0)\partial_j E_i + \dots \qquad (2.13)$$

In a flat-space plasma to zeroth order in the expansion of Eqs. (2.12) and (2.13), only the time derivatives of the magnetic fields are kept. To first order the spatial derivatives of the magnetic field can then be used as an input to deduce the electric fields. The first derivatives of the electric fields (obtained to first order) can be used to deduce the second spatial derivatives of the magnetic fields and so on. If we combine the general relativistic gradient expansion with the drift approximation, the essential step is the generalization of the two-fluid description and of the usual MHD reduction to the situation where the fully

inhomogeneous geometry is parametrized as in Eq. (2.1). This analysis will lead automatically to the correct dynamical variables whose explicit form can be compared, for instance, with the corresponding variables deduced in the context of conformally flat geometries which are usually assumed in more conventional perturbative expansions.

III. CHARGED FLUIDS AND GRADIENT EXPANSION

The electromagnetic interaction affects the evolution of the electron-photon-ion system, while it affects only indirectly the evolution of the weakly interacting species. Since large-scale magnetic fields gravitate, the relativistic fluctuations of the geometry are modified by their presence via Eqs. (2.6), (2.7), and (2.8). The purpose of the present section is to deduce the usual MHD reduction to lowest order in the gradient expansion, i.e. by allowing an underlying geometry with potentially large inhomogeneities over sufficiently large length scales. The obtained equations generalize previous perturbative results (see, in particular, [9]) and will be cross-checked in the appropriate limits. The Maxwell equations in a four-dimensional curved space-time can be written as

$$\nabla_\mu F^{\mu\nu} = 4\pi j^\nu, \qquad \nabla_\mu \tilde{F}^{\mu\nu} = 0, \qquad (3.1)$$

where ∇_μ is the covariant derivative defined with respect to the four-dimensional metric $g_{\mu\nu}$, while $\tilde{F}^{\mu\nu}$ denotes the dual field strength. In terms of the ADM decomposition of Eq. (2.1), the field strengths and their duals are

$$F^{0i} = -\frac{\mathcal{E}^i}{N^2}, \qquad F^{ij} = -\gamma_{mk}\eta^{ijk}\frac{\mathcal{B}^m}{N}, \qquad (3.2)$$

$$\tilde{F}^{0i} = -\frac{\mathcal{B}^i}{N^2}, \qquad \tilde{F}^{ij} = \gamma_{mk}\eta^{ijk}\frac{\mathcal{E}^m}{N}. \qquad (3.3)$$

The totally antisymmetric Levi-Civita tensor η^{ijk} is defined as

$$\eta^{ijk} = \frac{\epsilon^{ijk}}{\sqrt{\gamma}}, \qquad \eta_{ijk} = \sqrt{\gamma}\epsilon_{ijk}, \qquad (3.4)$$

where ϵ_{ijk} is the Levi-Civita symbol in flat space. The Maxwell field strengths with doubly covariant indices are given in Appendix A [see Eq. (A27)] and simplify when the shift vector vanishes. Defining with e the electron charge, the total current appearing in Eq. (3.1) is the sum of the currents of the electrons and of the ions,[7]

$$j^\nu = j^\nu_{(e)} + j^\nu_{(i)}, \qquad j^\nu_{(e)} = -e\tilde{n}_e u^\nu_e, \qquad j^\nu_{(i)} = e\tilde{n}_i u^\nu_i, \qquad (3.5)$$

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[7]In Eq. (3.5) as well as in other equations, the (lowercase) roman subscripts label the species (for instance, electrons or ions) while the (lowercase) italic subscripts (or superscripts) must be understood as (spatial) tensor indices.

where $\tilde{n}_e$ and $\tilde{n}_i$ denote, respectively, the concentrations of electrons and ions. The generic four-velocity u^μ satisfies $g^{\mu\nu}u_\mu u_\nu = 1$, which means that

$$u^0 = \frac{\cosh y}{N}, \qquad u^i = \frac{v^i \cosh y}{N}, \qquad \cosh y = \frac{1}{\sqrt{1 - \frac{v^2}{N^2}}}, \qquad (3.6)$$

where $v^2 = \gamma_{ij}v^i v^j$. Note that, from now on, we shall deal, without loss of generality, with the case $N_i = 0$. For immediate convenience the notations of Eq. (3.6) can also be recast in a slightly different form by defining $\hat{v}^i = v^i/v$:

$$u^i = \hat{v}^i \sinh y,$$
$$u_i u^j = \hat{v}_i \hat{v}^j \sinh^2 y, \qquad (3.7)$$
$$u^i u_0 = \hat{v}^i N \cosh y \sinh y,$$

where $\hat{v}^i \hat{v}^j \gamma_{ij} = 1$. Bearing in mind Eqs. (3.5), (3.6), and (3.7), Eq. (3.1) demands the validity of the following system of equations:

$$\partial_i \left[\frac{\sqrt{\gamma}}{N}\mathcal{E}^i \right] = 4\pi e[n_i - n_e], \qquad \partial_i \left[\frac{\sqrt{\gamma}}{N}\mathcal{B}^i \right] = 0, \quad (3.8)$$

$$\partial_\tau \left[\frac{\sqrt{\gamma}}{N}\mathcal{B}^i \right] + \partial_j[\eta^{jki}\sqrt{\gamma}\gamma_{k\ell}\mathcal{E}^\ell] = 0, \qquad (3.9)$$

$$\partial_j[\sqrt{\gamma}\eta^{jki}\mathcal{B}^m\gamma_{km}] = 4\pi e[n_i v^i_{(i)} - n_e v^i_{(e)}] + \partial_\tau \left[\frac{\sqrt{\gamma}}{N}\mathcal{E}^i \right], \qquad (3.10)$$

where the electron and ion concentrations have been rescaled as

$$n_{(i)} = \tilde{n}_i \sqrt{\gamma}\cosh y_i, \qquad n_{(e)} = \tilde{n}_e \sqrt{\gamma}\cosh y_e. \qquad (3.11)$$

From the covariant conservation of the electron and ion currents (i.e. $\nabla_\mu j^\mu_{(e)} = 0$ and $\nabla_\mu j^\mu_{(i)} = 0$), the evolution equations for the rescaled electron and ion concentrations,

$$\partial_\tau n_e + \partial_k[n_e v^k_e] = 0, \qquad \partial_\tau n_i + \partial_k[n_i v^k_i] = 0, \quad (3.12)$$

imply that, to lowest order in the gradient expansion, n_e and n_i are constant. Equations (3.8), (3.9), and (3.10) assume a simpler form by an appropriate rescaling of the electric and magnetic fields:

$$\vec{\partial} \cdot \vec{E} = 4\pi e[n_i - n_e], \qquad \vec{\partial} \cdot \vec{B} = 0, \qquad (3.13)$$

$$\partial_\tau \vec{B} + \vec{\partial} \times \vec{E} = 0, \qquad (3.14)$$

$$\vec{\partial} \times \vec{B} = 4\pi e[n_i \vec{v}_i - n_e \vec{v}_e] + \partial_\tau \vec{E}, \qquad (3.15)$$

where the rescaled electric and magnetic fields are given by

$$E^i = \frac{\sqrt{\gamma}}{N}\mathcal{E}^i, \qquad B^i = \frac{\sqrt{\gamma}}{N}\mathcal{B}^i. \qquad (3.16)$$

The shorthand notation employed in Eqs. (3.13), (3.14), and (3.15) implies, for a generic vector A^i,

$$\vec{\partial} \cdot \vec{A} \equiv \partial_i A^i, \qquad (\vec{\partial} \times \vec{A})^i = \partial_j [N \gamma^{ik} \gamma^{jn} \eta_{nmk} A^m]. \tag{3.17}$$

The vectors appearing in Eqs. (3.13), (3.14), and (3.15) become three-dimensional Cartesian vectors in the limit when the background geometry is homogeneous, isotropic, and conformally flat, i.e.

$$N \to a(\tau), \qquad \gamma_{ij} = a^2(\tau) \delta_{ij}. \tag{3.18}$$

Using Eq. (3.18) in Eq. (3.11) we have that, for $y_e \ll 1$ and $y_i \ll 1$, the electron and ion concentrations[8] become $n_e = a^3 \tilde{n}_e$ and $n_i = a^3 \tilde{n}_i$. Using Eq. (3.18) in Eqs. (3.13), (3.14), and (3.15) the standard differential operators are recovered, i.e. $\vec{\partial} \to \vec{\nabla}, \vec{\partial} \times \vec{A} \to \vec{\nabla} \times \vec{A}$. In general terms, the system is not conformally invariant, as can be argued by looking at the form of Eq. (3.17) and by appreciating that indices are raised and lowered in terms of γ_{ij}. Various discussions obtained in the limit defined by Eq. (3.18) [9,10] (see also [7,8]) can be generalized to the fully inhomogeneous situation. Without dwelling on all possible generalizations we shall focus our attention only on a consistent implementation of the two-fluid and one-fluid descriptions.

Neglecting, for a moment, the terms responsible for the momentum exchange between electrons and ions, the covariant conservation of the energy-momentum tensor of the charged species implies

$$\nabla_\mu T^{\mu\nu}_{(e)} = j^{(e)}_\alpha F^{\nu\alpha}, \qquad \nabla_\mu T^{\mu\nu}_{(i)} = j^{(i)}_\alpha F^{\nu\alpha}, \tag{3.19}$$

where

$$T^{\mu\nu}_{(e)} = \rho_e u^\mu_e u^\nu_e, \qquad T^{\mu\nu}_{(i)} = \rho_i u^\mu_i u^\nu_i. \tag{3.20}$$

Using Eq. (A20) and choosing the free index of Eq. (3.19) to be timelike, the explicit evolution of the energy density of the electrons can be obtained, and it is

$$\partial_\tau [\rho_e \cosh^2 y_e] + N^2 \partial_k \left[\frac{\rho_e}{N} \cosh y_e \sinh y_e \hat{v}^k_e \right]$$
$$- NK \rho_e \cosh^2 y_e + [3\nabla_k N + N\Gamma^j_{kj}] \rho_e \cosh y_e \sinh y_e \hat{v}^k_e$$
$$- NK_{kj} [\rho_e \hat{v}^k_e \hat{v}^j_e \sinh^2 y_e + \Pi^{kj}_e] = -j^{(e)}_k \mathcal{E}^k, \tag{3.21}$$

where the contribution of the anisotropic stress of the electrons Π^{ij}_e has also been included for completeness. The same expression also holds in the case of the ions by flipping the sign of the electric charge and by replacing $e \to i$ in the various subscripts (i.e. $\rho_e \to \rho_i$, $y_e \to y_i$, ... and so on and so forth). By choosing the free index of Eq. (3.19) to be spacelike, the resulting equation is

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$$\partial_\tau [\rho_e \cosh y_e \sinh y_e \hat{v}^k_e] + \partial_m [\rho_e \sinh^2 y_e \hat{v}^m_e \hat{v}^k_e + \Pi^{mk}]$$
$$- 2NK^k_m \rho_e \sinh y_e \cosh y_e \hat{v}^m_e + \nabla^k N \rho_e \cosh^2 y_e$$
$$- NK \rho_e \sinh y_e \cosh y_e \hat{v}^k_e + [\nabla_m N + N\Gamma^n_{nm}]$$
$$\times [\rho_e \hat{v}^m_e \hat{v}^k_e \sinh^2 y_e + \Pi^{km}] + N\Gamma^k_{mn} [\rho \hat{v}^m \hat{v}^n s^2(y) + \Pi^{mn}]$$
$$= j^{(e)}_0 \frac{\mathcal{E}^k}{N} - \gamma_{mq} j^{(e)}_n \eta^{knm} \mathcal{B}^q + \mathcal{C}_{ei} + \mathcal{C}_{e\gamma}, \tag{3.22}$$

where $\mathcal{C}_{ei}$ and $\mathcal{C}_{e\gamma}$ denote the collision terms of the electrons with ions and photons. Similarly, the evolution equation for the ion velocity field can be obtained by replacing $e \to i$ in the relevant subscripts, in full analogy with what has already been suggested, after Eq. (3.21), for the evolution equations of the energy density. Equations (3.21) and (3.22) can be expanded in gradients, and, to lowest order, the evolution of the electron and ion energy densities can be derived from Eq. (3.21):

$$\partial_\tau \rho_e - NK \rho_e = -en_e \frac{\vec{v}_e \cdot \vec{E}}{\gamma},$$
$$\partial_\tau \rho_i - NK \rho_i = en_i \frac{\vec{v}_i \cdot \vec{E}}{\gamma}, \tag{3.23}$$

where, generically, the following notation will be employed throughout the paper:

$$\vec{F} \cdot \vec{G} = \gamma_{mn} F^m G^n, \qquad (\vec{F} \times \vec{G})^k = \frac{\gamma_{in} \gamma_{m\ell}}{N} F^n G^m \eta^{i\ell k}. \tag{3.24}$$

The reduction from the two-fluid to the one-fluid description follows the standard steps of the flat-space-time case (see, e.g., [44]) but with the difference that the terms stemming from the fully inhomogeneous nature of the underlying geometry will be consistently taken into account. The idea of the one-fluid reduction is to pass from a description holding for the two (or more) separate species to a one-fluid system where the dynamical variables are global, like the total current, the baryon energy density, the center-of-mass velocity of the electron-ion system and so on and so forth. The sum of the ion and of the electron energy densities is usually referred to as the baryon density, and its evolution equation is obtained by summing, term by term, the two equations appearing in Eq. (3.23):

$$\partial_\tau \rho_b - NK \rho_b = \frac{\vec{J} \cdot \vec{E}}{\gamma}, \tag{3.25}$$

where ρ_b and $\vec{J}$ are, respectively, the baryon density and the total current:

$$\rho_b = \rho_e + \rho_i, \qquad \vec{J} = e(n_i \vec{v}_i - n_e \vec{v}_e). \tag{3.26}$$

In Eq. (3.25), the contribution of the electric field has been kept since it multiplies the total current which will turn out to be related with spatial gradients of the magnetic fields as well as with the spatial gradients of the geometry, possibly multiplied by terms containing the magnetic field itself. From Eq. (3.22) the evolution equation for the electron velocity can be written as

$$\partial_\tau v_{\rm e}^k + N\partial^k N - \mathcal{G}_j^k v_{\rm e}^j$$

$$= -\frac{e\tilde{n}_{\rm e}N^2}{\rho_{\rm e}\sqrt{\gamma}}[E^k + (\vec{v}_{\rm e}\times\vec{B})^k] + N\Gamma_{\rm ei}(v_{\rm i}^k - v_{\rm e}^k)$$

$$+ \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm e}}N\Gamma_{\rm e\gamma}(v_\gamma^k - v_{\rm e}^k), \tag{3.27}$$

where

$$\mathcal{G}_j^k = \left[\frac{\partial_\tau N}{N}\delta_j^k + 2NK_j^k\right]. \tag{3.28}$$

In Eq. (3.27) the collision terms have been included and the shorthand notation of Eq. (3.24) has been used for the vector product. Similarly, the evolution equation for the ion velocity is given by

$$\partial_\tau v_{\rm i}^k + N\partial^k N - \mathcal{G}_j^k v_{\rm i}^j$$

$$= \frac{e\tilde{n}_{\rm i}N^2}{\rho_{\rm i}\sqrt{\gamma}}[E^k + (\vec{v}_{\rm i}\times\vec{B})^k] + N\Gamma_{\rm ie}(v_{\rm e}^k - v_{\rm i}^k)$$

$$+ \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm i}}N\Gamma_{\rm i\gamma}(v_\gamma^k - v_{\rm i}^k). \tag{3.29}$$

By summing up Eq. (3.27) (multiplied by the electron mass) and Eq. (3.29) (multiplied by the ion mass) the evolution equation for the center-of-mass velocity of the electron-ion system,

$$v_{\rm b} = \frac{m_{\rm e}v_{\rm e}^k + m_{\rm i}v_{\rm i}^k}{m_{\rm e} + m_{\rm i}}, \tag{3.30}$$

can be obtained, and the evolution equations for the baryon-lepton-photon system are

$$\partial_\tau \rho_\gamma = \frac{4}{3}KN\rho_\gamma - \frac{4}{3}N\partial_k\left(\frac{\rho_\gamma}{N}v_\gamma^k\right), \tag{3.31}$$

$$\partial_\tau v_{\rm b}^k = \mathcal{G}_j^k v_{\rm b}^j - N\partial^k N + \frac{(\vec{J}\times\vec{B})^k N^2}{\gamma\rho_{\rm b}(1 + m_{\rm e}/m_{\rm i})}$$

$$+ \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm b}}N\Gamma_{\gamma\rm e}(v_\gamma^k - v_{\rm b}^k), \tag{3.32}$$

$$\partial_\tau v_\gamma^k = \left[\mathcal{G}_j^k - \frac{NK}{3}\delta_j^k\right]v_\gamma^j - \frac{N^2}{4\rho_\gamma}\partial_m(\rho_\gamma\gamma^{mk})$$

$$- N\partial^k N + N\Gamma_{\gamma\rm e}(v_{\rm b}^k - v_\gamma^k), \tag{3.33}$$

where v_γ^k and ρ_γ denote, respectively, the photon velocity and the photon energy density. The possibility of describing the baryon-lepton-photon fluid as a unique physical entity is a direct consequence of the fact that the electron-ion collision rate is much larger than the electron-photon (or ion-photon) rate. While the electron-photon rate increases with the temperature, the Coulomb rate decreases. The meeting point of the two rates occurs close to the MeV. Equations (3.31), (3.32), and (3.33) can then be used below the meeting point of the two rates.

Equation (3.29) (multiplied by $en_{\rm i}$) can be subtracted from Eq. (3.27) (multiplied by $en_{\rm e}$), leading to the generalized Ohm equation, i.e. the evolution equation of the total current:

$$\frac{\partial J^k}{\partial\tau} - \mathcal{G}_j^k J^j = -e[v_{\rm i}^k\partial_j(n_{\rm i}v_{\rm i}^j) - v_{\rm e}^k\partial_j(n_{\rm e}v_{\rm e})] - e(n_{\rm i} - n_{\rm e})N\partial^k N + \frac{\omega_{\rm pe}^2 + \omega_{\rm pi}^2}{4\pi}\vec{E} - en_{\rm i}\frac{N\partial_j[p_{\rm i}\gamma^{kj}]}{\rho_{\rm i}} + en_{\rm e}\frac{N\partial_j[p_{\rm e}\gamma^{kj}]}{\rho_{\rm e}}$$

$$+ en_{\rm e}N\Gamma_{\rm ei}\left(1 + \frac{m_{\rm e}}{m_{\rm i}}\right)\left[\frac{(n_{\rm i} - n_{\rm e})(m_{\rm e} + m_{\rm i})}{m_{\rm i}n_{\rm e} + n_{\rm i}m_{\rm e}}v_{\rm b}^k - \frac{(m_{\rm i} + m_{\rm e})}{e(n_{\rm i}m_{\rm e} + n_{\rm e}m_{\rm i})}J^k\right]$$

$$+ \frac{e^2 n_{\rm e}n_{\rm i}(m_{\rm e} + m_{\rm i})N^2}{\sqrt{\gamma}m_{\rm e}(n_{\rm i}m_{\rm e} + n_{\rm e}m_{\rm i})}\left(1 + \frac{m_{\rm e}}{m_{\rm i}}\right)(\vec{v}_{\rm b}\times\vec{B})^k + \frac{eN^2}{(n_{\rm i}m_{\rm e} + n_{\rm e}m_{\rm i})\sqrt{\gamma}}\left(n_{\rm i}\frac{m_{\rm e}}{m_{\rm i}} - n_{\rm e}\frac{m_{\rm i}}{m_{\rm e}}\right)(\vec{J}\times\vec{B})^k$$

$$+ \frac{4}{3}e\rho_\gamma\left\{\left(\frac{\Gamma_{\rm i\gamma}}{m_{\rm i}} - \frac{\Gamma_{\rm e\gamma}}{m_{\rm e}}\right)v_\gamma^k + \left[\frac{\Gamma_{\rm e\gamma}n_{\rm i}(m_{\rm e} + m_{\rm i})}{m_{\rm e}(m_{\rm i}n_{\rm e} + m_{\rm e}n_{\rm i})} - \frac{\Gamma_{\rm i\gamma}n_{\rm e}(m_{\rm e} + m_{\rm i})}{m_{\rm i}(m_{\rm i}n_{\rm e} + m_{\rm e}n_{\rm i})}\right]v_{\rm b}^k\right.$$

$$\left. - \left[\frac{\Gamma_{\rm e\gamma}}{e(m_{\rm i}n_{\rm e} + n_{\rm i}m_{\rm e})}\left(\frac{m_{\rm i}}{m_{\rm e}}\right) + \frac{\Gamma_{\rm i\gamma}}{e(m_{\rm i}n_{\rm e} + n_{\rm i}m_{\rm e})}\left(\frac{m_{\rm e}}{m_{\rm i}}\right)\right]J^k\right\}, \tag{3.34}$$

where Eq. (3.12) has been used and where the plasma frequencies for electrons and ions are defined, respectively, as

$$\omega_{\rm pe}^2 = \frac{4\pi e^2 n_{\rm e}N^2}{m_{\rm e}\sqrt{\gamma}}, \qquad \omega_{\rm pi}^2 = \frac{4\pi e^2 n_{\rm i}N^2}{m_{\rm i}\sqrt{\gamma}}. \tag{3.35}$$

Since the plasma is globally neutral, $n_{\rm e} = n_{\rm i} = n_0$ (with[9] $n_0 = \eta_{\rm b}n_\gamma$). Thus Eq. (3.35) greatly simplifies also because of the smallness of the ratio between the electron and ion mass:

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[9]Recall that $\eta_{\rm b}$ denotes the ratio between the baryonic concentration and the photon concentration, and it is given by $\eta_{\rm b} = 6.219 \times 10^{-10}(h_0^2\Omega_{\rm b0}/0.027\,73)(2.725\text{ K}/T_{\gamma0})^3$, where h_0 is the indetermination on the present value of the Hubble rate and $\Omega_{\rm b0}$ parametrizes the present value of the critical fraction of baryons; $T_{\gamma0}$ is the CMB temperature.

$$\frac{\partial J^k}{\partial \tau} + \left[N\Gamma_{ie}\delta_j^k + \frac{4\rho_\gamma \Gamma_{e\gamma}}{3n_0 m_e} \delta_j^k - \mathcal{G}_j^k \right] J^j$$

$$= -e[v_i^k \partial_j(n_i v_i^j) - v_e^k \partial_j(n_e v_e)] + \frac{\omega_{pe}^2}{4\pi} \left[E^k + (\vec{v}_b \times \vec{B})^k \right.$$

$$\left. + \frac{N}{en_0} \partial_j(p_e \gamma^{kj}) - \frac{(\vec{J} \times \vec{B})^k}{en_0} \right]. \quad (3.36)$$

In Eq. (3.36) the terms containing the spatial gradients have been kept to illustrate the analogies and the differences with the customary MHD discussions in flat space-time. The two terms appearing in the third line of Eq. (3.36) are, respectively, the thermoelectric term (containing the gradient of the pressure) and the Hall term (containing the vector product of the current and of the magnetic field). The thermoelectric term contains pressure gradients, and it is therefore of higher order in the gradient expansion. The terms containing the velocity field multiplied by its own gradient are even smaller than the thermoelectric term. The Hall term is of the same order as the thermoelectric term. Equation (3.36) can also be schematically written as

$$\partial_\tau J^k + (N\Gamma \delta_i^k - \mathcal{G}_i^k)J^i = S^k, \quad (3.37)$$

where $\Gamma = \Gamma_{ie} + [4\rho_\gamma/(3n_0 m_e)]\Gamma_{e\gamma})$ denotes the sum of the electron-ion and electron-photon rates. Equations (3.32) and (3.33) can be combined by noticing that, for sufficiently early times prior to decoupling, the baryon and photon velocities coincide. The resulting equation generalizes the standard evolution equation for the baryon-photon velocity which plays a role in the semianalytic treatment of the magnetized CMB anisotropies at small angular scales [9]. Another possible generalization concerns the propagation of electromagnetic waves in the plasma which is relevant for the Faraday rotation of the polarization plane of the CMB (see, e.g., [45] and also [46]). In conclusion, all the evolution equations customarily employed for the description of weakly coupled plasmas in the linearized approximation can be generalized to the case of a fully inhomogeneous geometry expressed in ADM variables.

The evolution equations derived in this section reproduce, in the conformally flat limit, the standard perturbative results of Ref. [9]. As an example, consider that Eqs. (3.31), (3.32), and (3.33) describe the evolution of the photon-baryon system to lowest order in the gradient expansion. All the quantities appearing in the latter equations depend both on $\vec{x}$ and τ and generalize the perturbative approach to the study of the relativistic fluctuations of the geometry in a cold plasma. To reproduce the perturbative results it suffices to identify

$$N(\vec{x}, \tau) = a(\tau)[1 + \phi(\vec{x}, \tau)],$$
$$\gamma_{ij}(\vec{x}, \tau) = a^2(\tau)[1 - 2\psi(\vec{x}, \tau)]\delta_{ij}, \quad (3.38)$$
$$\rho_\gamma(\vec{x}, \tau) = \bar{\rho}_\gamma(\tau)[1 + \delta_\gamma(\vec{x}, \tau)].$$

Provided ϕ, ψ, and δ_γ are perturbatively well defined, the choice of Eq. (3.38) corresponds to the case of the conformally Newtonian gauge discussed, for related reasons, in Appendix B. Using Eq. (3.38), Eqs. (3.31), (3.32), and (3.33) then become

$$\partial_\tau \delta_\gamma = 4\partial_\tau \psi - \tfrac{4}{3}\vec{\nabla} \cdot \vec{v}_\gamma, \quad (3.39)$$

$$\partial_\tau \vec{v}_\gamma = -\tfrac{1}{4}\vec{\nabla}\delta_\gamma - \vec{\nabla}\phi + a\Gamma_{\gamma e}(\vec{v}_b - \vec{v}_\gamma), \quad (3.40)$$

$$\partial_\tau \vec{v}_b = -\mathcal{H}\vec{v}_b + \frac{\vec{J} \times \vec{B}}{a^4 \rho_b} - \vec{\nabla}\phi + \frac{4}{3}\frac{\rho_\gamma}{\rho_b} a\Gamma_{\gamma e}(\vec{v}_\gamma - \vec{v}_b), \quad (3.41)$$

which coincide with the system discussed in [9]. For instance, Eqs. (3.39), (3.40), and (3.41) have been used to derive the appropriate initial conditions for magnetized CMB anisotropies in the tight-coupling approximation. In connection with Eq. (3.31) we should mention a general feature of the interplay between the gradient expansion and the more conventional perturbative expansion leading to Eqs. (3.39), (3.40), and (3.41): the term containing the velocity field in Eq. (3.31) is of higher order in the gradient expansion, but it is necessary to recover the correct linearized result. This occurrence simply signals that the gradient expansion and the standard perturbative expansion do not commute: the linearization of the equations written at the lowest order in the gradient expansion leads to a set of equations which is different from the one obtained by linearizing the exact equations. A useful compromise, as shown in Eq. (3.31), is to keep, in the lowest order of the gradient expansion, all those terms leading, after the linearization, to the standard perturbative equations in a given gauge.

IV. COUNTING GRADIENTS IN WEAKLY COUPLED PLASMAS

Let us consider, for the sake of concreteness, typical length scales of the order of (or larger than) the Hubble radius[10] (and hence larger than the Debye length) and typical time derivatives much smaller than the plasma frequency [see, e.g., Eq. (3.35)]. In weakly coupled plasmas the plasma parameter[11] g_{plasma} is, by definition, very small:

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[10] The definition of the Hubble radius r_H refers, strictly speaking, to the homogeneous and isotropic case and can be slightly ambiguous in the fully inhomogeneous situation, which is the one treated in the present paper. For the moment, r_H must be considered as the inverse of the trace of the extrinsic curvature (possibly evaluated in a specific gauge).

[11] The plasma parameter quantifies, by definition, the inverse of the number of particles present in the Debye sphere, i.e. the sphere whose radius is given by the Debye length.

$$g_{\text{plasma}} = \frac{1}{V_D n_0 x_e} = 24e^3 \sqrt{\frac{\zeta(3)}{\pi}} \sqrt{x_e \eta_{b0}}$$

$$= 2.308 \times 10^{-7} \sqrt{x_e} \left(\frac{h_0^2 \Omega_{b0}}{0.022\,73}\right)^{1/2}, \quad (4.1)$$

$$V_D = \frac{4}{3} \pi \lambda_D^3, \qquad \lambda_D = \sqrt{\frac{T}{8\pi e^2 n_0 x_e}},$$

where the reference temperature T has been taken to coincide with the photon temperature (recall that the electron and proton temperatures coincide up to a small quantity, which is the ratio of r_H^{-1} to the electron-photon rate [7,8]). Given the smallness of g_{plasma}, the electron-ion mean free path is much larger than the Debye scale, while the corresponding collision frequency is much smaller than the plasma frequency, i.e.

$$\frac{\lambda_D}{\lambda_{ei}} = \frac{g_{\text{plasma}}}{48\pi} \ln\Lambda_C, \qquad \frac{\Gamma_{ei}}{\omega_{pe}} = \frac{\ln\Lambda_C}{24\sqrt{2}\pi} g_{\text{plasma}}, \quad (4.2)$$

where $\Lambda_C = (18\sqrt{2}/g_{\text{plasma}})$ is the argument of the Coulomb logarithm. Since r_H greatly exceeds λ_{ei}, the space-time gradients of the system under consideration will be ordered as follows:

$$\nabla < \mathcal{O}\left(\frac{1}{r_H}\right) \ll \mathcal{O}\left(\frac{1}{\lambda_{ei}}\right) \ll \mathcal{O}\left(\frac{1}{\lambda_D}\right), \qquad \partial_\tau \ll \mathcal{O}(\omega_{pe}),$$
$$(4.3)$$

where ∇ and ∂_τ denote, respectively, the magnitude of a typical spatial gradient and the magnitude of a typical time gradient. Let us then consider Eq. (4.3) in conjunction with Eq. (3.37) and with Eq. (3.15), whose explicit form, in terms of the total current, can also be written as

$$(\vec{\partial} \times \vec{B})^k = 4\pi J^k + \partial_\tau E^k. \quad (4.4)$$

Because of the hierarchies established in Eq. (4.3) the time gradient of the total current must be negligible in comparison with the term containing the total rate. Similarly, the total current must be much larger than the displacement current, i.e.

$$\partial_\tau J^k \simeq \mathcal{O}(\mathcal{G}_i^k J^i) \ll N\Gamma J^k, \qquad 4\pi \vec{J}^k \gg \partial_\tau E^k. \quad (4.5)$$

Equations (3.37) and (4.3), together with Eq. (4.5), also imply the following pair of inequalities:

$$\partial_\tau J^k \ll \omega_{pe}^2 \partial_\tau E^k, \qquad \partial_\tau^2 J^k \ll \omega_{pe}^2 J^k. \quad (4.6)$$

But then Eq. (4.6) means that the Ohm equation of Eqs. (3.36) and (3.37) reduces to the following Ohm law:

$$J^k = \sigma[E^k + (\vec{v}_b \times \vec{B})^k], \qquad \sigma = \frac{\omega_{pe}^2}{4\pi N\Gamma_{ie}}, \quad (4.7)$$

where the conductivity σ is not bound to be homogeneous as in the case of previous treatments [9]. Equation (4.7) generalizes to the fully inhomogeneous situation the

standard result of the conformally flat limit already mentioned in Eq. (3.18) (see, e.g., [8], third reference). Note that, in the latter limit, the spatial derivatives of the extrinsic curvature and of the determinant of the metric are all vanishing; moreover, the total current, the electric and magnetic fields, and the conductivity are all rescaled through different powers of the scale factor with respect to their flat-space values. Consider now Eq. (4.7) written in its explicit form. Because of Eqs. (4.4), (4.5), and (4.6), the Ohmic electric field can be expressed as

$$E^i = \frac{1}{4\pi\sigma} \partial_j[N\gamma^{ik}\gamma^{jn}\eta_{mnk}B^m] - \frac{1}{N}\gamma_{fq}\gamma_{gp}\eta^{qpi}v_b^f B^g, \quad (4.8)$$

where the second equation reported in Eq. (3.13) has been used to simplify the obtained result. Equation (4.8) still implies, for weakly coupled plasmas, that in the drift approximation the electric fields depend upon the gradients of the magnetic field in the baryon rest frame. This means that the electric fields are higher order in the gradients. From Eq. (4.8), the generalized magnetic diffusivity equation can be derived, and the result is

$$\partial_\tau B^h + \frac{1}{4\pi}\partial_r\left\{\frac{N}{\sigma}\gamma^{hs}\gamma^{ru}\eta_{sui}\partial_j[N\gamma^{ik}\gamma^{jn}\eta_{nmk}B^m]\right\}$$
$$= \partial_r[\gamma^{hs}\gamma^{ru}\eta_{sui}\gamma_{fq}\gamma_{gp}v_b^f B^g \eta^{qpi}], \quad (4.9)$$

where, for immediate convenience, we preferred to avoid the shorthand notations employed before. The analog equation for the electric field is instead

$$\partial_\tau E^i = -\partial_\tau\left(\frac{\eta^{qpi}}{N}\gamma_{fq}\gamma_{gp}v_B^f B^g\right)$$
$$+ \frac{1}{4\pi\sigma}\partial_j[\partial_\tau(N\gamma^{ik}\gamma^{jn}\eta_{mnk}B^m)]. \quad (4.10)$$

In the conformally flat limit mentioned in Eq. (3.18), Eqs. (4.9) and (4.10) coincide, respectively, with their flat-space counterparts, namely,

$$\frac{\partial\vec{B}}{\partial\tau} = \vec{\nabla} \times (\vec{v}_b \times \vec{B}) + \frac{1}{4\pi\sigma}\nabla^2\vec{B}, \quad (4.11)$$

$$\frac{\partial\vec{E}}{\partial\tau} = -\frac{\partial}{\partial\tau}(\vec{v}_b \times \vec{B}) + \frac{1}{4\pi\sigma}\nabla^2\vec{E}, \quad (4.12)$$

where, according to Eq. (3.16), $\vec{B} = a^2\vec{\mathcal{B}}$ and $\vec{E} = a^2\vec{\mathcal{E}}$.

Let us now rewrite Eqs. (2.6), (2.7), and (2.8) in a more explicit form, which will turn out to be useful in the forthcoming sections:

$$\partial_\tau K - N\text{Tr}K^2 + \nabla^2 N$$
$$= N\ell_P^2\left[\frac{3p+\rho}{2} + (p+\rho)s^2(y) + \rho_B + \rho_E\right], \quad (4.13)$$

$$\nabla_i K - \nabla_k K_i^k = \ell_P^2 \Big[(p + \rho)\hat{v}_i s(y)c(y)$$
$$- \frac{1}{4\pi\gamma}\eta^{mnp}\gamma_{im}\gamma_{jn}\gamma_{pk}B^k E^j \Big], \quad (4.14)$$

$$\partial_\tau K_i^j - NKK_i^j - Nr_i^j + \nabla_i\nabla^j N$$
$$= N\ell_P^2 \Big[\frac{p - \rho}{2}\delta_i^j - (p + \rho)s^2(y)\hat{v}_i\hat{v}^j$$
$$- (p_B + p_E)\delta_i^j + \tilde{\Pi}_i^j + \Pi_i^j(E) + \Pi_i^j(B) \Big], \quad (4.15)$$

where the notations of Appendix A have been used [see, in particular, Eqs. (A25), (A24)–(A26), and (A32)–(A34)]. To zeroth order in the gradient expansion, the terms containing $s(y)$ are subleading since $y = v/N$ and $v^2 = \gamma_{ij}v^i v^j$. By combining Eq. (4.13) and the trace of Eq. (4.15) the terms $\partial_\tau K$ can be eliminated, and the resulting expression is the Hamiltonian constraint

$$K^2 - \text{Tr}K^2 + r = \ell_P^2[2(p + \rho)s^2(y) + 2\rho + (\rho_E + 3p_E)$$
$$+ (\rho_B + 3p_B)] \quad (4.16)$$

coinciding, as expected, with the expression already obtained in Eq. (2.9) in the light of the values of p_B and p_E. We recall that the velocity field, the energy density ρ, and the pressure p are global quantities given by the sum over the individual species:

$$p = \sum_a p_a, \qquad \rho = \sum_a \rho_a,$$
$$(p + \rho)v^k = \sum_a (p_a + \rho_a)v_a^k, \quad (4.17)$$

and obeying conservation equations which can be obtained, species by species, using the results of Eqs. (A20) and (A21) reported in Appendix A:

$$\partial_\tau\rho + N\partial_k\Big[\frac{(p + \rho)}{N}v^k\Big] - NK(p + \rho) = 0, \quad (4.18)$$

$$\partial_\tau[(p + \rho)v^i] + N^2\partial_k(p\gamma^{ik}) + N^2\partial_k\Pi^{ki} + (p + \rho)N\partial^i N$$
$$- 2NK_j^i(p + \rho)v^j - NK(p + \rho)v^i = 0. \quad (4.19)$$

As already mentioned after Eq. (4.8), the baryon rest frame is particularly useful for the treatment of the finite conductivity effects. In a perfectly conducting medium [i.e. $\sigma(\vec{x}, \tau) \to \infty$] the Ohmic electric field is perfectly screened. Owing to this occurrence, it is customary to define the plasma frame where the electric fields are set to zero. In the usual perturbative expansion defined, for instance, in Appendix B, the plasma frame coincides with the baryon rest frame. In the fully nonlinear case, however, the two concepts do not necessarily coincide. It is finally appropriate to mention that there are also nonlinearities associated with the Ohm law itself [see Eq. (3.36)], and an example along this direction is the so-called nonlinear Hall

effect. This effect comes by retaining the terms $\vec{J} \times \vec{B}$ in Eq. (3.36). The Hall term leads, in the magnetic diffusivity equation (3.36), to a term of the type $\vec{\nabla} \times [\vec{B} \times (\vec{\nabla} \times \vec{B})]/(en_0)$. The nonlinear Hall term can partially balance or even become greater than the dynamo term, under certain conditions [41]. The present framework paves the way for the consistent treatment of the gravitating counterpart of the nonlinear effects typical of cold plasmas (see, for instance, [21]).

V. ANTI-NEWTONIAN DRIFT APPROXIMATION

Consider, for simplicity, the case where $N = N(\tau)$; then $\gamma_{ij}(\vec{x}, \tau)$ corresponds to six unknown functions, and all the terms containing at least one spatial gradient of N vanish exactly. The spatial curvature is neglected since it contains two spatial gradients of γ_{ij}. Bearing in mind the definition of Eq. (2.11), Eqs. (4.13) and (4.15) can be written as

$$\partial_\tau K = N\text{Tr}K^2 + \frac{N\ell_P^2}{2}[(3\bar{p} + \bar{\rho}) - 2\rho_\Lambda + 2\rho_B], \quad (5.1)$$

$$\partial_\tau K = NK^2 + \frac{3\ell_P^2 N}{2}[\bar{p} - \bar{\rho} - 2\rho_\Lambda - 2p_B], \quad (5.2)$$

$$\partial_\tau\bar{K}_i^j = NK\bar{K}_i^j + \ell_P^2 N\Pi_i^j, \quad (5.3)$$

$$\partial_\tau\bar{\rho} = NK(\bar{p} + \bar{\rho}), \qquad \partial_\tau B^i = 0. \quad (5.4)$$

Concerning Eqs. (5.1), (5.2), (5.3), and (5.4), a few comments are in order. The Ohmic electric fields have been neglected since they are of higher order and, for the sake of simplicity, Π_i^j denotes the *total* anisotropic stress. Note that, in Eqs. (5.1), (5.2), (5.3), and (5.4), the total pressure and the total energy density have been separated, respectively, as $p = \bar{p} - \rho_\Lambda$ and $\rho = \bar{\rho} + \rho_\Lambda$, where ρ_Λ parametrizes the dark energy density contribution; $\bar{p}$ and $\bar{\rho}$ are the pressure and energy density of an ordinary fluid characterized by a (possibly inhomogeneous) barotropic index $w = \bar{p}/\bar{\rho}$. The approximations leading to Eqs. (5.1), (5.2), (5.3), and (5.4) define the anti-Newtonian limit. Eqs. (5.1) and (5.2) can also be written in more explicit terms by introducing the barotropic index and by recalling that, as in Eq. (2.11), $\bar{K}_i^j = K_i^j - (K/3)\delta_i^j$:

$$\partial_\tau K = NK^2 + \frac{3N\ell_P^2}{2}[(w - 1)\bar{\rho} - 2\rho_\Lambda - 2p_B], \quad (5.5)$$

$$\partial_\tau K = \frac{N}{3}K^2 + N\text{Tr}\bar{K}^2 + \frac{N\ell_P^2}{2}[(3w + 1)\bar{\rho} - 2\rho_\Lambda + 2\rho_B]. \quad (5.6)$$

By eliminating $\bar{\rho}$ between Eqs. (5.5) and (5.6) and by defining $dt = N(\tau)d\tau$, the resulting equation is given by

$$\partial_t K - \frac{w + 1}{2} K^2 + \frac{3(w - 1)}{4} \text{Tr} \bar{K}^2$$

$$= -\frac{3\ell_\text{P}^2}{2} \left[(w + 1)\rho_\Lambda + \frac{(3w - 1)}{3} \rho_\text{B} \right]. \quad (5.7)$$

The explicit solution of Eq. (5.3), $\bar{K}_i^j(\vec{x}, t)$, implies that

$$\bar{K}_i^j(\vec{x}, t) = \frac{\lambda_i^j(\vec{x})}{\sqrt{\gamma(\vec{x}, t)}} + \ell_\text{P}^2 \frac{\sigma_i^j(\vec{x}, t)}{\sqrt{\gamma(\vec{x}, t)}},$$

$$\sigma_i^j(\vec{x}, t) = \int^t \sqrt{\gamma(\vec{x}, t')} \Pi_i^j(\vec{x}, t') dt'. \quad (5.8)$$

By now introducing the rescaled variable $\mathcal{M}(\vec{x}, t) = [\gamma(\vec{x}, t)]^{(w+1)/4}$, and by recalling that $\rho_B(\vec{x}, t) = \rho_B(\vec{x})\gamma^{-2/3}$, Eq. (5.7) becomes

$$\partial_t^2 \mathcal{M} = \frac{3}{8}(w^2 - 1)\mathcal{M}^{(w-3)/(w+1)}[\text{Tr}\lambda^2 + \ell_\text{P}^4\text{Tr}\sigma^2$$

$$+ 2\ell_\text{P}^2\text{Tr}\lambda\sigma] + \frac{3(w + 1)^2\ell_\text{P}^2}{4}\rho_\Lambda(\vec{x})$$

$$\times \left\{ 1 + \frac{3w - 1}{3(w + 1)}\Lambda_\text{B}(\vec{x})\mathcal{M}^{-8/[3(w+1)]} \right\}\mathcal{M}, \quad (5.9)$$

where $\Lambda_B(\vec{x}) = \rho_B(\vec{x})/\rho_\Lambda(\vec{x})$. Defining the following auxiliary coefficients,

$$\mathcal{C}(\vec{x}, t) = \frac{3}{8}(w^2 - 1)[\text{Tr}\lambda^2 + \ell_\text{P}^4\text{Tr}\sigma^2 + 2\ell_\text{P}^2\text{Tr}\lambda\sigma],$$

$$\mathcal{D}(\vec{x}) = \frac{3(w + 1)^2\ell_\text{P}^2}{4}\rho_\Lambda(\vec{x}), \quad (5.10)$$

Eq. (5.9) becomes

$$\partial_t^2 \mathcal{M} = \mathcal{C}(\vec{x}, t)\mathcal{M}^{(w-3)/(w+1)}$$

$$+ \mathcal{D}(\vec{x})\left\{ 1 + \frac{3w - 1}{3(w + 1)}\Lambda_\text{B}(\vec{x})\mathcal{M}^{-8/[3(w+1)]} \right\}\mathcal{M}.$$

$$(5.11)$$

The general form of Eq. (5.11) does not have explicit analytic solutions. In various limits it is possible to integrate it once with respect to t in terms of space-dependent integration constants. The result, however, cannot be further integrated (or inverted) except in few special cases. Equation (5.11) can certainly be integrated numerically, but that study is beyond the scope of the present discussion.

It is useful to look at Eq. (5.11) by bearing in mind the usual assumptions of the separate universe picture, stipulating that any portion of the universe that is larger than the Hubble radius r_H but smaller than the physical wavelength on the perturbation will look like a separate unperturbed universe. Such a framework is justified in the case $\bar{K}_i^j = 0$. In the latter case, the extrinsic curvature can really be thought of as the inhomogeneous generalization of the Hubble parameter. Assuming, for consistency with this hypothesis, that large-scale electromagnetic fields are absent, Eqs. (5.1), (5.2), and (5.3) formally coincide with the

Friedmann-Lemaître equations. Consider next a slightly more complicated situation, namely, the one where $\rho_\Lambda = 0$ but $\bar{K}_i^j \neq 0$. In this case Eq. (5.11) can be explicitly integrated and $\mathcal{M}$ can be determined. Suppose, for the sake of simplicity, that $w = 0$ and that $\Lambda_\text{B}(\vec{x}) = 0$. In this case Eq. (5.11) can be solved exactly: $\mathcal{C}(\vec{x}, t)$ becomes independent of t since $\sigma_i^j(\vec{x}, t) \to 0$. The determinant of the metric becomes, in the cosmic time coordinate t,

$$\gamma(\vec{x}, t) = \left\{ H_i^2[t - t_i(\vec{x})]^2 - \frac{3}{8}\frac{\text{Tr}\lambda^2}{H_i^2} \right\}^2, \quad (5.12)$$

where H_i is a space-time constant while, as already mentioned, $\text{Tr}\lambda^2$ can have an arbitrary spatial dependence. From Eq. (5.12) we can argue that even if $\bar{K}_i^j \neq 0$, $\text{Tr}\lambda^2$ affects the final solution in such a way that it can be reabsorbed in the definition of the initial time coordinate or, put in different terms, $\bar{K}_i^j \neq 0$ but goes quickly to zero if the universe expands since $\bar{K}_i^j = \lambda_i^j/\sqrt{\gamma}$. If the magnetic anisotropic stress is taken into account, the full system must be solved consistently, and anti-Newtonian solutions do not seem sufficient even if general solutions of Eq. (5.11) are available. Therefore, it seems useful to explore a slightly different strategy and solve the system in the quasi-isotropic limit, where the contribution of the intrinsic curvature and of the magnetic anisotropic stress can be explicitly taken into account order by order.

VI. QUASI-ISOTROPIC MHD SOLUTION

In the MHD description, the fields E^i, B^i, and J^i defined in Eqs. (3.16) and (3.26) are solenoidal, i.e.

$$\partial_i E^i = 0, \qquad \partial_i B^i = 0, \qquad \partial_i J^i = 0, \quad (6.1)$$

so that flat-space MHD is recovered in the conformally flat limit of Eq. (3.18). Let us consider, as the first example of the present section, the following parametrization of the spatial geometry:

$$\gamma_{ij}(\vec{x}, \tau) = a^2(\tau)e^{-2\Psi(\vec{x}, \tau)}[\delta_{ij} + 2h_{ij}(\vec{x}, \tau)],$$

$$h_i^i = 0, \qquad N(\tau) = a(\tau), \quad (6.2)$$

where h_{ij} accounts for terms containing more than one spatial gradient. Using Eq. (6.2) in Eq. (2.2) the explicit form of the extrinsic curvature becomes

$$K_i^j = \left[\left(\frac{\partial_\tau \Psi}{a} - \frac{\mathcal{H}}{a} \right)\delta_i^j - \frac{\partial_\tau h_i^j}{a} \right]. \quad (6.3)$$

The conformally flat limit of Eqs. (6.2) and (6.3) corresponds to $\Psi \to 0$ and $h_{ij} \to 0$; following the same logic, the total fluid pressure and the total energy density can be separated, respectively, as

$$p(\vec{x}, \tau) = p^{(0)}(\tau) + p^{(1)}(\vec{x}, \tau),$$

$$\rho(\vec{x}, \tau) = \rho^{(0)}(\tau) + \rho^{(1)}(\vec{x}, \tau), \quad (6.4)$$

where $p^{(1)}(\vec{x}, \tau)$ and $\rho^{(1)}(\vec{x}, \tau)$ vanish in the conformally flat limit. Inserting Eqs. (6.2) and (6.3) into Eqs. (4.8) and (4.9) the Ohmic electric field and the magnetic diffusivity equation are, respectively,[12]

$$\vec{E} = \frac{1}{4\pi\sigma}\vec{\nabla}\times(e^{\Psi}\vec{B}) - e^{-\Psi}\vec{v}_{\rm b}\times\vec{B}, \qquad (6.5)$$

$$\partial_{\tau}\vec{B} + \frac{1}{4\pi\sigma}\vec{\nabla}\times[e^{\Psi}\vec{\nabla}\times(e^{\Psi}\vec{B})] = \vec{\nabla}\times(\vec{v}_{\rm b}\times\vec{B}). \qquad (6.6)$$

The terms containing h_{ij} have been neglected since they are of higher order. Inserting Eqs. (6.2) and (6.3) into Eq. (2.6) the following relation can be obtained,

$$\partial_{\tau}^2\Psi + \mathcal{H}\partial_{\tau}\Psi - \partial_{\tau}\mathcal{H}$$
$$= a^2\ell_{\rm P}^2\left\{\frac{\rho + 3p}{6} + \frac{(p+\rho)}{3}a^2e^{-2\Psi}s^2(y)\right.$$
$$\left. + \frac{e^{4\Psi}}{24\pi a^4}[E^2 + \tilde{E}^2 + B^2 + \tilde{B}^2]\right\}, \qquad (6.7)$$

where the following notations have been adopted:

$$E^2 = \delta_{ij}E^iE^j, \qquad \tilde{E}^2 = 2h_{ij}E^iE^j, \qquad B^2 = \delta_{ij}B^iB^j,$$
$$\tilde{B}^2 = 2h_{ij}B^iB^ju^2 = \gamma_{ij}u^iu^j$$
$$= a^2e^{-2\Psi}[s^2(y) + 2c^2(y)h_{ij}\hat{v}^i\hat{v}^j],$$
$$y = \frac{v}{a}, \qquad v^2 = \delta_{ij}v^iv^j. \qquad (6.8)$$

Note that $\tilde{E}^2 \ll E^2$ and $\tilde{B}^2 \ll B^2$ since $\tilde{E}$ and $\tilde{B}$ contain h_{ij} and are therefore of higher order in the expansion. Furthermore, in the resistive MHD limit (i.e. $\sigma \gg 1$) $\rho_e \ll \rho_B$ since

$$\rho_e = \frac{1}{8\pi a^4}\left\{\frac{e^{6\Psi}}{16\pi^2\sigma^2}[|\vec{\nabla}\times\vec{B}|^2 + |\vec{\nabla}\Psi\times\vec{B}|^2\right.$$
$$+ 2(\vec{\nabla}\Psi\times\vec{B})\cdot(\vec{\nabla}\times\vec{B})] - \frac{e^{4\Psi}}{2\pi\sigma}[(\vec{v}_{\rm b}\times\vec{B})$$
$$\left. \cdot(\vec{\nabla}\times\vec{B}) + (\vec{v}_{\rm b}\times\vec{B})\cdot(\vec{\nabla}\Psi\times\vec{B})] + e^{2\Psi}|\vec{v}_{\rm b}\times\vec{B}|^2\right\}. \qquad (6.9)$$

Equation (6.9) shows, in practice, that at the nonlinear level the baryon rest frame and the plasma frame are two complementary concepts. While in the baryon rest frame the electric energy density vanishes as σ^{-2} for large conductivity, if the conductivity is not large the contributions of the baryon velocity and of the gradients of the geometry are of higher order in the spatial gradients in comparison with the magnetic contributions. The resistive MHD limit

implies, as usual, the largeness of the magnetic Reynolds number defined as

$$R_{\rm m} = \frac{u}{\eta\nabla} \gg 1, \qquad \eta = \frac{1}{4\pi\sigma}, \qquad (6.10)$$

where η is the magnetic diffusivity, ∇^{-1} denotes the typical length scale associated with spatial gradients, and u is the modulus of the typical velocity of the plasma element. The velocity field itself is of higher order in comparison with ρ, p, $\rho_{\rm B}$ since, from the momentum constraint of Eq. (4.14), we can deduce

$$u^0u_i = \frac{1}{a^2\ell_{\rm P}^2(p^{(0)} + \rho^{(0)})}\left\{2\partial_i\partial_{\tau}\Psi + \partial_k\partial_{\tau}h_i^k\right.$$
$$\left. + \frac{\ell_{\rm P}^2e^{3\Psi}}{4\pi a^2}(\vec{E}\times\vec{B})_i\right\}, \qquad (6.11)$$

where, following the notations of Eq. (3.7), and to lowest order in the gradient expansion, the expression u^0u_i can also be written as $u^0u_i = e^{-2\Psi}v_ic^2(y)$. The term $\vec{E}\times\vec{B}$ can be written by using Eq. (6.5):

$$\vec{E}\times\vec{B} = e^{\Psi}\left[\frac{(\vec{\nabla}\times\vec{B})\times\vec{B} + (\vec{\nabla}\Psi\times\vec{B})\times\vec{B}}{4\pi\sigma}\right]$$
$$- e^{-\Psi}(\vec{v}_{\rm b}\times\vec{B})\times\vec{B}. \qquad (6.12)$$

Neglecting the Ohmic electric fields, Eq. (4.15) leads to the evolution equation for h_i^j:

$$\partial_{\tau}^2h_i^j + 2\mathcal{H}\partial_{\tau}h_i^j$$
$$= -e^{2\Psi}\left\{\left[\partial_i\partial^j\Psi - \frac{1}{3}\nabla^2\Psi\delta_i^j\right] + \partial_i\Psi\partial^j\Psi\right.$$
$$\left. - \frac{\delta_i^j}{3}(\nabla\Psi)^2\right\} - a^2\ell_{\rm P}^2\left\{-(p+\rho)\left(u_iu^j - \frac{u^2}{3}\delta_i^j\right)\right.$$
$$\left. + \frac{e^{4\Psi}}{4\pi a^4}\left(B_iB^j - \frac{B^2}{3}\delta_i^j\right)\right\}, \qquad (6.13)$$

where, using Eq. (6.2) in Eq. (2.3) and keeping only the leading contribution in the spatial gradients, $r_{ij}(\vec{x}, \tau)$ becomes, in explicit terms,

$$r_{ij}(\vec{x}, \tau) = [\nabla^2\Psi - (\nabla\Psi)^2]\delta_{ij} + \partial_i\partial_j\Psi + \partial_i\Psi\partial_j\Psi, \qquad (6.14)$$

where $\nabla^2\Psi = \delta^{ij}\partial_i\partial_j\Psi$ is the flat-space Laplacian and, similarly, $(\nabla\Psi)^2 = \delta^{ij}\partial_i\Psi\partial_j\Psi$. Finally, the trace of Eq. (4.15) leads, in the case of Eq. (6.2), to the following condition:

$$\partial_{\tau}^2\Psi + 5\mathcal{H}\partial_{\tau}\Psi - (2\mathcal{H}^2 + \partial_{\tau}\mathcal{H})$$
$$= \frac{2}{3}e^{2\Psi}[2\nabla^2\Psi - (\nabla\Psi)^2]$$
$$+ a^2\ell_{\rm P}^2\left\{\frac{p-\rho}{2} - \frac{p+\rho}{3}u^2 - \frac{e^{4\Psi}}{24\pi a^4}[B^2 + \tilde{B}^2]\right\}. \qquad (6.15)$$

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[12]The vectors appearing hereunder and in the remaining part of the present section are the standard three-dimensional vectors. Similarly, the vector products are standard vector products.

Using the parametrization of Eq. (6.4), Eqs. (6.7) and (6.15) imply the following relations:

$$\partial_\tau^2 \Psi + \mathcal{H} \partial_\tau \Psi = \frac{a^2 \ell_P^2}{6}(3p^{(1)} + \rho^{(1)}) + \frac{a^2 \ell_P^2}{3}\rho_B, \quad (6.16)$$

$$\partial_\tau^2 \Psi + 5\mathcal{H}\partial_\tau \Psi = \frac{2e^{2\Psi}}{3}[2\nabla^2 \Psi - (\nabla\Psi)^2]$$
$$+ a^2 \ell_P^2 \left[\frac{p^{(1)} - \rho^{(1)}}{2} - p_B\right], \quad (6.17)$$

$$3\mathcal{H}^2 = \ell_P^2 a^2 \rho^{(0)},$$
$$2(\mathcal{H}^2 - \partial_\tau \mathcal{H}) = \ell_P^2 a^2 (\rho^{(0)} + p^{(0)}). \quad (6.18)$$

Equations (6.13), (6.16), and (6.17) can be directly solved, and the result, in the plasma frame, reads

$$\Psi(\vec{x}, \tau) = q(\vec{x}) + \frac{2a^{3w+1}e^{2q(\vec{x})}}{3(3w + 5)H_i^2}\left\{\nabla^2 q(\vec{x}) - \frac{[\nabla q(\vec{x})]^2}{2}\right\}$$
$$- \frac{\ell_P^2 a^{3w-1}e^{4q(\vec{x})}}{60\pi(3w + 1)H_i^2}B^2(\vec{x}), \quad (6.19)$$

$$h_i^j(\vec{x}, \tau) = -\frac{2a^{3w+1}e^{2q(\vec{x})}}{(3w+1)(3w+5)H_i^2}\left\{\left[\partial_i \partial^j q(\vec{x}) - \frac{\nabla^2 q(\vec{x})}{3}\delta_i^j\right]\right.$$
$$\left. + \partial_i q(\vec{x})\partial^j q(\vec{x}) - \frac{[\nabla q(\vec{x})]^2}{3}\delta_i^j\right\}$$
$$- \frac{\ell_P^2 a^{3w-1}e^{4q(\vec{x})}}{2\pi(3w-1)(3w+1)}\left[B_i(\vec{x})B^j(\vec{x}) - \frac{B^2(\vec{x})}{3}\delta_i^j\right], \quad (6.20)$$

where $q = q(\vec{x})$ is a generic function of the spatial coordinates encoding the dependence of the large-scale inhomogeneities. Note also that we used $H_i = \mathcal{H}_i/a_i$ at the initial reference time τ_i. Equations (6.19) and (6.20) show that the relative importance of the spatial gradients does depend upon the barotropic index.

The results obtained in the particular case of Eq. (6.2) can also be deduced in a more general parametrization, namely,

$$\gamma_{ik} = a^2(\tau)[\alpha_{ik}(\vec{x}) + \beta_{ik}(\vec{x}, \tau)],$$
$$\gamma^{kj} = \frac{1}{a^2(\tau)}[\alpha^{kj}(\vec{x}) - \beta^{kj}(\vec{x}, \tau)], \quad N(\tau) = a(\tau), \quad (6.21)$$

where $\beta_{ij}(\vec{x}, \tau)$ contains the contribution of the gradients while $\alpha_{ij}(\vec{x})$ is fully inhomogeneous but does not contain any gradient. The conformally flat limit of Eq. (6.21) does correspond to $\alpha_{ij} \to \delta_{ij}$ and $\beta_{ij} \to 0$. The indices of β_{ij} are raised and lowered by using α_{ij}. According to Eq. (2.2), the extrinsic curvature and its contractions become

$$K_i^j = -\frac{1}{a}\left(\mathcal{H}\delta_i^j + \frac{\partial_\tau \beta_i^j}{2}\right),$$
$$K = -\frac{1}{a}\left(3\mathcal{H} + \frac{1}{2}\partial_\tau \beta\right), \quad (6.22)$$
$$\mathrm{Tr}K^2 = \frac{1}{a^2}(3\mathcal{H}^2 + \mathcal{H}\partial_\tau \beta),$$

where, as in the previous case, $\mathcal{H} = \partial_\tau \ln a$. From the momentum constraint it also follows that

$$\nabla_k \partial_\tau \beta_i^k - \nabla_i \partial_\tau \beta = 2a\ell_P^2 \hat{v}_i c(y)s(y)(p + \rho). \quad (6.23)$$

The explicit form of the momentum constraint suggests looking for the solution in a separable form, namely,

$$\beta_i^j(\vec{x}, \tau) = g(\tau)\mu_i^j(\vec{x}) + f(\tau)\nu_i^j(\vec{x}). \quad (6.24)$$

Inserting Eqs. (6.21) and (6.22) into Eq. (4.13) and using the same parametrization of Eq. (6.4) for the inhomogeneous contributions of the pressure and of the energy density, the following pair of conditions can be obtained:

$$\partial_\tau\left(\frac{\partial_\tau \beta}{2a}\right) + \frac{\mathcal{H}}{a}\partial_\tau \beta = -a\ell_P^2\left(\frac{3p^{(1)} + \rho^{(1)}}{2} + \rho_B\right), \quad (6.25)$$

$$\partial_\tau \mathcal{H} = -\frac{a^2 \ell_P^2}{6}(\rho^{(0)} + 3p^{(0)}). \quad (6.26)$$

Supposing, for the sake of simplicity, that the barotropic index is constant, we have that

$$\rho^{(1)} = -\frac{1}{(3w + 1)}\frac{[\partial_\tau^2 \beta + \mathcal{H}\partial_\tau \beta]}{\ell_P^2 a^2} - \frac{2\rho_B}{(3w + 1)}, \quad (6.27)$$

where the contribution of the electric fields has been consistently neglected. Inserting now Eqs. (6.21) and (6.22) into Eq. (4.15) the following equations can be readily obtained:

$$\partial_\tau\left(\frac{\partial_\tau \beta_i^j}{2a}\right) + \mathcal{H}\frac{\partial_\tau \beta}{2a}\delta_i^j + \frac{3\mathcal{H}}{2a}\partial_\tau \beta_i^j + ar_i^j$$
$$= -a\ell_P^2\left[\frac{p^{(1)} - \rho^{(1)}}{2}\delta_i^j + \Pi_i^j(B) - p_B\delta_i^j\right], \quad (6.28)$$

$$\partial_\tau \mathcal{H} + 2\mathcal{H}^2 = -\frac{\ell_P^2 a^2}{2}(p^{(0)} - \rho^{(0)}). \quad (6.29)$$

Note that, from Eqs. (6.26) and (6.29), the standard Hubble parameter is given by $H = H_i a^{-3(w+1)/2}$, where H_i is an integration constant with the same meaning of the analog constant introduced in Sec. V. Using, then, Eq. (6.27) to eliminate $\rho^{(1)}$ from Eq. (6.28), we have that

$$\partial_\tau^2 \beta_i^j + 2\mathcal{H}\partial_\tau \beta_i^j + \delta_i^j\left(\frac{1 - w}{1 + 3w}\partial_\tau^2 \beta + 2\frac{1 + w}{1 + 3w}\mathcal{H}\partial_\tau \beta\right)$$
$$= \frac{4}{3}a^2 \ell_P^2\left(\frac{3w - 1}{3w + 1}\right)\rho_B - 2a^2 \ell_P^2 - 2a^2 r_i^j \Pi_i^j(B). \quad (6.30)$$

Equation (6.30) can be solved by positing

$$\beta_i^j(\vec{x}, \tau) = g(\tau)\mu_i^j(\vec{x}) + f(\tau)\nu_i^j(\vec{x}). \qquad (6.31)$$

Then, inserting Eq. (6.31) into Eq. (6.30) and assuming $w \neq 1/3$, we obtain

$$g(\tau) = a^{3w+1}, \qquad f(\tau) = a^{3w-1}, \qquad (6.32)$$

$$\mu_i^j(\vec{x}) = -\frac{4}{H_i^2(3w+5)(3w+1)} \\ \times \left[P_i^j(\vec{x}) + \frac{3w^2 - 6w - 5}{4(9w+5)} P(\vec{x})\delta_i^j \right], \quad (6.33)$$

$$\nu_i^j(\vec{x}) = -\frac{\ell_P^2}{\pi H_i^2(3w-1)(3w+1)} \\ \times \left[B_i(\vec{x})B^j(\vec{x}) + \frac{9w^2 - 9w - 8}{9(6w+5-3w^2)} B^2(\vec{x})\delta_i^j \right], \\ (6.34)$$

where $P_i^j(\vec{x}) = r_i^j(\vec{x}, \tau)a^2(\tau)$ accounts for the intrinsic curvature computed from $\alpha_{ij}(\vec{x})$ and, by definition,

$$B_i(\vec{x})B^j(\vec{x}) = \frac{\alpha_{mi}(\vec{x})}{\alpha(\vec{x})} B^m(x)B^j(\vec{x}), \\ B^2(\vec{x}) = \frac{\alpha_{mn}(\vec{x})}{\alpha(\vec{x})} B^m(\vec{x})B^n(\vec{x}), \quad (6.35)$$

with $\alpha(\vec{x}) = \det(\alpha_{ij})$. Consequently, $\gamma_{ij}(\vec{x}, \tau)$ can be written as

$$\gamma_{ij}(\vec{x}, \tau) = a^2(\tau)[\alpha_{ij}(\vec{x}) + \mu_{ij}(\vec{x})a^{3w+1} + \nu_{ij}(\vec{x})a^{3w-1}]. \\ (6.36)$$

If $w = 1/3$

$$\gamma_{ij}(\vec{x}, \tau) = \tau^2[\alpha_{ij}(\vec{x}) + \mu_{ij}(\vec{x})\tau^2 + \nu_{ij}(\vec{x})\ln\tau], \quad (6.37)$$

where, in the latter case,

$$\mu_i^j(\vec{x}) = -\frac{1}{3H_i^2}\left[P_i^j(\vec{x}) - \frac{5}{24}P(\vec{x})\delta_i^j \right], \\ \nu_i^j(\vec{x}) = -\frac{\ell_P^2}{2\pi H_i^2}\left[B_i(\vec{x})B^j(\vec{x}) - \frac{B^2(\vec{x})}{3}\delta_i^j \right]. \quad (6.38)$$

If $w < -1/3$ the contributions of the spatial curvature and of the magnetic field become progressively subleading even if the terms proportional to ν_{ij} are diluted faster. If $-1/3 < w < 1/3$ the contribution of the spatial curvature is progressively increasing, while the contribution associated with the magnetic fields decreases. Finally, if $w > 1/3$ both terms increase but at a different rate. The solutions illustrated in the present section generalize the perturbative treatment, where the large-scale magnetic fields are taken as a supplementary component in the linearized equations of the relativistic metric perturbations. The linearized evolution of the curvature perturbations is usually described

not in terms of γ_{ij} but rather in terms of an appropriate gauge-invariant combination. It is therefore necessary to consider the interplay between the two different descriptions in the case where fully inhomogeneous magnetic fields contribute to the curvature perturbations which also receive an independent contribution from the standard adiabatic mode.

VII. NONLINEAR ADIABATIC MODE IN THE PLASMA FRAME

The initial conditions or the calculation of the CMB observables in the presence of large-scale magnetic fields are set within linear perturbation theory expressed either in a specific gauge or within a suitable gauge-invariant treatment [7,8]. In contrast with earlier studies, it has been recently clarified that different kinds of initial conditions are contemplated, ranging from the magnetized adiabatic mode to the various magnetized entropy modes. Nonlinear generalizations of the magnetized curvature perturbations will now be discussed.

In the standard perturbative treatment of large-scale inhomogeneities, the curvature perturbations on comoving orthogonal hypersurfaces (conventionally denoted by $\mathcal{R}$) and the curvature perturbations on uniform density hypersurfaces (conventionally denoted by ζ) are often used to parametrize the large-scale curvature inhomogeneities. Both variables are gauge invariant and can therefore be expressed in any gauge, such as the longitudinal gauge or the synchronous gauge [39,40]. In the longitudinal gauge (see Appendix B for an explicit definition in terms of the ADM variables) and in the plasma frame, $\mathcal{R}$ and ζ can be written as[13]

$$\bar{\mathcal{R}}(\vec{x}, \tau) = -\psi - \frac{\mathcal{H}(\mathcal{H}\phi + \partial_\tau\psi)}{\mathcal{H}^2 - \partial_\tau\mathcal{H}}, \qquad (7.1)$$

$$\bar{\zeta}(\vec{x}, \tau) = -\psi + \frac{\delta_s^{(L)}\rho + \delta_s\rho_B}{3(\rho_t + p_t)}, \qquad (7.2)$$

where $\delta_s^{(L)}\rho$ denotes the scalar fluctuation of the energy density of the fluid sources in the longitudinal gauge; $\delta_s\rho_B = B^2(\vec{x})/[8\pi a^4(\tau)]$ denotes the fluctuation of the energy density associated with the fully inhomogeneous magnetic field, while ρ_t and p_t are, respectively, the total energy density and pressure determining the (homogeneous and isotropic) background geometry.

By computing the difference $(\bar{\zeta} - \bar{\mathcal{R}})$ and by comparing the obtained result with the Hamiltonian constraint [see Eq. (B5) of Appendix B], the difference $(\bar{\zeta} - \bar{\mathcal{R}})$ becomes

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[13]The notation $\bar{\mathcal{R}}$ and $\bar{\zeta}$ (as opposed to $\mathcal{R}$ and ζ) is meant to distinguish the quantities defined in perturbation theory from their counterparts defined to a given order in the gradient expansion.

$$\bar{\zeta} - \bar{\mathcal{R}} = \frac{2\nabla^2 \psi}{3\ell_P^2 a^2 (p_t + \rho_t)}, \qquad (7.3)$$

which is negligible in the limit of typical length scales much larger than the Hubble radius at the corresponding epoch. The variables $\bar{\zeta}$ and $\bar{\mathcal{R}}$ are gauge invariant: their numerical value does not change for two coordinate systems connected by infinitesimal coordinate transformations of the type

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \epsilon^\mu, \qquad \epsilon_\mu = a^2(\tau)(\epsilon_0, -\partial_i \epsilon), \quad (7.4)$$

where both ϵ_0 and ϵ depend on the space-time point.[14] A coordinate transformation of the type (7.4) will change metric fluctuations according to the Lie derivative in the direction of the vector ϵ^μ; consider, for the sake of concreteness, the transformation from the synchronous coordinate system to the longitudinal one:

$$\begin{aligned} \psi^{(S)} &\rightarrow \psi^{(L)} = \psi^{(S)} + \mathcal{H}\epsilon_0, \\ F^{(S)} &\rightarrow F^{(L)} = F^{(S)} - \epsilon, \end{aligned} \qquad (7.5)$$

$$\begin{aligned} B^{(S)} &\rightarrow B^{(L)} = B^{(S)} + \epsilon_0 - \partial_\tau \epsilon, \\ \phi^{(S)} &\rightarrow \phi^{(L)} = \phi^{(S)} - \partial_\tau \epsilon_0 - \mathcal{H}\epsilon_0. \end{aligned} \qquad (7.6)$$

In the synchronous gauge $\phi^{(S)} = 0$ and $B^{(S)} = 0$, while, by definition of the longitudinal gauge, $F^{(L)} = 0$ and $B^{(L)} = 0$ (see, e.g., [40]). Consequently, the standard relation between the variables appearing in $\bar{\zeta}$ and $\bar{\mathcal{R}}$ can be written as

$$\begin{aligned} \psi &= -\xi + \mathcal{H}\partial_\tau F, \\ \phi &= -\partial_\tau^2 F - \mathcal{H}\partial_\tau F, \\ \delta_s^{(L)} &= \delta_s^{(S)} - \partial_\tau \rho_t \partial_\tau F. \end{aligned} \qquad (7.7)$$

By inserting Eq. (7.7) into Eqs. (7.1) and (7.2), the standard synchronous expression of $\bar{\mathcal{R}}$ and $\bar{\zeta}$ can be readily obtained as

$$\bar{\mathcal{R}} = \xi + \frac{\mathcal{H}\partial_\tau \xi}{\mathcal{H}^2 - \partial_\tau \mathcal{H}}, \qquad \bar{\zeta} = \xi + \frac{\delta_s^{(S)}\rho + \delta_s \rho_B}{3(\rho_t + p_t)}, \qquad (7.8)$$

and it coincides, as expected, with previous discussions (see, e.g., [40]). Because of gauge invariance, the evolution equations for $\bar{\zeta}$ and $\bar{\mathcal{R}}$ can be derived and discussed in any gauge, for instance, by linearizing the covariant conservation of the energy-momentum tensor of the sources. Let us consider, for the sake of concreteness, the matter-radiation transition, and let us define the fluctuation of the total pressure in terms of the sound speed and of the nonadiabatic pressure fluctuation δp_{nad}:

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[14]As explained in Appendix B, the first part of the present section assumes the standard perturbative treatment and, therefore, the underlying background geometry is taken to be conformally flat as in the case of the vanilla ΛCDM scenario.

$$\delta_s p = \bar{c}_{st}^2 \delta_s \rho + \delta p_{\text{nad}}, \qquad \bar{c}_{st}^2 = \frac{\partial_\tau p_t}{\partial_\tau \rho_t} = \frac{4}{3(3\alpha + 4)}, \qquad (7.9)$$

where $\alpha = a(\tau)/a_{\text{eq}}$ and the sound speed, as shown in the case of the matter-radiation transition, is fully homogeneous. The result for the evolution of $\bar{\zeta}$, when the plasma effects are carefully included, has been given in the second paper of Ref. [47],

$$\begin{aligned} \partial_\tau \bar{\zeta} + \frac{\mathcal{H}\delta p_{\text{nad}}}{(p_t + \rho_t)} &= \frac{\vec{E} \cdot \vec{J}}{3a^4(p_t + \rho_t)} + \frac{\mathcal{H}\delta_s \rho_B (3\bar{c}_{st}^2 - 1)}{3(\rho_t + p_t)} \\ &\quad + \frac{\mathcal{H}\delta_s \rho_E}{3(\rho_t + p_t)} \mathcal{F}(\alpha, \bar{c}_{st}) - \frac{\theta_t}{3}, \end{aligned} \qquad (7.10)$$

where the following quantities have been introduced:

$$\mathcal{F}(\alpha, \bar{c}_{st}) = \frac{[3\bar{c}_{st}^2 g_1^2 \alpha^2 + g_2^2(3\bar{c}_{st}^2 - 2)]}{g_1^2 \alpha^2 + g_2^2}, \qquad \theta_t = \vec{\nabla} \cdot \vec{v}_t; \qquad (7.11)$$

g_1 and g_2 are two numerical constants which depend upon the ΛCDM parameters which can be explicitly computed (see the second paper of [47]). For the fiducial set of ΛCDM parameters, $g_1 \ll g_2 \simeq 10^{-5}$. Neglecting the electric fields, Eq. (7.11) can be easily solved by direct integration since, for length scales much larger than the Hubble radius, Eq. (7.11) reads

$$\frac{\partial \bar{\zeta}}{\partial \ln \alpha} = -\frac{\delta_s p_{\text{nad}}}{(p_t + \rho_t)} + \left(\bar{c}_{st}^2 - \frac{1}{3} \right) \frac{\delta_s \rho_B}{(p_t + \rho_t)}, \qquad (7.12)$$

with the result that, in the case of the magnetized adiabatic mode (i.e. $\delta p_{\text{nad}} = 0$), the solution of Eq. (7.12) becomes

$$\bar{\zeta}(\vec{x}, \tau) = \bar{\zeta}_*(\vec{x}) - \frac{3}{4} R_\gamma \Omega_B(\vec{x}) \frac{\alpha}{3\alpha + 4}. \qquad (7.13)$$

In the nonlinear case it does not make sense to separate the energy density and the pressure in a background value supplemented by the corresponding fluctuations. Still, it will be possible to define the sound speed in terms of the barotropic index, i.e.

$$c_{st}^2(\vec{x}, \tau) = \frac{\partial_\tau p}{\partial_\tau \rho} = w + \frac{\partial_\tau w}{NK(p + \rho)}. \qquad (7.14)$$

Equation (7.14) coincides with the second relation of Eq. (7.9) in the fully homogeneous case. However, when the description of the geometry is given in fully inhomogeneous terms, the two definitions lead to different results.

Let us now see how the curvature perturbations can be generalized to the nonlinear level. A possible nonlinear generalization of the variable ζ is given by

$$\zeta_i = \frac{\nabla_i \ln \sqrt{\gamma}}{3} + \frac{NK}{3\partial_\tau \rho} \nabla_i (\rho + \rho_B), \qquad (7.15)$$

while a complementary generalization is

$$\zeta_i = \frac{\nabla_i \ln\sqrt{\gamma}}{3} + \frac{\nabla_i(\rho + \rho_B)}{3(p + \rho)}. \qquad (7.16)$$

Note that Eqs. (7.15) and (7.16) are equivalent to zeroth order in the gradient expansion since $\partial_\tau \rho = NK(p + \rho)$. However, for practical reasons, the definition (7.16) will be preferred. With similar logic, the nonlinear generalization of $\mathcal{R}$ can be written as

$$\mathcal{R}_i = \frac{\nabla_i \ln\sqrt{\gamma}}{3} + \frac{\nabla_i(K^2 - \mathrm{Tr}K^2)}{6\ell_P^2(\rho + p)}. \qquad (7.17)$$

The variable given in Eq. (7.15) is inspired by the one defined in Ref. [48] in the case of an energy-momentum tensor dominated by a single scalar field. The pair of variables defined in Eqs. (7.16) and (7.17) can be compared to their linearized counterpart by expressing the extrinsic curvature in a specific gauge. For instance, in the longitudinal gauge, using the results of Appendix B [in particular, Eq. (B2)] Eqs. (7.16) and (7.17) become

$$\zeta_i \to \nabla_i \left[-\psi + \frac{\delta_s^{(L)}\rho + \delta_s \rho_B}{3(\rho_t + p_t)} \right],$$
$$\mathcal{R}_i \to \nabla_i \left[-\psi - \frac{\mathcal{H}(\mathcal{H}\phi + \partial_\tau \psi)}{\mathcal{H}^2 - \partial_\tau \mathcal{H}} \right], \qquad (7.18)$$

which also implies, because of Eqs. (7.1) and (7.2), $\zeta_i^{(L)} \to \nabla_i \bar{\zeta}$ and $\mathcal{R}_i^{(L)} \to \nabla_i \bar{\mathcal{R}}$. By taking the difference of ζ_i and $\mathcal{R}_i$ and by using the Hamiltonian constraint of Eq. (2.9), we obtain

$$\zeta_i - \mathcal{R}_i = \frac{\nabla_i r}{6\ell_P^2(p + \rho)} - \frac{\nabla_i(p + \rho)u^2}{3(p + \rho)}. \qquad (7.19)$$

By expanding the right-hand side of Eq. (7.19) to first order in the (longitudinal) metric fluctuations, it can be verified that the obtained result coincides with the spatial gradient of Eq. (7.3). As far as gauge invariance is concerned, the approach followed here is similar to the one invoked in related contexts (see, e.g., [48,49]): Eqs. (7.16) and (7.17) can be shown to be gauge invariant to a given order in the gradient expansion. Suppose, indeed, that we evaluate Eqs. (7.16) and (7.17) not in the longitudinal gauge but in a different gauge, for instance, the synchronous gauge. If the variables are truly nonlinear gauge-invariant variables, they must also be gauge invariant for infinitesimal gauge transformations of the kind discussed in Eqs. (7.5) and (7.6). With the help of Eq. (B9), the synchronous gauge expression of $\mathcal{R}_i$ can be written as

$$\mathcal{R}_i^{(S)} = \nabla_i \left[\xi + \frac{\mathcal{H}\partial_\tau \xi}{\mathcal{H}^2 - \partial_\tau \mathcal{H}} \right]$$
$$+ \frac{1}{3}\nabla_i \left[\nabla^2 F + \frac{\mathcal{H}}{\mathcal{H}^2 - \partial_\tau \mathcal{H}} \nabla^2 \partial_\tau F \right], \quad (7.20)$$

where the notation $\mathcal{R}_i^{(S)}$ reminds us that $\mathcal{R}_i$ is computed in the synchronous coordinate system. The first term in

square brackets of Eq. (7.20) clearly coincides with the spatial gradient of $\bar{\mathcal{R}}$ [see Eq. (7.1)]. The second term in square brackets in Eq. (7.20) therefore breaks gauge invariance, but it is of higher order in the gradients. The same kind of considerations can also be extended to the case of ζ_i. In summary, recalling Eq. (7.8) (i.e. the synchronous form of $\bar{\mathcal{R}}$ and $\bar{\zeta}$), we have that, in the synchronous parametrization,

$$\zeta_i^{(S)} = \nabla_i \bar{\zeta} + \frac{\nabla_i \nabla^2 F}{3} \equiv \zeta_i^{(L)} + \mathcal{O}(\nabla^3), \qquad (7.21)$$

$$\mathcal{R}_i^{(S)} = \nabla_i \bar{\mathcal{R}} + \frac{1}{3}\nabla_i \left[\nabla^2 F + \frac{\mathcal{H}}{\mathcal{H}^2 - \partial_\tau \mathcal{H}} \nabla^2 \partial_\tau F \right]$$
$$\equiv \mathcal{R}_i^{(L)} + \mathcal{O}(\nabla^3). \qquad (7.22)$$

Equations (7.21) and (7.22) are rather interesting and suggest that the variables ζ_i and $\mathcal{R}_i$ defined in Eqs. (7.16) and (7.17) can be computed in any suitable gauge. If the two gauges differ by an infinitesimal coordinate transformation, the terms leading to a breaking of gauge invariance are of higher order in the gradient expansion. In the specific example of Eqs. (7.21) and (7.22) the extra terms contain three gradients.

The observation of the previous paragraph fails, however, in the case of more general coordinate transformations. Still, the results suggested by Eqs. (7.21) and (7.22) can be generalized, along the lines of [48,49], to the case when the coordinate transformations are finite but still sufficiently well behaved. Consider, in particular, the following coordinate transformation:

$$X^i = X^i(\vec{x}, \tau), \qquad T = T(\vec{x}, \tau). \qquad (7.23)$$

As discussed in [48,49], along lines of constant X^i, the old coordinates change as $dx^\mu = \partial^\mu T ds$, where s is an arbitrary parameter and

$$X^i = f^i(\vec{x}) + \int \frac{\partial^i T}{\partial_\tau T \partial^\tau T} dT. \qquad (7.24)$$

If the new time coordinate T is nonsingular, the second term in the transformation of Eq. (7.24) can be discarded to leading order in the gradient expansion.

From Eqs. (7.16) and (7.17) the evolution equation of the nonlinear curvature perturbations can be obtained in explicit terms by using the governing equations and the covariant to the conservation equation to leading order in the spatial gradients. Let us therefore take the first time derivative of ζ_i, and let us drop the contribution of the electric fields which are subleading to zeroth order in the drift approximation; the result of this simple manipulation is

$$\partial_\tau \zeta_i = \frac{1}{3}\nabla_i\left(\frac{\partial_\tau\sqrt{\gamma}}{\sqrt{\gamma}}\right) + \frac{1}{3(p+\rho)}\nabla_i(\partial_\tau\rho + \partial_\tau\rho_{\rm B})$$

$$- \frac{\nabla_i(\rho + \rho_{\rm B})}{3(p+\rho)^2}[\partial_\tau p + \partial_\tau\rho]. \qquad (7.25)$$

But to first order in the gradient expansion, $\partial_\tau\rho = NK(p+\rho)$ and $\partial_\tau\rho_{\rm B} = 4NK\rho_{\rm B}/3$. Thus Eq. (7.25) becomes

$$\partial_\tau \zeta_i = \frac{KN}{3(p+\rho)}(\nabla_i p - c_{\rm st}^2\nabla_i\rho) + \frac{NK}{3}\left(\frac{1}{3} - c_{\rm st}^2\right)\frac{\nabla_i\rho_{\rm B}}{(\rho+p)}$$

$$+ \frac{4}{9}\frac{\rho_B}{p+\rho}\nabla_i(NK). \qquad (7.26)$$

Equation (7.26) generalizes Eq. (7.10), as it can be easily appreciated by considering various specific limits. Suppose, for instance, that the barotropic index w is constant both in time and in space and that $\rho_{\rm B} = 0$. Then, $c_{\rm st}^2 = w$ and Eq. (7.26) implies $\partial_\tau\zeta_i = 0$. This is the case of the single adiabatic mode in the absence of magnetic fields.

In general terms the solution of Eq. (7.26) represents a complicated self-consistent problem since both $c_{\rm st}^2$ and w will be both space and time dependent; moreover, always from a general point of view, $\nabla_i(NK)$ and $\nabla_i\rho_{\rm B}$ can be of the same order. To simplify the situation let us make the following (not completely realistic) assumption:

$$\frac{\nabla_i(NK)}{NK} \ll \frac{\nabla_i\rho_{\rm B}}{\rho_{\rm B}}, \qquad w(\alpha) = \frac{1}{3(\alpha+1)}, \qquad (7.27)$$

where, as before, $\alpha = a/a_{\rm eq}$. Equation (7.26) can then be solved directly, and the result is

$$\zeta_i(\vec{x}, \alpha) = \bar{\zeta}_i(\vec{x}) - \frac{3\alpha}{4(3\alpha+4)}\frac{\nabla_i\rho + \nabla_i\rho_{\rm B}}{\rho}, \qquad (7.28)$$

where the integration constant has been matched with the value of ζ_i determined in the linearized approximation. The solution (7.28) is interesting but it assumes an expression for the evolution of the barotropic index which is only justified in the homogeneous and isotropic cases.

A much safer approach for the computation of the curvature perturbations is to first obtain the quasi-isotropic MHD solutions at the wanted order in the gradients (as already done in Sec. VI) and then to evaluate explicitly $\mathcal{R}_i$ and ζ_i. Bearing in mind that, in the notations of Sec. VI, $\sqrt{\gamma} = a^3(\tau)\sqrt{\alpha(\vec{x})}[1 + \beta(\vec{x}, \tau)/2]$, the explicit expression of $\mathcal{R}_i(\vec{x}, \tau)$ can be computed, and it is

$$\mathcal{R}_i(\vec{x}, \tau) = \frac{1}{6}\frac{\partial_i\alpha}{\alpha} + \frac{1}{6}[g(\tau)\partial_i\mu + f(\tau)\partial_i\nu]$$

$$+ \frac{\mathcal{H}}{6(\mathcal{H}^2 - \partial_\tau\mathcal{H})}[\partial_\tau g\partial_i\mu + \partial_\tau f\partial_i\nu]. \quad (7.29)$$

Recalling the explicit expressions of $\mu(\vec{x})$ and $\nu(\vec{x})$, we have that

$$\mathcal{R}_i(\vec{x}, \tau) = \frac{1}{6}\frac{\partial_i\alpha}{\alpha} - \frac{\partial_i P}{18H_{\rm i}^2(w+1)}a^{3w+1}$$

$$- \frac{4(9w+7)(9w+1)}{3(3w-1)(3w+1)(6w+5-3w^2)(w+1)}$$

$$\times \partial_i\Omega_{\rm B}a^{3w-1}, \qquad (7.30)$$

where now $\Omega_{\rm B}(\vec{x}) = \ell_{\rm P}^2 B^2(\vec{x})/(24\pi H_{\rm i}^2)$. By using the explicit expression of the Hamiltonian constraint, the variable $\zeta_i(\vec{x}, \tau)$ is only sensitive, by construction, to the gradients of the magnetic fields, i.e.

$$\zeta_i(\vec{x}, \tau) = \mathcal{R}_i(\vec{x}, \tau) + \frac{\partial_i P}{6\ell_{\rm P}^2 a^2(\rho+p)}$$

$$= \frac{1}{6}\frac{\partial_i\alpha}{\alpha} - \frac{4(9w+7)(9w+1)}{3(3w-1)(3w+1)(6w+5-3w^2)(w+1)}$$

$$\times \partial_i\Omega_{\rm B}a^{3w-1}. \qquad (7.31)$$

Both in the case of Eq. (7.30) and in the case of Eq. (7.31), the leading order result is fully inhomogeneous, and it accounts for the large-scale curvature perturbations connected, in this specific case, to the adiabatic solution. The spatial gradients affecting $\mathcal{R}_i$ are induced both by the spatial curvature and by the magnetic inhomogeneities. Conversely, the spatial gradients of the curvature affect ζ_i only to higher orders because of a cancellation arising when the solution for $\mathcal{R}_i$ is inserted into the fully inhomogeneous form of the Hamiltonian constraint of Eq. (7.19). Note that in the case $w = 1/3$, Eq. (6.38) implies that ν_i^j is traceless. This implies that the contribution of the magnetic fields to the curvature perturbation vanishes to first order in the gradient expansion in a radiation-dominated universe, while the effect of ordinary spatial inhomogeneities does contribute. Consequently, it has been shown that, while it is difficult to solve explicitly the nonlinear generalization of the evolution equation for the magnetized curvature perturbations, the same techniques leading to the quasi-isotropic solutions can be used to derive the expression of the nonlinear generalization of the magnetized curvature perturbations.

VIII. CONCLUDING REMARKS

The general relativistic gradient expansion has been combined with the standard tenets of the drift approximation used in the description of cold plasmas. Nonlinear effects are typical of both general relativistic dynamics and of magnetohydrodynamics in flat space-time. It is then natural to treat them in a unified perspective, where the standard linearized approximation is not assumed from the very beginning. A fully nonlinear system has been derived in terms of the ADM variables and discussed in specific physical limits. The evolution of magnetic and curvature inhomogeneities has been treated and solved on the same footing. The present findings have also been contrasted with the standard linearized description of magnetized

curvature perturbations, both in the presence of nonadiabatic pressure fluctuations and in the case of adiabatic initial conditions. After introducing the nonlinear magnetized adiabatic mode, an explicit evolution equation for the magnetized curvature perturbations has been derived to leading order in the spatial gradients. While the results of the present investigation pave the way for more thorough scrutiny of nonlinear effects in gravitating plasmas prior to photon decoupling, they also fill an existing gap in the present literature. Indeed, the treatment of magnetized plasmas usually rests upon equations written in homogeneous and isotropic backgrounds supplemented by the corresponding relativistic fluctuations of the geometry and of the fluid sources. Here our modest attempt has been to suggest and partially develop a description which is independent of the background but valid to a given order in the spatial gradients.

APPENDIX A: ELECTROMAGNETIC FIELDS AND ADM DECOMPOSITION

The explicit forms of the Christoffel symbols in terms of the ADM variables of Eq. (2.1) are

$$\Gamma_{00}^0 = \frac{\partial_\tau N}{N} + \frac{N^{k(3)}\nabla_k N}{N} - \frac{N^m N^n}{N} K_{mn},$$

$$\Gamma_{0i}^0 = \frac{\nabla_i N}{N} - \frac{N^m}{N} K_{mi},$$

$$\Gamma_{00}^i = N^{(3)}\nabla^i N + \partial_\tau N^i - 2NN^m K_m^i + N^{m(3)}\nabla_m N^i$$
$$\quad - \frac{N^i}{N}(\partial_\tau N + N^{m(3)}\nabla_m N - N^m N^n K_{mn}), \quad (A1)$$

$$\Gamma_{ij}^0 = -\frac{1}{N} K_{ij},$$

$$\Gamma_{i0}^j = -N^j \frac{{}^{(3)}\nabla_i N}{N} + \frac{N^j N^m}{N} K_{mi} - NK_i^j + {}^{(3)}\nabla_i N^j,$$

$$\Gamma_{mn}^i = {}^{(3)}\Gamma_{mn}^i + \frac{N^i}{N} K_{mn}.$$

In Eq. (A1) the indices are lowered and raised by using γ_{ij} so that, for instance, $K_i^j = \gamma^{kj}K_{ki}$; ${}^{(3)}\nabla_i$ is the covariant derivative defined with respect to the spatial metric γ_{ij}, and the corresponding Christoffel symbol is

$$^{(3)}\Gamma_{mn}^i = \tfrac{1}{2}\gamma^{ij}(-\partial_j\gamma_{mn} + \partial_n\gamma_{jm} + \partial_m\gamma_{nj}). \quad (A2)$$

Recalling Eqs. (2.2) and (2.3) the traces K, $\text{Tr}K^2$, and r are defined as

$$K = K_i^i, \qquad \text{Tr}K^2 = K_i^j K_j^i, \qquad r = \gamma^{ij}r_{ij}. \quad (A3)$$

From Eq. (A1) the components of the Ricci tensor and the Ricci scalar read

$$R_0^0 = \frac{\partial_\tau K}{N} - \text{Tr}K^2 + \frac{\nabla^2 N}{N} - \frac{N^m}{N}\nabla_m K + \frac{N^q}{N}\mathcal{L}_q, \quad (A4)$$

$$R_i^0 = \frac{1}{N}\mathcal{L}_i, \quad (A5)$$

$$R_i^j = \frac{1}{N}\partial_\tau K_i^j - KK_i^j - r_i^j + \frac{1}{N}\nabla_i\nabla^j N - \frac{N^m}{N}\nabla_m K_i^j$$
$$\quad + \frac{1}{N}\nabla_m N^j K_i^m - \frac{1}{N}\nabla_i N^m K_m^j - \frac{N^j}{N}\mathcal{L}_i, \quad (A6)$$

where, for the sake of conciseness, the following notations have been adopted:

$$^{(3)}\nabla_i = \nabla_i, \qquad \mathcal{L}_i = (\nabla_i K - \nabla_k K_i^k). \quad (A7)$$

From Eqs. (A4)–(A7), the components of the Einstein tensors can be easily obtained, and they are

$$G_0^0 = \frac{1}{2}(K^2 + r - \text{Tr}K^2) + \frac{N^q}{N}\mathcal{L}_q, \quad (A8)$$

$$G_i^0 = \frac{1}{N}(\nabla_i K - \nabla_k K_i^k), \quad (A9)$$

$$G_i^j = \frac{1}{N}\partial_\tau(K_i^j - K\delta_i^j) - K\left(K_i^j - \frac{K}{2}\delta_i^j\right) + \frac{\text{Tr}K^2}{2}\delta_i^j$$
$$\quad - \left(r_i^j - \frac{r}{2}\delta_i^j\right) + \frac{1}{N}(\nabla_i\nabla^j N - \nabla^2 N\delta_i^j)$$
$$\quad - \frac{N^m}{N}\nabla_m(K_i^j - K\delta_i^j) + \frac{1}{N}\nabla_m N^j K_i^m$$
$$\quad - \frac{1}{N}\nabla_i N^m K_m^j - \frac{N^j}{N}\mathcal{L}_i. \quad (A10)$$

For a perfect relativistic fluid the components of the energy-momentum tensor with covariant indices are

$$T_{00} = (p + \rho)u_0 u_0 - p(N^2 - N_k N^k),$$
$$T_{i0} = (p + \rho)u_0 u_i + pN_i, \quad (A11)$$
$$T_{ij} = (p + \rho)u_i u_j + p\gamma_{ij}.$$

The condition $g^{\mu\nu}u_\mu u_\nu = 1$ implies, in terms of the ADM variables,

$$[u_0 - N^k u_k]^2 = N^2(1 + u^2), \qquad u^2 = \gamma^{ij}u_i u_j. \quad (A12)$$

Equations (A11) and (A12) are written in general terms, and therefore, we shall also have that, in general,

$$T_0^0 = \rho + (p + \rho)u\sqrt{1 + u^2}\left[\frac{N^k u_k}{Nu} + \frac{u}{\sqrt{1 + u^2}}\right], \quad (A13)$$

$$T_i^0 = \frac{(p + \rho)}{N}\sqrt{1 + u^2}u_i, \quad (A14)$$

$$T_i^j = -p\delta_i^j - (p + \rho)u_i u^j - \frac{p + \rho}{N}\sqrt{1 + u^2}u_i N^j. \quad (A15)$$

From Eqs. (A9) and (A14) the explicit forms of the Hamiltonian and of the momentum constraints (i.e. $G_0^0 = \ell_P^2 T_0^0$ and $G_i^0 = \ell_P^2 T_i^0$) become, respectively,

$$K^2 + r - \text{Tr}K^2 = 2\ell_{\text{P}}^2[\rho + (\rho + p)u^2],$$

$$\mathcal{L}_i = \ell_{\text{P}}^2(p + \rho)\sqrt{1 + u^2}u_i. \tag{A16}$$

Note that the term proportional to $\mathcal{L}_q$ in G_0^0 vanishes exactly with the terms containing $N^k u_k$ in Eq. (A13) once the momentum constraint is imposed. In the case $N_i = 0$ and within the parametrization of Eqs. (3.6) and (3.7), the relevant component of the fluid energy-momentum tensor can be written as

$$T^{00} = \frac{1}{N^2}[\rho\cosh^2 y + p\sinh^2 y],$$

$$T^{0i} = \frac{p + \rho}{N}\sinh y \cosh y \hat{v}^i, \tag{A17}$$

$$T^{ij} = (p + \rho)\hat{v}^i\hat{v}^j\sinh^2 y + p\gamma^{ij}.$$

The covariant conservation of the energy-momentum tensor can be written in terms of the Christoffel symbols obtained in Eq. (A1),

$$\nabla_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_k T^{k0} + (2\Gamma_{00}^0 + \Gamma_{0k}^k)T^{00}$$
$$+ (3\Gamma_{0k}^0 + \Gamma_{kj}^j)T^{0k} + \Gamma_{kj}^0 T^{kj},$$

$$\nabla_\mu T^{\mu i} = \partial_0 T^{0i} + \partial_k T^{ki} + \Gamma_{00}^i T^{00} + 2\Gamma_{0j}^i T^{0j}$$
$$+ (\Gamma_{00}^0 + \Gamma_{0k}^k)T^{i0} + \Gamma_{jk}^i T^{jk} + (\Gamma_{j0}^0 + \Gamma_{jk}^k)T^{ij}, \tag{A19}$$

which can also be explicitly written, in ADM variables, as

$$\nabla_\mu T^{\mu 0} = \frac{1}{N^2}\left\{\partial_\tau[ps^2(y) + \rho c^2(y)]\right.$$
$$+ N^2\partial_k\left[\frac{p + \rho}{N}c(y)s(y)\hat{v}^k\right]$$
$$- NK[ps^2(y) + \rho c^2(y)] + [3\nabla_k N + N\Gamma_{kj}^j]$$
$$\times (p + \rho)s(y)c(y)\hat{v}^k - NK_{kj}[(p + \rho)\hat{v}^k\hat{v}^j s^2(y)$$
$$\left. + p\gamma^{kj} + \Pi^{kj}]\right\}, \tag{A20}$$

$$\nabla_\mu T^{\mu i} = \frac{1}{N}\partial_\tau[(p + \rho)s(y)c(y)\hat{v}^i]$$
$$+ \partial_k[(p + \rho)s^2(y)\hat{v}^i\hat{v}^k + p\gamma^{ik} + \Pi^{ki}]$$
$$+ \frac{\nabla^i N}{N}[\rho c^2(y) + ps^2(y)] - 2K_j^i(p + \rho)s(y)c(y)\hat{v}^j$$
$$- K(p + \rho)s(y)c(y)\hat{v}^i + \Gamma_{kj}^i[(p + \rho)\hat{v}^k\hat{v}^j s^2(y)$$
$$+ p\gamma^{kj} + \Pi^{kj}] + \left(\frac{\nabla_j N}{N} + \Gamma_{jk}^k\right)$$
$$\times [(p + \rho)\hat{v}^i\hat{v}^j s^2(y) + p\gamma^{ij} + \Pi^{ij}], \tag{A21}$$

where the shorthand notation $c(y) = \cosh y$ and $s(y) = \sinh y$ has been adopted and where Π^{ij} denotes the possible contribution of the anisotropic stress which has been included for completeness. Note that, in Eqs. (A20) and (A21), there is no potential ambiguity since, when the shift vector vanishes, Γ_{ij}^k coincides with $^{(3)}\Gamma_{ij}^k$. This is not the case, in general, as the last equality of Eq. (A1) clearly shows.

Finally, the components of the electromagnetic energy-momentum tensor $\mathcal{T}_\mu^\nu$ become, in ADM variables,

$$\mathcal{T}_0^0 = \frac{1}{8\pi\sqrt{\gamma}}\left[q_{mn}\frac{E^m E^n}{N} + \frac{\gamma_{nk}P_{ijm}\eta^{ijk}}{2}B^m B^n\right.$$
$$\left. - \left(\frac{p_{nm}}{N} + \frac{\gamma_{nk}\eta^{ijk}}{2}Q_{ijm}\right)B^n E^m\right], \tag{A22}$$

$$\mathcal{T}_i^0 = \frac{1}{4\pi N\sqrt{\gamma}}[Q_{ijk}E^k E^j - P_{ijk}B^k E^j], \tag{A23}$$

$$\mathcal{T}_i^j = \frac{1}{4\pi\sqrt{\gamma}}\left\{\left(\frac{q_{im}}{N}E^m E^j - \gamma_{ab}P_{imn}\eta^{jmb}B^n B^a\right)\right.$$
$$+ \left(Q_{imn}\gamma_{ab}\eta^{jmb}E^n B^a - p_{mi}\frac{B^m E^j}{N}\right)$$
$$+ \frac{1}{4}\delta_i^j\left[P_{mnk}\gamma_{ab}\eta^{mnb}B^k B^a + \frac{2}{N}(p_{ab}B^a E^b - q_{ab}E^a E^b)\right.$$
$$\left.\left. - Q_{mnk}\gamma_{ab}\eta^{mnb}E^k B^a\right]\right\}. \tag{A24}$$

In Eqs. (A22)–(A24) the following auxiliary tensors have been introduced,

$$q_{ki} = \frac{N_k N_i + \gamma_{ik}(N^2 - N_m N^m)}{N\sqrt{\gamma}},$$

$$p_{ki} = \frac{N_k\gamma_{ij}\gamma_{mn}\eta^{mjn}}{\sqrt{\gamma}}, \tag{A25}$$

$$Q_{ijk} = \frac{N_j\gamma_{ik} - N_i\gamma_{jk}}{N\sqrt{\gamma}}, \qquad P_{ijk} = \frac{\gamma_{im}\gamma_{jn}\gamma_{pk}\eta^{mnp}}{\sqrt{\gamma}}. \tag{A26}$$

In terms of the tensors given in Eqs. (A25) and (A26) the covariant components of the field strengths can be written as

$$F_{0i} = q_{ki}E^k - p_{ki}B^k, \qquad F_{ij} = Q_{ijk}E^k - P_{ijk}B^k, \tag{A27}$$

while the contravariant components of the field strength have already been reported in Eqs. (3.2) and (3.3) in terms of $\mathcal{E}^k = (N/\sqrt{\gamma})E^k$ and of $\mathcal{B}^k = (N/\sqrt{\gamma})B^k$. In the gauge where the shift vector vanishes, it is easy to show, within the present decomposition, that

$$q_{ki} = \frac{N}{\sqrt{\gamma}}\gamma_{ki}, \qquad p_{ki} = 0, \qquad Q_{ijk} = 0,$$

$$P_{ijk} = \frac{\gamma_{im}\gamma_{jn}\gamma_{pk}}{\sqrt{\gamma}}\eta^{mnk}, \tag{A28}$$

and Eqs. (A22)–(A24) reduce to

$$\mathcal{T}^0_0 = \frac{\gamma_{mn}}{8\pi\gamma}(E^m E^n + B^m B^n), \tag{A29}$$

$$\mathcal{T}^0_i = -\frac{\gamma_{im}\gamma_{jn}\gamma_{pk}\eta^{mnk}}{4\pi\gamma N}B^p E^j, \tag{A30}$$

$$\mathcal{T}^j_i = \frac{1}{4\pi\gamma}\Big[\gamma_{im}E^m E^j + \gamma_{im}B^m B^j$$
$$- \frac{\delta^j_i}{2}\gamma_{mn}(E^m E^n + B^m B^n)\Big]. \tag{A31}$$

In terms of the shorthand notation employed in the bulk of the paper, the components reported in Eqs. (A29)–(A31) can also be written as

$$\mathcal{T}^0_0 = \frac{1}{8\pi\gamma}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}) \equiv \rho_B + \rho_E, \tag{A32}$$

$$\mathcal{T}^0_i = -\frac{\gamma_{im}}{4\pi\gamma}(\vec{E}\times\vec{B})^m, \tag{A33}$$

$$\mathcal{T}^j_i = \frac{1}{4\pi\gamma}[\gamma_{im}E^m E^j + \gamma_{im}B^m B^j - \frac{\delta^j_i}{2}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})]$$
$$\equiv -(p_E + p_B)\delta^j_i + \Pi^j_i(E) + \Pi^j_i(B), \tag{A34}$$

where $p_e = \rho_e/3$ and $p_B = \rho_B/3$; $\Pi^j_i(E)$ and $\Pi^j_i(B)$ denote, respectively, the electric and the magnetic anisotropic stress.

APPENDIX B: RELATIONS TO THE BARDEEN FORMALISM

The Bardeen formalism [39] is one of the main tools customarily employed for a quantitative assessment of the impact of large-scale magnetic fields on the CMB anisotropies. It is therefore useful to describe the relation of the methods described in the present paper to the Bardeen approach. In the linearized theory of cosmological perturbations the fluctuations can be separated in scalar, vector, and tensor modes as

$$\delta g_{\mu\nu} = \delta_s g_{\mu\nu} + \delta_v g_{\mu\nu} + \delta_t g_{\mu\nu}. \tag{B1}$$

The scalar modes of the geometry are parametrized in terms of four independent functions. The vector modes are parametrized in terms of two pure vectors W_i and Q_i obeying $^{(3)}\nabla_i Q^i = 0$ and $^{(3)}\nabla_i W^i = 0$. Finally, the tensor modes are parametrized in terms of a rank-two tensor h_{ij} which is both divergenceless and traceless. Overall, before gauge fixing, the number of independent functions amounts to 10. At the nonlinear level, the decomposition of Eq. (B1) is meaningless, but still the more general description discussed in this paper contains, as special cases, the gauge-dependent approaches to the magnetized CMB anisotropies [8,9]. For instance, the choice of the

conformally Newtonian gauge (often dubbed the longitudinal gauge) corresponds to $N_i = 0$, $N(\vec{x}, \tau) = a(\tau)[1 + \phi(\vec{x}, \tau)]$, and $\gamma_{ij}(\vec{x}, \tau) = a^2(\tau)[1 - 2\psi(\vec{x}, \tau)]\delta_{ij}$. The extrinsic curvature and $\sqrt{\gamma}$ are therefore given by

$$K^j_i = \Big[-\frac{\mathcal{H}}{a} + \frac{1}{a}(\partial_\tau\psi + \mathcal{H}\phi)\Big]\delta^j_i, \quad \sqrt{\gamma} = a^3(1 - 3\psi). \tag{B2}$$

From Eqs. (A8)–(A10), the various components of the Einstein tensors can be obtained by keeping only the terms which are linear in the metric fluctuations. For instance, Eq. (A8) implies

$$G^0_0(\vec{x}, \tau) = \bar{G}^0_0(\tau) + \delta_s G^0_0(\vec{x}, \tau), \tag{B3}$$

$$\bar{G}^0_0(\tau) = 3\frac{\mathcal{H}^2}{a^2},$$
$$\delta_s G^0_0(\vec{x}, \tau) = \frac{2}{a^2}[\nabla^2\psi - 3\mathcal{H}(\mathcal{H}\phi + \psi')]. \tag{B4}$$

It is useful to remark that, in Eq. (B4), $|\delta_s G^0_0(\vec{x}, \tau)| < \bar{G}^0_0(\tau)$, as implied by the validity of the perturbative approximation. Conversely, in the gradient expansion, what matters is not the absolute magnitude of the perturbation in comparison with the background, but rather the number of gradients defining the various orders of the expansion. Following the same procedure of Eq. (B4) the evolution equations in the longitudinal gauge can be explicitly obtained from Eqs. (A9) and (A10):

$$\nabla^2\psi - 3\mathcal{H}(\mathcal{H}\phi + \partial_\tau\psi) = \frac{\ell^2_P}{2}a^2[\delta_s\rho + \delta_s\rho_B + \delta_s\rho_E], \tag{B5}$$

$$2\nabla^2(\mathcal{H}\phi + \partial_\tau\psi) + \ell^2_P\Big[(p_t + \rho_t)\theta_t + \frac{\vec{\nabla}\cdot(\vec{E}\times\vec{B})}{4\pi a^4}\Big] = 0, \tag{B6}$$

$$2\Big[\partial^2_\tau\psi + \mathcal{H}(\partial_\tau\phi + 2\partial_\tau\psi) + (\mathcal{H}^2 + 2\partial_\tau\mathcal{H})\phi$$
$$+ \frac{1}{2}\nabla^2(\phi - \psi)\Big]\delta^j_i + \partial_i\partial^j(\phi - \psi)$$
$$= \ell^2_P[(\delta_s p + \delta_s p_e + \delta_s p_B)\delta^j_i - \tilde{\Pi}^j_i$$
$$- \Pi^j_i(E) - \Pi^j_i(B)], \tag{B7}$$

where $\theta_t = \vec{\nabla}\cdot\vec{v}_t$. In analog terms the evolution equations can be obtained in a different gauge either by performing the appropriate gauge transformation on both sides of Eqs. (B6) and (B7), or by using again the general form of the Ricci (or Einstein) tensors reported in Appendix A. The synchronous gauge equations can be obtained by positing

$$N(\tau) = a(\tau),$$

$$\gamma_{ij}(\vec{x}, \tau) = a^2(\tau)[(1 + 2\xi)\delta_{ij} + 2\partial_i \partial_j F].$$

(B8)

In the case of the synchronous gauge condition the extrinsic curvature and $\sqrt{\gamma}$ read

$$K_i^j = \frac{1}{a}[-\mathcal{H}\delta_i^j - \partial_\tau \xi \delta_i^j - \partial_i \partial^j \partial_\tau F],$$

$$\sqrt{\gamma} = a^3(1 + 3\xi + \nabla^2 F).$$

(B9)

Finally, not only can the evolution of the scalar modes be readily obtained but also those for the vector and for the tensor modes. For instance, from Eq. (A6), in the gauge

$$N(\tau) = a(\tau), \qquad \gamma_{ij} = a^2(\tau)\delta_{ij},$$

$$N_i(\vec{x}, \tau) = a^2(\tau)Q_i(\vec{x}, \tau),$$

(B10)

the vector fluctuation of the Ricci tensor can be written

$$R_i^j(\vec{x}, \tau) = \bar{R}_i^j(\tau) + \delta_v R_i^j(\vec{x}, \tau),$$

$$R_i^j(\tau) = -\frac{1}{a^2}(\partial_\tau \mathcal{H} + 2\mathcal{H}^2)\delta_i^j,$$

$$\delta_v R_i^j(\vec{x}, \tau) = \frac{1}{2a^2}\{[\partial_i \partial_\tau Q^j + \partial^j \partial_\tau Q_i] + 2\mathcal{H}[\partial_i Q^j + \partial^j Q_i]\}.$$

(B11)

In the bulk of the paper we do not dwell on the derivation of the linearized results. It will, however, be understood that they are easily obtainable from the general expressions reported in Appendix A by going through the same steps outlined in the specific examples sketched in this appendix.

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