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Gravitational lens optical scalars in terms of energy-momentum distributions

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This is a general work on gravitational lensing. We present new expressions for the optical scalars and the deflection angle in terms of the energy-momentum tensor components of matter distributions. Our work generalizes standard references in the literature where normally stringent assumptions are made on the sources. The new expressions are manifestly gauge invariant, since they are presented in terms of curvature components. We also present a method of approximation for solving the lens equations, that can be applied to any order.

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I. INTRODUCTION

There being so many excellent publications that have covered the study of gravitational lensing, our justification for another general article on the subject comes from the fact that we present new expressions for the optical scalars and deflection angle for a wide variety of matter distributions in terms of the matter components.

Gravitational lensing has became a significant tool to make progress in our knowledge on the matter content of our Universe. In particular, there is a large number of works that use gravitational lensing techniques in order to know how much mass is in galaxies or clusters of galaxies. One of the most exciting results was to reaffirm the need for some kind of dark matter, that appears to interact with the barionic matter only through gravitation.

The question in which there is not yet general agreement is on the very nature of this dark matter. The most common conception is that it is based on collisionless particles [1], and where the pressures are negligible. However, in the context of cosmological studies, one often recurs to models of dark matter in terms of scalar fields [2–5]. There is also the possibility that dark matter was described in terms of spinor fields [6].

One method to study the nature of dark matter consists in observing the deformation of images of galaxies behind a matter distribution that is the source of a gravitational lens.

The fact that gravitational lensing can be useful for the study of the nature of dark matter has been emphasized many times, in particular in respect to the question of its equation of state [7–9].

In many astrophysical situations, the gravitational effects on light rays are weak, and the source and observer are far away from the lens; therefore, they are studied under the formalism of weak field and thin gravitational lenses. The basic and familiar variables in this discussion are shown in Fig. 1.

In this framework, the lens equation reads

$$\beta^a = \theta^a - \frac{d_{ls}}{d_s}\alpha^a. \tag{1}$$

The differential of this equation can be written as

$$\delta\beta^a = A^a{}_b\delta\theta^b, \tag{2}$$

where the matric $A^a{}_b$ is in turn expressed by

$$A^a{}_b = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}, \tag{3}$$

where the optical scalars κ, γ_1,and γ_2 are known as convergence κ and shear components $\{\gamma_1, \gamma_2\}$, and have the information of distortion of the image of the source due to the lens effects.

It is somehow striking that in most astronomical works on gravitational lensing, it is assumed that the lens scalars and deflection angle can be obtained from a Newtonian-like potential function. These expressions, although are easy to use, have some limitations:

(i) They neglect more general distribution of energy-momentum tensor T_{ab}; in particular, they only take into account the timelike component of this tensor. In this way, they severely restrict the possible

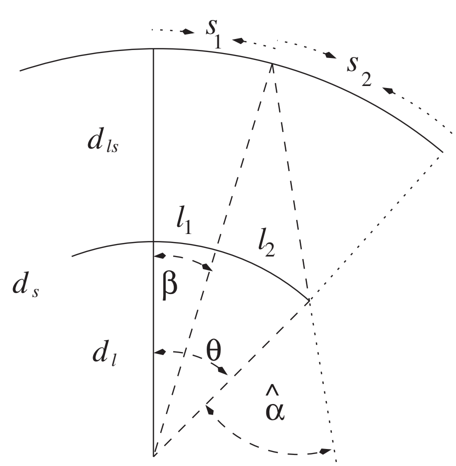

FIG. 1. This graph shows the basic and familiar angular variables in terms of a simple flat background geometry. The letter s denote sources, the letter l denotes lens and the observer is assumed to be situated at the apex of the rays.

candidates to dark matter that can be studied with these expressions.

(ii) They are not expressed in terms of gauge invariant quantities.

(iii) Since these expressions are written in terms of a potential function, it is not easily seen how different components of T_{ab} contribute in the generation of these images.

Moreover, most of them assume from the beginning that thin lens is a good approximation.

In other cases in which the thin lens approximation is not used [10], the results are presented in a way in which gauge invariance is not obvious; however, see [11].

In this paper, we extend the work appearing in standard references on gravitational lensing[12–15] and present new expressions that do not suffer from the limitations mentioned above. In particular, we present gauge invariant expressions for the optical scalars and deflection angle for some general class of matter distributions. In this first work on the subject, we study weak field gravitational lensing over a flat background.

In Sec. II, we present the general setting, starting from the geodesic deviation equation, where we fix some of our notation, and obtain gauge invariant expressions for the lens scalars. We also present a method of approximation for solving the lens equations that can be applied to any order. In Secs. III and IV , we will restrict the study to axially symmetric lenses, and we present expressions for the lens scalars and deflection angle in the thin lens case. In Sec. V, we concentrate on the spherically symmetric case, and after a general study of this geometry, we obtain expressions in terms of the energy-momentum distribution of these optical quantities. We end with a summary final Sec. VI and a couple of appendices.

II. INTEGRATED EXPANSION AND SHEAR

A. General equations: The geodesic deviation equation

Let us consider the general case of a null geodesic starting from the position p_s (source) and ending at p_o (observer). Let us characterize the tangent vector as $\ell = \frac{\partial}{\partial\lambda}$, so that

$$\ell^b \nabla_b \ell^a = 0; \tag{4}$$

that is, λ is an affine parameter.

We can now consider also a continuous set of nearby null geodesics. This congruence of null geodesics can be constructed in the following way. Let S be a two dimensional spacelike surface (the source image) such that the null vector ℓ is orthogonal to S. Next, we can generalize ℓ to be a vector field in the vicinity of the initial geodesic in the following way: let the function u be defined so that it is constant along the congruence of null geodesics emanating orthogonally to S and reaching the observing point p_o. Then, without loss of generality, we can assume that

$$\ell_a = \nabla_a u, \tag{5}$$

which implies that the congruence has zero twist.

We can complete to a set of null tetrad, so that m^a and $\bar{m}^a$ are tangent to S. At other points, m^a is chosen so that it is tangent to the surfaces $u = \text{constant}$ and $\lambda = \text{constant}$. Then, a deviation vector at the source image can be expressed by

$$\varsigma^a = \varsigma\bar{m}^a + \bar{\varsigma}m^a. \tag{6}$$

In order to propagate this deviation vector along the null congruence, one requires that its Lie derivate vanishes along the congruence; that is,

$$\mathfrak{L}_\ell \varsigma^a = 0, \tag{7}$$

which is equivalent to

$$\ell^b \nabla_b \varsigma^a - \varsigma^b \nabla_b \ell^a = 0. \tag{8}$$

From this, it can be proved that

$$\ell(\ell_b \varsigma^b) = 0. \tag{9}$$

The expansion and shear of the congruence are defined [16] respectively by

$$\theta = \frac{1}{2}\nabla_a \ell^a \tag{10}$$

and

$$|\sigma| = \sqrt{\frac{1}{2}\nabla_{(a}\ell_{b)}\nabla^a \ell^b - \theta^2}, \tag{11}$$

whose relation to the spin coefficient[17] quantities ρ and σ is given by

$$\theta = -\frac{1}{2}(\rho + \bar{\rho}) \tag{12}$$

and

$$|\sigma|^2 = \sigma\bar{\sigma}, \tag{13}$$

with certain abuse of the notation on the first appearance of $|\sigma|$.

Let us now calculate the covariant derivative of ς^a in the direction of ℓ,

$$\ell^b \nabla_b \varsigma^a = \ell(\varsigma)\bar{m}^a + \varsigma \ell^b \nabla_b \bar{m}^a + \text{c.c.}, \tag{14}$$

while

$$\varsigma^b \nabla_b \ell^a = \varsigma \bar{m}^b \nabla_b \ell^a + \text{c.c.}, \tag{15}$$

where c.c. means complex conjugate.

Let us note that

$$\bar{m}^b \nabla_b \ell^a = (-\beta' + \bar{\beta})\ell^a - \bar{\sigma} m^a - \rho \bar{m}^a \tag{16}$$

and

$$\ell^b \nabla_b \bar{m}^a = (\bar{\epsilon} - \epsilon)\bar{m}^a - \bar{\kappa} n^a - \tau' \ell^a, \tag{17}$$

where we are using the GHP [17] notation for the spin coefficients. In our case, one has

$$\kappa = 0 \tag{18}$$

since ℓ is geodesic. Notice that the Lie derivative of vector m in the direction of ℓ is

$$[\ell, m]^a = (\bar{\rho} + \epsilon - \bar{\epsilon})m^a + \sigma\bar{m}^a + (\bar{\beta}' - \beta - \bar{\tau}')\ell^a. \tag{19}$$

Then, since the Lie transport of m in the direction of ℓ should not have any ℓ component, because m are always tangent to the surfaces $u =$ constant and $\lambda =$ constant, one obtains that

$$\tau' = \beta' - \bar{\beta}. \tag{20}$$

Therefore, from Eq. (8), one has

$$\ell(\varsigma)\bar{m}^a + \varsigma(\bar{\epsilon} - \epsilon)\bar{m}^a + \varsigma(\bar{\sigma}m^a + \rho\bar{m}^a) + \text{c.c.} = 0, \tag{21}$$

which implies

$$0 = \ell(\varsigma) + \varsigma(\bar{\epsilon} - \epsilon) + \varsigma\rho + \bar{\varsigma}\sigma. \tag{22}$$

Using the GHP notation, one can write the previous equation as

$$0 = \text{Þ}(\varsigma) + \varsigma\rho + \bar{\varsigma}\sigma,$$

where Þ is the well behaved derivation of type $\{1, 1\}$ in the direction of ℓ. In order to have simple relations in terms of coordinate derivatives in the direction of λ, the complex phase of m and $\bar{m}$ can be chosen so that $\epsilon = 0$, so that finally one has

$$\ell(\varsigma) = \frac{\partial\varsigma}{\partial\lambda} = -\varsigma\rho - \bar{\varsigma}\sigma. \tag{23}$$

We see then that ρ determines the instantaneous expansion, and σ determines the instantaneous shear of the congruence.

Let us recall from the GHP equations [17] that

$$\ell(\rho) = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}, \tag{24}$$

and

$$\ell(\sigma) = (\rho + \bar{\rho})\sigma + \Psi_0. \tag{25}$$

Defining the matrix P from

$$P = \begin{pmatrix} \rho & \sigma \\ \bar{\sigma} & \bar{\rho} \end{pmatrix}, \tag{26}$$

one has

$$\ell(P) = P^2 + Q; \tag{27}$$

where Q is given by

$$Q = \begin{pmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{pmatrix} \tag{28}$$

with

$$\Phi_{00} = -\frac{1}{2}R_{ab}\ell^a\ell^b \tag{29}$$

and

$$\Psi_0 = C_{abcd}\ell^a m^b \ell^c m^d. \tag{30}$$

Defining $\mathcal{X}$ by

$$\mathcal{X} = \begin{pmatrix} \varsigma \\ \bar{\varsigma} \end{pmatrix}, \tag{31}$$

the equation for ς can be written as

$$\ell(\mathcal{X}) = -P, \tag{32}$$

so that

$$\ell(\ell(\mathcal{X})) = -Q\mathcal{X}, \tag{33}$$

which it only involves curvature quantities.

B. Approximation method for solving the geodesic deviation equation

Although the last equation can be integrated numerically without problems, it is sometimes convenient to have at hand some method for approximated solutions. So, next we present an approximation scheme that can be applied to any order one wishes to obtain; however, we will concentrate on the linear approximation since in weak field lens studies it is consistent to consider linear effects of the curvature on geodesic deviations.

Let us first transform to a first order differential equation, defining $\mathcal{V}$ to be

$$\mathcal{V} \equiv \frac{d\mathcal{X}}{d\lambda} \tag{34}$$

and

$$\mathbf{X} \equiv \begin{pmatrix} \mathcal{X} \\ \mathcal{V} \end{pmatrix}, \tag{35}$$

one obtains

$$\ell(\mathbf{X}) = \frac{d\mathbf{X}}{d\lambda} = \begin{pmatrix} \mathcal{V} \\ -Q\mathcal{X} \end{pmatrix} = A\mathbf{X} \tag{36}$$

with

$$A \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -Q & 0 \end{pmatrix}. \tag{37}$$

Equation (36) can be reexpressed in integral form, which gives

$$\mathbf{X}(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda')\mathbf{X}(\lambda')d\lambda'. \tag{38}$$

One can define the sequence

$$\mathbf{X}_1(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda')\mathbf{X}_0 d\lambda', \tag{39}$$

$$\mathbf{X}_2(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} A(\lambda')\mathbf{X}_1(\lambda')d\lambda', \quad (40)$$

and so on.

Assuming that Q is in some sense small, one expects that this sequence will converge and therefore provide for the solution.

Let us observe that

$$\mathbf{X}_2(\lambda) = \mathbf{X}_1(\lambda) + \int_{\lambda_0}^{\lambda} A(\lambda') \int_{\lambda_0}^{\lambda'} A(\lambda'')d\lambda''d\lambda'\mathbf{X}_0 \quad (41)$$

and that

$$A(\lambda')A(\lambda'') = A'A'' = \begin{pmatrix} -Q'' & 0 \\ 0 & -Q' \end{pmatrix}, \quad (42)$$

where we are using the notation $Q' = Q(\lambda')$. Similarly, one has

$$A'A''A''' = \begin{pmatrix} 0 & -Q'' \\ Q'Q''' & 0 \end{pmatrix} \quad (43)$$

and

$$A'A''A'''A'''' = \begin{pmatrix} Q''Q'''' & 0 \\ 0 & Q'Q''' \end{pmatrix}. \quad (44)$$

So, one can see that only at the fourth product of matrices A's one has complete second order of matrices Q's.

Returning to the sequence, the third element in first order is given by

$$\mathbf{X}_3(\lambda) = \mathbf{X}_0 + \int_{\lambda_0}^{\lambda} \begin{pmatrix} 0 & \mathbb{I} \\ -Q' & 0 \end{pmatrix} d\lambda'\mathbf{X}_0 + \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda'} \begin{pmatrix} -Q'' & 0 \\ 0 & -Q' \end{pmatrix} d\lambda''d\lambda'\mathbf{X}_0 + \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda'}\int_{\lambda_0}^{\lambda''} \begin{pmatrix} 0 & -Q'' \\ 0 & 0 \end{pmatrix} d\lambda'''d\lambda''d\lambda'\mathbf{X}_0. \quad (45)$$

Working out each term, one can see that

$$\mathbf{X}_3(\lambda) = \begin{pmatrix} \mathbb{I} - \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda'} Q''d\lambda''d\lambda' & (\lambda - \lambda_0)\mathbb{I} - \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda'} (\lambda'' - \lambda_0)Q''d\lambda''d\lambda' \\ -\int_{\lambda_0}^{\lambda} Q'd\lambda' & \mathbb{I} - \int_{\lambda_0}^{\lambda}(\lambda' - \lambda_0)Q'd\lambda' \end{pmatrix}\mathbf{X}_0, \quad (46)$$

where one can check that the second row is just the derivative of the first row.

Let us note that in this equation, one has not yet determined whether the position designated by λ is to the future or the past of the position designated by λ_0, so that one can use this approximated expression for both cases, keeping the same direction for the vector ℓ. If one changes the direction of the vector ℓ, then one has to take into account that $\mathcal{V}$ changes to $-\mathcal{V}$. In particular, it is easy to see that (46) is invariant under interchange of $\lambda \to -\lambda$ and $\mathcal{V} \to -\mathcal{V}$.

Note, also, that the double integral that appears in the first row and second column can by written by doing an integration by parts as

$$\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda'} (\lambda'' - \lambda_0)Q(\lambda'')d\lambda''d\lambda' = \int_{\lambda_0}^{\lambda} (\lambda - \lambda')(\lambda' - \lambda_0)Q(\lambda')d\lambda'. \quad (47)$$

In the following, we will make use of this equality.

C. The integrated shear and expansion

Now, in order to integrate the geodesic deviation equation, we must choose the correct initial conditions. In the case of light rays belonging to the past null cone of the observer and intersecting S at the source, these initial conditions are $\mathcal{X} = 0$ and $\mathcal{V} \neq 0$; thus, one can think the beam starts backwards in time from the observer position, and so initially has vanishing departure, but with nonzero expansion and shear.

Therefore, in the linear approximation one has

$$\mathcal{X}(\lambda) = \left((\lambda - \lambda_0)\mathbb{I} - \int_{\lambda_0}^{\lambda} (\lambda - \lambda')(\lambda' - \lambda_0)Q'd\lambda' \right)\mathcal{V}(\lambda_0) \quad (48)$$

and

$$\mathcal{V}(\lambda) = \left(\mathbb{I} - \int_{\lambda_0}^{\lambda} (\lambda' - \lambda_0)Q'd\lambda' \right)\mathcal{V}(\lambda_0). \quad (49)$$

In these integrations, λ_0 indicates the position at the observer and from now on, λ_s will indicate the position at the source.

We observe from the first expression, that if the metric were flat ($Q = 0$), in order to get a deviation vector constructed from $\mathcal{X}_1$, defined as $\mathcal{X}$ evaluated at $\lambda_s = \lambda_0 + d_s$, one must choose as initial condition

$$\mathcal{V}(\lambda_0) = \frac{1}{(\lambda_s - \lambda_0)}\mathcal{X}(\lambda_s = \lambda_0 + d_s) = \frac{1}{d_s}\mathcal{X}_1. \quad (50)$$

Let us remark that we have just fixed the scale of the affine parameter λ to coincide with the measure of spacelike distances.

But, in the case of the presence of a gravitational lens, if an observer sees an image of "size" $\mathcal{X}_o$, which means $\mathcal{X}_o \equiv d_s\mathcal{V}_o$ (since actually what is observed is $\mathcal{V}_o = \mathcal{V}(\lambda_0)$), then it should be produced by a source of size $\mathcal{X}_s = \mathcal{X}(\lambda_s)$, as described by Eq. (48) and depicted in Fig. 2.

In order to simplify the notation, we set from now on $\lambda_0 = 0$ and $\lambda_s = d_s$, then Eq. (48) reduces to

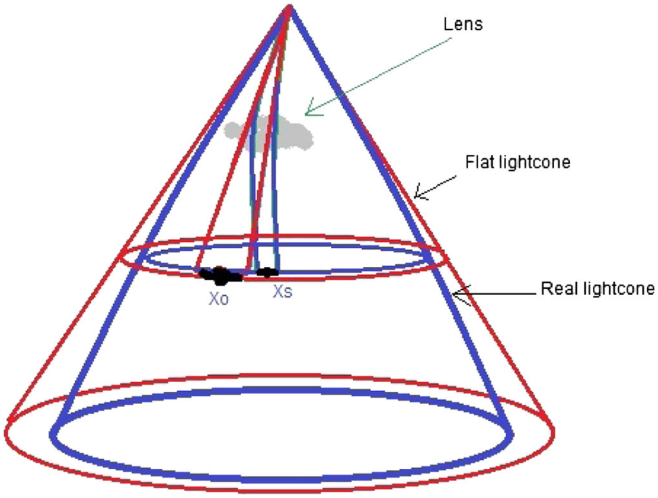

FIG. 2 (color online). An object of typical dimension $d_s \varsigma_s^a$ it appears to the observer to have a size $d_s \varsigma_o^a$.

$$\mathcal{X}_s = \left(\mathbb{I} - \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')Q' d\lambda'\right)\mathcal{X}_o. \quad (51)$$

Note also that although a generic complex displacement should be $\varsigma = |\varsigma| e^{i\phi}$, for our purposes it is enough to consider a complex displacement ς of unit modulus; namely, $\varsigma = e^{i\varphi}$. Then, from Eq. (51) one would have

$$\varsigma_s(\varphi) = \Bigg[1 - \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Phi_{00}(\lambda')d\lambda' - \left(\frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Psi_0(\lambda')d\lambda'\right)e^{-2i\varphi}\Bigg]e^{i\varphi}, \quad (52)$$

where one can see that for the flat case one has $\varsigma(\lambda, \varphi) = e^{i\varphi}$. From this equation, it is also observed that the expansion is only governed by the integration of Φ_{00}, and that the shear is only given by the integration of Ψ_0.

D. Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation

In order to compare with the standard representation of the lens scalar, we note that the original deviation vector in the source will be given by Eq. (51), i.e.

$$\begin{pmatrix} \varsigma_s \\ \bar{\varsigma}_s \end{pmatrix} = \left(\mathbb{I} - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s} Q' d\lambda'\right)\begin{pmatrix} \varsigma_o \\ \bar{\varsigma}_o \end{pmatrix}; \quad (53)$$

if we make the following decomposition into real and imaginary part,

$$\varsigma_o = \varsigma_{oR} + i\varsigma_{oI}, \quad (54)$$

$$\varsigma_s = \varsigma_{sR} + i\varsigma_{sI}, \quad (55)$$

$$\Psi_0 = \Psi_{0R} + i\Psi_{0I}, \quad (56)$$

we obtain from Eq. (53) that

$$\begin{aligned} \varsigma_{sR} &= \left(1 - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s}(\Phi'_{00} + \Psi'_{0R})d\lambda'\right)\varsigma_{oR} \\ &\quad - \left(\int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s}\Psi'_{0I}d\lambda'\right)\varsigma_{oI}, \\ \varsigma_{sI} &= \left(1 - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s}(\Phi'_{00} - \Psi'_{0R})d\lambda'\right)\varsigma_{oI} \\ &\quad - \int_0^{d_s} \frac{\lambda'(d_s - \lambda')}{d_s}\Psi'_{0I}d\lambda'\varsigma_{oR}. \end{aligned} \quad (57)$$

Note also that in principle, the integration must be made through the actual geodesic followed by a photon in its path from the source to observer. However, the last expressions are valid only in the limit where the linear approximation is valid. If one considers a linear perturbation from flat spacetime, then the curvature components Φ_{00} and Ψ_0 would be already of linear order. Then, in the context of weak field gravitational lensing, it is consistent to consider a null geodesic in flat spacetime, since the actual null geodesic can be thought of as a null geodesic in flat spacetime plus some corrections of higher orders.

We choose, then, a null geodesic coming from a source located at a distance d_s from the observer, and select a Cartesian coordinate system where this geodesic will propagate along the y negative direction. As we mentioned previously, one can actually integrate the equations either along the physical direction or one can integrate to the past from a null geodesic that starts at the observer position. We make this second choice.

We also need a null tetrad $\{l^a, m^a, \bar{m}^a, n^a\}$, adapted to this geodesic:

$$\begin{aligned} l^a &= (-1, 0, 1, 0), \qquad & m^a &= \frac{1}{\sqrt{2}}(0, i, 0, 1), \\ \bar{m}^a &= \frac{1}{\sqrt{2}}(0, -i, 0, 1), \qquad & n^a &= \frac{1}{2}(-1, 0, -1, 0). \end{aligned} \quad (58)$$

Now, in order to compare with the usual expressions for the lens scalars κ, γ_1, and γ_2, let us recall that they are defined via the relation Eq. (2); but since it is a linear relation, one can relate the deviation vectors by the same matrix, namely

$$\varsigma_s^i = A^i_j \varsigma_o^j, \quad (59)$$

where $\{\varsigma_s^i, \varsigma_o^i\}$ are the spatial vectors associated with $\{\varsigma_s, \varsigma_o\}$ respectively. In this expression, it is necessary to determine the meaning of the indices (i, j) of the two dimensional space of the images. In order to observe the natural Cartesian orientation, we identify the first component of the two dimensional space with the z one of the complete system, and the second component of the two dimensional space with the x one. We need, then, to know the components of the spatial vectors ς_o^a generated by ς_o and similarly by ς_s in a Cartesian like coordinate system. In the case of ς_o^a, it is given by

$$\varsigma_o^a = \varsigma_o \bar{m}^a + \bar{\varsigma}_o m^a = \frac{1}{\sqrt{2}}(\varsigma_o(0, -i, 0, 1) + \bar{\varsigma}_o(0, i, 0, 1))$$
$$= \frac{1}{\sqrt{2}}(0, i(\bar{\varsigma}_o - \varsigma_o), 0, (\varsigma_o + \bar{\varsigma}_o)) = \frac{2}{\sqrt{2}}(0, \varsigma_{oI}, 0, \varsigma_{oR}); \quad (60)$$

and a similar expression is obtained for ς_s^a.

Therefore, by replacing into Eq. (59), we obtain

$$\varsigma_{sR} = (1 - \kappa - \gamma_1)\varsigma_{oR} - \gamma_2 \varsigma_{oI}, \quad (61)$$

$$\varsigma_{sI} = -\gamma_2 \varsigma_{oR} + (1 - \kappa + \gamma_1)\varsigma_{oI}; \quad (62)$$

which, by comparing with Eq. (57), implies that

$$\kappa = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Phi'_{00} d\lambda', \quad (63)$$

$$\gamma_1 = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Psi'_{0R} d\lambda', \quad (64)$$

$$\gamma_2 = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Psi'_{0I} d\lambda'. \quad (65)$$

Let us emphasize that these expressions for the weak field lens quantities are explicitly gauge invariant, since they are given in terms of the curvature components, which are gauge invariant. This is in contrast to the usual treatment of weak field gravitational lensing found in the literature, which use, for example, equation (2.17) of reference [13] as the source for the calculation of the lens scalars.

Note that the last two equations can be written as:

$$\gamma_1 + i\gamma_2 = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Psi'_0 d\lambda'. \quad (66)$$

As a final comment to this section, it is important to remark that these expressions are valid for any weak field gravitational lens on a perturbed flat spacetime, without restriction on the size of the lens compared with the other distances. However, if we make use of the hypothesis of the thin lens, these equations can be further simplified, as we will show below.

III. THE AXIALLY SYMMETRIC LENS (INCLUDING THE SPHERICALLY SYMMETRIC CASE)

When one observes an astrophysical system, very often one needs to extract information out of the bulk of the matter distribution, which normally involves making some model assumptions on the nature of the distribution. So, very often one considers spherically symmetric models or the less restrictive case of axially symmetric distribution. In this latter case, the axis coincides with the line passing through the central region of the distribution and the observer.

In this section, then, we consider the case of an axially symmetric gravitational lens without introducing further assumptions on the extend of the lens. Later, we will consider thin lenses.

Using the same setting as in the last section, one is studying the motion of a photon which travels along the negative y direction, with impact parameter J and angle ϑ from the z axis. Then, one notes that the component Φ_{00} is a spin zero real quantity, and it only depends on the (J, y) coordinates. Meanwhile, the component Ψ_0 is a spin two complex quantity and it has the functional dependence

$$\Psi_0 = |\Psi_0| e^{2i\vartheta + \text{phase}} \quad (67)$$

where the phase is gauge dependent. For reasons that will become more clear during the study of spherically symmetric systems, we define the real quantities $\psi_0(J, y)$ from

$$\Psi_0(J, y, \vartheta) = -\psi_0(J, y)e^{2i\vartheta}. \quad (68)$$

From this, we deduce that the optical scalars have the following dependence:

$$\kappa(J) = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\Phi_{00}(J, \lambda')d\lambda' \quad (69)$$

and

$$\gamma_1 + i\gamma_2 = -\frac{1}{d_s}e^{2i\vartheta}\int_0^{d_s} \lambda'(d_s - \lambda')\psi_0(J, \lambda')d\lambda'. \quad (70)$$

This invites us to also define the real quantity $\gamma(J, y)$ from

$$\gamma_1 + i\gamma_2 = -\gamma e^{2i\vartheta} \quad (71)$$

so that one simply has

$$\gamma(J, y) = \frac{1}{d_s}\int_0^{d_s} \lambda'(d_s - \lambda')\psi_0(J, \lambda')d\lambda'. \quad (72)$$

IV. THE THIN LENS APPROXIMATION

A. The general case

Now, we will consider the case of a lens whose size is small compared with the distances to the source and the observer. Let there be again a Cartesian coordinate system such that the lens can be thought to be localized around the plane $y = 0$.

Then, as it was indicated in the last section, in the linear approximation we can replace the actual null geodesic by one in a flat spacetime. Then, considering a null geodesic as in the previous section, coming from a source located at a distance d_s from the observer, and at a distance d_{ls} from the lens, coming parallel to the y axis, but in the negative direction, we will use J to represent the impact parameter and ϑ to denote the angle of the trajectory as measured from the z axis in the (z, x) plane. We choose the scale of the affine parameter λ such that the geodesic is described by

$$(x(\lambda), y(\lambda), z(\lambda)) = (x_0, \lambda - d_l, z_0); \tag{73}$$

i.e., $\lambda = 0$ indicates the position of the observer, and $\lambda = d_s$ the position of the source.

Then, if we represent generically by C each one of the scalars $\{\Phi_{00}, \Psi_0\}$ that appears in the expressions for the lens scalars, we have that by doing an integration by parts we obtain the relation

$$\begin{aligned}\int_0^{d_s} \lambda'(d_s - \lambda')C(\lambda')d\lambda' &= \lambda'(d_s - \lambda')\tilde{C}(\lambda')|_0^{d_s} - \int_0^{d_s}(d_s - 2\lambda')\tilde{C}(\lambda')d\lambda' \\ &= -\int_0^{d_s}(d_s - 2\lambda')\tilde{C}(\lambda')d\lambda'\end{aligned} \tag{74}$$

where

$$\tilde{C}(\lambda') = \int_0^{\lambda'} C(\lambda'')d\lambda''. \tag{75}$$

Then, if we assume a thin lens, the scalars C will be sharply peaked around $\lambda = d_l$, where it is located, and $\tilde{C}(\lambda)$ can be approximated by

$$\tilde{C}(\lambda') \cong \begin{cases} 0 & \forall \lambda < d_l - \delta \\ \hat{C} & \forall \lambda \geq d_l + \delta \end{cases} \tag{76}$$

where $\delta \ll d_l$, $\delta \ll d_{ls}$, and $\delta \ll d_s$. Therefore, we obtain

$$\begin{aligned}\int_0^{d_s}(d_s - 2\lambda')\tilde{C}(\lambda')d\lambda' &\cong \hat{C}\int_{d_l}^{d_s}(d_s - 2\lambda')d\lambda' \\ &= \hat{C}d_l(d_l - d_s) = -\hat{C}d_l d_{ls},\end{aligned} \tag{77}$$

where we have neglected terms of order $O(\frac{\delta}{d_l})$. Obviously, all this also holds for the particular case of a delta Dirac distribution for the curvature components; however, our relaxed notion of thin lens is entirely expressed by the (76) behavior.

Finally, we conclude that in the thin lens approximation, the expressions for the lens scalars are reduced to

$$\kappa = \frac{d_l d_{ls}}{d_s}\hat{\Phi}_{00}, \tag{78}$$

$$\gamma_1 + i\gamma_2 = \frac{d_l d_{ls}}{d_s}\hat{\Psi}_0, \tag{79}$$

where

$$\hat{\Phi}_{00} = \int_0^{d_s} \Phi_{00} d\lambda, \qquad \hat{\Psi}_0 = \int_0^{d_s} \Psi_0 d\lambda, \tag{80}$$

are the projected curvature scalars along the line of sight.

We again emphasize that these expressions for the lens scalars are explicitly gauge invariants.

B. The axially symmetric case (which includes the spherically symmetric case)

1. The lens scalars in terms of projected Ricci and Weyl Scalars

For axially symmetric lens (and in fact spherically symmetric lens), the projected curvature scalars are given by

$$\hat{\Phi}_{00}(J) = \int_0^{d_s} \Phi_{00}(\lambda')d\lambda' \tag{81}$$

$$\hat{\Psi}_0(J) = -e^{2i\vartheta}\hat{\psi}_0(J), \tag{82}$$

where one can see that

$$\hat{\psi}_0(J) = -e^{-2i\vartheta}\int_0^{d_s} \Psi_0(\lambda')d\lambda'. \tag{83}$$

The reason for the minus sign choice is that in many common astrophysical situations one would find $\hat{\psi}_0(J) > 0$.

By replacing in Eqs. (78) and (79), we obtain for the lens scalars

$$\kappa = \frac{d_{ls}d_l}{d_s}\hat{\Phi}_{00}(J), \tag{84}$$

$$\gamma_1 = -\frac{d_{ls}d_l}{d_s}\hat{\psi}_0(J)\cos(2\vartheta), \tag{85}$$

$$\gamma_2 = -\frac{d_{ls}d_l}{d_s}\hat{\psi}_0(J)\sin(2\vartheta), \tag{86}$$

which implies that

$$\gamma = \frac{d_{ls}d_l}{d_s}\hat{\psi}_0(J). \tag{87}$$

These equations can be compared to those of reference [18] where they use different notations but similar content.

2. Deflection angle in terms of projected Ricci and Weyl Scalars

We wish now to express the deflection angle in terms of the curvature scalars.

From Eq. (2), we know that

$$A^i_j = \frac{d\beta^i}{d\theta^j} = \delta^i_j - \frac{d_{ls}}{d_s}\frac{d\alpha^i}{d\theta^j} = \delta^i_j - \frac{d_{ls}d_l}{d_s}\frac{d\alpha^i}{dx^j}; \tag{88}$$

where, in the last equality, we have used that in the thin lens approximation $\frac{d}{d\theta^i} \approx d_l\frac{d}{dx^i}$.

We define the components of $\alpha^i = (\alpha^1, \alpha^2)$ as

$$(\alpha^i) = \alpha(J)\left(\frac{z_0}{J}, \frac{x_0}{J}\right) \tag{89}$$

since, as we mentioned above, we are respecting the Cartesian orientation in the two dimensional space of the images. We then obtain that the shears and convergence can be written as

$$\kappa = \frac{1}{2}\frac{d_{ls}d_l}{d_s}\left(\frac{d\alpha^1}{dz_0} + \frac{d\alpha^2}{dx_0}\right), \tag{90}$$

$$\gamma_1 = \frac{1}{2}\frac{d_{ls}d_l}{d_s}\left(\frac{d\alpha^1}{dz_0} - \frac{d\alpha^2}{dx_0}\right), \tag{91}$$

$$\gamma_2 = \frac{d_{ls}d_l}{d_s}\frac{d\alpha^1}{dx_0} = \frac{d_{ls}d_l}{d_s}\frac{d\alpha^2}{dz_0}. \tag{92}$$

Noting that

$$x_0 = J\sin(\vartheta), \qquad z_0 = J\cos(\vartheta), \tag{93}$$

we obtain

$$\kappa = \frac{1}{2}\frac{d_{ls}d_l}{d_s}\left(\frac{d\alpha}{dJ} + \frac{\alpha(J)}{J}\right), \tag{94}$$

$$\gamma_1 = \frac{1}{2}\frac{d_{ls}d_l}{d_s}\cos(2\vartheta)\left(\frac{d\alpha}{dJ} - \frac{\alpha(J)}{J}\right), \tag{95}$$

$$\gamma_2 = \frac{1}{2}\frac{d_{ls}d_l}{d_s}\sin(2\vartheta)\left(\frac{d\alpha}{dJ} - \frac{\alpha(J)}{J}\right). \tag{96}$$

It is interesting to note that

$$\kappa - \gamma_1\cos(2\vartheta) - \gamma_2\sin(2\vartheta) = \frac{d_l d_{ls}}{d_s}\frac{\alpha(J)}{J}, \tag{97}$$

from which, using Eqs. (84)–(86), it is deduced that

$$\alpha(J) = J(\hat{\Phi}_{00}(J) + \hat{\psi}_0(J)). \tag{98}$$

It is worthwhile to remark that this constitutes an equation for the bending angle expressed in terms of the gauge invariant curvature components in a very simple compact form. We do not have knowledge of a previous presentation of this equation.

It is also important to emphasize that we have derived the expression for the deflection angle from the information contained in the calculation of the optical scalars, coming from the geodesic deviation equation.

Note that, at first sight, it seems that if we reconstruct the lens scalars using Eqs. (94)–(96), from expression (98) for the deflection angle, we would obtain some condition on the bending angle when compared with Eqs. (84)–(86); however, this is only an apparent inconsistency. Let us see this in more detail. To begin with, by replacing Eq. (98) into Eqs. (94)–(96), we obtain

$$\kappa = \frac{d_{ls}d_l}{2d_s}\left[2(\hat{\Phi}_{00}(J) + \hat{\psi}_0(J)) + J\frac{d(\hat{\Phi}_{00} + \hat{\psi}_0)}{dJ}\right], \tag{99}$$

$$\gamma_1 = \frac{d_{ls}d_l}{2d_s}\cos(2\vartheta)J\left(\frac{d\hat{\Phi}_{00}}{dJ} + \frac{d\hat{\psi}_0}{dJ}\right), \tag{100}$$

$$\gamma_2 = \frac{d_{ls}d_l}{2d_s}\sin(2\vartheta)J\left(\frac{d\hat{\Phi}_{00}}{dJ} + \frac{d\hat{\psi}_0}{dJ}\right). \tag{101}$$

Proceeding with the calculation, we now use one of the Bianchi identities, as expressed in the GHP formalism [17], namely

$$Þ\Psi_1 - ð'\Psi_0 + ð\Phi_{00} - Þ\Phi_{01} = 0 \tag{102}$$

where, in our case, $Þ = l^a\partial_a$, $ð = m^a\partial_a$, and $ð' = \bar{m}^a\partial_a$, i.e.

$$Þ = -\frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \qquad ð = \frac{1}{\sqrt{2}}\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right), \\ ð' = \frac{1}{\sqrt{2}}\left(-i\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right); \tag{103}$$

that is, we are using the flat null tetrad system. The expression for the edth operator is correct due to the fact that the intrinsic two dimensional metric in the space (x, z) is constant, and therefore $ð = m$.

Let us now change to a polar coordinate system in the two dimensional subspace, so that

$$\frac{\partial}{\partial x} = \frac{\partial J}{\partial x}\frac{\partial}{\partial J} + \frac{\partial\vartheta}{\partial x}\frac{\partial}{\partial\vartheta}, \qquad \frac{\partial}{\partial z} = \frac{\partial J}{\partial z}\frac{\partial}{\partial J} + \frac{\partial\vartheta}{\partial z}\frac{\partial}{\partial\vartheta}; \tag{104}$$

with

$$\frac{\partial\vartheta}{\partial x} = \frac{\cos(\vartheta)}{J}, \qquad \frac{\partial\vartheta}{\partial z} = -\frac{\sin(\vartheta)}{J}. \tag{105}$$

In this case, the metric of the two dimensional space (J, ϑ) is not constant, so that in principle the edth operator acting on a quantity f of type (p, q) should be [17]

$$ðf = m(f) + (-p\beta + q\beta')f; \tag{106}$$

but a direct calculation in the (J, ϑ) frame gives all spin coefficients zero. Therefore, in this frame we also have $ðf = m(f)$. Then, we get

$$ð = \frac{1}{\sqrt{2}}e^{i\vartheta}\left(\frac{\partial}{\partial J} + \frac{i}{J}\frac{\partial}{\partial\vartheta}\right), \\ ð' = \frac{1}{\sqrt{2}}e^{-i\vartheta}\left(\frac{\partial}{\partial J} - \frac{i}{J}\frac{\partial}{\partial\vartheta}\right). \tag{107}$$

If we now project the Bianchi identity on the line of sight direction, i.e. by integrating along the y-direction, we obtain

$$\Psi_1|_0^{d_s} - ð'\hat{\Psi}_0 + ð\hat{\Phi}_{00} - \Phi_{01}|_0^{d_s} = 0; \tag{108}$$

which, assuming $\Psi_1 \approx 0$ and $\Phi_{01} \approx 0$ far away from the lens, it implies

$$ð'(\hat{\psi}_0 e^{2i\vartheta}) = -ð(\hat{\Phi}_{00}). \tag{109}$$

From this, one finds

$$\frac{d\hat{\psi}_0}{dJ} + 2\frac{\hat{\psi}_0}{J} = -\frac{d\hat{\Phi}_{00}}{dJ}. \quad (110)$$

Then, by replacing this relation into Eqs. (99)–(101) we obtain Eqs. (84)–(86), as anticipated.

The Bianchi identities have not been used very often in the context of gravitational lenses; however, we note that in references [19,20] they have used them to obtain a Poisson like equation in order to determine the matter distribution.

For the study of the errors committed in the use of the thin lens approximation, one can read [21].

V. DETAILED STUDY OF STATIONARY SPHERICALLY SYMMETRIC LENSES

Up to now, we have presented gauge invariant expressions for the deviation angle and the optical scalars in terms of the curvature components of the null tetrad adapted to the motion of the photons. In order to obtain expressions that use information of the structure of the sources, one has to work with frames adapted to the geometry of the matter distribution which forms the gravitational lens. Therefore, in this section we study the case of stationary spherically symmetric sources.

A. Spacetime geometry in standard coordinate system

The metric

For stationary spherically symmetric spacetime, the line element can be expressed by

$$ds^2 = a(r)dt^2 - b(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (111)$$

It is convenient to define $\Phi(r)$ and $M(r)$ from

$$a(r) = e^{2\Phi(r)}, \quad (112)$$

and

$$b(r) = \frac{1}{1 - \frac{2M(r)}{r}}. \quad (113)$$

The more general distribution of energy-momentum compatible with spherical symmetry is described by an energy-momentum tensor given by

$$T_{tt} = \varrho e^{2\Phi(r)}, \quad (114)$$

$$T_{rr} = \frac{P_r}{(1 - \frac{2M(r)}{r})}y, \quad (115)$$

$$T_{\theta\theta} = P_t r^2, \quad (116)$$

$$T_{\varphi\varphi} = P_t r^2 \sin(\theta)^2, \quad (117)$$

where we have introduced the notion of radial component P_r and tangential component P_t.

The Einstein field equations

$$G_{ab} = -8\pi T_{ab}, \quad (118)$$

in terms of the previous variables are

$$\frac{dM}{dr} = 4\pi r^2 \varrho, \quad (119)$$

$$r^2\frac{d\Phi}{dr} = \frac{M + 4\pi r^3 P_r}{1 - \frac{2M(r)}{r}}, \quad (120)$$

$$r^3\left(\frac{d^2\Phi}{dr^2} + \left(\frac{d\Phi}{dr}\right)^2\right)\left(1 - \frac{2M}{r}\right) + r^2\frac{d\Phi}{dr}\left(1 - \frac{M}{r} - \frac{dM}{dr}\right) - r\frac{dM}{dr} + M = 8\pi r^3 P_t. \quad (121)$$

The conservation equation is

$$\frac{dP_r}{dr} = -(\varrho + P_r)\frac{d\Phi}{dr} - \frac{2}{r}(P_r - P_t). \quad (122)$$

B. Geometry with respect to a null system

The tetrad

For our purpose, it is more convenient to use a null coordinate system to describe the spherically symmetric geometry. Let us introduce, then, a function

$$u = t - r^*, \quad (123)$$

where r^* is chosen so that u is null. Then, by inspection of Eq. (111), one can see that

$$du = dt - \frac{dr^*}{dr}dr = dt - \sqrt{\frac{b}{a}}dr, \quad (124)$$

since then one has

$$ds^2 = adu^2 + 2\sqrt{ab}dudr - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (125)$$

It is natural to define the principal null direction $\tilde{\ell}_P$ from

$$\tilde{\ell}_P = du, \quad (126)$$

which implies that the vector is

$$\tilde{\ell}_P^a = g^{ab}du_b = \frac{1}{\sqrt{ab}}\left(\frac{\partial}{\partial r}\right)^a \quad (127)$$

where we have used that

$$(g^{ab}) = \frac{2}{\sqrt{ab}}\frac{\partial}{\partial u}\frac{\partial}{\partial r} - \frac{1}{b}\frac{\partial}{\partial r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi}\right). \quad (128)$$

Let us define the null tetrad

$$\tilde{\ell}_P = \mathbb{A}\frac{\partial}{\partial r}, \quad (129)$$

$$\tilde{n}_P = \frac{\partial}{\partial u} + U\mathbb{A}\frac{\partial}{\partial r}, \tag{130}$$

with the complex null vector

$$\tilde{m}_P = \frac{\sqrt{2}P_0}{r}\frac{\partial}{\partial \zeta} \tag{131}$$

in terms of the stereographic coordinate ζ.

Therefore, one has

$$\mathbb{A} = \frac{1}{\sqrt{ab}}, \tag{132}$$

and

$$U = -\frac{1}{2b\mathbb{A}^2} = -\frac{a}{2}. \tag{133}$$

It is worthwhile to note that we have chosen to keep using r as a coordinate, which measures the surfaces of the symmetric spheres. Instead, one could have chosen to use an affine coordinate $\tilde{r}$ so that one would have $\tilde{\ell} = \frac{\partial}{\partial \tilde{r}}$; but then, the surfaces of the symmetric spheres would be some function of $\tilde{r}$, different from $4\pi\tilde{r}^2$.

C. The spin coefficients scalars and curvature components

For the spherically symmetric metric, the non vanishing spin coefficients are

$$\tilde{\rho} = -\frac{\mathbb{A}}{r}, \tag{134}$$

$$\tilde{\rho}' = -\frac{U\mathbb{A}}{r}, \tag{135}$$

$$\beta = \frac{1}{\sqrt{2}r}\left(-\frac{\partial P_0}{\partial y_2} + i\frac{\partial P_0}{\partial y_3}\right), \tag{136}$$

$$\beta' = \frac{1}{\sqrt{2}r}\left(-\frac{\partial P_0}{\partial y_2} - i\frac{\partial P_0}{\partial y_3}\right), \tag{137}$$

$$\tilde{\epsilon}' = \frac{1}{2}\mathbb{A}\frac{d\mathbb{U}}{dr} \tag{138}$$

where we are using for the stereographic coordinate the decomposition $\zeta = \frac{1}{2}(y_2 + iy_3)$.

The curvature components that are different from zero are:

$$\tilde{\Phi}_{00} = -\frac{\mathbb{A}}{r}\frac{d\mathbb{A}}{dr}, \tag{139}$$

$$\tilde{\Phi}_{11} = -\frac{1}{4}\frac{d\mathbb{A}}{dr}\frac{dU}{dr}\mathbb{A} - \frac{1}{4}\nabla^2(U)\mathbb{A}^2 + \frac{1}{2r}\frac{dU}{dr}\mathbb{A}^2 + \frac{1}{r^2}\left(\frac{1}{2}\mathbb{A}^2 U + \frac{1}{4}\right), \tag{140}$$

$$\tilde{\Phi}_{22} = -\frac{\mathbb{A}U^2}{r}\frac{d\mathbb{A}}{dr}, \tag{141}$$

$$\tilde{\Lambda} = \frac{1}{12}\frac{d\mathbb{A}}{dr}\frac{dU}{dr}\mathbb{A} + \frac{1}{12}\nabla^2(U)\mathbb{A}^2 + \frac{1}{r}\left(\frac{1}{3}\frac{d\mathbb{A}}{dr}\mathbb{A}U + \frac{1}{6}\frac{dU}{dr}\mathbb{A}^2\right) + \frac{1}{r^2}\left(\frac{1}{6}\mathbb{A}^2 U + \frac{1}{12}\right), \tag{142}$$

and

$$\tilde{\Psi}_2 = -\frac{1}{6}\frac{d\mathbb{A}}{dr}\frac{dU}{dr}\mathbb{A} - \frac{1}{6}\nabla^2(U)\mathbb{A}^2 + \frac{1}{r}\left(\frac{1}{3}\frac{d\mathbb{A}}{dr}\mathbb{A}U + \frac{2}{3}\frac{dU}{dr}\mathbb{A}^2\right) + \frac{1}{r^2}\left(-\frac{1}{3}\mathbb{A}^2 U - \frac{1}{6}\right). \tag{143}$$

Note that from (124) and (126) one has that

$$\tilde{\ell}_P = dt - \sqrt{\frac{b}{a}}dr, \tag{144}$$

and therefore

$$\tilde{\ell}_P^{\,a} = \frac{1}{a}\left(\frac{\partial}{\partial t}\right)^a + \sqrt{\frac{1}{ab}}\left(\frac{\partial}{\partial r}\right)\Bigg|_t^a. \tag{145}$$

Also, let us note that

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial t}; \tag{146}$$

which then implies that

$$\tilde{n}_P = \frac{1}{2}\frac{\partial}{\partial t} - \frac{1}{2}\sqrt{\frac{a}{b}}\left(\frac{\partial}{\partial r}\right)\Bigg|_t^a. \tag{147}$$

In these last equations, $(\frac{\partial}{\partial r})|_t^a$ is meant at constant t, as opposite to the previous equations in which $\frac{\partial}{\partial r}$ was meant at constant u.

D. Spinor Ricci components in terms of energy-momentum components in the nonisotropic case

The spinor Ricci components can be written in terms of the energy-momentum distribution as

$$\tilde{\Phi}_{00} = \frac{4\pi}{a}(\varrho + P_r), \tag{148}$$

$$\tilde{\Phi}_{11} = \pi(\varrho - P_r + 2P_t), \tag{149}$$

$$\tilde{\Phi}_{22} = a\pi(\varrho + P_r), \tag{150}$$

$$\tilde{\Lambda} = \frac{\pi}{3}(\varrho - P_r - 2P_t). \tag{151}$$

These expressions are exact for the spherically symmetric spacetime. If one needs linear expressions around flat spacetime, one must set $a = 1$.

Note that one has

$$\tilde{\Phi}_{22} = U^2 \tilde{\Phi}_{00}. \tag{152}$$

Using the expressions for $\tilde{\Phi}_{11}$ and $\tilde{\Lambda}$, one can prove that

$$\tilde{\Phi}_{11} + 3\tilde{\Lambda} = \frac{\mathbb{A}}{r}\frac{d(U\mathbb{A})}{dr} + \frac{1}{r^2}\left(U\mathbb{A}^2 + \frac{1}{2}\right) = 2\pi(\varrho - P_r). \tag{153}$$

Also, from the relation of the null tetrad components with the old variables, one can obtain that

$$U\mathbb{A}^2 + \frac{1}{2} = \frac{M(r)}{r}. \tag{154}$$

This equation gives U in terms of $\mathbb{A}$ and M.

Using this in the expression for $\tilde{\Phi}_{00}$, one obtains

$$\frac{1}{\mathbb{A}}\frac{d\mathbb{A}}{dr} = \frac{4\pi r(\varrho + P_r)}{(\frac{2M}{r} - 1)}. \tag{155}$$

This is a useful equation only involving $\mathbb{A}$, which allows its calculation in terms of the components of the energy-momentum tensor.

The contracted Bianchi identity (2.37) of [17] for spherically symmetric metrics is

$$Þ\tilde{\Phi}_{11} + Þ'\tilde{\Phi}_{00} + 3Þ\tilde{\Lambda} = (\tilde{\rho}' + \bar{\tilde{\rho}}')\tilde{\Phi}_{00} + 2(\tilde{\rho} + \bar{\tilde{\rho}})\tilde{\Phi}_{11}, \tag{156}$$

or, explicitely,

$$\frac{d\tilde{\Phi}_{11}}{dr} + \mathbb{A}U\frac{d\tilde{\Phi}_{00}}{dr} + 2\frac{dU}{dr}\mathbb{A}\tilde{\Phi}_{00} + 3\frac{d\tilde{\Lambda}}{dr} = -2\frac{\mathbb{A}U}{r}\tilde{\Phi}_{00} - 4\frac{\mathbb{A}}{r}\tilde{\Phi}_{11}, \tag{157}$$

which gives the conservation equation in the form

$$\frac{dP_r}{dr} = -(\varrho + P_r)\frac{m_g(r)}{r^2} - \frac{2}{r}(P_r - P_t) \tag{158}$$

where we are using

$$m_g(r) = \frac{r^2}{2}\frac{d\ln U}{dr}. \tag{159}$$

E. Simple relation for Weyl component $\tilde{\Psi}_2$

Let us observe that

$$\tilde{\Psi}_2 + 2\tilde{\Lambda} = \frac{\mathbb{A}}{r}\frac{d(\mathbb{A}U)}{dr}. \tag{160}$$

Then, from Eq. (153), one can deduce that

$$\tilde{\Psi}_2 = \frac{4\pi}{3}(\varrho - P_r + P_t) - \frac{M}{r^3}. \tag{161}$$

This is a very simple relation for $\tilde{\Psi}_2(r)$ in terms of the energy density $\varrho(r)$, the spacelike components, and the mass function $M(r)$. Our expression generalizes those of reference [22] for the case of anisotropic energy-momentum tensor.

F. The bending angle and lens scalars in terms of energy-momentum components, curvature components, and $M(r)$

1. Relation between the scalars curvatures in the two different tetrads

In order to express the function $\alpha(J)$ in terms of the curvature scalars defined with the spherically symmetric tetrad, we need to know how the tetrads transform between them. To do so, let us recall that at linear order, we only need the transformation between the flat tetrad $\{l^a, m^a, \bar{m}^a, n^a\}$ adapted to the null geodesic coming from the source, and a flat tetrad $\{\tilde{l}^a, \tilde{m}^a, \bar{\tilde{m}}^a, \tilde{n}^a\}$ obtained from $\{\tilde{l}^a_P, \tilde{m}^a_P, \bar{\tilde{m}}^a_P, \tilde{n}^a_P\}$ by setting $a = b = 1$. Then, using standard spherical coordinates we have

$$\tilde{l}^a = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = (1, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)), \tag{162}$$

$$\tilde{n}^a = (1, -\sin(\theta)\cos(\phi), -\sin(\theta)\sin(\phi), -\cos(\theta)), \tag{163}$$

$$\tilde{m}^a = ð_0(\tilde{l}^a) = \frac{1}{\sqrt{2}}(0, -\cos(\theta)\cos(\phi) + i\sin(\phi), -\cos(\theta)\sin(\phi) - i\cos(\phi), \sin(\theta)), \tag{164}$$

$$\bar{\tilde{m}}^a = \bar{ð}_0(\tilde{l}^a) = \frac{1}{\sqrt{2}}(0, -\cos(\theta)\cos(\phi) - i\sin(\phi), -\cos(\theta)\sin(\phi) + i\cos(\phi), \sin(\theta)). \tag{165}$$

In these expressions, we use the symbols $ð_0$ and $\bar{ð}_0$ to denote the edths operators of the spheres of symmetry, with unit radius.

The transformation tetrad will be of the form

$$l^a = c_{l\tilde{n}}\tilde{l}^a - c_{l\bar{\tilde{m}}}\tilde{m}^a - c_{l\tilde{m}}\bar{\tilde{m}}^a + c_{l\tilde{l}}\tilde{n}^a, \tag{166}$$

$$n^a = c_{n\tilde{n}}\tilde{l}^a - c_{n\bar{\tilde{m}}}\tilde{m}^a - c_{n\tilde{m}}\bar{\tilde{m}}^a + c_{n\tilde{l}}\tilde{n}^a, \tag{167}$$

$$m^a = c_{m\tilde{n}}\tilde{l}^a - c_{m\bar{\tilde{m}}}\tilde{m}^a - c_{m\tilde{m}}\bar{\tilde{m}}^a + c_{m\tilde{l}}\tilde{n}^a, \tag{168}$$

$$\bar{m}^a = c_{\bar{m}\tilde{n}}\tilde{l}^a - c_{\bar{m}\bar{\tilde{m}}}\tilde{m}^a - c_{\bar{m}\tilde{m}}\bar{\tilde{m}}^a + c_{\bar{m}\tilde{l}}\tilde{n}^a; \tag{169}$$

where the notation is $c_{l\tilde{n}} = l^a\tilde{n}_a$, and so on. From these relations, we can construct the transformation of the Ricci and Weyl scalars, but it is more convenient and easy to work with the spinor diad associated to the tetrad. Then, the transformation of the dyads will be

$$o^A = A\tilde{o}^A + B\tilde{\iota}^A, \qquad \iota^A = C\tilde{o}^A + D\tilde{\iota}^A, \tag{170}$$

together to the condition that o^A, ι^A conform a spinorial base, i.e.,

$$AD - BC = 1, \tag{171}$$

From the relations given in Appendix A, we get

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\sqrt{1 - \sin(\theta)\sin(\phi)}e^{i((\eta+\eta'+\pi)/2)}, \\ B &= \sqrt{1 + \sin(\theta)\sin(\phi)}e^{i((\eta'-\eta+\pi)/2)}, \\ C &= \frac{1}{2}\sqrt{1 + \sin(\theta)\sin(\phi)}e^{i((\eta-\eta'+\pi)/2)}, \\ D &= \frac{1}{\sqrt{2}}\sqrt{1 - \sin(\theta)\sin(\phi)}e^{-i((\eta'+\eta+\pi)/2)}; \end{aligned} \tag{172}$$

where η and η' satisfies

$$e^{i\eta} = \frac{-\cos(\theta)\sin(\phi) + i\cos(\phi)}{\sqrt{1 - \sin^2(\theta)\sin^2(\phi)}}, \tag{173}$$

$$\begin{aligned} e^{i\eta'} &= \frac{\cos(\theta) + i\sin(\theta)\cos(\phi)}{\sqrt{1 - \sin^2(\theta)\sin^2(\phi)}} = \frac{z + ix}{J} \\ &= \cos(\vartheta) + i\sin(\vartheta), \end{aligned} \tag{174}$$

and in the last equality, the fact that $J = r\sqrt{1 - \sin^2(\theta)\sin^2(\phi)}$, was used (see Appendix A). Note, then, that, $\eta' = \vartheta$.

The general transformation between tetrads induces the following transformation on the curvature scalar Φ_{00} and Ψ_0:

$$\begin{aligned} \Phi_{00} &= \Phi_{ABA'B'}o^A o^B o^{A'} o^{B'} \\ &= A^2\bar{A}^2\tilde{\Phi}_{00} + 2A^2\bar{A}\,\bar{B}\,\tilde{\Phi}_{01} + A^2\bar{B}^2\tilde{\Phi}_{02} \\ &\quad + 2A\bar{A}^2B\tilde{\Phi}_{10} + 4A\bar{A}B\bar{B}\tilde{\Phi}_{11} + 2AB\bar{B}^2\tilde{\Phi}_{12} \\ &\quad + \bar{A}^2B^2\tilde{\Phi}_{20} + 2B^2\bar{B}\,\bar{A}\,\tilde{\Phi}_{21} + B^2\bar{B}^2\tilde{\Phi}_{22}, \end{aligned} \tag{175}$$

$$\begin{aligned} \Psi_0 &= \Psi_{ABCD}o^A o^B o^C o^D \\ &= A^4\tilde{\Psi}_0 + 4A^3\tilde{\Psi}_1 + 6A^2B^2\tilde{\Psi}_2 + 4AB^3\tilde{\Psi}_3 + B^4\tilde{\Psi}_4. \end{aligned} \tag{176}$$

In the spherically symmetric case, these transformations simplify considerably and finally, at linear order, one has $\tilde{\Phi}_{22} = \frac{1}{4}\tilde{\Phi}_{00}$ so that

$$\Psi_0 = 3\frac{J^2}{r^2}\tilde{\Psi}_2(r)e^{2i\vartheta}, \tag{177}$$

$$\Phi_{00} = \frac{2J^2}{r^2}\left(\tilde{\Phi}_{11} - \frac{1}{4}\tilde{\Phi}_{00}\right) + \tilde{\Phi}_{00}. \tag{178}$$

2. The deflection angle in terms of spherically symmetric components of the curvature

From (98), the function $\alpha(J)$ expressed in terms of the spherically symmetric null tetrad reads:

$$\alpha(J) = J\int_{-d_l}^{d_{ls}}\left[-\frac{3J^2}{r^2}\tilde{\Psi}_2 + \frac{2J^2}{r^2}\left(\tilde{\Phi}_{11} - \frac{1}{4}\tilde{\Phi}_{00}\right) + \tilde{\Phi}_{00}\right]dy. \tag{179}$$

Note that in this case, the integration is on the coordinate y, instead of using arbitrary affine parameter. Also, note that $r = \sqrt{J^2 + y^2}$.

This constitutes an important explicit relation for the bending angle in terms of the curvature as seen in an spherically symmetric frame, which is the natural frame for the sources of the gravitational lens.

3. Expressions for the bending angle in terms of energy-momentum components and $M(r)$

Using Eqs. (148), (149), (161), and (179) we get an expression for the bending angle in terms of the mass, energy density, and spacelike components of the energy-momentum tensor, namely

$$\begin{aligned} \alpha(J) = J\int_{-d_l}^{d_{ls}}\Big[&\frac{3J^2}{r^2}\left(\frac{M(r)}{r^3} - \frac{4\pi}{3}\varrho(r)\right) \\ &+ 4\pi(\varrho(r) + P_r(r))\Big]dy. \end{aligned} \tag{180}$$

This is a new and useful relation for the deflection angle in terms of the physical fields which are the sources of the gravitational lens. It is also worth mentioning that this expression for the bending angle can also be deduced from the geodesic equation using standard techniques, as it is shown in Appendix B.

It is curious that the bending angle does not depend explicitly on the tangential spacelike components of the energy-momentum tensor.

4. The optical scalars in terms of spherically symmetric components of the curvature

From Eq. (84) and (87) one obtains

$$\kappa(J) = \frac{d_{ls}d_l}{d_s}\int_{-d_l}^{d_{ls}}\left[\frac{2J^2}{r^2}\left(\tilde{\Phi}_{11} - \frac{1}{4}\tilde{\Phi}_{00}\right) + \tilde{\Phi}_{00}\right]dy, \tag{181}$$

and

$$\gamma(J) = -\frac{d_{ls}d_l}{d_s}\int_{-d_l}^{d_{ls}}\left[3\frac{J^2}{r^2}\tilde{\Psi}_2(r)\right]dy. \tag{182}$$

These expressions give the optical scalars in terms of gauge invariant expressions for the curvature components adapted to the symmetry of the matter distribution, which is the source of the gravitational lens.

5. *Expressions for the lens scalars in terms of energy-momentum components and $M(r)$*

In a similar way, the lens scalars, in terms of the spherically symmetric physical fields, are given by

$$\kappa = \frac{4\pi d_l d_{ls}}{d_s}\int_{-d_l}^{d_{ls}}\left[\rho + P_r + \frac{J^2}{r^2}(P_t - P_r)\right]dy$$
$$\gamma = \frac{d_l d_{ls}}{d_s}\int_{-d_l}^{d_{ls}}\frac{J^2}{r^2}\left[\frac{3M}{r^3} - 4\pi(\rho + P_t - P_r)\right]dy. \tag{183}$$

These new expressions let us see explicitly the contributions of different components of the energy-momentum tensor on the optical scalars. One can see that a couple of terms disappear in the isotropic case, in which $P_r = P_t$.

Our expressions are valid for generic energy-momentum distributions, not usually considered in the literature are the possible implications of non vanishing spacelike components of T_{ab}. In future works, we will consider the implications of models with nontrivial energy-momentum tensors on observed gravitational lenses.

G. Two simple examples

In order to show the application of our treatment of gravitational lenses, we will consider next two standard models that are often used in representing the source of gravitational lenses.

1. *A monopole mass (Schwarzschild)*

As a simple example, let there be a monopole distribution characterized by a mass M; therefore, a simple computation gives $\tilde{\Phi}_{00} = 0$, and $\tilde{\Psi}_2 = -\frac{M}{r^3}$. Then, by considering that the observer and the source are far away, one can replace in the extremes of the integration (as is usually made) $d_s \to \infty$ and $d_l \to \infty$, then

$$\hat{\Phi}_{00} = 0, \tag{184}$$

$$\hat{\psi}_0 = -2\int_0^\infty \frac{3J^2}{r^2}\tilde{\Psi}_2 dy = \frac{4M}{J^2}, \tag{185}$$

and by replacing into Eqs. (84), (87), and (98), we readily obtain the well known results

$$\alpha(J) = \frac{4M}{J}, \tag{186}$$

$$\kappa = 0, \tag{187}$$

$$\gamma = \frac{d_l d_{ls}}{d_s}\frac{4M}{J^2}. \tag{188}$$

2. *The isothermal profile*

One simple model of dark matter that is used to explain the rotation curves of galaxies is the isothermal profile, which is defined by the density function

$$\rho = \frac{v_c^2}{4\pi r^2}, \tag{189}$$

where v_c is the circular velocity.

Since $v_c \ll c$, the pressures in this model are negligible. Then we obtain

$$\hat{\Phi}_{00} = \int_{-\infty}^{\infty}\frac{v_c^2}{r^2}dy = \frac{v_c^2\pi}{J}, \tag{190}$$

$$\hat{\psi}_0 = \int_{-\infty}^{\infty}\frac{2J^2 v_c^2}{r^4}dy = \frac{v_c^2\pi}{J}. \tag{191}$$

From these relations follow the well known results:

$$\alpha = 2\pi v_c^2, \tag{192}$$

$$\kappa = \frac{d_l d_{ls}}{d_s}\frac{v_c^2\pi}{J}, \tag{193}$$

$$\gamma = \frac{d_l d_{ls}}{d_s}\frac{\pi v_c^2}{J}. \tag{194}$$

VI. FINAL COMMENTS

Several works on gravitational lensing reach up to the expressions that relate the optical scalars with the curvature components in terms of the tetrad adapted to the motion of the photons; we have here also presented expressions for the bending angle in terms of the curvature components. Furthermore, we have presented above expressions for the optical scalars and deflection angle directly in terms of the matter components of the sources of the gravitational lens, valid for an extended class of matter distributions. In order to do that, one has to assume some structure for the source, so that in this first work on the subject, we have treated the first natural model of spherical symmetry for the sources. But in Sec. III, we have presented expressions that are valid also for spheroidal distributions, since we only required axissymmetry along the line of sight.

Our expressions circumvent several deficiencies: gauge dependence, lack of explicit expressions, neglect of spacelike components of the energy-momentum tensor, etc. It is probably worthwhile to remark that since the function $M(r)$ is determined in terms of the $\varrho(r)$ by Eq. (119), all our expressions are explicit expressions in terms of the energy-momentum components of the matter generating the gravitational lens. As a trivial check of our equations, we have presented two simple examples for which the optical scalars and deflection angle are readily obtained.

The extension of this study to sources with different structure and to the cosmological background will be presented elsewhere.

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APPENDIX A: TETRAD TRANSFORMATIONS

First, we note that

$$\frac{J^2}{r^2}=\frac{J^2}{J^2+y^2}=\frac{J^2}{J^2+r^2\sin(\theta)^2\sin(\phi)^2}, \qquad \text{(A1)}$$

and solving this for J^2/r^2 we find

$$\frac{J^2}{r^2}=1-\sin(\theta)^2\sin(\phi)^2, \qquad \text{(A2)}$$

The complete equations that satisfy the spinorial components are given by

$$A\bar{A}=c_{l\tilde{n}}=\frac{1}{2}(-1+\sin(\theta)\sin(\phi)), \qquad \text{(A3)}$$

$$A\bar{B}=-c_{l\tilde{\bar{m}}}=-\frac{1}{\sqrt{2}}(\cos(\theta)\sin(\phi)-i\cos(\phi)), \qquad \text{(A4)}$$

$$B\bar{B}=c_{l\tilde{l}}=(-1-\sin(\theta)\sin(\phi)), \qquad \text{(A5)}$$

$$C\bar{C}=c_{n\tilde{n}}=\frac{1}{4}(-1-\sin(\theta)\sin(\phi)), \qquad \text{(A6)}$$

$$C\bar{D}=-c_{n\tilde{\bar{m}}}=\frac{1}{2\sqrt{2}}(\cos(\theta)\sin(\phi)-i\cos(\phi)), \qquad \text{(A7)}$$

$$A\bar{C}=c_{m\tilde{n}}=-\frac{1}{2\sqrt{2}}(-\cos(\theta)-i\sin(\theta)\cos(\phi)), \qquad \text{(A8)}$$

$$A\bar{D}=-c_{m\tilde{\bar{m}}}=\frac{1}{2}(\sin(\theta)+\sin(\phi)-i\cos(\theta)\cos(\phi)), \qquad \text{(A9)}$$

$$B\bar{C}=-c_{m\tilde{m}}=\frac{1}{2}(\sin(\theta)-\sin(\phi)-i\cos(\theta)\cos(\phi)), \qquad \text{(A10)}$$

$$B\bar{D}=c_{m\tilde{l}}=-\frac{1}{\sqrt{2}}(\cos(\theta)+i\sin(\theta)\cos(\phi)), \qquad \text{(A11)}$$

$$D\bar{D}=c_{n\tilde{l}}=\frac{1}{2}(-1+\sin(\theta)\sin(\phi)); \qquad \text{(A12)}$$

and its complex conjugates together to the condition

$$AD-BC=1. \qquad \text{(A13)}$$

APPENDIX B: DEFLECTION ANGLE IN TERMS OF T_{ab} FROM GEODESIC EQUATION

The four velocity vector of the particle has modulus

$$e^{2\Phi}\left(\frac{dt}{d\lambda}\right)^2-\frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\varphi}{d\lambda}\right)^2=\kappa; \qquad \text{(B1)}$$

where λ is an affine parameter of the geodesic, and we have already made use of the symmetry that allows us to study just the motion in the equatorial plane $\theta=\frac{\pi}{2}$. The constant κ has values 1 for massive particles and 0 for massless particles. This choice for κ sets the unit for the affine parameter for the massive particle case; however the unit for the massless case remains undetermined.

There are also two integrals of motion. J is a constant of motion associated to the existence of a rotational Killing vector which can be expressed by

$$J=r^2\frac{d\varphi}{d\lambda}; \qquad \text{(B2)}$$

E is another constant of motion associated to the existence of a timelike Killing vector, which can be expressed by

$$E=e^{2\Phi}\frac{dt}{d\lambda}. \qquad \text{(B3)}$$

Then Eq. (B1) takes the form

$$e^{-2\Phi}E^2-\frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\lambda}\right)^2-\frac{J^2}{r^2}=\kappa; \qquad \text{(B4)}$$

or

$$\left(\frac{dr}{d\lambda}\right)^2+\left(\frac{J^2}{r^2}-e^{-2\Phi}E^2\right)\left(1-\frac{2M}{r}\right)=-\kappa(1-\frac{2M}{r}); \qquad \text{(B5)}$$

which can also be expressed as:

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{J^2}{r^2}-\frac{J^2}{r^2}\frac{2M}{r}-\kappa\frac{2M}{r}-E^2e^{-2\Phi}\left(1-\frac{2M}{r}\right)=-\kappa. \qquad \text{(B6)}$$

It is observed that the choice of the affine parameter λ is related to the definitions of the constants of motion J and E. Since Φ tends to zero in the asymptotic region, it is natural to take λ so that $E=1$. This is equivalent to say that in the asymptotic region one has $dt=d\lambda$.

In this way there is no more freedom in the choice of units for J. For an incident photon traveling in the $-y$ direction, with coordinate $x=x_0$, the Newtonian expression for the angular momentum, for a unit mass particle gives $J=rv\sin(\varphi+\frac{\pi}{2})=r\cos(\varphi)=x_0$; that is with this choice of affine parameter, J has the meaning of asymptotic impact parameter x_0.

For convenience in the algebraic manipulation, let us define

$$a_1(\Phi) \equiv 1 - e^{-2\Phi}; \tag{B7}$$

so that $e^{-2\Phi} = 1 - a_1(\Phi)$ in the above equation.

Therefore, for a photon, one can express (B6) by

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - J^2\frac{2M}{r^3} + \frac{2M}{r} + \left(1 - \frac{2M}{r}\right)a_1(\Phi) = 1. \tag{B8}$$

The corresponding potential for the motion of a photon is

$$V_\ell = -J^2\frac{M}{r^3} + \frac{a_1(\Phi)}{2} + \frac{M}{r} - \frac{M}{r}a_1(\Phi); \tag{B9}$$

which, we remark, is an exact expression.

If one considers only linear departures from the flat metric, one whould replace $e^{-2\Phi} \approx (1 - 2\Phi)$; and so one would obtain

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} + 2\left[-\frac{\kappa M}{r} - \frac{J^2M}{r^3} + E^2\left(\frac{M}{r} + \Phi\right)\right] = E^2 - \kappa. \tag{B10}$$

For the massless case, by choosing the parametrization λ so that $\frac{dt}{d\lambda} \to 1$ for $r \to \infty$ one has $E = 1$, and therefore, one defines

$$V_\ell(r) = -\frac{J^2M}{r^3} + \frac{M}{r} + \Phi. \tag{B11}$$

The motion of a photon can be deduced from the Lagrangian

$$L = \frac{1}{2}\left(\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2\right) - V_\ell(r) = \frac{1}{2}\left(\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2\right) - V_\ell(r); \tag{B12}$$

where in the last equality we have used Cartesian like coordinate system with $r = \sqrt{x^2 + y^2}$. This system obviously has the integral of motion (B2) with Lagrangian energy $\mathcal{E} = \frac{E^2}{2}$.

The equations of motion are:

$$\frac{dv_x}{d\lambda} = -\frac{x}{r}\frac{dV_\ell}{dr}, \tag{B13}$$

$$\frac{dv_y}{d\lambda} = -\frac{y}{r}\frac{dV_\ell}{dr}; \tag{B14}$$

with the velocity notation $v_x = \frac{dx}{d\lambda}$ an $dv_y = \frac{dy}{d\lambda}$.

Let us assume the initial conditions: $x = x_0$, $y \gg 2M$, $v_x = 0$ and $v_y = -1$. Then, in this case, x_0 is the impact parameter, so that $J = x_0$.

After passing through the gravitational lens, the trajectory will be deflected so that, in the asymptotic region one would have $v_x|_\infty = \delta v$ and $v_y|_\infty = -\sqrt{1 - (\delta v)^2}$; since the photon must travel at the velocity of light. Then the bending angle can be calculated from

$$\alpha = -\arctan\frac{v_x|_\infty}{v_y|_\infty} \approx -\frac{v_x|_\infty}{v_y|_\infty} \approx -\delta v. \tag{B15}$$

The variation in the velocity can be calculated from

$$\delta v = -\int_{\lambda_o}^{\lambda_f}\frac{x}{r}\frac{dV_\ell}{dr}d\lambda = -\int_{-d_l}^{d_{ls}}\frac{x}{r}\frac{dV_\ell}{dr}dy; \tag{B16}$$

where we have taken $d\lambda = dy$.

The coordinate system has origin at the center of the spherical symmetry. In the approximation of a lens contained in a plane, the center is in this plane.

To consider the equation of motion of a massless particle in the more general case, we also use the equations of motion in the Cartesian like coordinate system, where now the potential is given by (B11); so that

$$\frac{dV_\ell}{dr} = 2\frac{J^2}{r^3}\frac{M}{r} + \left(1 - \frac{J^2}{r^2}\right)\left(\frac{\frac{dM}{dr}}{r} - \frac{M}{r^2}\right) + \frac{d\Phi}{dr} = 3\frac{J^2M}{r^4} - \frac{M}{r^2} + \left(1 - \frac{J^2}{r^2}\right)4\pi r\rho + \frac{M + 4\pi r^3P_r}{r^2(1 - \frac{2M}{r})}; \tag{B17}$$

and in the linear regime one has

$$-\frac{dV_\ell}{dr} = \frac{3J^2}{r}\left(-\frac{M(r)}{r^3} + \frac{4\pi}{3}\rho(r)\right) - 4\pi r(\rho(r) + P_r(r)). \tag{B18}$$

Finally the deflection angle is given by

$$\alpha = -\delta v = \int_{-d_l}^{d_{ls}}\frac{x_0}{r}\frac{dV_\ell}{dr}dy = \int_{-d_l}^{d_{ls}}\frac{x_0}{r}\left[-\frac{3J^2}{r}\left(-\frac{M(r)}{r^3} + \frac{4\pi}{3}\rho(r)\right) + 4\pi r(\rho(r) + P_r(r))\right]dy; \tag{B19}$$

where it is understood that $r = \sqrt{x_0^2 + y^2}$.

This coincide with expression (180) appearing above.

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