# Entropy, confinement, and chiral symmetry breaking

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This paper studies the way in which confinement leads to chiral symmetry breaking (CSB) through a gap equation. We argue that a combination of entropic effects, related to fluctuations of Wilson loops with massless constituents, and an Abelian gauge invariance of the confinement action as expressed in terms of the usual confining effective propagator $8\pi K_F \delta_{\mu\nu}/k^4$, in effect removes infrared singularities coming from use of this propagator in a standard gap equation ($K_F$ is the string tension). Beginning from an Abelian gauge-invariant description of CSB that differs from this standard gap equation, we show how to extract a corresponding gap equation that incorporates both entropic effects and Abelian gauge invariance by replacement of the confining propagator with $8\pi K_F \delta_{\mu\nu}/(k^2 + m^2)^2$. Here the finite mass $m$ turns out to be $\approx M(0)$ [$M(p^2)$ is the running quark mass], based on an extension of an old calculation of the author. This massive propagator gives semiquantitatively two critical properties of confinement: (1) a negative contribution to the confining potential coming from entropy; (2) an infrared cutoff required by Abelian gauge invariance. Entropic effects lead to a $\bar{q}q$ condensate and contribute a negative term $\sim -K_F/M(0)$, essential for a massless pion, to the pion Hamiltonian. The resulting gap equation leads to $M^2(0) \approx K_F/\pi$. We argue that one-gluon exchange is not strong enough in the IR to drive quark CSB, but in any case is necessary to get the correct renormalization-group ultraviolet behavior. We find the standard renormalization-group result with the improvement that the prefactor (related to $\langle\bar{q}q\rangle$) can be calculated from the confining solution. Finally, we briefly point out the Minkowski-space virtues of using a principal-part propagator to describe confinement.

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## I. INTRODUCTION

The usual picture of chiral symmetry breaking (CSB) in QCD is a beautiful and correct one: An isoscalar excitation receives a vacuum expectation value, and is accompanied by massless Goldstone pions leading to a running constituent quark mass $M(p^2)$, with $M(0) \neq 0$, and expectation values such as $\langle\bar{q}q\rangle$. However, from the viewpoint of QCD there are still several obscure points, in spite of decades of work. In this paper we concentrate on the following questions, for the simplest case of zero temperature and density, the only case we consider here (for finite temperature and density, see [1] and references therein):

(1) Is CSB produced by simple one-gluon exchange, as in the old Johnson-Baker-Willey (JBW) [2] gap equation and its QCD variants, or is it produced by confinement?

(2) If, as we argue here, confinement is not only sufficient for CSB (see, e.g., [3,4]) but also necessary, how does one bypass, in the gap equation, the usual infrared singularities of confining forces? How does one enforce an Abelian gauge invariance, stemming from a standard description of confinement via an effective vectorial propagator, in the gap equation?

(3) A further confinement issue: If one-gluon exchange is too weak for CSB and confinement is necessary, the big terms in the pion Hamiltonian—the kinetic energy and the confining term—are positive. Where is the negative term that can yield a massless pion?

(4) How does one connect the infrared-dominated confinement dynamics to ultraviolet-dominated renormalization-group (RG) behavior, as correctly but incompletely described by one-gluon exchange?

Many papers have been written on one or another of these questions, but the present approach differs from those known to the author. There are a number of papers [3,5–10] that make some attempt to model area-law confinement as it might arise in QCD (as opposed to purely phenomenological effective propagators, NJL models, and so on); all of them make one approximation or another, and ours is no exception. This paper differs from most of the cited papers by attempting to maintain covariance and avoiding the use of special gauges, such as Coulomb gauge, within the context of a Euclidean gap equation with confining forces.

Our answers to the questions we have asked depend on two major points, one having to do with entropy and the other with taming infrared (IR) singularities in the confining potential, as related to a certain Abelian gauge invariance of the effective vectorlike propagator that we use to describe confinement. [This has nothing to do with the real gluon propagator, and the gauge invariance has nothing to do with color-$SU(3)$ gauge invariance.] The first point is that, in the Wilson loops needed for equations describing CSB, these loops describe small- or zero-mass excitations. As a consequence the loops have quite substantial space-time fluctuations, or in other words, large entropy. In these

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Wilson loops large transverse separation of the quark and antiquark is highly improbable because of the consequent area-law action penalty; the loops are convoluted and ramified, looking more like a bush of lines than a well-defined loop. The entropy is a negative contribution to the effective action of the loops, which is the basis for the answer to question 3.

The second point is that the effective action describing confinement is an (approximate) area-law action, written in terms of integrals over the closed contours of participating Wilson loops. Because the loops are closed, there is an Abelian gauge invariance; because of the Abelian gauge invariance, we show that there is no IR singularity from confining forces in amplitudes formed from closed Wilson loops. The cancelling of would-be IR singularities is another way of saying that large $q\bar{q}$ separations are improbable. However, a JBW-like gap equation contains only an open quark line, and is generally both Abelian gauge-dependent and singular in the IR. We show that one can nevertheless construct an effectively gauge-invariant gap equation of JBW form, with a special effective propagator that both tames IR singularities and incorporates entropic effects, that reflects the physics of a related closed-loop amplitude involving both the usual quark and a heavy quark. The static potential $V(r)$ of this confining propagator is negative for all $r$, has a negative term at $r = 0$ reflecting entropic effects, plus a linearly rising term $K_F r$ for $r \leq M(0)^{-1}$ that flattens out at larger distances. This large-distance weakening of the linearly rising potential is irrelevant, since larger $q\bar{q}$ separations are so improbable.

To understand these points we begin with an exposition of the general structure of the quark gap equation to be studied in this paper.

### A. The gap equation

This gap equation is an approximate and simplified form of the standard Schwinger-Dyson equation for the Dirac trace of the inverse quark propagator, as shown pictorially in Fig. 1. It was first introduced in QED [2] and later applied many times to CSB in QCD. Although simplified, it is a suitable arena in which to address all the issues raised in our list of questions. One important aspect of such a gap equation, as shown long ago [11], is that a CSB-violating solution to the gap equation implies a solution to the homogeneous Bethe-Salpeter equation for a massless pion; this equation at zero pion momentum is essentially the same as the gap equation, except for a crucial change of sign because the pion amplitude has a $\gamma_5$ factor. However, this relationship between CSB and a massless pion does not reveal some interesting differences. The pion Hamiltonian $H_\pi$ needs, as already mentioned, a negative term to compensate the positive kinetic energy and linearly rising potential. But the Euclidean quark gap equation shows no sign of this negative term.

FIG. 1. The JBW equation. The line with a cross corresponds to the $B(p^2)$ term in the inverse propagator $S^{-1}(p^2) = \not{p}A(p^2) + B(p^2)$. See text for details.

We write the Euclidean inverse propagator for a quark with zero bare mass as

$$S^{-1}(p) = \not{p}A(p^2) + B(p^2). \tag{1}$$

The first approximation, commonly used with gap equations, is to set $A(p^2) = 1$ and then to rename $B$ as $M(p^2)$, the running quark mass. If there is no CSB then $M(p^2) \equiv 0$. CSB corresponds to positive solutions, necessarily vanishing at large $p$, with $M(0) > 0$. The generic JBW equation that we will use, corresponding to Fig. 1, is

$$M(p^2) = \frac{1}{(2\pi)^4} \int d^4k \gamma_\mu D^{\text{eff}}_{\mu\nu}(p-k)\gamma_\nu \frac{M(k^2)}{k^2 + M^2(k^2)}, \tag{2}$$

where $D^{\text{eff}}_{\mu\nu}(p-k)$ can be either a massive-gluon propagator, multiplied by an appropriate running charge, or an effective propagator for confinement as described in Sec. IV, or a sum of these (Sec. VI).

As we will soon see this gap equation is both IR singular and does not have Abelian gauge invariance, for standard choices of the effective propagator in it. Our goal is to find a particular $D^{\text{eff}}_{\mu\nu}(p-k)$ such that gauge-invariant physics is properly represented in the JBW-like gap equation above; our technique will be to find an appropriate gap equation imbedded in a gauge-invariant amplitude that, precisely because of gauge invariance, has no IR singularities.

### B. An effective propagator approximating an area law

A standard choice for an effective confining propagator is

$$D^{\text{eff}}_{\mu\nu}(k) = 8\pi K_F \delta_{\mu\nu} \frac{1}{k^4}, \tag{3}$$

with $K_F$ the string tension. Its space-time Fourier transform needs IR regulation. We introduce an infinitesimal mass $\xi$ to define this Fourier transform:

$$D^{\text{eff}}_{\mu\nu}(x) = \frac{8\pi K_F \delta_{\mu\nu}}{(2\pi)^4} \int d^4k \frac{e^{ik\cdot x}}{(k^2+\xi^2)^2} = \delta_{\mu\nu} \frac{K_F \ln(\xi^2 x^2)}{2\pi} + \ldots, \tag{4}$$

where the omitted terms are irrelevant in the limit $\xi \to 0$. The significance of this particular choice is that the effective quark action from this propagator is

$$I = \frac{K_F}{4\pi} \oint dx_\mu \oint dy_\nu \delta_{\mu\nu} \ln[\xi^2(x-y)^2], \tag{5}$$

where the integral is over a (sum of) closed quark world lines. Consider the case of a single quark self-energy, so that the two contours are the same. If this contour is flat (i.e., two-dimensional) then the integral

$$\frac{1}{4\pi}\oint dx_\mu \oint dy_\nu \delta_{\mu\nu} \ln[\xi^2(x-y)^2] \tag{6}$$

is precisely the area $A$ of the flat surface bounded by the contour, as one sees by integrating by parts twice and using $\Box \ln x^2 = 4\pi\delta(x)$ in two dimensions. The action for such a flat contour, then, is the expected area-law action $K_F A$. It is not exactly the area for a nonflat contour, but it has the properties of confinement needed for our discussion. (See further discussion in [12].) In any dimension it is independent of the cutoff $\xi$, because the action has Abelian gauge invariance, and a change of $\xi$ is a gauge transformation.

The propagator of Eq. (4) is perfectly valid for use in the closed-loop equation of Sec. III, in which the IR singularities all cancel. We will see applications of this cancellation in Sec. I C immediately below, and in Sec. III. But our goal is to construct a JBW-like gap equation with open lines, which will require another effective propagator.

### C. Abelian gauge invariance, its violation, and infrared singularities

The Abelian gauge invariance associated with the effective propagator means that in coordinate space any terms $\sim\partial/\partial x_\mu \partial/\partial y_\nu \Lambda(x-y)$ can be added to the propagator without changing the action, because the integral over a closed loop of such a derivative term vanishes. A change of regulator mass $\xi$ has $\Lambda \sim (x-y)^2$. Unfortunately, the gap equation Eq. (2), interpreted naively, does not have this Abelian gauge invariance; it certainly changes if $(p-k)_\mu(p-k)_\nu$ terms are added. Not only that, but in a general covariant gauge it has IR singularities no matter what the behavior of $M(p^2)$ is for small momentum.

There are (at least) two ways to use this Abelian gauge invariance and to get rid of the IR singularities. One is to deal only with quantities relevant to closed loops; an example is the pion wave function and related Hamiltonian $H_\pi$ (for the sake of brevity we speak in nonrelativistic terms). We begin with the static potential found from the regulated effective propagator, with the result:

$$V(r) = \int_{-\infty}^{\infty} dt D^{\rm eff}_{00}(t,r) = -\frac{K_F}{\xi} + K_F r + \mathcal{O}(\xi r). \tag{7}$$

This needs regulation because the infinite integral over time is equivalent to an open quark world line. As we will review, and extend, in Sec. III it was shown long ago [3] there is a cancelling divergence in the one-loop quark self-mass $M$ such that $2M + V(r)$ has no divergences and was claimed to be just the linearly rising potential $K_F r$. However, it was not noted at the time, as we do note here, that there is a finite term in the quark self-mass, such that the sum is actually

$$2M + V(r) = K_F r - \frac{3K_F}{\pi M}. \tag{8}$$

That this new term is negative is crucial in generating a zero-mass pion with confining effects. We interpret the negative sign as a hallmark of entropic effects, as discussed in Sec. II.

As an aside, the effective quark Hamiltonian $H_q$ that is more or less equivalent to the gap equation has no negative term; it looks something like

$$H_q = \frac{1}{r} + K_F r, \qquad M = \langle H_q \rangle. \tag{9}$$

Roughly speaking, the negative entropic term tends to be cancelled by the quark mass $M$ that should be part of $H_q$ whose eigenvalue is $M$. We minimize on $r$ to find that the two terms in $H_q$ are equal, so that $M \sim K_F \ell$ where $\ell$ is the minimizing value of $r$. Later we will interpret $\ell$ as the scale length for entropic fluctuations.

The second way to reinstate Abelian gauge invariance is close in spirit to the pinch technique (PT) [13,14], which is used to extract gauge-invariant off-shell Green's functions (such as a quark propagator) from some gauge-invariant quantity such as the $S$ matrix. Similarly, with the aid of a fictitious heavy quark called $\chi$, with mass $M_\chi^2 \gg K_F$, it is possible to construct a singularity-free Abelian gauge-invariant dynamics for quark CSB, as detailed in Sec. III from the (integrand of) two-loop graphs with one confining propagator [Eq. (3)] for the Green's function $G_{\chi q} = \int d^4x \exp[iv\cdot x]\langle|T[\bar{\chi}q(x)\bar{q}\chi(0)]|\rangle$. This Green's function is gauge invariant, which as we will see, using techniques given in [3,15], means that there are no IR singularities. Even though the sum of these closed-loop graphs is IR-finite, the goal is to find a JBW-like gap equation that is also IR-finite. This will require using an effective propagator in the JBW-like gap equation that is cut off with a physical mass $m$ that is not to be sent to zero:

$$D^{\rm eff}_{\mu\nu}(k) \to 8\pi K_F \delta_{\mu\nu}/(k^2+m^2)^2, \tag{10}$$

with $m \sim M$. Using a cutoff propagator in the gap equation gives very nearly the same physics that would come from using the standard confining propagator in the closed-loop Green's function $G_{\chi q}$.

There are actually two distinct physical effects embodied in the finite mass $m$. The first is to mimic the cutoff coming from gauge invariance in a closed-loop equation. The second is that there is a negative term in the static potential at $r = 0$ that we identify in Sec. II with entropic effects.

The bottom line is that using the massive effective propagator of Eq. (10) in the standard gap equation of Eq. (2) both represents entropic effects and the cutoff inherited from a closed-loop gap equation. This gap equation is solved both semianalytically and numerically in Sec. V, which yields a running quark mass $M(p^2)$ with

$M^2(0) \approx K_F/\pi$ and vanishing in the UV as $1/p^4$. This UV behavior is wrong; it is corrected in Sec. VI by adding one-gluon exchange (see, e.g., [16]) to the confining gap equation. In the UV one finds not only the correct RG behavior $M(p^2) \sim (\ln p^2)^a/p^2$ (here $a$ is a constant given in [16]) but also a specific numerical coefficient determined by confining effects. (The RG by itself cannot determine this constant.) The constant is, as shown in [17], equivalent to knowledge of the condensate $\langle \bar{q}q \rangle$, but we will not pursue here an accurate determination of this condensate value.

Of course, in the UV the gluon propagator is massless. But for dealing with the IR effects of gluons we now know [13,14,18] that the QCD gluon has a dynamical mass that tames all IR singularities both in the gap equation and in the QCD running charge. In the IR massive one-gluon exchange gives a small ($\sim 10\%$) increase in $M(0)$. That $M(0)$ increases is consistent with the old idea that gluon exchange, if strong enough, could give rise to CSB by itself; that the increase is not so large suggests that perhaps it is not strong enough. We argue in Sec. IV that one-gluon exchange, interpreted in light of present-day knowledge of the massive-gluon propagator and running charge in the IR, is not strong enough by itself to give CSB for quarks, but almost certainly is strong enough to give it for adjoint fermions, coupled more strongly in QCD by a factor 9/4 than quarks are; the argument, based on [19], is given in Sec. IV. Lattice simulations confirm [20] that adjoint fermions undergo CSB without confinement, and thus presumably the mechanism is gluon exchange. Other authors claim that one-gluon exchange can drive CSB for quarks (see, for example, a very recent study of the full Schwinger-Dyson equation for the Landau-gauge quark propagator [21]), but just barely. It is likely that the question of whether or not one-gluon exchange can drive CSB will be resolved by careful studies of how CSB behaves near the deconfinement transition temperature, which is known from lattice studies to be close to the CSB-restoration temperature [22]. [See, however, a recent preprint [23] claiming that in $SU(3)$ removal of confining center vortices does not restore chiral symmetry, as it is known to do for $SU(2)$ [4].]

### D. A Minkowski-space formulation

Throughout this paper we use a Euclidean formulation of CSB. It is, of course, of interest to know how to understand CSB in Minkowski space. This is not completely obvious, since one runs into issues such as continuing $\theta(x^2)$ from Minkoski space to Euclidean space. We will not explore this in any detail here, but simply point out that it is inappropriate to use a Feynman propagator in the area-law action $I$ of Eq. (5). This action must be real or it is hard to see how to interpret it as referring to a geometrical area. We suggest using instead the principal-part propagator, used long ago by Wheeler and Feynman in a version of QED. Not only is this real, but it is finite at $x_\mu = 0$, unlike the Feynman propagator that has a logarithmic singularity. This feature cures certain singularities in the gap equation in somewhat the same way that our Euclidean cutoff $m$ does. A more complete discussion of the principal-part propagator approach will be given separately.

## II. ENTROPY

We argue that entropic effects (embodied in large space-time fluctuations in worldlines of pions that are composites of massless quarks) may well be a major source of negative terms necessary for a massless pion with area-law confinement (see [3] for an early description of entropic contributions). These entropic effects come from the masslessness of the quarks and of the (Goldstone) pion, with the consequence that a pionic $q\bar{q}$ Wilson loop with a large longitudinal separation between initial and final points is highly ramified or branched. Anywhere along the perimeter, the large transverse separation of $q$ from $\bar{q}$ is exponentially disfavored because of the consequent large area-law action penalty; separation of more than about $M(0)^{-1}$ practically never occurs. A linearly rising potential is irrelevant for larger separations. Because the long, twisting, and narrow branches are made of massless constituents having a massless bound state, their action per unit length is too small to overcome entropy. These branches signal the formation of a $\langle \bar{q}q \rangle$ condensate. (Formation of the condensate as branches of a $q\bar{q}$ Wilson loop may have something to do with the light-cone interpretation [24] of condensates as objects localized with respect to hadrons; we take no position on this possibility.) On general grounds we show that the entropic effects should contribute a term $\sim -K_F/M(0)$ to the pion mass, and that if only kinetic energy and linearly rising potential terms are kept, masslessness of the pion is assured if $M(0)$ has a specific value $\sim K_F^{1/2}$. That this entropic term is negative is crucial, since other negative terms, such as one-gluon exchange or hyperfine structure, may not be large enough to give the pion a zero mass (or in other words, to yield CSB). (We have nothing new to say about the rho meson, whose mass we attribute in the standard way to chromomagnetic hyperfine splitting.)

It is rather easy to understand the basic properties of area-law dynamics for heavy quarks $\chi$, where heavy means that the quark mass $M_\chi$ obeys $M_\chi^2 \gg K_F$. (We assume these quarks are quenched and that there are no other matter fields.) Such quarks move in essentially classical paths, nearly straight lines, even for times $T$ that can be large compared to the QCD time scale, so that the spatial separation $R$ of a $\chi\bar{\chi}$ pair can be specified in advance with small changes coming from the quark dynamics. An area law simply means that the expectation value of the $R \times T$ Wilson loop describing this configuration is about $\exp[-K_F RT]$. There is no question of a condensate of these quarks, since their paths (in Minkowski space) have essentially no backward moving segments, necessary for a

$\langle\bar{\chi}\chi\rangle$ condensate. Moreover, the entropy of the flux sheets confining these quarks is not large compared to the action (or else the 't Hooft criterion would be violated, yielding both confinement and dual confinement).

For light quarks $q$, with (current) mass obeying $M_q^2 \ll K_F$, things can be very different. (Again assume the quenched case, with no other matter fields; string breaking is impossible.) Suppose we fix the time $T$ between the initial and final $q\bar{q}$ configurations, each for simplicity taken to be at the spatial origin. What kind of paths occur in the path integral for a quantity such as $\langle T[\bar{q}\gamma_5 q(0)\bar{q}\gamma_5 q(T,\vec{0})]\rangle$, and what kinds of areas do these paths have?

The answer is that the paths are highly ramified (branched), as shown in Fig. 2. (For clarity we do not show the fine-scale movement of the quarks about each other, which is sketched in Fig. 3.) We can no longer specify the average spatial separation, now called $\ell$, of these light quarks, which must be calculated. Recall from the discussion around Eq. (9) that $\ell \sim M/K_F$ (see also [12] for further discussion and references). An average Wilson loop resembles a highly branched polymer, with most of the branches representing pions with mass proportional to $M_q$. Note that going backwards in time is not at all hindered, so that there is no bar to forming a nonzero value of $\langle\bar{q}q\rangle$. A typical configuration will have an overall length

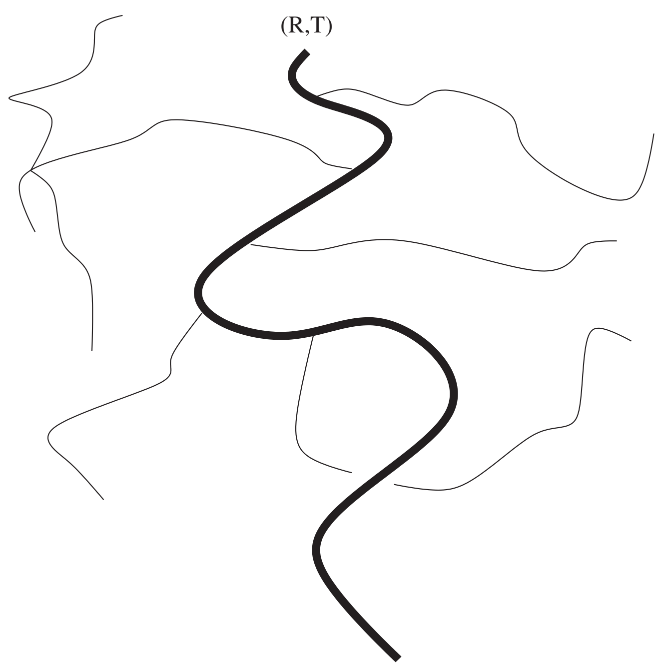


FIG. 2. A schematic of a Wilson loop for a pion characteristic of light-quark dynamics; the thick line symbolizes the original $q\bar{q}$ loop, and the thin lines symbolize pions. Because the loops are narrow we do not show their individual $q$ and $\bar{q}$ lines. A $\gamma_5$ is understood at each end point and every three-vertex.

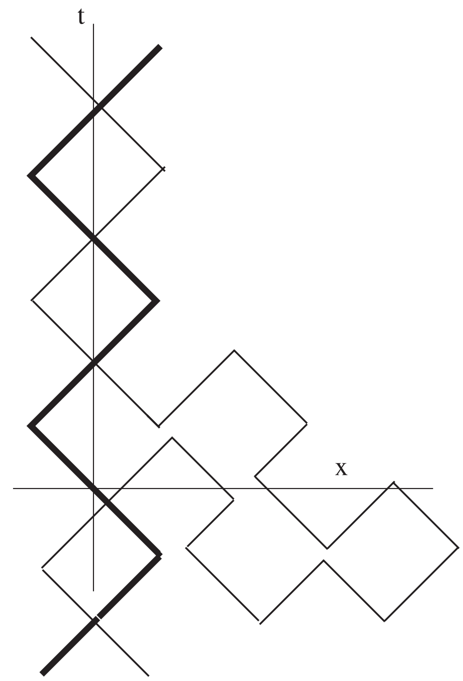


FIG. 3. A schematic closeup of the $q\bar{q}$ Wilson loop in Minkowski space, showing the propagation of massless particles along their light cones under the influence of a confining force (after [3]). Very massive quarks would only propagate in the forward timelike direction, but light quarks can have spacelike legs, signifying the formation of a condensate. The length scale in this figure is the entropy scale length $\ell$.

$L \sim T^2/\ell$, but any particular branch will have a separation between the $q$ line and the $\bar{q}$ line of $\mathcal{O}(\ell)$. Here the correlation length $\ell$ is related to the CSB-generated constituent quark mass $M$ by $K_F\ell \simeq M$ (see, for example, an elementary discussion in [12]). So the area of the Wilson-loop is $\mathcal{O}(L\ell)$. In terms of the overall length $L$, this looks like a perimeter law, although the actual area can be large because of ramification.

The upshot of this discussion is that when $K_F \gg M_q^2$, pionic Wilson-loop configurations that look sheet-like (that is, for some $q\bar{q}$ separation $R$ that is large compared to $K_F^{-1/2}$) are highly suppressed. This is because the available configurational entropy of the flux sheets cannot overcome the action penalty in the exponentially small area-law contribution to the Wilson-loop vacuum expectation value. But because the action penalty for forming pions from light quarks is small, entropy can dominate. Roughly speaking, the entropic term to be added to the area-law term gives a result of the form:

$$\langle W\rangle \sim \exp\left[-K_F T^2 + \frac{T^2}{\ell^2}\ln(2d-1)\right] \qquad (11)$$

in space-time dimension $d$. Since the overall length $L$ scales as $T^2/\ell$ this can also be written:

$$\langle W \rangle \sim \exp\left[-K_F L\ell + \frac{L}{\ell}\ln(2d-1)\right]. \quad (12)$$

(We use a standard approximation for the entropy, as counting the ways links of length $\ell$ can extend themselves on a hypercubic lattice with no backtracking. There are other terms in the action coming from quark-pion vertex effects, and other terms in the entropy coming from counting the ways the Wilson loop is ramified; we do not discuss them here.) With $\ell \sim M/K_F$ the entropic term contributes a term $\sim -K_F/M$ to the action density, or energy. The Wilson loop will ramify until the two terms are approaching equality, at which point other physical effects take over.

Even though rho mesons (for example) are made of virtually massless quarks, they are heavy in part because of QCD hyperfine interactions to the point that their fluctuation entropy cannot overcome their action. A standard estimate of hyperfine splitting is

$$M_\rho - M_\pi \approx \frac{32\pi\alpha_s(0)}{9M^2}|\psi(0)|^2, \quad (13)$$

which is about 700 MeV for parameters that we use in this paper.

## III. ABELIAN GAUGE INVARIANCE, INFRARED CUTOFFS, AND ENTROPY

### A. Extracting a JBW gap equation from a gauge-invariant Green's function

Here is a sketch of how to extract, from a closed-loop gauge-invariant amplitude, a JBW gap equation that is nonsingular and that has all the physical effects (entropy, IR cutoff) represented with reasonable quantitative accuracy. This extraction is in somewhat the spirit of the pinch technique [13,14], in which (color-) gauge-invariant Green's functions are extracted from some gauge-invariant object such as the $S$ matrix. The PT is used to enforce gauge invariance in off-shell Green's functions for non-Abelian gauge theories. After many a long calculation the pinch technique turns out to be entirely equivalent to calculating the off-shell Green's functions in the background field Feynman gauge. We use the fictitious heavy quark $\chi$ specifying that it is a chiral singlet, coupled to ordinary quarks by an interaction such as $\lambda[\bar{\chi}(S + i\gamma_5 P)q + \text{H.c.}]$, where the fictitious color-singlet scalar $S$ and pseudoscalar $P$ transform inversely to $q$ under chiral transformations $[(S + i\gamma_5 P) \to \exp(-i\alpha\gamma_5) \times (S + i\gamma_5 P)]$. With the help of these fictitious particles we construct Green's functions involving only closed fermion loops, hence showing Abelian gauge invariance.

Consider a gauge-invariant heavy-light Green's function such as the $S$ proper self-energy (irrelevant factors omitted):

$$G_{\chi q}(v) = \langle 0| \int d^4x e^{iv\cdot x} T(\bar{\chi}q(0)\bar{q}\chi(x))|0\rangle. \quad (14)$$

The quark gap equation will be extracted from equating integrands in the expression for $G_{\chi q}$ having no confining propagator to that with one confining propagator. Of course, to equate integrands of equal integrals is far from unique, but we find a gap equation whose satisfaction assures the satisfaction of CSB as described with Abelian gauge invariance. The one-loop graph $G_{\chi q,1}$ (with no confining propagator) has the form:

$$G_{\chi q,1} = \int d^4p\, \mathrm{Tr} S_q(p) S_\chi(v+p), \quad (15)$$

where $S_{\chi,q}$ is, respectively, the $\chi$, $q$ propagator. The gauge-invariant description of CSB is simply to equate the integrand of $G_{\chi q,1}$ at one loop to the integrand of the same Green's function $G_{\chi q,2}$ at two loops, as shown in Fig. 4, at the same time going to the limit of large $M_\chi$. As appropriate for an effective propagator, we keep only graphs with a single confining propagator line. Because of Abelian gauge invariance this can be the usual massless confining propagator of Eq. (3). In both expressions for $G_{\chi q}$ we wish to isolate the term linear in $M(p^2)$, the quark running mass, in the quark propagator $S_q(p)$ in the integrand of the $p$ integration, in the limit of heavy $\chi$ mass $M_\chi$. We write this amplitude as (up to an irrelevant factor)

$$G_{\chi q,2}(v) = \int d^4p \int d^4k \frac{T_{\mu\nu}(p,k,v)8\pi K_F[\delta_{\mu\nu} + \xi k_\mu k_\nu]}{k^4}, \quad (16)$$

where $T_{\mu\nu}(p,k,v)$ is the sum of graphs in Fig. 4 with the gluon lines (one of momentum $k$ and the other of momentum $-k$) opened. The terms $\sim k_\mu k_\nu$ come from a choice of gauge. One can show either by direct calculation of graphs or by using the techniques of [15] that $T_{\mu\nu}$ is conserved and so the gauge terms must all cancel: $k_\mu T_{\mu\nu} = 0$. (It is helpful but not essential to set $v = 0$ in the graphical demonstration.) A mass gap, which is implied for the light quark because we seek CSB, plus conservation imply that $T_{\mu\nu}$ must vanish at least quadratically at $k = 0$, and so there is no IR singularity in the sum of graphs. After Dirac-matrix traces, a possible form for $T_{\mu\nu}$ is

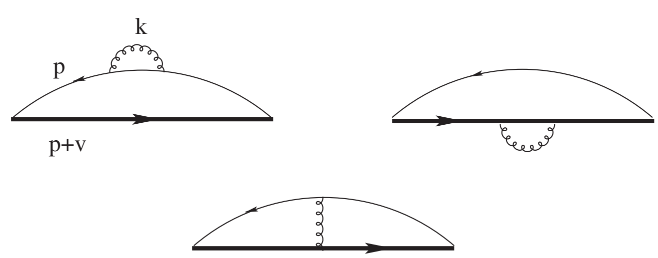


FIG. 4. Two-loop graphs whose sum has Abelian gauge invariance for the confining propagator. The thick line is the $\chi$ field; the thin line is a quark of momentum $p$; the curly line is the area-law propagator of momentum $k$.

$$T_{\mu\nu}(p,k,q) = [k^2\delta_{\mu\nu} - k_\mu k_\nu]T_1 + [k^2 p_\mu p_\nu - k\cdot p(k_\mu p_\nu + k_\nu p_\mu) + \delta_{\mu\nu}(p\cdot k)^2]T_2 + \ldots. \quad (17)$$

The omitted terms may have $v_\alpha$ factors in the numerator, but these are irrelevant. A good way to ignore them is to evaluate $G_{\chi q,2}(v)$ at zero momentum.

Given the decomposition of Eq. (17) it is straightforward but lengthy to isolate the terms not dependent on the $\chi$ field in the scalars $T_i$ in the heavy $M_\chi$ limit. (We do not go into the details of the calculation, which is roughly equivalent to evaluating the two-loop photon propagator in QED.) The result of dropping $\chi$-related terms is a gap equation that is rather similar to the original one of Eq. (2), except that there is a factor quadratic in $k$, as in the decomposition of Eq. (17), not present in the original JBW gap equation that would come from combining Eqs. (2) and (3). There should be a compensating quadratic term in $k$ in a denominator, so that the UV behavior in $k$ is unchanged. The simplest form for such a factor, after taking angular averages, is to multiply the confining propagator by $k^2/(k^2+m_0^2)$, where $m_0 \sim M(0)$. The result is a new propagator, called $D^{\text{eff},1}_{\mu\nu}(k)$, that gives a nonsingular gap equation:

$$D^{\text{eff},1}_{\mu\nu}(k) \to 8\pi K_F \delta_{\mu\nu}\frac{1}{k^2(k^2+m_0^2)}. \quad (18)$$

Purely to simplify certain technical details, explained below, we choose instead the massive propagator of Eq. (10) in the new gap equation. The technical reason for preferring this propagator is that we can construct certain features of the gap equation using this propagator from that of a massive-gluon propagator (which we invoke in Sec. IV below) by differentiation with respect to $m$. For convenience we repeat the confining propagator of Eq. (10):

$$D^{\text{eff}}_{\mu\nu}(k) = 8\pi K_F \delta_{\mu\nu}/(k^2+m^2)^2. \quad (19)$$

By setting $m_0 = 2m$ one finds that the static potentials associated with Eqs. (18) and (19) are very nearly equal for relatively small $mr$, with the first two terms in a power series expansion being exactly equal. The static potential for the propagator in Eq. (18) is

$$V^{(1)}(r) = \frac{-K_F}{2m^2 r}(1-e^{-2mr}) \approx \frac{-K_F}{m} + K_F r + \ldots, \quad (20)$$

and for the propagator of Eq. (19) it is

$$V(r) = \frac{-K_F}{m}e^{-mr} \approx \frac{-K_F}{m} + K_F r + \ldots. \quad (21)$$

At any distance, including $mr \gg 1$, the two potentials differ by less than 10% of $K_F/m$. Note that without regard to any of the detailed considerations of this section one could invoke propagators and potentials of this general class simply on the grounds that they yield the entropic effects of Sec. II, as exemplified in the negative terms, while respecting the arguments there that the $q\bar{q}$ separation is limited. Of course the second term is the standard linearly rising potential. The $r$ dependence at distances $\geq 1/m$ should not matter much because separation of the $q$ and the $\bar{q}$ beyond this distance is improbable.

### B. An estimate of the entropic term

We will estimate $m$ from the negative term found in a simple one-loop calculation of the quark self-mass with the regulated effective propagator first introduced as describing confinement, in Eq. (4). The result was already stated in Eq. (8). Evaluate this one-loop self-mass from the integral:

$$M(\not{p} = iM) = \frac{8\pi K_F}{(2\pi)^4}\int d^4k \frac{[4M - 2i(\not{p}-\not{k}]}{(k^2+\mu^2)^2[(p-k)^2+M^2]}\Bigg|_{\not{p}=iM} \to \frac{K_F}{2\mu} - \frac{3K_F}{2\pi M} + \mathcal{O}(\mu). \quad (22)$$

In the sum $2M + V(r)$ we equate the negative term to the negative term at $r = 0$ in the static potential of Eq. (21) and find

$$m = \frac{\pi M}{3} \approx M. \quad (23)$$

In effect, although the $-K_F/m$ term in $V(r)$ cancels, it reappears in finite form as $-3K_F/(\pi M)$. At first it may seem odd that the mass operator gives a negative term to the sum $2M + V(r)$, but something must do so, or it will be impossible to find a zero-mass pion (in the present crude approximation). It might also seem odd that entropic effects are found in Feynman graphs, but this is a standard feature of the description of polymer condensates in condensed matter physics (see [25] and earlier references therein), in which the configurational entropy of the polymers becomes a negative (mass)$^2$ term in a conventional scalar field theory.

### C. How can the pion be massless with confinement?

We can now propose an answer to one of the questions raised in the introduction: How can a pion be massless with confining forces? [For purposes of the following heuristic discussion, we assume that one-gluon effects are too weak to produce CSB by themselves, and omit writing them.] Look at the relativistic pseudo-Hamiltonian

$$H_\pi = p + 2M + V(r) \to p + K_F r - \frac{3K_F}{\pi M}. \quad (24)$$

Substitute $p \to 1/r$ and minimize on $r$ to find a variational approximation

$$\langle H_\pi \rangle = 2K_F^{1/2} - \frac{3K_F}{\pi M}. \quad (25)$$

There is a zero-mass bound state when $M = 3K_F^{1/2}/(2\pi)$, or about 220 MeV. Of course the estimate coming from Eq. (25) is only qualitative, and in the real world there are other negative terms to be included, including gluon exchange and hyperfine structure, but these are not the dominant negative contributions. (If they were, the one-gluon JBW equation would have yielded CSB, but it is perhaps plausible that it does not.)

The approximation $m \approx M$ is not necessarily highly accurate, so in the Euclidean confining gap equation using the massive effective propagator of Eq. (36) we set $m = \alpha M$ for a range of $\alpha \approx 1$. But first, both to set the stage for a Euclidean phenomenology of confinement CSB and to illustrate the argument that one-gluon exchange may not be enough for quark CSB, we briefly review the arguments of [19] concerning one-gluon exchange.

## IV. GLUON EXCHANGE IN THE JBW EQUATION

In addition to these studies of confinement we briefly explore the usual Euclidean JBW equation for one-gluon exchange, but with a massive-gluon [19]. One reason to do this is to set the stage for techniques used in the confining JBW equation. Another is that within the general framework of the gap equation we use for confinement, but with a one-gluon propagator and running charge rather than the effective confining propagator, it appears that one-dressed-gluon exchange does not yield CSB for quarks [19], but would do so for adjoint fermions (coupled more strongly by a factor of 9/4 in QCD), as shown in lattice simulations [20]. This one-gluon result is based on the nonperturbative generation of a dynamical gluon mass [13,14] $m_g$ of about $2\Lambda$ where $\Lambda \approx 300$ MeV is the QCD mass scale. The main effect of the dynamical mass is to reduce the zero-momentum strong coupling $\alpha_s(0)$ to about 0.4–0.5 (with no quarks). The conclusion that one-gluon CSB does not give CSB is far from unassailable, because of approximations in the gap equation itself and possible inaccuracies in the claimed dynamical mass and running charge. Indeed, a preprint that came out as this paper was being written up [21], based on an extensive and complicated study of the gap equation in the Landau gauge and using Landau-gauge form factors from lattice simulations as input, claims that inclusion of rather subtle ghost effects does lead to gluonic CSB without confinement. As these authors acknowledge, the CSB mechanism they find is not very strong, and they also confirm that CSB is absent within the approximations of Ref. [19], which has no explicit quark-gluon vertex corrections or ghost contributions.

In our opinion the fate of gluonic CSB remains to be determined. There are lattice simulations [4] claiming that in $SU(2)$ lattice-gauge theory, confinement by center vortices is both necessary and sufficient for CSB, suggesting that standard gluonic effects are not the CSB driving mechanism. There are also simulations (for example, [22]) showing that the CSB phase transition temperature is quite close (not necessarily identical) to the deconfinement phase transition temperature. Another recent preprint [23] muddies the waters further, claiming that in $SU(3)$ lattice-gauge simulations removal of center vortices removes confinement but not CSB, contradicting the $SU(2)$ simulation results of [4]. We do not know how this puzzle will be resolved.

Long ago the author argued that the infrared singularities of QCD, coming from asysmptotic freedom, had to be cured by the generation of a dynamical gluon mass [13]. A one-dressed-loop PT approximation showed that “wrong-sign” asymptotic freedom problems were cured by the generation of a dynamical gluon mass of about 600–700 MeV or so. In recent years such a dynamical mass has been abundantly confirmed by lattice simulations and more sophisticated PT treatments; see [14]. As for the running charge, [13] gives the following approximation for Euclidean (spacelike) momenta:

$$\bar{g}^2(k^2) = \frac{1}{b\ln[(k^2 + 4m_g^2)/\Lambda^2]}, \qquad \alpha_s(0) = \frac{1}{4\pi b\ln(4m_g^2/\Lambda^2)}, \tag{26}$$

where $m_g$ is the dynamical gluon mass, $\Lambda$ the QCD mass scale, and, for gauge group $SU(N)$ with $N_f$ flavors, $b$ is the one-loop coefficient in the beta-function:

$$b = \frac{11N - 2N_f}{48\pi^2}. \tag{27}$$

(We use this running charge only for spacelike momenta, so the singularity for timelike momentum is irrelevant. There is a more complicated modified form [26] that is free of singularities in the timelike regime, and it agrees rather well with the above form for spacelike momenta.) In this paper we use, consistent with lattice determinations, phenomenology, and more sophisticated treatments of the PT Schwinger-Dyson equations [14], $m_g = 2\Lambda \approx 600$ MeV.

Let us accept these PT results, although the final word has yet to be said on their quantitative accuracy, and ask what they have to say about one-gluon CSB. The *linearized* massive-gluon JBW equation is, in Landau gauge,

$$M(p^2) = \frac{C_2}{(2\pi)^4}\int d^4k \frac{3\bar{g}^2 M(k)}{[(p-k)^2 + m_g^2]k^2}, \tag{28}$$

where $C_2$ is the quark Casimir eigenvalue [4/3 in $SU(3)$] and $m_g$ the gluon mass. It turns out that accounting for a gluon mass does two things: (1) It makes the gap equation finite at zero-momentum; (2) and more important, it bounds the IR running coupling. If the mass is too small, the running charge gets unacceptably large, as judged by phenomenology and solutions to the PT Schwinger-Dyson equations.

In the IR we can replace the running charge $\bar{g}^2$ by its zero-momentum value, which we call simply $g^2$, and then integrate over angles, with the result:

$$\int d\Omega_k \frac{1}{(p-k)^2 + m_g^2} \equiv K(k;p) = \frac{4\pi^2}{p^2 + k^2 + m_g^2 + [(p^2 + k^2 + m_g^2)^2 - 4p^2k^2]^{1/2}}. \tag{29}$$

However, this kernel does not yield a simple differential equation. Earlier, it was proposed [19,26] to approximate the angular integral by

$$K(k;p) \approx 2\pi^2\left[\frac{\theta(p^2-k^2)}{p^2+m_g^2} + \frac{\theta(k^2-p^2)}{k^2+m_g^2}\right] \equiv \tilde{K}(k,p). \tag{30}$$

Numerically the approximate kernel $\tilde{K}$ is, on the average, about 20–30% larger than the true kernel for IR momenta ($\leq m_g$), but it approaches the true kernel in the UV. We ignore this IR discrepancy, because it is in the direction to reinforce our conclusion that one-gluon exchange is too weak for CSB, and because the primary use of the one-gluon JBW equation will be for large momenta. Using the approximate kernel $\tilde{K}$ and the appropriate arguments for the running charge yields an integral equation:

$$M(p^2) = \frac{3C_2g^2}{16\pi^2}\int dk^2 \frac{k^2M(k^2)}{k^2+M^2(k^2)}\left[\frac{\theta(p^2-k^2)}{p^2+m_g^2} + (k \leftrightarrow p)\right]. \tag{31}$$

There is a corresponding differential equation:

$$M''(p^2) + \frac{2M'(p^2)}{p^2+m_g^2} + \frac{\lambda M(p^2)}{(p^2+m_g^2)^2} = 0, \qquad \lambda = \frac{3C_2g^2}{16\pi^2}. \tag{32}$$

This is nothing but the original JBW equation, with the variable $p^2 + m_g^2$ in place of $p^2$. It has power-law solutions

$$M(p^2) = \mathrm{const}(p^2+m_g^2)^{\nu_\pm}, \qquad \nu_\pm = \tfrac{1}{2}\{-1 \pm [1-4\lambda]^{1/2}]. \tag{33}$$

If the zero-momentum coupling is too small, there is no CSB. The standard analysis is that the critical coupling is the point at which the square root in Eq. (33) turns imaginary, and that there is CSB for couplings larger than this critical value. Then one-gluon CSB, in this approximation, requires

$$\alpha_s(0) \equiv \frac{g^2}{4\pi} \geq \frac{\pi}{3C_2}. \tag{34}$$

With $C_2 = 4/3$ this yields $\alpha_s(0) \geq 0.8$, approximately, somewhat greater than the value 0.5 given by Eq. (26). Taking account of the difference between the true massive kernel and the approximate kernel would change the critical value of $\alpha_s(0)$ to about one. It seems likely, then, that one-gluon exchange is not strong enough to drive CSB.

There are two results of this subsection: First, the suggestion that ordinary one-gluon exchange is too small to drive CSB for quarks (but likely large enough for adjoing fermions, where the critical coupling is only 4/9 as large, because of the Casimir eigenvalue [19]). Second, one can derive our confining gap equation by differentiating with respect to $m_g^2$, and replacing the coupling $g^2$ by $-8\pi K_F$.

## V. THE CONFINING GAP EQUATION

For convenience we repeat the gap equation of Eq. (2):

$$M(p^2) = \frac{1}{(2\pi)^4}\int d^4k\gamma_\mu D^{\mathrm{eff}}_{\mu\nu}\gamma_\nu(p-k)\frac{M(k^2)}{k^2+M^2(k^2)}, \tag{35}$$

and the massive effective confining propagator (having nothing to do with the true QCD gluon propagator):

$$D_{\mathrm{eff}}(k)_{\mu\nu} \equiv \delta_{\mu\nu}D_{\mathrm{eff}}(k), \qquad D_{\mathrm{eff}}(k) = \frac{8\pi K_F}{(k^2+m^2)^2}, \tag{36}$$

with a finite value of $m$.

Clearly, we can find the $S$-wave projection of the gap equation by differentiating the massive kernel of Eq. (29) with respect to the gluon mass. But (as with the one-gluon equation) the resulting gap equation can only be studied numerically. As we did with the massive one-gluon exchange in Sec. IV immediately above, we will instead differentiate the approximate massive one-gluon $S$-wave kernel $\tilde{K}$ of Eq. (30) with respect to $m_g^2$, replace the gluon mass $m_g$ by $m$, and make some other obvious changes. The accuracy of this procedure relative to the direct use of Eq. (29) is perhaps 20–30%. This yields a Euclidean gap equation:

$$\begin{aligned} M(p^2) &= \frac{2K_F}{\pi(p^2+m^2)^2}\int_0^{p^2} dk^2 \frac{k^2M(k^2)}{k^2+M^2(k^2)} \\ &\quad + \frac{2K_F}{\pi}\int_{p^2}^{\infty} dk^2 \frac{k^2M(k^2)}{(k^2+m^2)^2(k^2+M^2(k^2))} \\ &= J_>(p^2) + J_<(p^2), \end{aligned} \tag{37}$$

where $J_>(p^2)$ is the integral from 0 to $p^2$. The corresponding differential equation is

$$M(p^2)'' + \frac{3M(p^2)'}{p^2+m^2} + \frac{4K_F}{\pi}\left[\frac{p^2M(p^2)}{(p^2+m^2)^3[p^2+M^2(p^2)]}\right] = 0. \tag{38}$$

This is really a family of differential equations, as we see by writing them in nondimensional form. We write

$$M(p^2) = Mf(u \equiv p^2/M^2),\quad M \equiv M(p^2 = 0),\quad m \equiv \alpha M, \tag{39}$$

to find

$$f''(u) + \frac{3f'(u)}{u + \alpha^2} + \frac{4K_F}{\pi M^2}\left[\frac{uf(u)}{(u + \alpha^2)^3[u + f^2(u)]}\right] = 0. \tag{40}$$

The boundary conditions are $f(0) = 1$, $f'(0) = 0$. From Eq. (23) we expect $\alpha \approx 1$. Suppose for the sake of argument that $\alpha$ is fixed even if $M$ changes; this is reasonable, given that $m$ is not an externally imposed quantity but one which actually should scale with $M$. Then note that the dimensionless coupling parameter $K_F/M^2$ appearing in the differential equation is not known if only the differential equation is available. It has to be determined from the integral equation at zero momentum:

$$M^2 = \frac{2K_F}{\pi}\int_0^\infty du \frac{uf(u)}{(u + \alpha^2)^2[u + f^2(u)]} \equiv \frac{2K_F}{\pi} I(\alpha). \tag{41}$$

Either the integral or the differential equation yields a large-momentum falloff $M(p^2) \sim 1/p^4$. That the UV falloff is faster than what is expected from the OPE and the renormalization group is not a problem; the required behavior $M(p^2) \sim (\ln p^2)^a/p^2$ [16] follows from one-gluon exchange, which we take up in the next section. While there is no need for an area law to give the actual UV behavior, it is necessary that the area-law UV behavior be no slower than prescribed by the OPE and the renormalization group. The correct RG behavior will be reinstated by adding the one-gluon terms of Eq. (28).

For small momentum one finds

$$f(u) = \left[1 - \frac{2K_F}{3\pi M^2}\left(\frac{u^3}{\alpha^6}\right) + \ldots\right] \tag{42}$$

showing that the running mass changes but little near zero momentum.

It remains to determine $M$, a quantity that comes entirely from $J_<$. Given the family of solutions to the differential equation, we can now estimate the mass by imposing the integral condition of Eq. (41). It turns out numerically that $M$ does not change very rapidly with $m$ in the vicinity of $m = M$. Calculations for the range $0.8 < \alpha < 1$ yield mass values in the range $M^2 = (0.6 - 1)K_F/\pi$, with smaller $\alpha$ corresponding to larger $M$. [The limit $\alpha$ or $m = 0$ gives $M$ diverging like $\ln(1/m)$]. We show the case $\alpha = 0.9$, for which $M \approx 0.9\sqrt{K_F/\pi} \approx 230$ MeV, in Fig. 5. Note that $p^2/M^2 \approx 15$ corresponds to $p^2 \approx 1$ GeV$^2$.

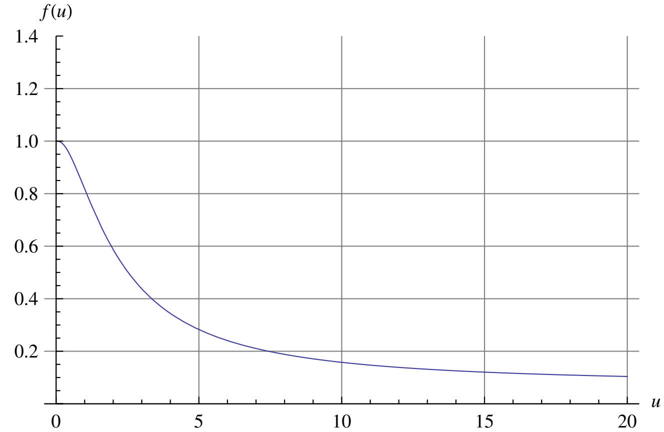


FIG. 5 (color online). Numerical calculation of the running quark mass $f(u)$ vs. $u$, for $M^2 = 0.8K_f/\pi$ and $\alpha = 0.9$.

As we have seen, at fixed $\alpha$ the only dimensionless strength parameter in this zero-temperature, zero-density problem is $K_F/M^2$. Unlike the dimensionless coupling constant of one-gluon exchange, this parameter cannot be tuned, since $M$ ultimately is determined by an integral equation. So there can be no phase transition between CSB and not-CSB at some finite value of this parameter. If one tries to find a phase with no CSB, so that $M$ somehow approaches zero, the dimensionless strength parameter becomes infinitely strong, and there surely must be CSB.

## VI. A GAP EQUATION WITH BOTH AREA LAW AND ONE-GLUON TERMS

One-gluon exchange determines the UV asymptotic behavior through the running charge, so we reinstate $\bar{g}^2$ instead of the zero-momentum value $g^2$ in the one-gluon equation. Then the area-law plus one-gluon integral equation is

$$\begin{aligned} M(p^2) = &\int_0^{p^2} dk^2 \frac{k^2M(k^2)}{k^2 + M^2(k^2)}\left[\frac{2K_F}{\pi(p^2 + M^2)^2} + \frac{3C_2\bar{g}(p^2)^2}{16\pi^2(p^2 + m_g^2)}\right] + \int_{p^2}^\infty dk^2 \frac{k^2M(k^2)}{(k^2 + M^2(k^2))} \\ &\times\left[\frac{2K_F}{\pi(k^2 + M^2)^2} + \frac{3C_2\bar{g}(k^2)^2}{16\pi^2(k^2 + m_g^2)}\right] \\ \equiv &J_>(p^2) + J_<(p^2) + K_>(p^2) + K_<(p^2), \end{aligned} \tag{43}$$

where $J_>$, $J_<$ are defined as before [see Eq. (37)] and $K_>$, $K_<$ refer, respectively, to the integrals from 0 to $p^2$ and from $p^2$ to infinity of the one-gluon kernel. Note that for QCD the one-gluon term begins to dominate the confining term at a momentum $p^2$ of order $2K_F/\alpha_s(0)$.

One-gluon corrections to confinement are of two types. The first correction is in the IR; at zero momentum it comes entirely from $K_<$. Let us now call the mass coming solely from confining effects $M_c$. The correction to $M_c$ is approximately

$$M = M_c \frac{1}{[1 - aI_1]},$$
$$I_1 = \int_0^\infty du \frac{u f(u)}{(u + \gamma)(u + f^2(u)) \ln[\beta(u + 4\gamma)]}, \tag{44}$$

where $\beta = M^2/\Lambda^2$, $\gamma = m_g^2/M^2$, and $a$ is the Lane constant

$$a = \frac{3C_2}{16\pi^2 b} = \frac{9C_2}{11N - 2N_f}. \tag{45}$$

As expected, one-gluon corrections increase $M$, since one-gluon effects work in the direction of producing CSB. For the case shown in Fig. 5, plus $\beta \approx 0.7$, $\beta\gamma = 4$, we find $I_1 \approx 0.22$, and with ([for no quarks, $SU(3)$] $a = 4/11$ the mass is increased by a factor of 1.1, approximately, or to about $\sqrt{K_F/\pi} \approx 250$ MeV.

The second, and more important to us, is the UV correction, where the one-gluon term dominates. There is one useful simplification: The term $K_<$ [last term on the right of Eq. (43)] is nonleading by one power of $\ln p^2$ in the UV, and we will drop it. Define $M_c(p^2)$ as the solution to the pure confinement equation (37) with a kernel $J_c(p; k)$:

$$M_c(p^2) = \int J_c(p;k) M_c(k^2),$$
$$J_c(p;k) = \frac{2K_F k^2}{\pi}\Bigg[\theta(p^2 - k^2) \times \frac{1}{(p^2 + m^2)^2[k^2 + M_c^2(k^2)]^2} + (k \leftrightarrow p)\Bigg], \tag{46}$$

where

$$\int \equiv \int_0^\infty dk^2. \tag{47}$$

We write the solution to Eq. (43), without the $K_<$ term, as $M(p^2) = M_c(p^2) + Q(p^2)$. We are only interested in the UV behavior of this equation, which then can be linearized in $Q$, and takes the form:

$$M_c(p^2) + Q(p^2) = \int J_c(p; k)[M_c(k^2) + Q(k^2)] + \int K_>(p; k)[M_c(k^2) + Q(k^2)], \tag{48}$$

where the running mass in the denominator of $K_>$ is $M_c(k^2)$:

$$K_>(p;k) = \frac{ak^2\theta(p^2 - k^2)}{p^2 + m_g^2}\Bigg[\ln\Bigg[\frac{p^2 + 4m_g^2}{\Lambda^2}\Bigg][k^2 + M_c^2(k^2)]\Bigg]^{-1}. \tag{49}$$

In Eq. (48) the $M_c$ on the left cancels against the $J_c M_c$ term on the right, leaving

$$Q = J_c Q + K_>(M_c + Q) \tag{50}$$

using a streamlined matrix notation, with momentum arguments and the integral sign suppressed [this should not lead to confusion of $M_c$ as we now use it with the mass as defined in Eq. (46)]. We will solve this in the UV as a power series in $K_>$. It will turn out that $Q(p^2) \sim (\ln p^2)^{a-1})/p^2$ in the UV, and it is straightforward to see that the term $J_c Q$ vanishes more rapidly than this, by a power of $p^2$. So we drop the $J_c Q$ term in Eq. (50). This leaves

$$Q = K_> M_c + K_> Q = \frac{1}{1 - K_>} K_> M_c = (1 + K_> + (K_>)^2 + \ldots) K_> M_c. \tag{51}$$

Defining the inverse $(1 - K_>)^{-1}$ is slightly subtle, because the integral of $K_>$ over a function depends on how rapidly the function vanishes in the UV. In particular, the confining solution $M_c$ vanishes like $1/p^4$ in the UV, which means that the function $K_> M_c$ vanishes like $1/p^2 \ln p^2$

$$K_> M_c(p^2) = \frac{a}{[p^2 + m_g^2]\ln[\frac{p^2 + 4m_g^2}{\Lambda^2}]} \times \int_0^{p^2} dk^2 k^2 M_c(k^2)[k^2 + M_c^2(k^2)]^{-1}. \tag{52}$$

Because of the rapid vanishing of $M_C(k^2)$ in the UV we can take the upper limit in the integral from $p^2$ to infinity. Then

$$K_> M_c(p^2) \equiv \frac{aI_c}{[p^2 + m_g^2]\ln[\frac{p^2 + 4m_g^2}{\Lambda^2}]} \tag{53}$$

with

$$I_c = \int_0^\infty dk^2 k^2 M_c(k^2)[k^2 + M_c^2(k^2)]^{-1}. \tag{54}$$

But for any function $F(p^2)$ that behaves like $1/p^2 \ln p^2$ at infinity the integral $K_> F$ behaves like $(\ln \ln p^2)/p^2 \ln p^2$, and for functions $F$ going like $(\ln \ln p^2)^N/p^2$, $K_> F$ behaves like $(\ln \ln p^2)^{N+1}/N p^2 \ln p^2$. One can then define the inverse $(1 - K_>)^{-1}$ by the pedestrian means of summing the series in Eq. (51), with the result (valid in the UV):

$$Q(p^2) = \frac{aI_c}{p^2 \ln p^2} \exp[a \ln \ln p^2] = \frac{aI_c}{p^2 \ln p^2} [\ln p^2]^a. \tag{55}$$

It is clear that the meaning of $\ln \ln p^2$ in the UV is

$$\ln \ln p^2 \to \ln\Bigg[\frac{g^2}{\bar{g}^2(p^2)}\Bigg] \tag{56}$$

in view of

$$\frac{g^2}{\bar{g}^2(p^2)} = 1 + bg^2 \ln\Bigg[1 + \frac{p^2}{4m_g^2}\Bigg], \tag{57}$$

which follows from Eq. (26). Then the result for the UV behavior coming from combining confinement and one-gluon terms is

$$M(p^2) \to \frac{3C_2\bar{g}^2(p^2)I_c}{16\pi^2 p^2}\left[\frac{g^2}{\bar{g}^2(p^2)}\right]^a. \quad (58)$$

This UV behavior is what the RG dictates, and by adding in confinement effects we are able to give the prefactor $aI_c$. From the work of [17] this allows an estimate of the $\langle\bar{q}q\rangle$ condensate, although we will not pursue that further here because of various complications (see, for example, [27]). Finally, we can change $M_c(p^2)$ to $M_{1c}(p^2)$, which is $M_c(p^2)$ modified by the IR one-gluon corrections [cf. Eq. (44)] to incorporate the IR corrections from one-gluon exchange.

## VII. SUMMARY

We suggest that when chiral symmetry is broken, leading to a running quark mass $M(p^2)$, the ensuing massless Goldstone bosons contribute to an entropy-driven condensate such as $\langle\bar{q}q\rangle$ by ramifying a large number of branches from a basic "trunk" Wilson loop that itself shows large fluctuations. Even if the Wilson loop represents quenched quarks, and so is incapable of breaking, configurations with the $q$ and the $\bar{q}$ far apart are quite improbable compared to ramified configurations where they are only separated by a distance of order $M(0)^{-1} \sim K_F^{-1/2}$, because the associated area-law action is large compared to the entropy. This means that in a linearly rising potential $K_F r$, separations with $r$ rather larger than $K_F^{-1/2}$ are not probed, and the potential at such large distances is irrelevant. This is the physical interpretation of the effects of a cutoff in Green's functions that have an Abelian gauge invariance associated with the confining propagator. From such a Green's function we formulate a nonsingular confining gap equation of JBW-type, with a mass $m$ in the confining effective propagator treated not as a regulator to be set to zero, but as a finite mass $\sim M(0)$ that can be estimated from processes that respect the Abelian gauge invariance. The static potential coming from the confining effective propagator rises linearly only out to a finite distance, and has a negative term at the origin that we identify with entropic effects. We estimate $m \approx M(0)$ through comparison with an extension of an old calculation (having Abelian gauge invariance) in which $m$ could be properly treated as a regulator mass, to be sent to zero after cancellations. The extended calculation replaces the regulator by a specific and physical mass, whose dynamical effects are equivalent to keeping $m$ as finite and of this value in the JBW equation.

We also studied one-gluon effects within the same framework, thereby finding the correct large-momentum behavior of QCD as known from the RG, but with a calculable prefactor. The final result, including IR one-gluon enhancements, is a quark mass $M(0) \approx 250$ MeV. This is somewhat smaller than the commonly quoted value of 300 MeV, which is largely based, not on true dynamical estimates, but on assuming that the sum of quark masses represents most of the mass of the hadron in question. Our mass has a different interpretation, given its entropic underpinnings.

Much remains to be done, both in formulating and solving more elaborate (and more accurate) forms of the gap equation, including in Minkowski space, where it is appropriate to use a principal-part confining propagator, relating these to pion dynamics, and perhaps making some progress in understanding entropic effects quantitatively.

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