QCD sum rule determination of the charm-quark mass

S. Bodenstein,[1] J. Bordes,[2] C. A. Dominguez,[1,3] J. Peñarrocha,[2] and K. Schilcher[4]

[1]*Centre for Theoretical & Mathematical Physics, University of Cape Town, Rondebosch 7700, South Africa*
[2]*Departamento de Física Teórica, Universitat de Valencia, and Instituto de Física Corpuscular, Centro Mixto Universitat de Valencia-CSIC*
[3]*Department of Physics, Stellenbosch University, Stellenbosch 7600, South Africa*
[4]*Institut für Physik, Johannes Gutenberg-Universität Staudingerweg 7, D-55099 Mainz, Germany*

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QCD sum rules involving mixed inverse moment integration kernels are used in order to determine the running charm-quark mass in the $\overline{\text{MS}}$ scheme. Both the high and the low energy expansion of the vector current correlator are involved in this determination. The optimal integration kernel turns out to be of the form $p(s) = 1 - (s_0/s)^2$, where s_0 is the onset of perturbative QCD. This kernel enhances the contribution of the well known narrow resonances, and reduces the impact of the data in the range $s \simeq 20$–$25\ \text{GeV}^2$. This feature leads to a substantial reduction in the sensitivity of the results to changes in s_0, as well as to a much reduced impact of the experimental uncertainties in the higher resonance region. The value obtained for the charm-quark mass in the $\overline{\text{MS}}$ scheme at a scale of 3 GeV is $\bar{m}_c(3\ \text{GeV}) = 987 \pm 9$ MeV, where the error includes all sources of uncertainties added in quadrature.

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Progress on the theoretical [1–12], as well as on the experimental, information [13–16] on the vector current correlator has allowed for a considerable improvement on the accuracy of QCD sum rule determinations of the charm-quark mass [17–19]. The analysis of [17] is based on inverse (Hilbert) moment QCD sum rules, requiring QCD knowledge of the vector correlator in the low energy, as well as in the high energy region. In [18], an alternative approach was used which involves only QCD information at short distances, together with (a) a simple integration kernel $p(s) = 1 - s/s_0$ (local constraint), and (b) Legendre-type polynomial kernels (global constraint). In this paper, we describe an improved analysis based on the use of direct as well as inverse moment kernels of the form $p(s) = 1 - (s_0/s)^n$, with $n \geq 1$. These kernels enhance considerably the impact of the well-known narrow resonances, as compared with e.g. a simple kernel $p(s) = 1/s^2$, or $p(s) = 1 - s/s_0$. They also provide a welcome stronger suppression of the contribution of data in the range $s \simeq 20$–$25\ \text{GeV}^2$. In comparison with simple inverse moments without pinching, this means that results are less sensitive to assumptions about the onset of perturbative QCD (PQCD), as well as to the treatment of the higher resonance data. For instance, changes in s_0 in the range $s_0 \simeq 15$–$23\ \text{GeV}^2$ lead to a variation in $\bar{m}_c(3\ \text{GeV})$ of only 4 MeV (for $n = 2$) as opposed to a variation of 14 MeV for $p(s) = 1/s^2$, as used in [17].

We consider the vector current correlator

$$\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iqx}\langle 0|T(V_\mu(x)V_\nu(0))|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2), \tag{1}$$

where $V_\mu(x) = \bar{c}(x)\gamma_\mu c(x)$. From the residue theorem in the complex s-plane ($-q^2 \equiv Q^2 \equiv s$) it follows

$$\int_0^{s_0} p(s)\frac{1}{\pi}\text{Im}\Pi(s)ds = -\frac{1}{2\pi i}\oint_{C(|s_0|)} p(s)\Pi(s)ds + \text{Res}[\Pi(s)p(s), s=0], \tag{2}$$

where $p(s)$ is an integration kernel, and

$$\text{Im}\,\Pi(s) = \frac{1}{12\pi}R_c(s), \tag{3}$$

with $R_c(s)$ the standard R-ratio for charm production. The PQCD expansion of $\Pi(s)$ at short distances can be written as

$$\Pi(s)|_{\text{PQCD}} = e_c^2\sum_{n=0}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \Pi^{(n)}(s), \tag{4}$$

where $e_c = 2/3$ is the charm-quark electric charge, and

$$\Pi^{(n)}(s) = \sum_{i=0}\left(\frac{\bar{m}_c^2}{s}\right)^i \Pi_i^{(n)}, \tag{5}$$

and $\bar{m}_c \equiv \bar{m}_c(\mu)$ is the running charm-quark mass in the $\overline{\text{MS}}$-scheme. Up to order $\mathcal{O}\,[\alpha_s^2(\bar{m}_c^2/s)^6]$ the function $\Pi(s)_{\text{PQCD}}$ has been calculated in [1], and exact results for $\Pi_0^{(3)}$ and $\Pi_1^{(3)}$ have been found in [2]. The function $\Pi_2^{(3)}$ is known exactly up to a constant [3]. At five-loop order $\mathcal{O}(\alpha_s^4)$ the full logarithmic terms for $\Pi_0^{(4)}$ may be found in [5], and for $\Pi_1^{(4)}$ in [6]. Since there is incomplete knowledge at this loop-order, we shall use the available information as a measure of the truncation error in PQCD. The low energy expansion of the vector correlator around $s = 0$ can be written as

$$\Pi_{\text{PQCD}}(s) = \frac{3e_c^2}{16\pi^2}\sum_{n\geq 0}\bar{C}_n z^n, \tag{6}$$

where $z = s/(4\bar{m}_c^2)$. The coefficients $\bar{C}_n$ can be expanded in powers of $\alpha_s(\mu)$

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_m) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 (\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_m + \bar{C}_n^{(22)} l_m^2) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 (\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_m + \bar{C}_n^{(32)} l_m^2 + \bar{C}_n^{(33)} l_m^3) + \ldots \quad (7)$$

where $l_m \equiv \ln(\bar{m}_c^2(\mu)/\mu^2)$. Up to three-loop level, the coefficients up to $n = 30$ of $\bar{C}_n$ are known [8,9]. At four-loop level, we have $\bar{C}_0$ and $\bar{C}_1$ from [8,10], $\bar{C}_2$ from [9], and $\bar{C}_3$ from [11]. We will choose $p(s)$ so that no coefficients $\bar{C}_4$ and above contribute to the residue at $s = 0$, $\mathrm{Res}[\Pi(s)p(s), s = 0]$.

Apart from the quark mass, the fundamental QCD parameters are the running strong coupling $\alpha_s(\mu^2)$, and the gluon condensate. For the strong coupling, we use the world average from [20], which agrees with lattice QCD results [21], $\alpha_s(M_Z^2) = 0.1184 \pm 0.0007$. However, we will consider other values when comparing results for $\bar{m}_c$ with other analyses. In the nonperturbative sector, the leading power correction in the operator product expansion involves the gluon condensate, i.e. $\langle(\alpha_s/\pi)G^2\rangle$ whose value has been extracted [22] from the ALEPH data on τ-decays. While the gluon condensate is renormalization group invariant, its determination from QCD sum rules involves a difference between integrals of PQCD and integrated experimental data. This leads to an unavoidable dependence of the gluon condensate on the value of α_s used in the PQCD expression of the correlator. Extrapolating the results of [22] to include current values of α_s [20,23] leads to $\langle(\alpha_s/\pi)G^2\rangle = (0.01 \pm 0.01)$ GeV^4. This large uncertainty in the value of the gluon condensate has only a very small impact on our results for $\bar{m}_c$.

Turning to the experimental data, we follow closely the analysis of [17]. For the first two narrow resonances, we use the latest data from the Particle Data Group [24], $M_{J/\psi} = 3.096916(11)$ GeV, $\Gamma_{J/\psi\to e^+e^-} = 5.55(14)$ keV, $M_{\psi(2s)} = 3.68609(4)$ GeV, $\Gamma_{\psi(2s)\to e^+e^-} = 2.35(4)$ keV. These two narrow resonances are followed by the open charm region where the contribution from the light quark sector R_{uds} needs to be subtracted from the total R-ratio R_{tot}. We perform this subtraction as in [25]. In the region 3.97 GeV $\leq \sqrt{s} \leq$ 4.26 GeV, we only use CLEO data [16] as they are the most precise. In connection with the three data sets from the BES collaboration[13–15], we assume that the systematic uncertainties are not fully independent and add them linearly, rather than in quadrature. However, we treat these data as independent from the CLEO data set [16], and thus add errors in quadrature. There is no data in the region $s = 25$–49 GeV^2, and beyond there is CLEO data up to $s \simeq 110$ GeV^2. The latter data is fully compatible with PQCD.

We discuss next the integration kernels $p(s)$ in Eq. (2), which we choose as

$$p(s) = 1 - \left(\frac{s_0}{s}\right)^n, \quad (8)$$

with $n \geq 1$. As discussed in [17], inverse moments $p(s) = 1/s^n$ should not involve too large values of n. In fact, the convergence of PQCD deteriorates with increasing n, the gluon condensate contribution increases sharply for $n > 2$, and the uncertainties in α_s and the renormalization scale μ have a greater impact on the total error of the charm-quark mass. On the other hand, direct kernels of the form $p(s) = s^n$, with $n \geq 1$, pose problems. Indeed, the high energy expansion of the vector correlator is incompletely known at $\mathcal{O}[\alpha_s^3]$, so that the greater the value of n, the greater is the contribution of the higher order mass corrections at this order in PQCD. Already with $n = 1$, one would need a Pade approximation for the term $\mathcal{O}[\alpha_s^3\bar{m}_c^6]$. Hence, in order to avoid any approximation up to this order in PQCD we restrict ourselves to direct moments with $n = 0$, and include inverse powers in an attempt to enhance the contribution of the well-known narrow resonances, J/ψ and $\psi(2S)$, and at the same time suppressing the broad resonance region. We found that Eq. (8) with $n = 2$ is the optimal kernel as explained next. In Fig. 1, we show the experimental data for the ratio $R(s)$ together with the kernel Eq. (8) with $n = 2$ and $s_0 \simeq 23$ GeV^2, and the simple kernel $p(s) = 1/s^2$ normalized such that both kernels coincide at the peak of the second narrow resonance $\psi(2S)$, i.e. $s \simeq 13.6$ GeV^2. One can easily appreciate that in comparison with the latter, the former kernel leads to a welcome enhancement of the weight of the J/ψ,

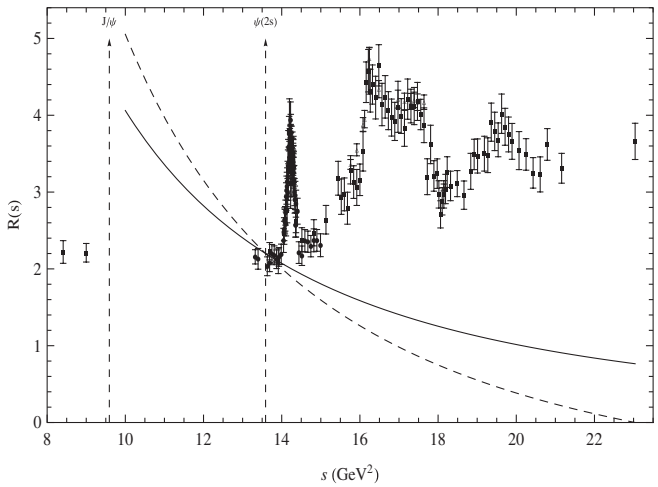

FIG. 1. Experimental data for the total $R(s)$ ratio together with the optimal integration kernel, Eq. (8), with $n = 2$ (dash curve), and $p(s) = 1/s^2$ (solid curve) normalized to coincide with the former at the position of the $\psi(2S)$ peak.

as well as to a strong suppression of the broad resonance region, and particularly the region near the onset of the continuum. Quantitatively, the ratio of the area under the hadronic spectral function weighted with $p(s)$, in the narrow resonance region, $\mathcal{I}_{\rm nr}$, and in the broad resonance region and beyond, $\mathcal{I}_{\rm br}$, is $\mathcal{I}_{\rm nr}/\mathcal{I}_{\rm br} = 3.6$ for $p(s) = 1/s^2$, and $\mathcal{I}_{\rm nr}/\mathcal{I}_{\rm br} = 7.7$ for Eq. (8) with $n = 2$. Other values of n lead to slightly less enhancement. In addition, the kernel Eq. (8) with $n = 2$ leads to final results for $\bar{m}_c$ which are fairly insensitive to the choice of s_0. For instance, in the range $s_0 \simeq 15.0$–$23.0\ \mathrm{GeV}^2$, $\bar{m}_c(3\ \mathrm{GeV})$ changes by about 4.0 MeV, while using the kernel $p(s) = 1/s^2$ it changes by 14.0 MeV.

Proceeding to our determination, we list in Table I the results for $\bar{m}_c(3\ \mathrm{GeV})$ at different orders in perturbation theory, and using two integration kernels. The results for the kernel $p(s) = 1/s^2$ differ from [17] as we use now a slightly different value of the strong coupling, of the gluon condensate and of the $\psi(2S)$ parameters as given above, and we include the CLEO data [16]. The various errors associated to the final value of $\bar{m}_c(3\ \mathrm{GeV})$ are given in Table II. Results from fixed order perturbation theory are essentially the same as using contour improved perturbation theory to integrate around the circle of radius s_0 in the complex s-plane. In Fig. 2, we show the results for $\bar{m}_c(3\ \mathrm{GeV})$ as a function of s_0 for the kernel $p(s) = 1/s^2$, and in Fig. 3 for the kernel Eq. (8) with $n = 2$, the latter exhibiting improved stability. The convergence pattern in α_s of the PQCD integral as a function of $\bar{m}_c$ can be studied by computing

$$\mathcal{I}(s_0) = -\frac{1}{2\pi i}\oint_{C(|s_0|)} p(s)\Pi(s)ds + \mathrm{Res}[\Pi(s)p(s), s = 0], \tag{9}$$

TABLE I. Results for the charm-quark mass at different orders in PQCD, and for two integration kernels. The results for $p(s) = 1/s^2$ are obtained using slightly different values of the QCD parameters, and a different integration procedure as in [17].

	$\bar{m}_c(3\ \mathrm{GeV})$ (in MeV)			
Kernel	$\bar{m}_c^{(0)}$	$\bar{m}_c^{(1)}$	$\bar{m}_c^{(2)}$	$\bar{m}_c^{(3)}$
s^{-2}	1129	1021	998	995
$1-(s_0/s)^2$	1146	1019	991	987

TABLE II. The various uncertainties due to the data (EXP), the value of α_s ($\Delta\alpha_s$), changes of $\pm 35\%$ in the renormalization scale around $\mu = 3$ GeV ($\Delta\mu$), the value of the gluon condensate (NP), and due to variations in s_0 (s_0).

		Uncertainties (in MeV)					
Kernel	$\bar{m}_c(3\ \mathrm{GeV})$	EXP	$\Delta\alpha_s$	$\Delta\mu$	NP	s_0	Total
s^{-2}	995	9	3	1	1	14	17
$1-(s_0/s)^2$	987	7	4	1	1	4	9

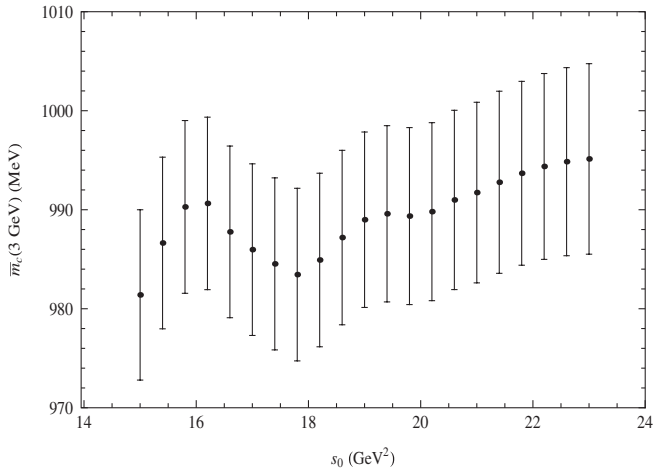

FIG. 2. Results for $\bar{m}_c(3\ \mathrm{GeV})$ as a function of s_0 for the kernel $p(s) = 1/s^2$. The variation of $\bar{m}_c(3\ \mathrm{GeV})$ in this range is up to 14 MeV.

with these integrals being functions of both $\bar{m}_c$ and α_s. Using $\bar{m}_c(3\ \mathrm{GeV}) = 987$ MeV, and Eq. (8) with $n = 2$ we find reasonable convergence, i.e. $\mathcal{I}^{(0)} = 91.4\ \mathrm{GeV}^2$, $\mathcal{I}^{(1)} = 62.0\,\mathrm{GeV}^2$, $\mathcal{I}^{(2)} = 57.0\,\mathrm{GeV}^2$, and $\mathcal{I}^{(3)} = 56.3\,\mathrm{GeV}^2$, where the upper index in $\mathcal{I}^{(j)}$ indicates the power of α_s.

Our final result using the optimal kernel, Eq. (8), with $n = 2$ is

$$\bar{m}_c(3\ \mathrm{GeV}) = 987 \pm 9\ \mathrm{MeV}, \tag{10}$$

in good agreement within errors with the result from inverse moment QCD sum rules [17], other recent determinations [18,19,25,26], as well as lattice QCD [21]. Translated into a scale invariant mass, the above result gives $\bar{m}_c(\bar{m}_c) = 1278 \pm 9$ MeV for the value used here for the strong coupling.

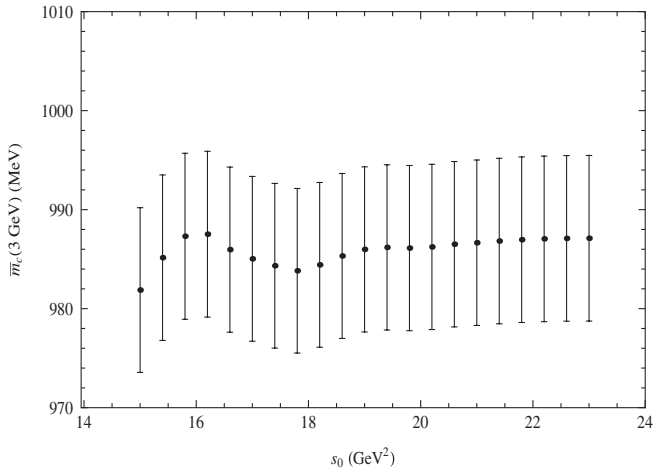

FIG. 3. Results for $\bar{m}_c(3\ \mathrm{GeV})$ as a function of s_0 for the kernel Eq. (8) with $n = 2$. The variation of $\bar{m}_c(3\ \mathrm{GeV})$ in this range is up to 4 MeV.

A recent analysis [19] using inverse moment sum rules finds $m_c(3\ \mathrm{GeV}) = 998 \pm 29$ MeV. The relatively large error arises mostly from the unconventional choice of different renormalization scales for the quark mass and for the strong coupling. If these scales are taken to be the same, then there is a substantial cancellation of the scale dependence in inverse moment sum rules leading to a smaller uncertainty. However, this does not affect sum rules with kernels involving positive powers of s, as e.g. in [18]. By using different kernels, different moments, different renormalization procedures and different integration methods, one can produce a variety of quark mass values with different errors. Following the philosophy used in most previous determinations of the heavy quark masses, we choose the procedure which leads to the smallest final error, provided the central values are all compatible.

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[1] K. G. Chetyrkin, R. Harlander, J. H. Kühn, and M. Steinhauser, Nucl. Phys. **B503**, 339 (1997).
[2] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. B, Proc. Suppl. **189**, 49 (2009).
[3] K. G. Chetyrkin, R. Harlander, and J. H. Kühn, Nucl. Phys. **B586**, 56 (2000).
[4] Y. Kiyo, A. Maier, P. Maierhöfer, and P. Marquard, Nucl. Phys. **B823**, 269 (2009).
[5] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **101**, 012002 (2008).
[6] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. B, Proc. Suppl. **135**, 243 (2004).
[7] K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Phys. Lett. B **371**, 93 (1996); Nucl. Phys. B **482**, 213 (1996); **505**, 40 (1997).
[8] R. Boughezal, M. Czakon, and T. Schutzmeier, Phys. Rev. D **74**, 074006 (2006); Nucl. Phys. B, Proc. Suppl. **160**, 160 (2006).
[9] A. Maier, P. Maieröfer, and P. Marquard, Nucl. Phys. **B797**, 218 (2008); Phys. Lett. B **669**, 88 (2008).
[10] K. G. Chetyrkin, J. H. Kühn, and C. Sturm, Eur. Phys. J. C **48**, 107 (2006).
[11] A. Maier, P. Maierhöfer, P. Marquard, and A. V. Smirnov, Nucl. Phys. **B824**, 1 (2010).
[12] A. H. Hoang, V. Mateu, and S. Mohammad Zebarjad, Nucl. Phys. **B813**, 349 (2009).
[13] J. Z. Bai *et al.* (BES 2000), Phys. Rev. Lett. **84**, 594 (2000).
[14] J. Z. Bai *et al.* (BES 2002), Phys. Rev. Lett. **88**, 101802 (2002).
[15] J. Z. Bai *et al.* (BES 2006), Phys. Rev. Lett. **97**, 262001 (2006).
[16] D. Cronin-Hennessy *et al.* (CLEO 2009), Phys. Rev. D **80**, 072001 (2009).
[17] J. H. Kühn, M. Steinhauser, and C. Sturm, Nucl. Phys. **B778**, 192 (2007); K. G. Chetyrkin *et al.*, Phys. Rev. D **80**, 074010 (2009).
[18] S. Bodenstein, J. Bordes, C. A. Dominguez, J. Peñarrocha, and K. Schilcher, Phys. Rev. D **82**, 114013 (2010).
[19] B. Dehnadi, A. H. Hoang, V. Mateu, S. Mohammad Zebarjad, arXiv: 1102.2264.
[20] S. Bethke, Eur. Phys. J. C **64**, 689 (2009).
[21] C. T. H. Davies *et al.* (HPQCD Collab.), Phys. Rev. D **78**, 114507 (2008); C. McNeile (PQCD Collab.), Phys. Rev. D **82**, 034512 (2010).
[22] C. A. Dominguez and K. Schilcher, J. High Energy Phys. 01 (2007) 093.
[23] A. Pich, arXiv:1101.2107.
[24] K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).
[25] A. Hoang and M. Jamin, Phys. Lett. B **594**, 127 (2004).
[26] A. Signer, Phys. Lett. B **672**, 333 (2009).