Constraints on the four-generation quark mixing matrix from a fit to flavor-physics data

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In the scenario with four quark generations, we perform a fit using flavor-physics data and determine the allowed values—preferred central values and errors—of all of the elements of the 4×4 quark-mixing matrix. In addition to the direct measurements of some of the elements, we include in the fit the present measurements of several flavor-changing observables in the K and B systems that have small hadronic uncertainties, and also consider the constraints from the vertex corrections to $Z \to b\bar{b}$. The values taken for the masses of the fourth-generation quarks are consistent with the measurements of the oblique parameters and perturbativity of the Yukawa couplings. We find that $|\tilde{V}_{tb}| \geq 0.98$ at 3σ, so that a fourth generation cannot account for any large deviation of $|V_{tb}|$ from unity. The fit also indicates that all the new-physics parameters are consistent with zero, and the mixing of the fourth generation with the other three is constrained to be small: we obtain $|\tilde{V}_{ub'}| < 0.06$, $|\tilde{V}_{cb'}| < 0.027$, and $|\tilde{V}_{tb'}| < 0.31$ at 3σ. Still, this does allow for the possibility of new-physics signals in B_d, B_s and rare K decays.

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I. INTRODUCTION

There is no unequivocal theoretical argument which restricts the number of quark generations to three as in the standard model (SM). An additional fourth generation (SM4) is one of the simplest extensions of the SM, and retains all of its essential features: it obeys all the SM symmetries and does not introduce any new ones. At the same time, it can give rise to many new effects, some of which may be observable even at the current experiments [1]. Even though the fourth-generation quarks may be too heavy to have been produced at the pre-LHC colliders, they may still affect low-energy measurements through their mixing with the lighter quarks. The up-type quark t' would contribute to $b \to s$ and $b \to d$ transitions at the 1-loop level, while the down-type quark b' would contribute similarly to $c \to u$ and $t \to c$.

The addition of a fourth generation to the SM leads to a 4×4 quark-mixing matrix CKM4, which is an extension of the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix in the SM. The parametrization of this unitary matrix requires six real parameters and three phases. The additional phases can lead to increased CP violation, and can provide a natural explanation for the deviations from the SM predictions seen in some measurements of CP violation in the B-meson system [2–7]. A heavy fourth generation can play a crucial role in the dynamical generation of the electroweak (EW) symmetry breaking [8]. Also, the large Yukawa couplings of the fourth-generation quarks, together with the possible large phases, can help efficient EW baryogenesis [9].

The EW precision measurements of the oblique parameters S and T imply strong correlations between the masses of the fourth-generation quarks [10,11]. The parameter space of fourth-generation masses with minimal contributions to S and T, and in agreement with all experimental constraints, is [11,12]

$$m_{t'} \geq 400 \text{ GeV},$$
$$m_{t'} - m_{b'} \simeq \left(1 + \frac{1}{5}\frac{m_H}{115 \text{ GeV}}\right) \times 50 \text{ GeV}, \qquad (1)$$

where $m_{t'}$, $m_{b'}$ and m_H are the masses of t', b', and the Higgs boson H, respectively. On the other hand, the perturbativity of the Yukawa coupling implies that $m_{t'} \lesssim \sqrt{2\pi}\langle v\rangle \approx 600$ GeV, where $\langle v\rangle$ is the vacuum expectation value of the Higgs. Arguments based on the unitarity of partial S-wave scattering amplitudes for color-singlet, elastic, same-helicity t'-$\bar{t}'$ scattering at tree-level restrict $m_{t'} \lesssim \sqrt{4\pi/3}\langle v\rangle \approx 500$ GeV [13,14]. Thus, the fourth-generation quark masses are constrained to a narrow band, which increases the predictivity of the SM4.

The quark-mass bounds above may be somewhat relaxed with the introduction of heavy fourth-generation leptons, which help in partially cancelling out the effect of the fourth generation on the S and T parameters. Even in the absence of any quark-lepton cancellation, the EW precision measurements restrict [11,12]

$$m_{l'} - m_{\nu'} \simeq (30\text{–}60) \text{ GeV},$$

where $m_{l'}$ and $m_{\nu'}$ are the masses of the fourth-generation charged lepton l' and neutrino ν', respectively. Thus even in the absence of any fine-tuned cancellations, there is a significant allowed range for the masses of the fourth-generation fermions, which is, in fact, not beyond the reach of the LHC. The invisible decay width of the Z boson

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constrains the mass of the fourth-generation neutrino to be greater than 45 GeV. Though one would need a special mechanism leading to a massive fourth-generation neutrino and three ultralight SM neutrinos, phenomenologically this is perfectly allowed.

In order to make concrete SM4 predictions, the first step is to determine the elements of CKM4. This involves not only fixing the values of the new parameters, but also reevaluating those of the SM. This is because not all elements of the CKM matrix are measured directly. For example, the bounds on $|V_{td}|$ and $|V_{ts}|$ are obtained from decays involving loops, and these are rather weak. And though $|V_{tb}|$ is measured in the tree-level decay $t \to bW$, its value is not that precise: the direct measurement at the Tevatron from single top production gives $|V_{tb}| = 0.88 \pm 0.07$ [15–17]. Now, $|V_{tb}| = 1$ is predicted in the SM to an accuracy of 10^{-3}. Although the Tevatron value is consistent with the SM prediction, it can also be as small as 0.67 at 3σ. Thus, the values of the elements V_{tq} $(q = d, s, b)$ are not obtained through measurements. Rather, they are mainly determined using the unitarity of the 3×3 CKM matrix [18]. However, the assumption of the unitarity of the 3×3 matrix is clearly invalid in the four-generation scenario, and relaxing it allows a much larger range of values for the elements $|V_{tq}|$. For example, a large deviation of $|V_{tb}|$ from unity is claimed to be possible in the SM4 [19–23].

We parametrize the CKM4 with 9 parameters, and perform a combined fit to these parameters using flavor-physics data. In addition to the direct measurements of the CKM4 matrix elements, the fit includes observables that have small hadronic uncertainties: (i) R_{bb} and A_b from $Z \to b\bar{b}$, (ii) ϵ_K from $K_L \to \pi\pi$, (iii) the branching ratio of $K^+ \to \pi^+ \nu\bar{\nu}$, (iv) the mass differences in the B_d and B_s systems, (v) the time-dependent CP asymmetry in $B_d \to J/\psi K_S$, (vi) the measurement of the angle γ of the unitarity triangle from tree-level decays, (viii) the branching ratios of $B \to X_s \gamma$ and $B \to X_c e \bar{\nu}$, and (ix) the branching ratio of $B \to X_s \mu^+ \mu^-$ in the high-q^2 and low-q^2 regions. We do not include the oblique parameters in the fit, but simply take the values of the fourth-generation quark masses to be consistent with the EW precision data.

There have been several analyses of CKM4 in the past (e.g. see Refs. [5–7,21,23,24]). However, they all have a number of deficiencies compared to the present work. They do not perform a fit. Instead, at best, they present scatter plots showing the allowed ranges of the CKM4 matrix elements (or correlations between various observables). Of course, these plots cannot quantify what the errors on the elements are, nor the confidence level of the ranges. This information can be obtained only by performing a true fit. Also, some of them do not include all clean observables which can be affected by the fourth generation.

The paper is organized as follows. In Sec. II, we define the Dighe-Kim parametrization of the CKM4 matrix. In Sec. III, we present the observables which constrain the elements of CKM4, along with their experimental values. The results of the fit are presented in Sec. IV. We conclude in Sec. V with a discussion of the results.

II. CKM4 MATRIX: DIGHE-KIM PARAMETRIZATION

The CKM matrix in the SM is a 3×3 unitary matrix:

$$V_{\rm CKM3} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{2}$$

In the SM4, the CKM4 matrix is 4×4, and can be written as

$$V_{\rm CKM4} = \begin{pmatrix} \tilde{V}_{ud} & \tilde{V}_{us} & \tilde{V}_{ub} & \tilde{V}_{ub'} \\ \tilde{V}_{cd} & \tilde{V}_{cs} & \tilde{V}_{cb} & \tilde{V}_{cb'} \\ \tilde{V}_{td} & \tilde{V}_{ts} & \tilde{V}_{tb} & \tilde{V}_{tb'} \\ \tilde{V}_{t'd} & \tilde{V}_{t's} & \tilde{V}_{t'b} & \tilde{V}_{t'b'} \end{pmatrix}. \tag{3}$$

The above matrix can be described, with appropriate choices for the quark phases, in terms of 6 real quantities and 3 phases.

In this paper, we use the Dighe-Kim (DK) parametrization of the CKM4 matrix [25,26]. This allows us to treat the effects of the fourth generation perturbatively and explore the complete parameter space available. The DK parametrization defines

$$\begin{gathered} \tilde{V}_{us} \equiv \lambda, \qquad \tilde{V}_{cb} \equiv A\lambda^2, \qquad \tilde{V}_{ub} \equiv A\lambda^3 C e^{-i\delta_{ub}}, \\ \tilde{V}_{ub'} \equiv p\lambda^3 e^{-i\delta_{ub'}}, \qquad \tilde{V}_{cb'} \equiv q\lambda^2 e^{-i\delta_{cb'}}, \qquad \tilde{V}_{tb'} \equiv r\lambda, \end{gathered} \tag{4}$$

where λ is the sine of the Cabibbo angle, so that the CKM4 matrix takes the form

$$V_{\rm CKM4} = \begin{pmatrix} \# & \lambda & A\lambda^3 C e^{-i\delta_{ub}} & p\lambda^3 e^{-i\delta_{ub'}} \\ \# & \# & A\lambda^2 & q\lambda^2 e^{-i\delta_{cb'}} \\ \# & \# & \# & r\lambda \\ \# & \# & \# & \# \end{pmatrix}. \tag{5}$$

The elements denoted by "#" can be determined uniquely from the unitarity condition $V^\dagger_{\rm CKM4} V_{\rm CKM4} = I$. They can be calculated in the form of an expansion in powers of λ such that each element is accurate up to a multiplicative factor of $[1 + \mathcal{O}(\lambda^3)]$.

The matrix elements $\tilde{V}_{ud}$, $\tilde{V}_{cd}$ and $\tilde{V}_{cs}$ retain their SM values,

$$\begin{aligned} \tilde{V}_{ud} &= 1 - \frac{\lambda^2}{2} + \mathcal{O}(\lambda^4), \\ \tilde{V}_{cd} &= -\lambda + \mathcal{O}(\lambda^5), \\ \tilde{V}_{cs} &= 1 - \frac{\lambda^2}{2} + \mathcal{O}(\lambda^4), \end{aligned} \tag{6}$$

whereas the values of the matrix elements V_{td}, V_{ts} and V_{tb} are modified due to the presence of the additional quark generation:

$$\begin{aligned}\tilde{V}_{td} &= A\lambda^3(1 - Ce^{i\delta_{ub}}) + r\lambda^4(qe^{i\delta_{cb'}} - pe^{i\delta_{ub'}}) \\ &\quad + \frac{A}{2}\lambda^5(-r^2 + (C + Cr^2)e^{i\delta_{ub}}) + \mathcal{O}(\lambda^6), \\ \tilde{V}_{ts} &= -A\lambda^2 - qr\lambda^3 e^{i\delta_{cb'}} + \frac{A}{2}\lambda^4(1 + r^2 - 2Ce^{i\delta_{ub}}) \\ &\quad + \mathcal{O}(\lambda^5), \\ \tilde{V}_{tb} &= 1 - \frac{r^2\lambda^2}{2} + \mathcal{O}(\lambda^4). \end{aligned} \tag{7}$$

In the limit $p = q = r = 0$, only the elements present in the 3×3 CKM matrix retain nontrivial values, and the above expansion corresponds to the Wolfenstein parametrization [27] with $C = \sqrt{\rho^2 + \eta^2}$ and $\delta_{ub} = \tan^{-1}(\eta/\rho)$.

The remaining new CKM4 matrix elements are

$$\begin{aligned}\tilde{V}_{t'd} &= \lambda^3(qe^{i\delta_{cb'}} - pe^{i\delta_{ub'}}) + Ar\lambda^4(1 + Ce^{i\delta_{ub}}) \\ &\quad + \frac{\lambda^5}{2}(pe^{i\delta_{ub'}} - qr^2e^{i\delta_{cb'}} + pr^2e^{i\delta_{ub'}}) + \mathcal{O}(\lambda^6), \\ \tilde{V}_{t's} &= q\lambda^2 e^{i\delta_{cb'}} + Ar\lambda^3 \\ &\quad + \lambda^4\left(-pe^{i\delta_{ub'}} + \frac{q}{2}e^{i\delta_{cb'}} + \frac{qr^2}{2}e^{i\delta_{cb'}}\right) + \mathcal{O}(\lambda^5), \\ \tilde{V}_{t'b} &= -r\lambda + \mathcal{O}(\lambda^4), \\ \tilde{V}_{t'b'} &= 1 - \frac{r^2\lambda^2}{2} + \mathcal{O}(\lambda^4). \end{aligned} \tag{8}$$

III. CONSTRAINTS ON THE CKM4 MATRIX ELEMENTS

In order to obtain constraints on the CKM4 matrix elements, we perform a χ^2 fit for all 9 CKM4 parameters using the CERN minimization code MINUIT [28]. The fit is carried out for $m_{t'} = 400$ GeV and 600 GeV. The b' mass is fixed by the relation $m_{t'} - m_{b'} = 55$ GeV [see Eq. (1)]. We include both experimental errors and theoretical uncertainties in the fit. In the following subsections, we discuss the various observables used as constraints, and give their experimental values.

A. Direct measurements of the CKM elements

The values of CKM elements obtained from the measurement of the tree-level weak decays are independent of the number of generations. Hence, they apply to the 3×3 and 4×4 matrices. The elements $|\tilde{V}_{ud}|$, $|\tilde{V}_{us}|$, $|\tilde{V}_{ub}|$, $|\tilde{V}_{cd}|$, $|\tilde{V}_{cs}|$ and $|\tilde{V}_{cb}|$ have all been directly measured. We use the following measurements [18] to constrain the CKM4 parameters:

$$\begin{aligned}|\tilde{V}_{ud}| &= (0.97418 \pm 0.00027), \\ |\tilde{V}_{cd}| &= (0.23 \pm 0.011), \\ |\tilde{V}_{us}| &= (0.2255 \pm 0.0019), \\ |\tilde{V}_{cs}| &= (1.04 \pm 0.06), \\ |\tilde{V}_{ub}| &= (3.93 \pm 0.36) \times 10^{-3}, \\ |\tilde{V}_{cb}| &= (41.2 \pm 1.1) \times 10^{-3}. \end{aligned} \tag{9}$$

B. Unitarity of the CKM4 matrix

Constraints on the CKM4 matrix elements can be obtained by using the unitarity of the CKM4 matrix. Through a variety of independent measurements, the SM 3×3 submatrix has been found to be approximately unitary. We therefore expect all the CKM4 matrix elements which involve both the fourth-generation and light quarks to be relatively small.

Using the measurements of $|V_{ud}|$, $|V_{us}|$ and $|V_{ub}|$, the first row of the CKM4 matrix gives

$$\begin{aligned}|\tilde{V}_{ub'}|^2 &= 1 - (|\tilde{V}_{ud}|^2 + |\tilde{V}_{us}|^2 + |\tilde{V}_{ub}|^2) \\ &= 0.0001 \pm 0.0011. \end{aligned} \tag{10}$$

Using the measurements of $|\tilde{V}_{cd}|$, $|\tilde{V}_{cs}|$ and $|\tilde{V}_{cb}|$, the second row gives

$$\begin{aligned}|\tilde{V}_{cb'}|^2 &= 1 - (|\tilde{V}_{cd}|^2 + |\tilde{V}_{cs}|^2 + |\tilde{V}_{cb}|^2) \\ &= -0.136 \pm 0.125. \end{aligned} \tag{11}$$

Similarly, from the first column of CKM4, we have

$$\begin{aligned}|\tilde{V}_{td}|^2 + |\tilde{V}_{t'd}|^2 &= 1 - (|\tilde{V}_{ud}|^2 + |\tilde{V}_{cd}|^2) \\ &= -0.002 \pm 0.005. \end{aligned} \tag{12}$$

Finally, the second column of CKM4 implies

$$\begin{aligned}|\tilde{V}_{ts}|^2 + |\tilde{V}_{t's}|^2 &= 1 - (|\tilde{V}_{us}|^2 + |\tilde{V}_{cs}|^2) \\ &= -0.134 \pm 0.125. \end{aligned} \tag{13}$$

C. Vertex corrections to $Z \to b\bar{b}$

Including the QCD and QED corrections, the decay rate for $Z \to b\bar{b}$ is given by [29] =

$$\begin{aligned}&\Gamma(Z \to q\bar{q}) \\ &= \frac{\alpha m_Z}{16\sin^2\theta_W\cos^2\theta_W}(|a_q|^2 + |v_q|^2)(1 + \delta_q^{(0)})(1 + \delta_{\rm QED}^q) \\ &\quad \times (1 + \delta_{\rm QCD}^q)(1 + \delta_\mu^q)(1 + \delta_{t\rm QCD}^q)(1 + \delta_b). \end{aligned} \tag{14}$$

Here, $a_q = 2I_3^q$ and $v_q = (2I_3^q - 4|Q_q|\sin^2\theta_W)$ are the axial and vector coupling constants, respectively.

The δ terms are corrections due to various higher-order loops: =

(i) $\delta_q^{(0)}$ contains small electroweak corrections not absorbed in $\sin^2\theta_W$. Their effect is at most at the 0.5% level.

(ii) $\delta_{\rm QED}^q$ represents small final-state QED corrections that depend on the charge of final fermion. It is very small: 0.2% for the charged leptons, 0.8% for the u-type quarks and 0.02% for the d-type quarks [29].

(iii) $\delta_{\rm QCD}$ includes the QCD corrections common to all quarks; it is given by [29]

$$\delta_{\rm QCD} = \frac{\alpha_s}{\pi} + 1.41\left(\frac{\alpha_s}{\pi}\right)^2, \tag{15}$$

where α_s is the QCD coupling constant taken at the m_Z scale: $\alpha_s = \alpha_s(m_Z^2) = 0.12$.

(iv) δ_μ^q contains the kinematical effects of the external fermion masses, including some mass-dependent QCD radiative corrections. It is only important for the b quark (0.5%), and to a lesser extent for the τ lepton (0.2%) and the c quark (0.05%) [29,30].

(v) The correction $\delta_{t\rm QCD}^q$ consists of QCD contributions to the axial part of the decay and originates from doublets with large mass splitting [29,31]. In the presence of the fourth generation, it is given by [32]

$$\delta_{t\rm QCD}^q = -\frac{a_q}{v_q^2 + a_q^2}\left(\frac{\alpha_s}{\pi}\right)^2 \times [a_t f(\mu_t) + a_{t'} f(\mu_{t'}) + a_{b'} f(\mu_{b'})], \tag{16}$$

where

$$f(\mu_f) \approx \log\left(\frac{4}{\mu_f^2}\right) - 3.083 + \frac{0.346}{\mu_f^2} + \frac{0.211}{\mu_f^4}, \tag{17}$$

with $\mu_f^2 = 4m_f^2/m_Z^2$.

(vi) δ_b is nonzero only for $q = b$ and is due to the $Zb\bar{b}$ vertex loop corrections. In the presence of the fourth generation, it is given by [32]

$$\delta_b \approx 10^{-2}\left[\left(-\frac{m_t^2}{2m_Z^2} + 0.2\right)|\tilde{V}_{tb}|^2 + \left(-\frac{m_{t'}^2}{2m_Z^2} + 0.2\right)|\tilde{V}_{t'b}|^2\right]. \tag{18}$$

In order to isolate the large mass dependences appearing in the $Zb\bar{b}$ vertex δ_b, one takes the following ratio [29]:

$$R_{bb} = \left(1 + \frac{2}{R_s} + \frac{1}{R_c} + \frac{1}{R_u}\right)^{-1}, \tag{19}$$

where

$$R_s \equiv \frac{\Gamma(Z\to b\bar{b})}{\Gamma(Z\to s\bar{s})} \approx 0.9949(1+\delta_b),$$
$$R_c \equiv \frac{\Gamma(Z\to b\bar{b})}{\Gamma(Z\to c\bar{c})} \approx 0.9960\frac{(1+v_b^2)}{(1+v_c^2)}\frac{(1+\delta_{t\rm QCD}^b)}{(1+\delta_{t\rm QCD}^c)}(1+\delta_b),$$
$$R_u \equiv \frac{\Gamma(Z\to b\bar{b})}{\Gamma(Z\to u\bar{u})} \approx 0.9955\frac{(1+v_b^2)}{(1+v_c^2)}\frac{(1+\delta_{t\rm QCD}^b)}{(1+\delta_{t\rm QCD}^c)}(1+\delta_b). \tag{20}$$

Using Eqs. (16)–(20), we get (for $m_{t'} = 400$–600 GeV)

$$R_{bb} = \left[1 + \frac{3.584}{(1+\delta_b)}\right]^{-1}. \tag{21}$$

The data give [33]

$$R_{bb} = 0.216 \pm 0.001, \tag{22}$$

which, through Eq. (18), determines a linear combination of $|\tilde{V}_{tb}|^2$ and $|\tilde{V}_{t'b}|^2$. This constrains the combination $r\lambda$ of the CKM4 parameters.

We also consider constraints from the forward-backward (FB) asymmetry in $Z\to b\bar{b}$. The $Z\to b\bar{b}$ interaction Lagrangian is

$$\mathcal{L} = \frac{g}{\cos\theta_W}\bar{b}\gamma^\mu(g_{bL}P_L + g_{bR}P_R)bZ_\mu, \tag{23}$$

where $P_{L(R)}$ are the chirality projection operators, and

$$g_{bL} = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_W + \delta g_{bL}^t + \delta g_{bL}^{t'}, \tag{24}$$

$$g_{bR} = \frac{1}{3}\sin^2\theta_W + \delta g_{bR}^t + \delta g_{bR}^{t'}. \tag{25}$$

Here, the δ's represent the radiative corrections due to the t and t' quarks. The FB asymmetry in $Z\to b\bar{b}$ allows us to determine the asymmetry parameter[1]

$$A_b = \frac{g_{bL}^2 - g_{bR}^2}{g_{bL}^2 + g_{bR}^2}. \tag{26}$$

Within both the SM and SM4, only the g_{bL} terms receive corrections proportional to $m_{t,t'}^2$ at the loop level. We have [22,35–38]

$$\delta g_{bL}^t = \frac{\alpha}{16\pi\sin^2\theta_W\cos^2\theta_W}\frac{m_t^2}{m_Z^2}|\tilde{V}_{tb}|^2, \tag{27}$$

$$\delta g_{bL}^{t'} = \frac{\alpha}{16\pi\sin^2\theta_W\cos^2\theta_W}\frac{m_{t'}^2}{m_Z^2}|\tilde{V}_{t'b}|^2, \tag{28}$$

[1]The measured FB asymmetry $A_{\rm FB}^{0,b}$ is related to the asymmetry parameter A_b via $A_{\rm FB}^{0,b} \approx (3/4)A_eA_b$, where A_e is the corresponding asymmetry parameter for the electron [34]. We only consider the parameter A_b since A_e, and hence $A_{\rm FB}^{0,b}$ itself, would involve contribution from the fourth-generation lepton sector, while we would like to restrict ourselves to the quark sector in this paper.

$$g^t_{bR} = 0, \tag{29}$$

$$g^{t'}_{bR} = 0. \tag{30}$$

The data give [33]

$$A_b = 0.923 \pm 0.020, \tag{31}$$

which, through Eq. (26), constrains the combination $r\lambda$ of the CKM4 parameters.

D. The K system

Here, we present observables in various K decays with the addition of a fourth generation.

1. Indirect CP violation in $K_L \to \pi\pi$

Indirect CP violation in $K_L \to \pi\pi$ is described by the parameter ϵ_K, given by [6,39]

$$\epsilon_K = \frac{\kappa_\epsilon e^{i\phi_\epsilon}}{\sqrt{2}(\Delta M_K)_{\rm exp}} \, {\rm Im}(M^K_{12}). \tag{32}$$

$(\Delta M_K)_{\rm exp}$ is the K_L-K_S mass difference. The parameters $\phi_\epsilon = (43.51 \pm 0.05)^\circ$ and $\kappa_\epsilon = 0.92 \pm 0.02$ [39] include an additional effect from ${\rm Im}(A_0)$, where $A_0 \equiv A(K \to (\pi\pi)_{I=0})$. M^K_{12} is the off-diagonal element in the dispersive part of the amplitude for K^0-$\bar{K}^0$ mixing:

$$(M^K_{12})^* = \frac{\langle \bar{K}^0 | \mathcal{H}^{\Delta S=2}_{\rm eff} | K^0 \rangle}{2m_K}. \tag{33}$$

The calculation of M^K_{12} in the SM4 gives [40]

$$\begin{aligned} M^K_{12} = \left(\frac{G_F^2 M_W^2}{12\pi^2}\right) m_K \hat{B}_K f_K^2 [&\eta_c (\tilde{V}^*_{cd}\tilde{V}_{cs})^2 S(x_c) \\ &+ 2\eta_{ct}(\tilde{V}^*_{cd}\tilde{V}_{cs})(\tilde{V}^*_{td}\tilde{V}_{ts}) S(x_c, x_t) + \eta_t (\tilde{V}^*_{td}\tilde{V}_{ts})^2 S(x_t) \\ &+ 2\eta_{ct'}(\tilde{V}^*_{cd}\tilde{V}_{cs})(\tilde{V}^*_{t'd}\tilde{V}_{t's}) S(x_c, x_{t'}) \\ &+ 2\eta_{tt'}(\tilde{V}^*_{td}\tilde{V}_{ts})(\tilde{V}^*_{t'd}\tilde{V}_{t's}) S(x_t, x_{t'}) \\ &+ \eta_{t'} (\tilde{V}^*_{t'd}\tilde{V}_{t's})^2 S(x_{t'})]. \end{aligned} \tag{34}$$

For the decay constant and bag parameter, we take $f_K = (155.8 \pm 1.7)$ MeV [41], $\hat{B}_K = 0.725 \pm 0.026$ [41]. The Inami-Lim functions $S(x)$ and $S(x, y)$ are [42]

$$\begin{aligned} S(x) &= \frac{4x - 11x^2 + x^3}{4(1-x)^2} - \frac{3}{2}\frac{x^3 \ln x}{(1-x)^3}, \\ S(x,y) &= xy\left\{\frac{\ln y}{y-x}\left[\frac{1}{4} + \frac{3}{2}\frac{1}{1-y} - \frac{3}{4}\frac{1}{(1-y)^2}\right] \right. \\ &\quad - \frac{\ln x}{y-x}\left[\frac{1}{4} + \frac{3}{2}\frac{1}{1-x} - \frac{3}{4}\frac{1}{(1-x)^2}\right] \\ &\quad \left. - \frac{3}{4}\frac{1}{(1-x)(1-y)}\right\}, \end{aligned} \tag{35}$$

where $x = m_q^2/M_W^2$ for all quarks q.

The predictions for the short-distance QCD factors are: $\eta_c = (1.51 \pm 0.24)$ [43], $\eta_{ct} = 0.47 \pm 0.04$ [44,45], $\eta_t = 0.58$ [46]. The values for η_{ct} and η_c have a sizeable uncertainty as they are sensitive to the light scale $\sim m_c$ where α_s is large. The QCD correction factor $\eta_{t'}$ is given by [47]

$$\eta_{t'} = (\alpha_s(m_t))^{6/23}\left(\frac{\alpha_s(m_{b'})}{\alpha_s(m_t)}\right)^{6/21}\left(\frac{\alpha_s(m_{t'})}{\alpha_s(m_{b'})}\right)^{6/19}. \tag{36}$$

$\alpha_s(\mu)$ is the running coupling constant at the scale μ at next-to-leading order[48]. Here, we assume $\eta_{tt'} = \eta_{t'}$ and $\eta_{ct'} = \eta_{ct}$.

Using Eqs. (32) and (34), we get

$$\begin{aligned} \epsilon_K = \frac{G_F^2 M_W^2 f_K^2 m_K \hat{B}_K \kappa_\epsilon e^{i\phi_\epsilon}}{12\sqrt{2}\pi^2 (\Delta M_K)_{\rm exp}} {\rm Im}[&\eta_c (\tilde{V}^*_{cd}\tilde{V}_{cs})^2 S(x_c) \\ &+ 2\eta_{ct}(\tilde{V}^*_{cd}\tilde{V}_{cs})(\tilde{V}^*_{td}\tilde{V}_{ts}) S(x_c, x_t) + \eta_t (\tilde{V}^*_{td}\tilde{V}_{ts})^2 S(x_t) \\ &+ 2\eta_{ct'}(\tilde{V}^*_{cd}\tilde{V}_{cs})(\tilde{V}^*_{t'd}\tilde{V}_{t's}) S(x_c, x_{t'}) \\ &+ 2\eta_{tt'}(\tilde{V}^*_{td}\tilde{V}_{ts})(\tilde{V}^*_{t'd}\tilde{V}_{t's}) S(x_t, x_{t'}) \\ &+ \eta_{t'} (\tilde{V}^*_{t'd}\tilde{V}_{t's})^2 S(x_{t'})]. \end{aligned} \tag{37}$$

The measured value is $|\epsilon_K| = (2.32 \pm 0.007) \times 10^{-3}$ [18]. This mainly puts constraints on the combinations $\tilde{V}^*_{td}\tilde{V}_{ts}$ and $\tilde{V}^*_{t'd}\tilde{V}_{t's}$, which, to leading order in λ, depend on $A^2[1 - Ce^{i\delta_{ub}}]$ and $q^2[1 - (p/q)e^{i(\delta_{cb'} - \delta_{ub'})}]$, respectively.

2. $K^+ \to \pi^+ \nu\bar{\nu}$

The flavor-changing neutral-current quark-level transition $\bar{s} \to \bar{d}\nu\bar{\nu}$ is responsible for the decay $K^+ \to \pi^+\nu\bar{\nu}$. Unlike other K decays, $K^+ \to \pi^+\nu\bar{\nu}$ is dominated by the short-distance interactions. The long-distance (LD) contribution to ${\rm Br}(K^+ \to \pi^+\nu\bar{\nu})$ is about 3 orders of magnitude smaller than that of the short-distance [49,50]. As the decay $K^+ \to \pi^+\nu\bar{\nu}$ occurs via loops containing virtual heavy particles, it is sensitive to the fourth-generation quark t'.

The effective Hamiltonian for the decay $K^+ \to \pi^+\nu\bar{\nu}$ in the SM4 can be written as

$$\begin{aligned} \mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}}\frac{\alpha}{2\pi s_W^2}\sum_{l=e,\mu,\tau}[&\tilde{V}^*_{cs}\tilde{V}_{cd} X^l_{\rm NL} + \tilde{V}^*_{ts}\tilde{V}_{td} X(x_t) \\ &+ \tilde{V}^*_{t's}\tilde{V}_{t'd} X(x_{t'})](\bar{s}d)_{V-A}(\bar{\nu}_l\nu_l)_{V-A}. \end{aligned} \tag{38}$$

The function $X(x)$ ($x \equiv m^2_{t,t'}/M_W^2$), relevant for the t and t' pieces, is given by

$$X(x) = \eta_X X_0(x), \tag{39}$$

where

$$X_0(x) = \frac{x}{8}\left[-\frac{2+x}{1-x} + \frac{3x-6}{(1-x)^2}\ln x\right]. \tag{40}$$

Above, η_X is the next-to-leading order QCD correction; its value is estimated to be 0.994 [40]. The function corresponding to $X(x)$ in the charm sector is $X^l_{\rm NL}$:

$$X^l_{\rm NL} = C_{\rm NL} - 4B^{(1/2)}_{\rm NL}, \tag{41}$$

where $C_{\rm NL}$ and $B^{(1/2)}_{\rm NL}$ correspond to the electroweak-penguin and box contributions, respectively. The explicit forms of $C_{\rm NL}$ and $B^{(1/2)}_{\rm NL}$ are given in Refs. [48,51].

The branching ratio of $K^+ \to \pi^+ \nu\bar{\nu}$ in the SM4 is given by

$$\begin{aligned}{\rm Br}(K^+ \to \pi^+\nu\bar{\nu}) &= \kappa_+\Bigg[\bigg(\frac{{\rm Im}(\tilde{V}^*_{ts}\tilde{V}_{td})}{\lambda^5}X(x_t) + \frac{{\rm Im}(\tilde{V}^*_{t's}\tilde{V}_{t'd})}{\lambda^5}X(x_{t'})\bigg)^2 \\ &\quad + \bigg(\frac{{\rm Re}(\tilde{V}^*_{cs}\tilde{V}_{cd})}{\lambda}P_0(X) + \frac{{\rm Re}(\tilde{V}^*_{ts}\tilde{V}_{td})}{\lambda^5}X(x_t) \\ &\quad + \frac{{\rm Re}(\tilde{V}^*_{t's}\tilde{V}_{t'd})}{\lambda^5}X(x_{t'})\bigg)^2\Bigg], \end{aligned} \tag{42}$$

where

$$\begin{aligned}\kappa_+ &= r_{K+}\frac{3\alpha^2{\rm Br}(K^+\to\pi^0 e^+\nu)}{2\pi^2 s^4_W}\lambda^8, \\ P_0(X) &= \frac{1}{\lambda^4}\bigg[\frac{2}{3}X^e_{\rm NL} + \frac{1}{3}X^\tau_{\rm NL}\bigg].\end{aligned} \tag{43}$$

We see that ${\rm Br}(K^+ \to \pi^+\nu\bar{\nu})$ is related to the experimentally well-known quantity ${\rm Br}(K^+ \to \pi^0 e^+\nu)$. r_{K+} summarizes the isospin-breaking corrections in relating $K^+ \to \pi^+\nu\bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$; its value is $r_{K+} = 0.901$. κ_+ is estimated to be $(5.36 \pm 0.026) \times 10^{-11}$ [52].

The measured value is ${\rm Br}(K^+ \to \pi^+\nu\bar{\nu}) = (1.7 \pm 1.1) \times 10^{-10}$ [18]. This mainly puts constraints on the combinations $\tilde{V}^*_{td}\tilde{V}_{ts}$ and $\tilde{V}^*_{t'd}\tilde{V}_{t's}$, which, to leading order in λ, depend on $A^2[1 - Ce^{i\delta_{ub}}]$ and $q^2[1 - (p/q)e^{i(\delta_{cb'}-\delta_{ub'})}]$, respectively.

E. The *B* system

Here, we present various observables in the B system with the addition of a fourth generation.

1. $B^0_{d,s}$-$\bar{B}^0_{d,s}$ mixing

Meson-antimeson mixing occurs in the SM through the box diagram, and is thus sensitive to new heavy particles appearing in the loop. Within the three-generation SM, the dominant contribution to B^0_q-$\bar{B}^0_q$ mixing ($q = d, s$) comes from the virtual top quark. The charm and the mixed top-charm contributions are negligibly small, and hence the analysis is simplified considerably. In the SM4, there is an additional contribution due to the virtual t' in the box diagram.

The mass difference ΔM_q is given by $\Delta M \simeq 2|M^q_{12}|$, where M^q_{12} is the virtual part of the box diagrams responsible for the mixing. For B^0_q–$\bar{B}^0_q$ mixing, M^q_{12} in the SM4 is given by

$$\begin{aligned}M^q_{12} &= \frac{G^2_F M^2_W}{12\pi^2}m_{B_q}\hat{B}_{bq}f^2_{B_q}[\eta_t(\tilde{V}^*_{tq}\tilde{V}_{tb})^2 S(x_t) \\ &\quad + \eta_{t'}(\tilde{V}^*_{t'q}\tilde{V}_{t'b})^2 S(x_{t'}) \\ &\quad + 2\eta_{tt'}(\tilde{V}^*_{tq}\tilde{V}_{tb})(\tilde{V}^*_{t'q}\tilde{V}_{t'b})S(x_t, x_{t'})],\end{aligned} \tag{44}$$

where $x_t = m^2_t/m^2_W$, $x_{t'} = m^2_{t'}/M^2_W$. The Inami-Lim functions $S(x)$ and $S(x, y)$ are given in Eq. (35). Here, we assume $\eta_{t'} = \eta_{tt'}$ for simplicity. The numerical values of the structure functions $S(x_{t'})$, $S(x_t, x_{t'})$ and the QCD correction factor $\eta_{t'}$ for various t' mass are given in Ref. [5]. In order to reduce the sizeable nonperturbative uncertainties due to the decay constant f_{B_q} and the bag parameter $\hat{B}_{bq}$, we consider the ratio $\Delta M_s/\Delta M_d$:

$$\frac{\Delta M_s}{\Delta M_d} = \frac{m_{B_s}}{m_{B_d}}\xi^2 \times M_R, \tag{45}$$

where

$$M_R = \frac{|\eta_t(\tilde{V}^*_{ts}\tilde{V}_{tb})^2 S(x_t) + \eta_{t'}(\tilde{V}^*_{t's}\tilde{V}_{t'b})^2 S(x_{t'}) + 2\eta_{tt'}(\tilde{V}^*_{ts}\tilde{V}_{tb})(\tilde{V}^*_{t's}\tilde{V}_{t'b})S(x_t, x_{t'})|}{|\eta_t(\tilde{V}^*_{td}\tilde{V}_{tb})^2 S(x_t) + \eta_{t'}(\tilde{V}^*_{t'd}\tilde{V}_{t'b})^2 S(x_{t'}) + 2\eta_{tt'}(\tilde{V}^*_{td}\tilde{V}_{tb})(\tilde{V}^*_{t'd}\tilde{V}_{t'b})S(x_t, x_{t'})|}, \tag{46}$$

with $\xi \equiv f_{B_s}\sqrt{\hat{B}_{bs}}/f_{B_d}\sqrt{\hat{B}_{bd}} = 1.243 \pm 0.028$ [41]. There is less uncertainty in ξ (~ 2–3%) than in $f_{B_q}\sqrt{\hat{B}_{bq}}$ (~ 7–8%).

The measured values are [18]

$$\begin{aligned}\Delta M_s &= (17.77 \pm 0.12)\ {\rm ps}^{-1}, \\ \Delta M_d &= (0.507 \pm 0.005)\ {\rm ps}^{-1},\end{aligned} \tag{47}$$

whose ratio is sensitive to $\tilde{V}^*_{tb}\tilde{V}_{ts}$, $\tilde{V}^*_{tb}\tilde{V}_{td}$, $\tilde{V}^*_{t'b}\tilde{V}_{t's}$, and $\tilde{V}^*_{t'b}\tilde{V}_{t'd}$. These correspond to the combinations of the CKM4 parameters A^2, $A^2(1 - Ce^{i\delta_{ub}})$, $qre^{i\delta_{cb'}}$, and $r(qe^{i\delta_{cb'}} - pe^{i\delta_{ub'}})$, respectively, to leading order in λ.

2. *CP* violation

CP violation in the quark sector is due to phases in the quark-mixing matrix. In the three-generation SM, the phase information in the CKM matrix is elegantly encapsulated in the unitarity triangle [18], whose interior angles are α, β and γ. In order to test the SM, these angles must be measured in as many ways as possible to test for consistency. Unknown strong QCD phases contaminate many of these methods; ways of removing these strong phases must be devised in order to cleanly measure the weak phases. In the SM4, many of the ways of eliminating the strong phases fail, since typically there are multiple amplitudes with different strong and weak phases. Here, we consider only those constraints from CP observables which are free from uncertainties due to the strong phases.

(i) $S_{J/\psi K_S}$: The coefficient of $\sin(\Delta M_d t)$ in the time-dependent indirect CP asymmetry in $B_d^0 \to J/\psi K_S$ is given by

$$S_{J/\psi K_S} = \sin 2\phi_{B_d}^{\rm tot}, \tag{48}$$

where $\phi_{B_d}^{\rm tot}$ is defined as

$$M_{12}^d = |M_{12}^d| e^{i2\phi_{B_d}^{\rm tot}}. \tag{49}$$

Thus, we have

$$S_{J/\psi K_S} = \frac{{\rm Im}(M_{12}^d)}{|M_{12}^d|}. \tag{50}$$

In the SM, this is $\sin 2\beta$, and is free of strong phases. It is thus a good observable to constrain the SM4 using Eq. (44). The measured value is [18]

$$S_{J/\psi K_S} = 0.672 \pm 0.024, \tag{51}$$

which is sensitive to $\tilde{V}_{tb}^* \tilde{V}_{td}$ and $\tilde{V}_{t'b}^* \tilde{V}_{t'd}$, i.e. to the parameter combinations $A^2(1 - C e^{i\delta_{ub}})$ and $r(q e^{i\delta_{cb'}} - p e^{i\delta_{ub'}})$, respectively.

(ii) Recently, the CDF and D0 collaborations measured indirect CP violation in $B_s^0 \to J/\psi \phi$ and found a 2.2σ deviation from the prediction of the SM [53]. At first sight, this seems to indicate a nonzero phase of B_s^0-$\bar{B}_s^0$ mixing, and many papers have been written exploring the contribution of particular new-physics models to this mixing (including the fourth generation [4]). However, there could be a significant contribution to this signal from new physics in the decay [54]. If this is the case, strong phases will play a role. For this reason, the constraints from this measurement are not included in the fit.

(iii) In the SM, $\gamma \equiv {\rm Arg}(-V_{ub}^* V_{ud})/(V_{cb}^* V_{cd})$. This phase can be probed in tree-level decays. By measuring several different decays, it is possible to remove the dependence on the strong phase and extract γ. The latest value is [18]

$$\gamma = (75.0 \pm 22.0)^\circ. \tag{52}$$

Because this angle is measured in tree-level decays, its value is unchanged with the addition of a fourth generation. Indeed, from Eqs. (4) and (6), we see that

$${\rm Arg}\left(-\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}}\right) \approx {\rm Arg}\left(-\frac{\tilde{V}_{ub}^* \tilde{V}_{ud}}{\tilde{V}_{cb}^* \tilde{V}_{cd}}\right) \approx \delta_{ub}. \tag{53}$$

Thus, the phase δ_{ub} can be constrained through the measurement of the weak phase γ, and this observable is included in the fit.

3. $B \to X_s \gamma$

The quark-level transition $\bar{b} \to \bar{s}\gamma$ induces the inclusive radiative decay $B \to X_s \gamma$. This decay can occur only at the loop level and hence is suppressed within the SM. It has been observed with a branching ratio of $(3.55 \pm 0.25) \times 10^{-4}$ [55], in good agreement with the next-to-next-to-leading order SM prediction of $(3.15 \pm 0.23) \times 10^{-4}$ [56]. Thus, $B \to X_s \gamma$ has a great potential to constrain new-physics models.

Within the SM, the effective Hamiltonian for the quark-level transition $\bar{b} \to \bar{s}\gamma$ can be written as

$$\mathcal{H}_{\rm eff} = \frac{4G_F}{\sqrt{2}} V_{ts} V_{tb}^* \sum_{i=1}^{8} C_i(\mu) Q_i(\mu), \tag{54}$$

where the form of the operators $O_i(\mu)$ and the expressions for calculating the Wilson coefficients $C_i(\mu)$ are given in Ref. [57]. The introduction of a fourth generation, in addition to the modifications $V_{ts} \to \tilde{V}_{ts}$ and $V_{tb} \to \tilde{V}_{tb}$, also changes the values of the Wilson coefficients $C_{7,8}$ via the virtual exchange of the t'-quark. They can be written as

$$C_{7,8}^{\rm tot}(\mu) = C_{7,8}(\mu) + \frac{\tilde{V}_{t'b}^* \tilde{V}_{t's}}{\tilde{V}_{tb}^* \tilde{V}_{ts}} C_{7,8}^{t'}(\mu). \tag{55}$$

The values of $C_{7,8}^{t'}$ can be calculated from the expressions for $C_{7,8}$ by replacing m_t by $m_{t'}$.

In order to reduce the large uncertainties arising from b-quark mass, we consider the following ratio:

$$R = \frac{{\rm Br}(B \to X_s \gamma)}{{\rm Br}(B \to X_c e \bar{\nu}_e)}.$$

In leading logarithmic approximation, this ratio can be written as [40]

$$R = \frac{|\tilde{V}_{tb}^* \tilde{V}_{ts}|^2}{|\tilde{V}_{cb}|^2} \frac{6\alpha |C_7^{\rm tot}(m_b)|^2}{\pi f(\hat{m}_c) \kappa(\hat{m}_c)}. \tag{56}$$

Here, the Wilson coefficient C_7 is evaluated at the scale $\mu = m_b$. The phase-space factor $f(\hat{m}_c)$ in ${\rm Br}(B \to X_c e \bar{\nu})$ is given by [58]

$$f(\hat{m}_c) = 1 - 8\hat{m}_c^2 + 8\hat{m}_c^6 - \hat{m}_c^8 - 24\hat{m}_c^4 \ln \hat{m}_c, \tag{57}$$

where $\hat{m}_c = m_c/m_b$. $\kappa(\hat{m}_c)$ is the 1-loop QCD correction factor [58]:

$$\kappa(\hat{m}_c) = 1 - \frac{2\alpha_s(m_b)}{3\pi}\left[\left(\pi^2 - \frac{31}{4}\right)(1-\hat{m}_c)^2 + \frac{3}{2}\right]. \quad (58)$$

The values of the branching ratios in R are $\mathrm{Br}(B \to X_s\gamma) = (3.55 \pm 0.25) \times 10^{-4}$ [18] and $\mathrm{Br}(B \to X_c e\bar{\nu}) = 0.1061 \pm 0.0016 \pm 0.0006$ [59]), the ratio being sensitive to $\tilde{V}^*_{tb}\tilde{V}_{ts}$ and $\tilde{V}^*_{t'b}\tilde{V}_{t's}$, i.e. to A^2 and $qre^{i\delta_{cb'}}$, respectively, to leading order in λ.

4. $B \to X_s l^+ l^-$

The effective Hamiltonian for the quark-level transition $\bar{b} \to \bar{s} l^+ l^-$ in the SM can be written as

$$\mathcal{H}_{\mathrm{eff}} = \frac{4G_F}{\sqrt{2}} V_{ts}V^*_{tb}\sum_{i=1}^{10} C_i(\mu)Q_i(\mu), \quad (59)$$

where the form of the operators Q_i and the expressions for calculating the coefficients C_i are given in Ref. [57]. The fourth generation, in addition to the modifications $V_{ts} \to \tilde{V}_{ts}$ and $V_{tb} \to \tilde{V}_{tb}$, changes the values of the Wilson coefficients $C_{7,8,9,10}$ via the virtual exchange of the t'. The Wilson coefficients in the SM4 can then be written as

$$C_i^{\mathrm{tot}}(\mu_b) = C_i(\mu_b) + \frac{\tilde{V}^*_{t'b}\tilde{V}_{t's}}{\tilde{V}^*_{tb}\tilde{V}_{ts}} C_i^{t'}(\mu_b), \quad (60)$$

where $i = 7, 8, 9, 10$. The new Wilson coefficients $C_i^{t'}(\mu_b)$ can easily be calculated by substituting $m_{t'}$ for m_t in the SM expressions involving the t quark.

The calculation of the differential decay rate gives

$$\frac{d\mathrm{Br}(B \to X_s l^+ l^-)}{dz} = \frac{\alpha^2 \mathrm{Br}(B \to X_c e\bar{\nu})}{4\pi^2 f(\hat{m}_c)\kappa(\hat{m}_c)} \frac{|\tilde{V}^*_{tb}\tilde{V}_{ts}|^2}{|\tilde{V}_{cb}|^2}(1-z)^2 D(z), \quad (61)$$

where

$$D(z) = (1+2z)(|C_9^{\mathrm{tot}}|^2 + |C_{10}^{\mathrm{tot}}|^2) + 4\left(1 + \frac{2}{z}\right)|C_7^{\mathrm{tot}}|^2 + 12\,\mathrm{Re}(C_7^{\mathrm{tot}} C_9^{\mathrm{tot}*}). \quad (62)$$

Here, $z \equiv q^2/m_b^2$ and $\hat{m}_q = m_q/m_b$ for all quarks q. The expressions for the phase-space factor $f(\hat{m}_c)$ and the 1-loop QCD correction factor $\kappa(\hat{m}_c)$ are given in Eqs. (57) and (58), respectively.

The theoretical prediction for the branching ratio of $B \to X_s l^+ l^-$ in the intermediate q^2 region (7 GeV$^2 \le q^2 \le 12$ GeV^2) is rather uncertain due to the nearby charmed resonances. The predictions are relatively cleaner in the low-q^2 (1 GeV$^2 \le q^2 \le 6$ GeV^2) and the high-q^2 (14.4 GeV$^2 \le q^2 \le m_b^2$) regions. Hence, we consider both low-q^2 and high-q^2 regions in the fit. The branching ratios are [60,61]

$$\mathrm{Br}(B \to X_s l^+ l^-)_{\mathrm{low}q^2} = (1.60 \pm 0.50) \times 10^{-6}, \quad (63)$$

$$\mathrm{Br}(B \to X_s l^+ l^-)_{\mathrm{high}q^2} = (0.44 \pm 0.12) \times 10^{-6}. \quad (64)$$

Both of these branching ratios are sensitive to $\tilde{V}^*_{tb}\tilde{V}_{ts}$ and $\tilde{V}^*_{t'b}\tilde{V}_{t's}$, i.e. to A^2 and $qre^{i\delta_{cb'}}$, respectively, to leading order in λ.

F. The D system

In principle, there can be constraints from D^0-$\bar{D}^0$ mixing. In the SM, this mixing arises due to d, s and b quarks in the box diagram. The b contribution is enhanced by a factor of $(m_b^2 - m_{s,d}^2)/(m_s^2 - m_d^2)$. On the other hand, it suffers a strong CKM suppression by a factor of $|V_{ub}V^*_{cb}|^2/|V_{us}V^*_{cs}|^2$ which is $\sim\lambda^8$. Thus, D^0-$\bar{D}^0$ mixing is dominated by the d- and s-quark contributions. As a result, the mixing is small within the SM and hence sensitive to new physics.

There have been attempts to constrain the CKM4 parameters using D^0-$\bar{D}^0$ mixing (for example, see Ref. [62]). However, precisely because the d and s quarks dominate, there can be large LD contributions to the mixing. At present, there is no definitive estimate of these LD effects. Because of this, we do not have an accurate enough prediction for D^0-$\bar{D}^0$ mixing, and this measurement cannot be incorporated in the fit at present.

IV. RESULTS OF THE FIT

We perform the fit to 9 CKM4 parameters, using the observables described in the previous section. We define

$$\chi^2_{\mathrm{total}} = \chi^2_{\mathrm{CKM}} + \chi^2_{\mathrm{UC}} + \chi^2_{Zbb} + \chi^2_{ZAb} + \chi^2_{|\epsilon_K|} + \chi^2_{K^+\to\pi^+\nu\bar{\nu}} + \chi^2_{\mathrm{mixing}} + \chi^2_{\sin 2\beta} + \chi^2_{\gamma} + \chi^2_{B\to X_s\gamma} + \chi^2_{\mathrm{incl\text{-}low}} + \chi^2_{\mathrm{incl\text{-}high}}, \quad (65)$$

where the exact definition of each χ^2 contribution is given in the Appendix. We perform the fit at two values of t' mass: $m_{t'} = 400$ GeV and $m_{t'} = 600$ GeV. In addition, we also perform a fit for the 4 parameters of the CKM matrix in the SM, in order to check for consistency with the standard fit. The results of these fits are summarized in Table I. It may be observed that the χ^2 per degree of freedom is small in each case, indicating that all the fits are good. The goodness-of-fit does not seem to depend much on the masses of the heavy quarks.

The fit for the SM is consistent with that obtained in Ref. [18]. As far as the parameters of the three-generation CKM matrix are concerned, their best-fit values are not affected much by the addition of a fourth generation. However, the allowed parameter space for C and δ_{ub} expands by almost a factor of 4. This is expected, since the constraint on $|\tilde{V}_{ub}|$ from the 3×3 unitarity is now relaxed.

On the other hand, the new real parameters p, q, r are consistent with zero, which is not surprising since the SM fit is a good one. This also is consistent with the

TABLE I. The results of the fit to the parameters of CKM and CKM4.

Parameter	SM	$m_{t'} = 400$ GeV	$m_{t'} = 600$ GeV
λ	0.227 ± 0.001	0.227 ± 0.001	0.227 ± 0.001
A	0.808 ± 0.021	0.801 ± 0.022	0.801 ± 0.022
C	0.38 ± 0.01	0.42 ± 0.04	0.42 ± 0.04
δ_{ub}	1.16 ± 0.06	1.24 ± 0.23	1.22 ± 0.24
p	-	1.45 ± 1.20	1.35 ± 1.56
q	-	0.16 ± 0.12	0.12 ± 0.07
r	-	0.30 ± 0.37	0.19 ± 0.27
$\delta_{ub'}$	-	1.21 ± 1.59	1.32 ± 1.76
$\delta_{cb'}$	-	1.10 ± 1.64	1.25 ± 1.81
χ^2/d.o.f.	6.64/14	6.01/11	6.06/11

observation that no meaningful constraints are obtained on the new phases $\delta_{ub'}$ and $\delta_{cb'}$: since vanishing p, q imply vanishing $\tilde{V}_{ub'}$, $\tilde{V}_{cb'}$, respectively, the phases of these two CKM4 elements have no significance.

For $m_{t'} = 400$ GeV, the maximum values of the parameters (p, q, r) are (2.65, 0.28, 0.67) to 1σ. For $m_{t'} = 600$ GeV, the 1σ upper bounds are (2.91, 0.19, 0.46). This indicates that these quantities are indeed $\mathcal{O}(1)$ or smaller, so that the expansion in λ in the DK parametrization is justified.

The magnitudes of CKM4 elements are of special interest, since the off-diagonal elements are indicative of the mixing between generations. Table II gives the allowed ranges for the magnitudes of CKM4 elements, obtained using the fit results in Table I. Clearly, the extension to four generations only expands the allowed ranges of the CKM parameters, while the allowed values of all of the new parameters of CKM4 (except $\tilde{V}_{t'b'}$) are consistent with zero.

The combinations of CKM4 matrix elements that control B_d-$\bar{B}_d$ and B_s-$\bar{B}_s$ mixing are $\tilde{V}^*_{tb}\tilde{V}_{td}$, $\tilde{V}^*_{tb}\tilde{V}_{ts}$, $\tilde{V}_{t'b}\tilde{V}_{t'd}$ and $\tilde{V}_{t'b}\tilde{V}_{t's}$. The allowed ranges of these quantities are given in Table III. It may be observed that here the fourth generation can have maximal impact. While $|\tilde{V}^*_{tb}\tilde{V}_{ts}|$ is

TABLE II. Magnitudes of the CKM4 elements obtained from the fit.

Magnitude	SM	$m_{t'} = 400$ GeV	$m_{t'} = 600$ GeV
$\vert\tilde{V}_{ud}\vert$	0.9743 ± 0.0002	0.9743 ± 0.0002	0.9743 ± 0.0002
$\vert\tilde{V}_{us}\vert$	0.227 ± 0.001	0.227 ± 0.001	0.227 ± 0.001
$\vert\tilde{V}_{ub}\vert$	$(3.55 \pm 0.17) \times 10^{-3}$	$(3.90 \pm 0.38) \times 10^{-3}$	$(3.91 \pm 0.39) \times 10^{-3}$
$\vert\tilde{V}_{ub'}\vert$	-	0.017 ± 0.014	0.016 ± 0.018
$\vert\tilde{V}_{cd}\vert$	0.227 ± 0.001	0.227 ± 0.001	0.227 ± 0.001
$\vert\tilde{V}_{cs}\vert$	0.9743 ± 0.0002	0.9743 ± 0.0002	0.9743 ± 0.0002
$\vert\tilde{V}_{cb}\vert$	0.042 ± 0.001	0.041 ± 0.001	0.041 ± 0.001
$\vert\tilde{V}_{cb'}\vert$	-	$(8.4 \pm 6.2) \times 10^{-3}$	$(6.0 \pm 3.8) \times 10^{-3}$
$\vert\tilde{V}_{td}\vert$	0.0086 ± 0.0003	0.009 ± 0.002	0.009 ± 0.001
$\vert\tilde{V}_{ts}\vert$	0.041 ± 0.001	0.041 ± 0.001	0.040 ± 0.001
$\vert\tilde{V}_{tb}\vert$	1	0.998 ± 0.006	0.999 ± 0.003
$\vert\tilde{V}_{tb'}\vert$	-	0.07 ± 0.08	0.04 ± 0.06
$\vert\tilde{V}_{t'd}\vert$	-	0.01 ± 0.01	0.01 ± 0.02
$\vert\tilde{V}_{t's}\vert$	-	0.01 ± 0.01	0.004 ± 0.010
$\vert\tilde{V}_{t'b}\vert$	-	0.07 ± 0.08	0.04 ± 0.06
$\vert\tilde{V}_{t'b'}\vert$	-	0.998 ± 0.006	0.999 ± 0.003

TABLE III. Combinations of CKM4 elements that control mixing in the B_d and B_s sectors.

Quantity	SM	$m_{t'} = 400$ GeV	$m_{t'} = 600$ GeV
$\vert\tilde{V}^*_{tb}\tilde{V}_{td}\vert$	0.0086 ± 0.0003	0.009 ± 0.002	0.009 ± 0.001
$\mathrm{Arg}(\tilde{V}^*_{tb}\tilde{V}_{td})$	$(-21.5 \pm 1.0)^\circ$	$(-30.4 \pm 10.3)^\circ$	$(-27.9 \pm 8.0)^\circ$
$\vert\tilde{V}^*_{tb}\tilde{V}_{ts}\vert$	0.041 ± 0.001	0.040 ± 0.001	0.040 ± 0.001
$\mathrm{Arg}(\tilde{V}^*_{tb}\tilde{V}_{ts})$	$(-178.86 \pm 0.06)^\circ$	$(-178.12 \pm 1.14)^\circ$	$(-178.12 \pm 0.57)^\circ$
$\vert\tilde{V}^*_{t'b}\tilde{V}_{t'd}\vert$	-	0.0010 ± 0.0015	0.0006 ± 0.0011
$\mathrm{Arg}(\tilde{V}^*_{t'b}\tilde{V}_{t'd})$	-	$(-107.1 \pm 106.5)^\circ$	$(-102.5 \pm 112.8)^\circ$
$\vert\tilde{V}^*_{t'b}\tilde{V}_{t's}\vert$	-	0.0005 ± 0.0010	0.0002 ± 0.0005
$\mathrm{Arg}(\tilde{V}^*_{t'b}\tilde{V}_{t's})$	-	$(37.8 \pm 120.3)^\circ$	$(40.1 \pm 174.1)^\circ$

little affected, the allowed range of $|\tilde{V}_{tb}^* \tilde{V}_{td}|$ is increased by up to a factor of 6–7. Moreover, the allowed range of the phase of $\tilde{V}_{tb}^* \tilde{V}_{td}$ is expanded by $\sim$10, while that of the phase of $\tilde{V}_{tb}^* \tilde{V}_{ts}$ is larger by a factor of $\sim$20 at $m_{t'} = 400$ GeV. Since the phases of $\tilde{V}_{t'b}^* \tilde{V}_{t'd}$ and $\tilde{V}_{t'b}^* \tilde{V}_{t's}$ are essentially unconstrained, they can influence B_d and B_s mixing to a large extent. In particular, the B_s-$\bar{B}_s$ mixing phase can be very large, as suggested by the recent measurements from $B_s \to J/\psi \phi$ decays [63]. The combinations $\tilde{V}_{ts}^* \tilde{V}_{td}$ and $\tilde{V}_{t's}^* \tilde{V}_{t'd}$ also contribute to rare K decays, and hence significant deviations of these quantities from the SM can also leave their imprints in the rare K decays.

V. DISCUSSION

In this paper, we consider the extension of the standard model (SM) to four generations. Using input from many flavor-physics processes, we perform a χ^2 fit to constrain the elements of the 4×4 CKM quark-mixing matrix (CKM4). The fit takes into account both experimental errors and theoretical uncertainties. Although we do not include the oblique parameters in our fit, we do take values for the masses of the fourth-generation quarks that are consistent with the oblique corrections.

At this stage, several comments are in order.

(i) The best-fit values of all three new real parameters of the CKM4 matrix are consistent with zero. Since the fit to the SM is also excellent—$\chi^2/\text{d.o.f.} = 6.64/14$, corresponding to a goodness-of-fit of 92%—we must conclude that the addition of a fourth generation is not necessary to get a better fit to the data.

(ii) We find $\tilde{V}_{tb} = 0.998 \pm 0.006$ for $m_{t'} = 400$ GeV and $\tilde{V}_{tb} = 0.999 \pm 0.003$ for $m_{t'} = 600$ GeV. Thus, at 3σ, we have $\tilde{V}_{tb} \geq 0.98$. Therefore, the SM4 cannot account for any large deviation of V_{tb} from unity.

(iii) In many previous analyses, it is mentioned that any mixing between the third and fourth generations is small. We find that this is indeed the case—the results of the fit constrain the matrix elements describing the mixing of the ordinary and fourth-generation quarks to be $|\tilde{V}_{ub'}| < 0.06$, $|\tilde{V}_{cb'}| < 0.027$, and $|\tilde{V}_{tb'}| < 0.31$ at 3σ.

(iv) However, the allowed parameter ranges still allow large deviations from the SM as far as the magnitudes and phases of the quantities $\tilde{V}_{tb}^* \tilde{V}_{td}$ and $\tilde{V}_{tb}^* \tilde{V}_{ts}$ are concerned. With additional new-physics contributions involving $\tilde{V}_{t'b} \tilde{V}_{t'd}$ and $\tilde{V}_{t'b} \tilde{V}_{t's}$, it may be possible to get significant new-physics signals in B_d and/or B_s mixing, which could be the most incisive probes of the fourth generation.

(v) The value of $|V_{ub}|$ required to explain the recent measurement of $\text{Br}(B^+ \to \tau^+ \nu_\tau)$ is 2.8σ larger than that obtained from the global fit to $|V_{ub}|$ otherwise, within the SM [64]. Our fit indicates that the best fit for $|\tilde{V}_{ub}|$ shifts to higher values with SM4, and the error on this quantity also increases. As a result, the SM4 may be able to account for this measurement much better than the SM.

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APPENDIX: THE χ^2 FUNCTION

We define our χ^2 function to be

$$\chi^2_{\text{total}} = \chi^2_{\text{CKM}} + \chi^2_{\text{UC}} + \chi^2_{Zbb} + \chi^2_{ZAb} + \chi^2_{|\epsilon_K|} + \chi^2_{K^+ \to \pi^+ \nu\bar{\nu}} + \chi^2_{\text{mixing}} + \chi^2_{\sin 2\beta} + \chi^2_{\gamma} + \chi^2_{B \to X_s \gamma} + \chi^2_{\text{incl-low}} + \chi^2_{\text{incl-high}}. \tag{A1}$$

The components of this function are defined below.

(i) For the direct measurements of the magnitudes of the elements,

$$\chi^2_{\text{CKM}} = \left(\frac{|\tilde{V}_{us}| - 0.2255}{0.0019}\right)^2 + \left(\frac{|\tilde{V}_{ud}| - 0.97418}{0.00027}\right)^2 + \left(\frac{|\tilde{V}_{cs}| - 1.04}{0.06}\right)^2 + \left(\frac{|\tilde{V}_{cd}| - 0.230}{0.011}\right)^2 + \left(\frac{|\tilde{V}_{ub}| - 0.00393}{0.00036}\right)^2 + \left(\frac{|\tilde{V}_{cb}| - 0.0412}{0.0011}\right)^2. \tag{A2}$$

$$\chi^2_{\text{UC}} = \left(\frac{|\tilde{V}_{ub'}|^2 - 0.0001}{0.0011}\right)^2 + \left(\frac{|\tilde{V}_{cb'}|^2 + 0.136}{0.125}\right)^2 + \left(\frac{(|\tilde{V}_{td}|^2 + |\tilde{V}_{t'd}|^2) + 0.002}{0.005}\right)^2 + \left(\frac{(|\tilde{V}_{ts}|^2 + |\tilde{V}_{t's}|^2) + 0.134}{0.125}\right)^2. \tag{A3}$$

(ii) For the $Z \to b\bar{b}$ decay,

$$\chi^2_{Zbb} = \left(\frac{R_{bb} - 0.216}{0.001}\right)^2, \tag{A4}$$

where R_{bb} is defined in Eq. (21) and

$$\chi^2_{ZAb} = \left(\frac{A_b - 0.923}{0.020}\right)^2, \tag{A5}$$

where A_b is defined in Eq. (26).

(iii) For K mixing,

$$\chi^2_{|\epsilon_K|} = \left(\frac{|\epsilon_K| - 0.00232}{0.00046}\right)^2, \tag{A6}$$

where ϵ_K is defined in Eq. (37). Here, experimental and theoretical errors are added in quadrature.

(iv) Next, we have

$$\chi^2_{K^+\to\pi^+\nu\bar{\nu}} = \left(\frac{\text{Br}(K^+ \to \pi^+\nu\bar{\nu}) - 1.7 \times 10^{-10}}{1.1 \times 10^{-10}}\right)^2, \tag{A7}$$

with $\text{Br}(K^+ \to \pi^+\nu\bar{\nu})$ as in Eq. (42).

(v) In B-meson mixing,

$$\chi^2_{\text{mixing}} = \left(\frac{M_R - 22.20}{1.04}\right)^2, \tag{A8}$$

with M_R as defined in Eq. (46) and

$$\frac{\Delta M_s}{\Delta M_d}\frac{m_{B_d}}{m_{B_s}}\frac{1}{\xi^2} = 22.40 \pm 1.04. \tag{A9}$$

(vi) For CP violation in $B_d \to J/\psi K_S$,

$$\chi^2_{\sin 2\beta} = \left(\frac{S_{J/\psi K_S} - 0.672}{0.024}\right)^2, \tag{A10}$$

with $S_{J/\psi K_S}$ defined as in Eq. (50).

(vii) For the CKM angle γ

$$\chi^2_\gamma = \left(\frac{\delta_{ub} - 75(\pi/180)}{22(\pi/180)}\right)^2. \tag{A11}$$

(viii) For the radiative decay

$$\chi^2_{B\to X_s\gamma} = \left(\frac{100R - 0.330}{0.041}\right)^2, \tag{A12}$$

with R as defined in Eq. (56). Here, experimental and theoretical errors are added in quadrature.

(ix) For the leptonic decay,

$$\chi^2_{\text{incl-low}} = \left(\frac{\text{Br}(B \to X_s l^+ l^-)_{\text{low}q^2} \times 10^6 - 1.6}{0.55}\right)^2. \tag{A13}$$

$\text{Br}(B \to X_s l^+ l^-)_{\text{low}q^2}$ has been obtained by integrating Eq. (61) within the limits (1 $\text{GeV}^2 \le q^2 \le$ 6 GeV^2). Here, experimental and theoretical errors are added in quadrature.

(x) Similarly,

$$\chi^2_{\text{incl-high}} = \left(\frac{\text{Br}(B \to X_s l^+ l^-)_{\text{high}q^2} \times 10^6 - 0.44}{0.14}\right)^2, \tag{A14}$$

where $\text{Br}(B \to X_s l^+ l^-)_{\text{high}q^2}$ has been obtained by integrating Eq. (61) within the limits (14.4 $\text{GeV}^2 \le q^2 \le m_b^2$).

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