# Metrics with Galilean conformal isometry

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The Galilean conformal algebra (GCA) arises in taking the nonrelativistic limit of the symmetries of a relativistic conformal field theory in any dimensions. It is known to be infinite dimensional in all spacetime dimensions. In particular, the 2d GCA emerges out of a scaling limit of linear combinations of two copies of the Virasoro algebra. In this paper, we find metrics in dimensions greater than 2 which realize the finite 2d GCA (the global part of the infinite algebra) as their isometry by systematically looking at a construction in terms of cosets of this finite algebra. We list all possible subalgebras consistent with some physical considerations motivated by earlier work in this direction and construct all possible higher-dimensional nondegenerate metrics. We briefly study the properties of the metrics obtained. In the standard one higher-dimensional ‘‘holographic’’ setting, we find that the only nondegenerate metric is Minkowskian. In four and five dimensions, we find families of nontrivial metrics with a rather exotic signature. A curious feature of these metrics is that all but one of them are Ricci-scalar flat.

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## I. INTRODUCTION

Nonrelativistic conformal theories have received a lot of recent attention in connection with the AdS/CFT conjecture, more generally, the gauge-gravity duality. The most popular of the versions of this nonrelativistic gauge-gravity duality has been the one studied in the context of the Schrödinger algebra. The Schrödinger algebra is the largest symmetry algebra of the free Schrödinger equation [1,2] and has been observed in cold atom systems at unitarity [3]. Gravity duals of a certain class of field theories possessing Schrödinger symmetry have been proposed in [4,5], and now there is extensive literature in this line of research, some of which can be found in the excellent review [6]. Another popular venue of research in this field has been in relation to spacetime with Lifshitz symmetry, as proposed in [7], which unlike the Schrödinger case, does not exhibit invariance under Galilean boosts and hence does not contain the Galilean group as part of the symmetry algebra.

In [8], a different direction to nonrelativistic AdS/CFT was proposed by focusing on a systematic limiting procedure of the relativistic symmetry group. The relativistic conformal algebra on the boundary was parametrically contracted to what is called the Galilean conformal algebra (GCA). One of the remarkable observations here was that the GCA could be given an infinite-dimensional lift for any spacetime dimensions. It was also observed that the GCA was important to the study of nonrelativistic hydrodynamics. Specifically, the finite-dimensional GCA is the symmetry algebra of the Euler equations, which is valid in cases where the fluid viscosity can be neglected. There have been further studies of the various aspects of the GCA in [9–12].

The gravity dual of the GCA was proposed initially to be a novel Newton-Cartan–like $AdS_2 \times R^d$ in [8]. The systematic limit, when performed on the parent AdS metric, leads to a degeneration. Hence the proposal was that when one looked for a standard one dimension higher holographic construction, there would be no nondegenerate spacetime metric and the theory described in terms of connections would be a geometrized version of Newtonian gravity. We should, at this point, remind the reader that in the case of the Schrödinger algebra, the gravity dual was found in a two-dimensional higher spacetime. The question of finding a metric with the Galilean conformal isometry in higher dimensions remains. Recently, in [13], a connection between asymptotically flat spaces and the GCA has been established. The 2d infinite-dimensional GCA was shown to be isomorphic to the Bondi-Metzner-Sachs (BMS) algebra [14] in three dimensions, which is the group of asymptotic isometries of flat three-dimensional space at null infinity [15]. The two different points of view are seemingly at loggerheads, and one of the issues that we address in this paper is this apparent confusion.

The basic philosophy behind constructing the gravity duals of nonrelativistic field theories is to realize the corresponding symmetry group as the isometry group of a spacetime metric. We attempt to find all possible higher (greater than two)-dimensional metrics possessing the Galilean conformal isometry by a process of coset construction. In the context of nonrelativistic Gauge-gravity duality, the authors of [16] have shown that under some ‘‘physical’’ conditions the metrics obtained by this method uniquely reproduce the holographic constructions with Schrödinger and Lifshitz isometries. This procedure has

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also been followed in [17], in relation to the aging algebra, an algebra of relevance to some nonequilibrium statistical mechanical systems without time translation symmetry. We conduct a case-by-case exhaustive study of all possible metrics that can arise out of this coset construction for the 2d GCA, using the finite part of the algebra. We look to implement the two physical conditions as outlined in [16] and then make our search more extensive by relaxing one of them. We find that when we are looking at metrics with one extra direction, the physical conditions do not lead to any nondegenerate spacetime metric in 3d, adding strength to the claim that the correct structure to look for is indeed a Newton-Cartan–like $AdS_2 \times R$. Interestingly, when one of the two physical conditions is relaxed, we obtain a flat 3d metric, in keeping with the connection discussed in [13]. We find other nondegenerate metrics for higher-dimensional spaces. Curiously, most of these metrics turn out to be Ricci-scalar flat, although (except for the Minkowskian one) they source nontrivial Ricci tensors.

The outline of the paper is as follows: we first review, in Sec. II, the Galilean conformal algebra, with special emphasis on the 2d GCA which shall be the focus of the paper. In Sec. III, we outline the procedure of constructing metrics on homogeneous coset spaces that we would use. Section IV contains the main results of the paper. We subdivide the section according to the dimension of the spacetime metric that we construct and make several comments. The main results are also summarized in a table in this section. We end with some concluding remarks. An appendix contains a list of all possible subalgebras for the 2d GCA.

## II. A REVIEW OF THE GCA

### A. GCA in arbitrary dimensions

The maximal set of conformal isometries of Galilean spacetime generates the infinite-dimensional Galilean conformal algebra [8]. The notion of Galilean spacetime is a little subtle since the spacetime metric degenerates into a spatial part and a temporal piece. Nevertheless, there is a definite limiting sense (of the relativistic spacetime) in which one can define the conformal isometries (see [18]) of the nonrelativistic geometry. Algebraically, the set of vector fields generating these symmetries is given by

$$\begin{aligned} L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t, \\ M_i^{(n)} &= t^{n+1}\partial_i, \\ J_a^{(n)} \equiv J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i), \end{aligned} \tag{2.1}$$

for integer values of $n$. Here $i = 1\ldots(d-1)$ range over the spatial directions. These vector fields obey the algebra

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}, \\ [L^{(m)}, J_a^{(n)}] &= -nJ_a^{(m+n)}, \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)}, \\ [L^{(m)}, M_i^{(n)}] &= (m-n)M_i^{(m+n)}. \end{aligned} \tag{2.2}$$

There is a finite-dimensional subalgebra of the GCA (also sometimes referred to as the GCA) which consists of taking $n = 0, \pm 1$ for the $L^{(n)}$, $M_i^{(n)}$ together with $J_a^{(0)}$. This algebra is obtained by considering the nonrelativistic contraction of the usual (finite-dimensional) global conformal algebra $SO(d, 2)$ (in $d > 2$ spacetime dimensions) (see, for example, [8,19]).

### B. GCA in 2d

In two spacetime dimensions, as is well known, the situation is special. The relativistic conformal algebra is infinite dimensional and consists of two copies of the Virasoro algebra. One expects this to be related to the infinite-dimensional GCA algebra [20]. In two dimensions the nontrivial generators in (2.2) are the $L_n$ and the $M_n$:

$$L^{(n)} = -(n+1)t^n x\partial_x - t^{n+1}\partial_t, \quad M^{(n)} = t^{n+1}\partial_x, \tag{2.3}$$

which obey

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}, \\ [M^{(m)}, M^{(n)}] &= 0, \\ [L^{(m)}, M^{(n)}] &= (m-n)M^{(m+n)}. \end{aligned} \tag{2.4}$$

These generators in (2.3) arise precisely from a nonrelativistic contraction of the two copies of the Virasoro algebra. To see this, let us remember that the nonrelativistic contraction consists of taking the scaling

$$t \longrightarrow t, \qquad x \longrightarrow \epsilon x, \tag{2.5}$$

with $\epsilon \longrightarrow 0$. This is equivalent to taking the velocities $v \sim \epsilon$ to zero (in units where $c = 1$). Consider the vector fields which generate (two copies of) the centerless Virasoro algebra in two dimensions:

$$\mathcal{L}^{(n)} = -z^{n+1}\partial_z, \qquad \bar{\mathcal{L}}^{(n)} = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{2.6}$$

In terms of space and time coordinates, $z = t + x$, $\bar{z} = t - x$. Expressing $\mathcal{L}_n$, $\bar{\mathcal{L}}_n$ in terms of $t$, $x$ and taking the above scaling limit in (2.5), we get the following combinations:

$$\begin{aligned} \mathcal{L}^{(n)} + \bar{\mathcal{L}}^{(n)} &= -t^{n+1}\partial_t - (n+1)t^n x\partial_x + \mathcal{O}(\epsilon^2); \\ \mathcal{L}^{(n)} - \bar{\mathcal{L}}^{(n)} &= -\frac{1}{\epsilon}t^{n+1}\partial_x + \mathcal{O}(\epsilon). \end{aligned} \tag{2.7}$$

Therefore we see that as $\epsilon \longrightarrow 0$ [20],

$$\mathcal{L}^{(n)} + \bar{\mathcal{L}}^{(n)} \longrightarrow L^{(n)}, \quad \epsilon(\mathcal{L}^{(n)} - \bar{\mathcal{L}}^{(n)}) \longrightarrow -M^{(n)}. \tag{2.8}$$

Let us now rewrite the (1 + 1)-dimensional (finite) algebra generated by $\{L^{(\pm 1)}, L^{(0)}\}$ and $\{M^{(\pm 1)}, M^{(0)}\}$. The nontrivial commutators resulting from (2.4) are given by

$$\begin{aligned} &[D,H]=H, \quad [D,K_0]=-K_0, \quad [D,K_1]=-K_1, \\ &[D,P]=P, \quad [K_0,H]=2D, \quad [B,H]=P, \\ &[K_1,H]=2B, \quad [K_0,B]=K_1, \quad [K_0,P]=2B, \end{aligned} \tag{2.9}$$

where we have made the following identifications:

$$\begin{aligned} &L^{(-1)} \equiv H, \qquad L^{(0)} \equiv D, \qquad L^{(+1)} \equiv K_0, \\ &M^{(-1)} \equiv P, \qquad M^{(0)} \equiv B, \qquad M^{(+1)} \equiv K_1. \end{aligned} \tag{2.10}$$

Here $H$ is the time translation generator, $D$ is the dilatation operator, $P$ is the spatial translation generator, $B$ is the Galilean boost, and $K_0$, $K_1$ are the two components of the special conformal generator. These identifications naturally arise when one considers the contraction of the relativistic conformal algebra [8]. In the rest of the paper we will entirely focus on the algebra written in (2.9) and not be concerned about the infinite-dimensional extension. We would look to realize this finite algebra as the isometries of spacetime metrics in dimensions greater than two. It is natural to expect that only the finite GCA would play the role of the true isometries and the other higher modes may correspond to asymptotic isometries of the metrics that we would obtain.[1] This is something that we will not address in the current paper.

## III. CONSTRUCTION OF METRICS ON COSET SPACES

Here we briefly review the construction of metrics on coset spaces that we will use in the rest of the paper. We closely follow the notation and conventions of [16]. We would like to consider a coset $\mathcal{M} = \mathcal{G}/\mathcal{H}$, where $\mathcal{G}$ is the Galilean conformal group and $\mathcal{H}$ is a subgroup of $\mathcal{G}$. The corresponding Lie algebras are denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and for each $g \in \mathfrak{g}$ there is a corresponding element denoted by $[g] \in \mathfrak{g}/\mathfrak{h}$. As vector spaces, we can always decompose

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{3.1}$$

The coset $\mathcal{M}$ is called a reductive coset if there exists a choice of $\mathfrak{m} \in \mathcal{M}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We will see that, generically, for the GCA we do not have such reductive cosets.

Our goal here will be to construct a $\mathcal{G}$-invariant metric on the homogeneous space $\mathcal{M}$. Given a Lie group the Cartan-Killing form is given by

$$\Omega_{ab} \equiv \frac{1}{I_{\rm adj}} f_{ac}{}^{d} f_{bd}{}^{c}, \tag{3.2}$$

where $f_{ab}{}^{c}$ are the structure constants and $I_{\rm adj}$ is the Dynkin index. For a semisimple Lie group the Cartan-Killing form is nondegenerate and therefore induces a nondegenerate $\mathcal{G}$-invariant metric on $\mathcal{M}$. However, the GCA is not a semisimple algebra, and thus the corresponding Killing form is degenerate. We would like to point out here that there is the possibility of constructing a nondegenerate 2-form over the whole group manifold via a procedure called "double extension." We will have more to say about this later.

Following [22], there exists a one-to-one correspondence between the $\mathcal{G}$-invariant metric on $\mathcal{M} = \mathcal{G}/\mathcal{H}$ and Ad($\mathcal{H}$)-invariant nondegenerate symmetric bilinear forms $\Omega$ on $\mathfrak{g}/\mathfrak{h}$. When $\mathcal{H}$ is connected, this invariance takes the following form:

$$\Omega_{[m][n]} f_{[k]p}{}^{[m]} + \Omega_{[k][m]} f_{[n]p}{}^{[m]} = 0, \tag{3.3}$$

where $[m], [n], \ldots$ are indices corresponding to $\mathfrak{m}$ and $p$ indicates the index corresponding to $\mathfrak{h}$. Given the structure constants for a particular choice of $\mathfrak{h}$ and $\mathfrak{m}$, we can solve for the bilinear $\Omega$ from this equation.

However, the existence or the uniqueness of a solution for $\Omega$ is not guaranteed, and we will observe later that for the GCA only a few choices for the subalgebra $\mathfrak{h}$ lead to a nondegenerate $\Omega$. Moreover, a typical solution of (3.3) does not fix $\Omega$ completely; rather, it gives a symmetric bilinear in terms of a bunch of arbitrary real numbers. This will therefore result in redundancies in the description of the $\mathcal{G}$-invariant (family of) metrics that we will eventually obtain.

Now let us choose an explicit coordinate basis as in [16]. First we fix a linear space decomposition (3.1) and denote that $t_m, t_n, \ldots$ are the bases of $\mathfrak{h}$ and $t_p, t_q, \ldots$ are the bases of $\mathfrak{m}$. Then an element $[g] \in \mathcal{G}/\mathcal{H}$ can be represented by

$$[g] = [\exp(x_m t_m) \exp(x_n t_n) \ldots] \quad \text{modulo } \mathcal{H}. \tag{3.4}$$

The Maurer-Cartan 1-form given by $J_g = g^{-1} dg$ can then be computed according to the linear space decomposition in (3.1),

$$J_g = e_m t_m + e_p t_p, \tag{3.5}$$

where $e_m$ and $e_p$ are the vielbein. The metric on the coset is then constructed by contracting the symmetric bilinear $\Omega$ with the vielbein,

$$G = \Omega^{pq} e_p e_q. \tag{3.6}$$

## IV. HOMOGENEOUS SPACES WITH 2D GALILEAN CONFORMAL ISOMETRY

In this section we will discuss and present the nontrivial homogeneous spaces (and the corresponding choice of the subalgebra) that we obtain via the coset construction. For the interested reader, we have presented a complete list of all possible subalgebras of the 2d GCA in the Appendix.

[1] For a brief review on asymptotic isometries, see e.g. [21].

Let us mention our guiding principles for the choices of subalgebra here. In [16], the authors uniquely determined the metrics for the Schrödinger and the Lifshitz algebras by imposing the following physical conditions:

(1) $\mathfrak{h}$ does not contain the translation generator $P$.

(2) $\mathfrak{h}$ contains the boost generator $B$.

As argued in [16], condition (1) is natural in the sense that $P$ would induce infinitesimal translations in the resulting geometry and should not be included in the stabilizer of a point in $\mathcal{G}/\mathcal{H}$. We shall strictly follow condition (1) in all our examples. Condition (2) is derived from the higher-dimensional analogue of Lorentz invariance. For a $d$-dimensional algebra, the authors of [16] proposed to keep $J_{ij}$, $B_i$ in $\mathfrak{h}$ to preserve Lorentz invariance in $d$ dimensions. We, however, do not believe in the sanctity of this condition in our analysis and will proceed to relax it in our exhaustive study.

### A. Three-dimensional Minkowski space

We begin by considering the case when $\dim\mathcal{M} = 3$. In this case, the only choice that gives a nondegenerate symmetric bilinear $\Omega$ (and therefore a nondegenerate metric) is the coset $\mathcal{M} = \mathcal{G}/\{H, D, K_0\}$. Note that in this case the subalgebra does not contain the boost generator $B$, and thus it falls under the category where we relax one of the physical conditions outlined above (and in [16]).

The structure constants are given by

$$f_{[i]H}{}^{[j]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad f_{[i]D}{}^{[j]} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_{[i]K_0}{}^{[j]} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \tag{4.1}$$

which gives

$$\Omega = \begin{pmatrix} 0 & -2\omega_{33} & 0 \\ -2\omega_{33} & 0 & 0 \\ 0 & 0 & \omega_{33} \end{pmatrix}, \qquad \omega_{ij} \in \mathbb{R}. \tag{4.2}$$

Now the coset element is parametrized as

$$[g] = [e^{x_P P} e^{x_1 K_1} e^{x_B B}], \tag{4.3}$$

which gives the following vielbein:

$$e_P = dx_P, \qquad e_{K_1} = dx_1, \qquad e_B = dx_B. \tag{4.4}$$

So, the resulting metric reads

$$ds_3^2 = \Omega^{pq} e_p e_q = \omega_{33}(-4dx_P dx_1 + dx_B). \tag{4.5}$$

This is the flat 3d Minkowski space.[2] As we remarked earlier based on the observation recently made in [13], this is a consequence of the isomorphism between the finite Galilean conformal group in $(1 + 1)$ dimensions and the Poincaré group in $(2 + 1)$ dimensions. The isomorphism actually extends beyond the finite GCA and encompasses the full infinite extension of the GCA on one side and the infinite-dimensional BMS group in three dimensions which is the asymptotic symmetric group of flat 3d spacetimes at null infinity [13].

We observe that the strict imposition of both the "physical conditions" above does not lead to any nondegenerate spacetime metric. As remarked in the Introduction, the original proposal for the dual gravitational description of a system with the GCA was given in terms of a Newton-Cartan–like AdS [8]. In the case of the three-dimensional bulk dual, the structure of the spacetime would be a fiber-bundled $\mathrm{AdS}_2 \times R$. The spacetime metric degenerates, and the dynamical quantities are the Christoffel symbols which "talk" to the separate metrics of the base $\mathrm{AdS}_2$ and the fibers. The imposition of Lorentz symmetry in two dimensions [condition (2)] in our present construction rules out a nondegenerate spacetime metric, and this is in keeping with the claim that the correct structure to look for is a Newton-Cartan–like $\mathrm{AdS}_2 \times R$.

Let us comment on a couple of things here about the flat metric that we have obtained. First, we know that if an $n$-dimensional manifold admits $\frac{1}{2}n(n + 1)$ Killing vectors, it must be a manifold of constant curvature. We were looking for spacetimes in three dimensions admitting the six-dimensional GCA as an isometry. So, we would have ended up with spacetimes of constant curvature. Our only choices are flat, de Sitter, or anti-de Sitter in three dimensions. That we get a flat spacetime is thus not a surprise.

Another point to note is that this seems to be the metric that is picked out by the method of contractions that gave rise to the GCA from the relativistic conformal algebra from the point of view of AdS/CFT [8], both on the boundary and in the bulk. To see this, let us remind ourselves that the $\mathrm{AdS}_3$ metric is obtained by the following coset construction (see e.g. [23]):

$$\mathrm{AdS}_3 = \frac{SL(2,R) \times SL(2,R)}{SL(2,R)_{\mathrm{diag}}}. \tag{4.6}$$

The above construction of the Minkowskian metric of the GCA is precisely the contraction of (4.6).[3] The finite GCA is obtained by contracting the global $SL(2,R) \times SL(2,R)$ of the Virasoro algebra, and $SL(2,R)_{\mathrm{diag}}$, the diagonal $SL(2,R)$ subgroup of the relativistic theory, is the parent of the $\{H, D, K_0\}$ subalgebra of the GCA.

### B. Four-dimensional metrics

Next we consider the case when $\dim\mathcal{M} = 4$. The first nontrivial case is the coset $\mathcal{M} = \mathcal{G}/\{B, D\}$, which obeys both the physical conditions outlined in [16]. In this case the structure constants are given by

[2] Clearly we can set $\omega_{33} = 1$ without any loss of generality.

[3] We would like to thank Rajesh Gopakumar for pointing this out to us.

$$f_{[i]B}{}^{[j]} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$f_{[i]D}{}^{[j]} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \tag{4.7}$$

which yields

$$\Omega = \begin{pmatrix} 0 & 0 & \omega_{13} & \omega_{14} \\ 0 & 0 & \omega_{14} & 0 \\ \omega_{13} & \omega_{14} & 0 & 0 \\ \omega_{14} & 0 & 0 & 0 \end{pmatrix}, \qquad \omega_{ij} \in \mathbb{R}. \tag{4.8}$$

This is nondegenerate as long as $\omega_{14} \neq 0$.

Since we get a nondegenerate bilinear, let us compute the vielbein in this case. We parametrize the coset element as

$$[g] = [e^{x_H H} e^{x_P P} e^{x_0 K_0} e^{x_1 K_1}], \tag{4.9}$$

which gives the following vielbein:

$$e_H = dx_H, \qquad e_P = dx_P, \qquad e_{K_0} = x_0^2 dx_H + dx_0,$$
$$e_{K_1} = 2x_0 x_1 dx_H + x_0^2 dx_P + dx_1. \tag{4.10}$$

For the sake of visualization, let us write down the full metric. We define $x_H = t$, $x_P = x$, $x_0 = y$, $x_1 = z$, $w_{31} = a$, $w_{41} = b$. The metric, then, can be written as

$$ds^2_{4(1)} = (2ay^2 + 4byz)dt^2 + 4by^2 dtdx + 2adtdy + 2bdtdz + 2bdxdy. \tag{4.11}$$

Note that here we have two arbitrary real numbers, $a$, $b$, which parametrize a family of metrics. This family of metrics has a vanishing Ricci scalar.

The other nontrivial result comes from taking the coset $\mathcal{M} = \mathcal{G}/\{B, \alpha_1 D + \alpha_2 K_1\}$. The structure constants are given by

$$f_{[i]B}{}^{[j]} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1/\alpha_2 & 0 \end{pmatrix},$$
$$f_{[i]\alpha_1 D + \alpha_2 K_1}{}^{[j]} = \alpha_1 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \tag{4.12}$$

which yields the following:

$$\Omega = \begin{pmatrix} 0 & 0 & \omega_{31} & \omega_{41} \\ 0 & 0 & 0 & -\frac{\alpha_1}{\alpha_2}\omega_{31} \\ \omega_{31} & 0 & 0 & 0 \\ \omega_{41} & -\frac{\alpha_1}{\alpha_2}\omega_{31} & 0 & 0 \end{pmatrix}, \qquad \omega_{ij} \in \mathbb{R}. \tag{4.13}$$

The above $\Omega$ is nondegenerate for $\alpha_1 \neq 0$. The vielbein are obtained to be

$$e_H = e^{-x_D} dx_H, \qquad e_P = e^{-x_D} dx_P,$$
$$e_D = dx_D - 2x_0 e^{-x_D} dx_H - \frac{\alpha_1}{\alpha_2} dx_P, \tag{4.14}$$
$$e_{K_0} = e^{-x_D} x_0^2 dx_H - x_0 dx_D + dx_0.$$

It can be checked that, without any loss of generality, we can set[4] $\omega_{31} = 1 = \alpha_2$. Hence we get a family of metrics parametrized by two real numbers, $\omega_{41}$ and $\alpha_1$. Again, for clarity, it is useful to write the metric down explicitly. We make the following redefinitions: $x_H = t$, $x_P = x$, $x_0 = y$, $e^{x_D} = r$, $\omega_{41} = a$,

$$ds^2_{4(2)} = \frac{2}{r^2}\{(1 - ay)drdt + (ay^2 - 2y)dt^2 - \alpha(r + y^2)dtdx + ardtdy + \alpha ydxdr - \alpha rdxdy\}. \tag{4.15}$$

It is trivial to check that this metric also has a vanishing Ricci scalar. This is the only nonreductive example that we encounter in the coset construction of the two-dimensional Galilean conformal symmetry.

Let us offer some comments regarding the signature of these four-dimensional metrics. It can be observed that the two distinct families of metrics we obtained take the following generic form:

$$ds^2 = 2\Omega_{13} e_1 e_3 + 2\Omega_{14} e_1 e_4 + 2\Omega_{2,(3/4)} e_2 e_{(3/4)}, \tag{4.16}$$

where $\Omega_{ij}$ are the corresponding matrix entries in (4.8) or (4.13) and $e_i$'s are the vielbein given in (4.10) or (4.14). If we introduce a local orthonormal frame $\{E_1, E_2, E_3, E_4\}$, where $E_i$'s are appropriate linear combinations of $e_i$'s, the particular form of the metric in (4.16) strongly suggests that the signature of the metric should be (2,2).[5] It is worth noting at this point that in [11], a geometric realization of the "exotic" Galilean conformal isometry in (2 + 1) dimensions was found in terms of an $AdS_7$ metric with (3,4) signature. [This is called exotic because there exists a central charge in the commutator of the boost generators. This is unique to (2 + 1) dimensions.]

### C. Five-dimensional metrics

Finally we present the five-dimensional metrics obtained via the coset construction. The first nontrivial case

[4]This is achieved by computing the Ricci tensor and observing that only the ratios $\omega_{41}/\omega_{31}$ and $\alpha_2/\alpha_1$ appear.

[5]It is easy to check that one cannot write $ds^2 = -E_1^2 + E_2^2 + E_3^2 + E_4^2$; however, one can write $ds^2 = -E_1^2 - E_2^2 + E_3^2 + E_4^2$.

is the coset $\mathcal{M} = \mathcal{G}/\{B\}$. This gives the following symmetric bilinear:

$$\Omega = \begin{pmatrix} \omega_{11} & 0 & \omega_{13} & \omega_{14} & \omega_{15} \\ 0 & 0 & 0 & \omega_{15} & 0 \\ \omega_{13} & 0 & \omega_{33} & \omega_{34} & 0 \\ \omega_{14} & \omega_{15} & \omega_{34} & \omega_{44} & 0 \\ \omega_{15} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \omega_{ij} \in \mathbb{R}, \qquad (4.17)$$

which is nondegenerate if $\omega_{15} \neq 0$ and $\omega_{33} \neq 0$, and without any loss of generality, we can set $\omega_{33} = 1 = \omega_{13} = \omega_{14} = \omega_{15} = \omega_{34}$. In this case we get the following vielbein:

$$\begin{aligned} e_H &= e^{-x_D} dx_H, \qquad e_P = e^{-x_D} dx_P, \\ e_D &= -2x_0 e^{-x_D} dx_H + dx_D, \\ e_{K_0} &= x_0^2 e^{-x_D} dx_H - x_0 dx_D + dx_0, \\ e_{K_1} &= 2x_0 x_1 e^{-x_D} dx_H + x_0^2 e^{-x_D} dx_P - x_1 dx_D + dx_1. \end{aligned} \qquad (4.18)$$

The resulting two-parameter family of metrics is Ricci-scalar flat. Clearly, this construction obeys both the physical conditions.

The only other nontrivial example in five dimensions is the coset $\mathcal{M} = \mathcal{G}/\{D\}$, which does not obey the physical condition (2). In this case we get

$$\Omega = \begin{pmatrix} 0 & 0 & \omega_{31} & \omega_{41} & 0 \\ 0 & 0 & \omega_{32} & \omega_{42} & 0 \\ \omega_{31} & \omega_{32} & 0 & 0 & 0 \\ \omega_{41} & \omega_{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_{55} \end{pmatrix}, \quad \omega_{ij} \in \mathbb{R}. \qquad (4.19)$$

This is also nondegenerate, provided $\omega_{55} \neq 0$ and $\omega_{32}\omega_{41} \neq \omega_{31}\omega_{42}$. The vielbein are given by

$$\begin{aligned} &e_H = dx_H, \quad e_P = dx_P, \quad e_{K_0} = x_0^2 dx_H + dx_0, \\ &e_{K_1} = (2x_0 x_1 + x_0^2 x_B) dx_H + x_0^2 dx_P + x_B dx_0 + dx_1, \\ &e_B = -2x_0 dx_P + dx_B. \end{aligned} \qquad (4.20)$$

This actually gives a four-parameter family of metrics. This family generically has a coordinate-dependent Ricci scalar which diverges as $R \sim x_1^2$ for $x_1 \to \infty$. If $\omega_{32} = 0$, we still get a nondegenerate metric but the Ricci scalar vanishes identically. On the other hand, if $\omega_{32} \neq 0$, then the Ricci scalar can vanish at a particular point in $x_1$.

As in the examples with four-dimensional metrics, it can also be argued that the existence of a local orthonormal frame and the precise structure of these five-dimensional metrics strongly suggest the signature (2,3).

Note that the two-dimensional GCA has six generators; hence a homogeneous space of five dimensions is constructed by choosing a subalgebra which consists of only one generator. This is a rather trivial choice which nonetheless yields a family of nontrivial metrics.

Finally, we summarize some of our results in Table I. Here $R$ denotes the curvature scalar defined by $R = g^{\mu\nu}R_{\mu\nu}$, $R^2_{\mu\nu} \equiv R^{\mu\nu}R_{\mu\nu}$, $R^2_{\mu\nu\rho\sigma} \equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, and finally the curvature of the Weyl tensor is defined as $C^2_{\mu\nu\rho\sigma} = C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$. The metrics that we obtain in this construction (except the Minkowski one) do yield a fairly nontrivial Ricci tensor. Thus it is not clear to us what matter fields will source such backgrounds. It is therefore not obvious that such matter fields will preserve the Galilean conformal isometry. Thus although our metrics do possess the desired isometry, the full background (the metric along with the matter fields sourcing it) may not.

Before we leave this section altogether, a few comments are in order: First, as we remarked earlier in this construction we get a family of metrics parametrized by arbitrary real numbers. The redundancy in this description does not fix the sign of these parameters and hence does not fix the signature of the metric. However, by assuming the existence of a local orthonormal frame, we seem to be able to fix the signature of these metrics, and they turn out to be rather nonstandard.

Second, note that once we know a metric with the Galilean conformal isometry in a given dimension, it is straightforward to construct a higher dimensional metric with the same isometry by fibering the lower-dimensional metric over a base manifold

$$ds^2 = f_1(\zeta) d\zeta^2 + f_2(\zeta) ds^2_{\mathrm{GCA}}, \qquad (4.21)$$

where $f_1(\zeta)$ and $f_2(\zeta)$ are two arbitrary functions and $ds^2_{\mathrm{GCA}}$ is the metric with the Galilean conformal isometry. This isometry acts nontrivially on the metric $ds^2_{\mathrm{GCA}}$ but has no natural action on the base manifold. However, a spacetime thus constructed is not a homogeneous space since the Galilean conformal isometry group does not act transitively on the whole manifold. Therefore the homogeneous

TABLE I. A summary of our results.

| Choice of subalgebra | dim$\mathcal{M}$ | Properties |
|---|---|---|
| $\langle B \rangle$ | 5 | $R = R^2_{\mu\nu} = R^2_{\mu\nu\rho\sigma} = 0 = C^2_{\mu\nu\rho\sigma}$ |
| $\langle D \rangle$ | 5 | $R \neq 0$, $R^2_{\mu\nu} \neq 0$, $R^2_{\mu\nu\rho\sigma} \neq 0$ and $C^2_{\mu\nu\rho\sigma} \neq 0$ singularity appears as $x_0, x_1, x_B \to \infty$ |
| $\langle B, D \rangle$ | 4 | $R = R^2_{\mu\nu} = R^2_{\mu\nu\rho\sigma} = 0 = C^2_{\mu\nu\rho\sigma}$ |
| $\langle B, \alpha_1 D + \alpha_2 K_1 \rangle$, $\alpha_{1,2} \neq 0$ | 4 | $R = R^2_{\mu\nu} = R^2_{\mu\nu\rho\sigma} = 0 = C^2_{\mu\nu\rho\sigma}$; a (nontrivial) nonreductive coset |
| $\langle H, D, K_0 \rangle$ | 3 | Minkowski |

spaces we obtained in four and five dimensions are not related in any obvious manner to the three-dimensional Minkowski space and are thus truly nontrivial.[6]

Finally, let us return to a point which was made in the initial sections. The 2d GCA has a degenerate Cartan-Killing form given by

$$\Omega \sim \begin{pmatrix} 0 & 0 & -2 & \\ 0 & 1 & 0 & \\ -2 & 0 & 0 & \\ & & & 0 \end{pmatrix}, \tag{4.22}$$

where the upper left $3 \times 3$ nondegenerate block comes from the $SL(2, R)$ subalgebra spanned by $\{L^{(\pm)}, L^{(0)}\}$. The rest of the matrix entries are all zeros.

However, the 2d GCA actually allows for a nondegenerate 2-form over the whole group manifold. The situation is similar to the well-known Nappi-Witten algebra [24] (the centrally extended 2d Euclidean algebra) or the Abelian extension of $d$-dimensional Euclidean algebra considered in e.g. [25]. The general construction of an invariant nondegenerate metric for nonsemisimple Lie algebra goes by the name of "double extension," as introduced in [26].[7] Below we briefly review this.

Let $\mathfrak{h}$ be any Lie algebra and $\mathfrak{h}^*$ be its dual. Let the basis for $\mathfrak{h}$ and $\mathfrak{h}^*$ be, respectively, denoted by $\{X_a\}$ and $\{X^a\}$ obeying the relation $\langle X_a, X^b\rangle = \delta_a^b$. Using the fact that $\mathfrak{h}$ acts on $\mathfrak{h}^*$ via the coadjoint representation, one can define the following Lie algebra structure on the vector space $\mathfrak{h} \oplus \mathfrak{h}^*$

$$[X_a, X_b] = f_{ab}{}^c X_c, [X^a, X^b] = 0, [X_a, X^b] = -f_{ac}{}^b X^c, \tag{4.23}$$

where $f_{ab}{}^c$ are the structure constants for the Lie algebra $\mathfrak{h}$. This defines a semidirect product of $\mathfrak{h}$ and $\mathfrak{h}^*$. It is now possible to define an invariant metric on this semidirect product algebra.

From the definition of the finite 2d GCA in (2.4) and the Lie algebra structure defined in (4.23), it is obvious that $X_a \equiv L^{(m)}$ and $X^a \equiv M^{(m)}$, where $m = 0, \pm$. Thus the GCA is isomorphic to the semidirect product of $SL(2, R)$ with its coadjoint representation. We can define a two-parameter family of invariant inner products in the following manner:

$$\langle X_a, X_b\rangle = \alpha\Omega_{ab}, \quad \langle X_a, X^b\rangle = \beta\delta_a^b, \quad \langle X^a, X^b\rangle = 0, \tag{4.24}$$

where $\alpha$ and $\beta$ are nonzero real numbers and $\Omega_{ab}$ is the nondegenerate Cartan-Killing form for $SL(2, R)$. This construction works for the semidirect product of any simple Lie algebra $g$ with its coadjoint representation. It is called the double extension of the trivial metric Lie algebra by $g$ [26].

## V. SUMMARY AND CONCLUSIONS

In this paper we have systematically constructed metrics in dimensions greater than two which realize the two-dimensional Galilean conformal algebra as their isometry. We classified all the relevant subalgebras of the 2d GCA and, in order to construct these metrics, looked at a formulation in terms of cosets. Though many choices of these cosets turned out to produce degenerate metrics, we were able to get some nontrivial higher-dimensional metrics. In three dimensions, we obtained a flat Minkowskian metric which we observed to be the contracted limit of the metric on $AdS_3$. In higher dimensions, viz. four and five, we found several families of metrics, all except one of which turned out to be Ricci-scalar flat.

It is curious that most of the metrics we have obtained are Ricci-scalar flat. It would be worthwhile trying to understand if there is any deeper reason behind this, or if it is a mere coincidence. One might also like to understand if there is any fundamental difference between these Ricci-scalar flat metrics and the family which is not Ricci-scalar flat, given that they were obtained by similar methods.

Despite the fact that these metrics (except the Minkowski one) seem to have a "wrong" signature, a further analysis may turn out to be useful in understanding their structure. It will be very interesting to determine the matter fields which source such backgrounds. However, since these metrics are neither Lorentzian nor Euclidean, it may be difficult to interpret such "matter fields" physically.

In the spirit of the gauge/gravity duality, one could try and reproduce the correlation functions of the 2d GCA [10,20] from a gravity analysis. This might actually be a challenging task, as there is little chance that modes would separate into normalizable and non-normalizable ones like in the usual AdS case. But if one is able to perform such computations, then one could claim that these metrics are actually holographically dual to the nonrelativistic field theories with the GCA as their symmetry algebra.

Another speculation made earlier was that the metrics obtained by this method might realize the infinite-dimensional GCA as asymptotic symmetries. It has been observed in [15] that the infinite BMS algebra in three dimensions, which is isomorphic to the 2d GCA [13], arises as the asymptotic symmetries of flat space at null infinity. So, this speculation indeed holds for our construction in three dimensions. The expectation is that the other metrics which have the finite 2d GCA as their isometries would also realize the infinite GCA in a manner similar to the BMS case. In [21], following the general scheme of calculating asymptotic symmetries outlined in [27], the authors constructed the asymptotic symmetry algebra for metrics with Schrödinger symmetry and found that the

[6]We would like to thank J. Simon and J. Figueroa-O'Farill for discussions related to this issue.

[7]We would like to thank J. Figueroa-O'Farill for explaining this issue to us and bringing this reference to our attention.

infinite extension of the Schrödinger algebra indeed emerges as the asymptotic symmetries of those metrics. The obstruction for applying the general formalism of [27] to the GCA was the absence of a spacetime metric. Now that, in this work, we have derived a number of metrics with the finite GCA as the isometry algebra, it should, in principle, be possible to carry out a similar analysis to [21] and check whether our speculation is indeed correct.

A natural direction of extending this analysis is to construct the metrics for the higher-dimensional GCAs by this method of cosets. But the problem of classifying relevant subalgebras quickly becomes intractable and the full analysis too unwieldy to attempt a case-by-case study. This would involve a mathematical machinery more elaborate and powerful than what we have used in the two-dimensional analysis. Another natural extension is to consider the super-GCA and construction of supercosets. A natural place to begin would again be two dimensions [28]. The size of the finite algebra would provide a challenge which, in this case, may be overcome by imposing strict physical conditions.

To conclude, let us remark on a point we have only fleetingly looked at in this paper. The existence of a nondegenerate 2-form on the full finite GCA is an avenue of potential fruitful research. Given that there is no field theory known for the GCA, it would be nice to use the construction of Nappi-Witten [24] and its generalizations [26] to construct a Wess-Zumino-Witten model with the GCA as its symmetry. It would also be useful to understand if the infinite extension of the GCA plays any interesting role in this context.

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## APPENDIX: THE LIST OF SUBALGEBRAS

Here we list the possible choices of the subalgebras for the finite part of GCA in (2.9). We begin by imposing the physical conditions imposed in [16] and then relaxing them. Just to remind the reader, the physical conditions are as follows:

(1) $\mathfrak{h}$ does not contain the translation generator $P$.

(2) $\mathfrak{h}$ contains the boost generator $B$.

However, we do not impose any constraint on the dimensionality of $\mathcal{M} = \mathcal{G}/\mathcal{H}$.

Let us therefore list the possible choices in descending order in dim$\mathcal{M}$:

(i) $\dim\mathcal{H} = 1$, $\dim\mathcal{M} = 5$:

$$\mathfrak{h} = \langle B\rangle, \qquad \mathfrak{m} = \langle H, P, D, K_0, K_1\rangle. \tag{A1}$$

More generally, however, we have

$$\mathfrak{h} = \langle \alpha_1 B + \alpha_2 H + \alpha_3 D + \alpha_4 K_0 + \alpha_5 K_1\rangle, \quad \alpha_1 \neq 0,$$
$$\mathfrak{m} = \langle H, P, D, K_0, K_1\rangle. \tag{A2}$$

(ii) $\dim\mathcal{H} = 2$, $\dim\mathcal{M} = 4$:

$$\mathfrak{h}_{(1)} = \langle B, K_1\rangle, \qquad \mathfrak{m}_{(1)} = \langle H, P, D, K_0\rangle, \tag{A3}$$

$$\mathfrak{h}_{(2)} = \langle B, D\rangle, \qquad \mathfrak{m}_{(2)} = \langle H, P, K_0, K_1\rangle. \tag{A4}$$

More generally, we can have

$$\mathfrak{h}_{(3)} = \langle B, \alpha_1 D + \alpha_2 K_1\rangle, \qquad \mathfrak{m}_{(3)} = \langle H, P, D, K_0\rangle, \qquad \alpha_2 \neq 0, \tag{A5}$$

$$\mathfrak{h}_{(4)} = \langle B, \alpha_1 D + \alpha_2 K_1\rangle, \qquad \mathfrak{m}_{(4)} = \langle H, P, K_0, K_1\rangle, \qquad \alpha_1 \neq 0. \tag{A6}$$

(iii) $\dim\mathcal{H} = 3$, $\dim\mathcal{M} = 3$:

$$\mathfrak{h}_{(1)} = \langle B, K_1, \alpha_1 D + \alpha_2 K_0\rangle, \qquad \mathfrak{m}_{(1)} = \langle H, P, K_0\rangle, \qquad \alpha_1 \neq 0, \tag{A7}$$

$$\mathfrak{h}_{(2)} = \langle B, K_1, K_0\rangle, \qquad \mathfrak{m}_{(2)} = \langle H, P, D\rangle. \tag{A8}$$

More generally, we have

$$\mathfrak{h}_{(3)} = \langle B, K_1, \alpha_1 D + \alpha_2 K_0 + \alpha_3 K_1\rangle, \qquad \mathfrak{m}_{(3)} = \langle H, P, K_0\rangle, \qquad \alpha_{1,3} \neq 0, \tag{A9}$$

$$\mathfrak{h}_{(4)} = \langle B, K_0, \alpha_1 D + \alpha_2 K_0 + \alpha_3 K_1\rangle, \qquad \mathfrak{m}_{(4)} = \langle H, P, D\rangle, \qquad \alpha_{2,3} \neq 0. \tag{A10}$$

(iv) $\dim\mathcal{H} = 4$, $\dim\mathcal{M} = 2$:

$$\mathfrak{h} = \langle B, K_1, D, K_0\rangle, \qquad \mathfrak{m} = \langle H, P\rangle. \tag{A11}$$

Let us now list the possibilities for relaxing condition (2); i.e. we consider $\mathfrak{h}$ not containing $B$. The choices are as follows:

(v) $\dim\mathcal{H} = 1$, $\dim\mathcal{M} = 5$:

$$\mathfrak{h}_{(1)} = \langle H\rangle, \qquad \mathfrak{m}_{(1)} = \langle P, D, K_0, K_1, B\rangle, \tag{A12}$$

$$\mathfrak{h}_{(2)} = \langle D\rangle, \qquad \mathfrak{m}_{(2)} = \langle H, P, K_0, K_1, B\rangle, \tag{A13}$$

$$\mathfrak{h}_{(3)} = \langle K_0 \rangle, \qquad \mathfrak{m}_{(3)} = \langle H, P, D, K_1, B \rangle, \tag{A14}$$

$$\mathfrak{h}_{(4)} = \langle K_1 \rangle, \qquad \mathfrak{m}_{(4)} = \langle H, P, D, K_0, B \rangle. \tag{A15}$$

(vi) $\dim \mathcal{H} = 2$, $\dim \mathcal{M} = 4$:

$$\mathfrak{h}_{(1)} = \langle H, D \rangle, \quad \mathfrak{m}_{(1)} = \langle P, K_0, K_1, B \rangle, \tag{A16}$$

$$\mathfrak{h}_{(2)} = \langle K_0, K_1 \rangle, \quad \mathfrak{m}_{(2)} = \langle H, P, D, B \rangle, \tag{A17}$$

$$\mathfrak{h}_{(3)} = \langle K_0, D \rangle, \quad \mathfrak{m}_{(3)} = \langle H, P, K_1, B \rangle, \tag{A18}$$

$$\mathfrak{h}_{(4)} = \langle K_1, D \rangle, \quad \mathfrak{m}_{(4)} = \langle H, P, K_0, B \rangle. \tag{A19}$$

More generally, we can have

$$\begin{aligned} &\mathfrak{h}_{(5)} = \langle D, \alpha_1 K_0 + \alpha_2 K_1 \rangle, \\ &\mathfrak{m}_{(5)} = \langle H, P, K_1, B \rangle, \qquad \alpha_1 \neq 0, \\ &\mathfrak{m}_{(5)} = \langle H, P, K_0, B \rangle, \qquad \alpha_2 = 0, \end{aligned} \tag{A20}$$

$$\begin{aligned} \mathfrak{h}_{(6)} &= \langle K_0, D + \alpha_1 K_1 \rangle, \\ \mathfrak{m}_{(6)} &= \langle H, P, K_1, B \rangle, \end{aligned} \tag{A21}$$

$$\begin{aligned} \mathfrak{h}_{(7)} &= \langle K_1, D + \alpha_1 K_0 \rangle, \\ \mathfrak{m}_{(7)} &= \langle H, P, K_0, B \rangle. \end{aligned} \tag{A22}$$

(vii) $\dim \mathcal{H} = 3$, $\dim \mathcal{M} = 3$:

$$\mathfrak{h}_{(1)} = \langle H, D, K_0 \rangle, \quad \mathfrak{m}_{(1)} = \langle P, K_1, B \rangle, \tag{A23}$$

$$\mathfrak{h}_{(2)} = \langle K_0, K_1, D \rangle, \quad \mathfrak{m}_{(2)} = \langle H, P, B \rangle. \tag{A24}$$

More generally, we can also have

$$\begin{aligned} &\mathfrak{h}_{(5)} = \langle K_0, D + \alpha_1 K_1, \beta_1 K_0 + \beta_2 K_1 \rangle, \\ &\mathfrak{m} = \langle H, P, B \rangle. \end{aligned} \tag{A25}$$

For the sake of completeness, below we list the possible subalgebras relaxing both conditions (1) and (2):

(i) $\dim \mathcal{H} = 1$, $\dim \mathcal{M} = 5$:

$$\mathfrak{h} = \langle P \rangle. \tag{A26}$$

(ii) $\dim \mathcal{H} = 2$, $\dim \mathcal{M} = 4$:

$$\mathfrak{h} = \langle P, B \rangle, \quad \langle P, K_1 \rangle, \quad \langle P, H \rangle, \quad \langle P, D \rangle. \tag{A27}$$

(iii) $\dim \mathcal{H} = 3$, $\dim \mathcal{M} = 3$:

$$\begin{aligned} &\mathfrak{h} = \langle P, B, K_1 \rangle, \quad \langle P, B, D \rangle, \quad \langle P, B, H \rangle, \\ &\langle P, K_1, D \rangle. \end{aligned} \tag{A28}$$


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