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Massive superstring scatterings in the Regge regime

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We calculate four classes of high-energy massive string scattering amplitudes of fermionic string theory at arbitrary mass levels in the Regge regime (RR). We show that all four leading order amplitudes in the RR can be expressed in terms of the Kummer function of the second kind. Based on the summation algorithm of a set of extended signed Stirling number identities, we show that all four ratios calculated previously by the method of decoupling of zero-norm states among scattering amplitudes in the Gross regime can be extracted from this Kummer function in the RR. Finally, we conjecture and give evidence that the existence of these four Gross regime ratios in the RR persists to subleading orders in the Regge expansion of all high-energy fermionic string scattering amplitudes.

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I. INTRODUCTION

The high-energy, fixed angle limit of string scattering amplitudes [1–3] had been used to probe the fundamental space-time symmetry of string theory [2]. In this approach, one needs to calculate an infinite number of massive string scattering amplitudes. By taking the high-energy limit of the calculation, a lot of mathematical simplifications result and many interesting characteristics of high-energy behavior of the theory can be obtained. There are two fundamental regimes of high-energy string scattering amplitudes. These are the fixed angle regime or Gross regime (GR), and the fixed momentum transfer regime or Regge regime (RR). These two regimes represent two different high-energy perturbation expansions of the scattering amplitudes, and contain complementary information of the theory. The high-energy string scattering amplitudes in the GR [1–3] were recently intensively reinvestigated for massive string states at arbitrary mass levels [4–13]. See also the developments in [14–16]. An infinite number of linear relations, or stringy symmetries, among string scattering amplitudes of different string states were obtained. Moreover, these linear relations can be solved for each fixed mass level, and ratios $T^{(N,2m,q)}/T^{(N,0,0)}$ among the amplitudes can be obtained. An important new ingredient of these calculations is the decoupling of zero-norm states (ZNS) [17–19] in the old covariant first quantized (OCFQ) string spectrum.

Another fundamental regime of high-energy string scattering amplitudes is in the RR [20–25]. See also [26–28]. An interesting breakthrough of the subject was made in 2008 [29] through the calculation of high-energy string scattering amplitudes for arbitrary mass levels in the RR. It turns out that both the saddle-point method and the method of decoupling of high-energy ZNS adopted in the calculation of GR do not apply to the case of RR. However, a direct calculation to get the complete form of the amplitudes is achievable and the general formula for the high-energy scattering amplitudes for each fixed mass level in the RR can be written down explicitly. It was found that the number of high-energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR calculated previously. In contrast to the case of scatterings in the GR, there is no linear relation among scatterings in the RR. Moreover, it was discovered that the leading order amplitudes at each fixed mass level in the RR can be expressed in terms of the Kummer function of the second kind. Furthermore, for those leading order high-energy amplitudes $A^{(N,2m,q)}$ in the RR with the same type of $(N, 2m, q)$ as those of GR,

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we can extract from them the ratios $T^{(N,2m,q)}/T^{(N,0,0)}$ in the GR by using this Kummer function. Mathematically, the proof was based on a set of summation algorithms for signed Stirling number identities derived by Mkauers in 2007 [30].

This new development of high-energy behavior of string theory enables one to express the ratios $T^{(N,2m,q)}/T^{(N,0,0)}$ (or symmetry) of string theory in terms of the Kummer function and thus may shed light on a deeper understanding of algebraic structure of stringy symmetries. Mathematically, the realization of the Stirling number identities by string theory brings an interesting bridge between string theory and combinatorial theory. It is thus important to probe the structure of more high-energy string scattering amplitudes in this context, and relate it to the Kummer function and more Stirling number identities.

In this paper, we will calculate four classes of high-energy massive string scattering amplitudes of fermionic string theory in the RR. We show that, as in the case of bosonic string, the leading order amplitudes in the RR can be expressed in terms of the Kummer function of the second kind. Based on the summation algorithm of a set of extended Stirling number identities (among them, one remains to be proved mathematically), we show that all four ratios calculated previously among scattering amplitudes in the GR can be extracted from this Kummer function in the RR. We point out that we failed to prove one of the Stirling number identities we used in the text. This identity will be taken as an identity predicted by string theory. We will also provide some numerical evidence in Appendix B to support our prediction. Hopefully a rigorous proof of it will be given in the near future. Finally, we conjecture and give evidence that the existence of these four GR ratios in the RR persists to all subleading orders in the Regge expansion of all four high-energy string scattering amplitudes for the even mass level with $(N+1) = \frac{M_2^2}{2} = \text{odd}$. For the odd mass levels with $(N+1) = \frac{M_2^2}{2} = \text{even}$, the existence of the GR ratios will be terminated and shows up only in the first $\frac{N+1}{2}+1$ terms in the Regge expansion of the amplitudes. This paper is organized as follows. In Sec. II, we briefly review the previous calculation of high-energy string scatterings in the GR. In Sec. III, we calculate four classes of fermionic string scatterings in the Regge limit. Section IV is devoted to the extraction of the ratios of high-energy amplitudes in the GR from the scattering amplitudes in the RR. We also give proofs of a set of Stirling number identities we used in the text. In Sec. V, we calculate the subleading order amplitudes and ratios. A conclusion is presented in Sec. VI. The exact kinematic relations of the Regge scatterings used in Sec. V are collected in Appendix A. In Appendix B, we give a numerical proof of the master identity Eq. (4.7) we used intensively in the text.

II. REVIEW OF FIXED ANGLE SCATTERINGS

In this section, we begin with a brief review of high-energy string scatterings in the fixed angle regime. That is in the kinematic regime

$$s, -t \to \infty, \frac{t}{s} \approx -\sin^2\frac{\theta}{2} = \text{fixed}\,(\text{but } \theta \neq 0) \qquad (2.1)$$

where s, t and u are the Mandelstam variables and θ is the CM scattering angle. It was shown [7,8] that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level $M_{2(B)}^2 = 2(N-1)$ are of the form

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q}(\alpha_{-1}^L)^{2m}(\alpha_{-2}^L)^q|0\rangle \qquad (2.2)$$

where the polarizations of the 2nd particle with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_{2(B)}}(E_2, k_2, 0) = \frac{k_2}{M_{2(B)}}$ as the momentum polarization, $e^L = \frac{1}{M_{2(B)}}(k_2, E_2, 0)$, the longitudinal polarization, and $e^T = (0, 0, 1)$ the transverse polarization. Note that

$$e^P = e^L \text{ in the GR,} \qquad (2.3)$$

and the scattering plane is defined by the spatial components of e^L and e^T. Polarizations perpendicular to the scattering plane are ignored because they are kinematically suppressed for four-point scatterings in the high-energy limit. One can use the saddle-point method to calculate the high-energy scattering amplitudes. For simplicity, we choose k_1, k_3 and k_4 to be tachyons. It turns out that all scattering amplitudes at each fixed mass level are proportional to each other, and the final result for the ratios of high-energy, fixed angle string scattering amplitude are [7,8]

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \left(-\frac{1}{M_{2(B)}}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!. \qquad (2.4)$$

The precise definition of $T^{(N,2m,q)}$ is as follows:

$$T^{(N,2m,q)} = \langle V_1(\partial X_2^T)^{N-2m-2q}(\partial X_2^L)^{2m}(\partial^2 X_2^L)^q e^{ik\cdot X_2}V_3V_4\rangle. \qquad (2.5)$$

In the above equation, vertices V_1, V_3 and V_4 can be arbitrary but fixed string states and their tensor indices are omitted. We use labels 1 and 2 for incoming particles and 3 and 4 for outgoing particles. In the center of mass frame, the scattering angle θ is defined to be the angle between $\vec{k}_1$ and $\vec{k}_3$. The ratios in Eq. (2.4) can also be obtained by using the decoupling of two types of ZNS in the spectrum

$$\text{Type I: } L_{-1}|x\rangle, \quad \text{where } L_1|x\rangle = L_2|x\rangle = 0, \qquad L_0|x\rangle = 0; \qquad (2.6)$$

$$\text{Type II: } \left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\tilde{x}\rangle, \quad \text{where } L_1|\tilde{x}\rangle = L_2|\tilde{x}\rangle = 0, \qquad (L_0 + 1)|\tilde{x}\rangle = 0. \tag{2.7}$$

While Type I states have zero-norm at any space-time dimension, Type II states have zero-norm only at $D = 26$. As examples, for $M_{2(B)}^2 = 4, 6$, we get [4,5]

$$T_{TTT}:T_{LLT}:T_{(LT)}:T_{[LT]} = 8:1:-1:-1, \tag{2.8}$$

$$\begin{array}{ccccccccccccccccc} T_{TTTT} & : & T_{TTLL} & : & T_{LLLL} & : & T_{TTL} & : & T_{LLL} & : & \tilde{T}_{LT,T} & : & \tilde{T}_{LP,P} & : & T_{LL} & : & \tilde{T}_{LL} \\ 16 & : & \frac{4}{3} & : & \frac{1}{3} & : & -\frac{4\sqrt{6}}{9} & : & -\frac{\sqrt{6}}{9} & : & -\frac{2\sqrt{6}}{3} & : & 0 & : & \frac{2}{3} & : & 0 \end{array}. \tag{2.9}$$

In the above two equations, the authors of [4,5] had used another basis (corresponding to states listed by Young diagrams) to define the amplitudes. For example

$$\mathcal{T}_{(LT)} = \langle V_1(\partial^2 X_2^{(L}\partial X_2^{T)} e^{ik\cdot X_2})V_3V_4\rangle, \qquad \mathcal{T}_{[LT]} = \langle V_1(\partial^2 X_2^{[L}\partial X_2^{T]} e^{ik\cdot X_2})V_3V_4\rangle. \tag{2.10}$$

These amplitudes are linear combination of $T^{(N,2m,q)}$ defined previously. We give one specific example here. One choice of the vertex of the spin three state at $M_{2(B)}^2 = 4$ is

$$(\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda} + \epsilon_{(\mu\nu)}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu})|0,k\rangle; \qquad \epsilon_{(\mu\nu)} = -\frac{3}{2}k^{\lambda}\epsilon_{\mu\nu\lambda}, \qquad k^{\mu}k^{\nu}\epsilon_{\mu\nu\lambda} = 0, \qquad \eta^{\mu\nu}\epsilon_{\mu\nu\lambda} = 0 \tag{2.11}$$

which is conformal invariant. In the high-energy limit, all components perpendicular to the scattering plane are of subleading order in energy and can be neglected. By using the helicity decomposition, and writing $\epsilon_{\mu\nu\lambda} = \sum_{\alpha,\beta,\delta} e_{\mu}^{\alpha} e_{\nu}^{\beta} e_{\lambda}^{\delta} u_{\alpha\beta\delta}$; $\alpha, \beta, \delta = P, L, T$, we can get [4,5]

$$\begin{aligned}(\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda} + \epsilon_{(\mu\nu)}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu})|0,k\rangle = [&u_{PLT}(6\alpha_{-1}^{PLT} + 6\alpha_{-1}^{(L}\alpha_{-2}^{T)}) + u_{TTP}(3\alpha_{-1}^{TTP} - 3\alpha_{-1}^{LLP} + 3\alpha_{-1}^{(T}\alpha_{-2}^{T)} - 3\alpha_{-1}^{(L}\alpha_{-2}^{L)}) \\ &+ u_{TTL}(3\alpha_{-1}^{TTL} - \alpha_{-1}^{LLL}) + u_{TTT}(\alpha_{-1}^{TTT} - 3\alpha_{-1}^{LLT})]|0,k\rangle\end{aligned} \tag{2.12}$$

where $\alpha_{-1}^{\mu\nu\lambda} \equiv \alpha_{-1}^{\mu}\alpha_{-1}^{\nu}\alpha_{-1}^{\lambda}$, etc. It can be shown that the ratios and amplitudes calculated in these two bases (one with Young tableaux and the other one no) were consistent with each other. However, the calculation for general mass levels is much easier to perform in the basis defined in Eq. (2.2). Note that Eq. (2.12) is valid also in the RR.

We now consider the fermionic string case. In this paper, we will only consider high-energy scattering amplitudes of string states with polarizations on the scattering plane. Some high-energy scatterings of string states with polarizations orthogonal to the scattering plane in the GR were discussed in [9]. It was shown that there are four types of high-energy string scattering amplitudes for states in the NS sector with even GSO parity which can be written down explicitly for the mass level $M_2^2 = 2(N+1)$ as (here we have replaced all e^L in [9] by e^P for the purpose of the following discussion in Secs. III and IV)

$$|N+1,2m,q\rangle \otimes |b_{-(1/2)}^T\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q+1}(\alpha_{-1}^P)^{2m} \times (\alpha_{-2}^P)^q (b_{-(1/2)}^T)|0,k\rangle, \tag{2.13}$$

$$|N+1,2m+1,q\rangle \otimes |b_{-(1/2)}^P\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q}(\alpha_{-1}^P)^{2m+1}(\alpha_{-2}^P)^q (b_{-(1/2)}^P)|0,k\rangle, \tag{2.14}$$

$$|N,2m,q\rangle \otimes |b_{-(3/2)}^P\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q}(\alpha_{-1}^P)^{2m}(\alpha_{-2}^P)^q \times (b_{-(3/2)}^P)|0,k\rangle, \tag{2.15}$$

$$\begin{aligned}|N-1,2m,q-1\rangle \otimes |b_{-(1/2)}^T b_{-(1/2)}^P b_{-(3/2)}^P\rangle &\equiv (\alpha_{-1}^T)^{N-2m-2q}(\alpha_{-1}^P)^{2m}(\alpha_{-2}^P)^{q-1}(b_{-(1/2)}^T) \\ &\times (b_{-(1/2)}^P)(b_{-(3/2)}^P)|0,k\rangle.\end{aligned} \tag{2.16}$$

Note that the number of α_{-1}^P operator in Eq. (2.14) is odd. In the OCFQ spectrum of open superstring, the solutions of physical state conditions include positive-norm propagating states and two types of zero-norm states. In the NS sector, the latter are [31]

$$\text{Type I: } G_{-(1/2)}|x\rangle, \quad \text{where } G_{1/2}|x\rangle = G_{3/2}|x\rangle = 0, \qquad L_0|x\rangle = 0; \tag{2.17}$$

$$\text{Type II: } (G_{-(3/2)} + 2G_{-(1/2)}L_{-1})|\tilde{x}\rangle, \quad \text{where } G_{1/2}|\tilde{x}\rangle = G_{3/2}|\tilde{x}\rangle = 0, \qquad (L_0 + 1)|\tilde{x}\rangle = 0. \tag{2.18}$$

While Type I states have zero-norm at any space-time dimension, Type II states have zero-norm only at $D = 10$. It was shown that [9], for each fixed mass level, all high-energy scattering amplitudes corresponding to states in Eqs. (2.13), (2.14), (2.15), and (2.16) are proportional to each other, and the ratios can be determined from the method of decoupling of two types of zero-norm states, Eqs. (2.17) and (2.18), or the method of Virasoro constraints in the high-energy limit. These ratios were calculated to be [9]

$$|N,2m,q\rangle\otimes|b^P_{-(3/2)}\rangle=\left(-\frac{1}{2M_2}\right)^{q+m}\frac{(2m-1)!!}{(-M_2)^m}|N,0,0\rangle\otimes|b^P_{-(3/2)}\rangle, \tag{2.19}$$

$$|N+1,2m+1,q\rangle\otimes|b^P_{-(1/2)}\rangle=\left(-\frac{1}{2M_2}\right)^{q+m}\frac{(2m+1)!!}{(-M_2)^{m+1}}|N,0,0\rangle\otimes|b^P_{-(3/2)}\rangle, \tag{2.20}$$

$$|N+1,2m,q\rangle\otimes|b^T_{-(1/2)}\rangle=\left(-\frac{1}{2M_2}\right)^{q+m}\frac{(2m-1)!!}{(-M_2)^{m-1}}|N,0,0\rangle\otimes|b^P_{-(3/2)}\rangle, \tag{2.21}$$

$$|N-1,2m,q-1\rangle\otimes|b^T_{-(1/2)}b^P_{-(1/2)}b^P_{-(3/2)}\rangle=\left(-\frac{1}{2M_2}\right)^{q+m}\frac{(2m-1)!!}{(-M_2)^m}|N,0,0\rangle\otimes|b^P_{-(3/2)}\rangle. \tag{2.22}$$

Note that, in order to simplify the notation, we have only shown the second state of the four-point functions to represent the scattering amplitudes on both sides of each equation above. This notation will be used throughout the paper whenever is necessary. Equations (2.19) to (2.22) are thus the SUSY generalization of Eq. (2.4) for the bosonic string. In the next section, in contrast to the ZNS method used in the GR, we will used a direct calculation method to calculate the scattering amplitudes for general mass levels in the RR. Furthermore, we can use these amplitudes to extract the ratios in Eqs. (2.19) to (2.22) calculated above.

III. FOUR CLASSES OF REGGE SCATTERINGS

We now turn to the discussion on high-energy string scatterings in the Regge regime. That is in the kinematic regime

$$s\to\infty,\qquad \sqrt{-t}=\text{fixed}\,(\text{but}\sqrt{-t}\neq\infty). \tag{3.1}$$

Instead of using (E,θ) as the two independent kinematic variables in the GR, we choose to use (s,t) in the RR. One of the reasons has been that $t\sim E\theta$ is fixed in the RR, and it is more convenient to use (s,t) rather than (E,θ). In the RR, to the lowest order, Eqs. (A13) to (A18) reduce to

$$e^P\cdot k_1=-\frac{1}{M_2}\left(\sqrt{p^2+M_1^2}\sqrt{p^2+M_2^2}+p^2\right)\simeq-\frac{s}{2M_2}, \tag{3.2a}$$

$$e^L\cdot k_1=-\frac{p}{M_2}\left(\sqrt{p^2+M_1^2}+\sqrt{p^2+M_2^2}\right)\simeq-\frac{s}{2M_2}, \tag{3.2b}$$

$$e^T\cdot k_1=0 \tag{3.2c}$$

and

$$e^P\cdot k_3=\frac{1}{M_2}\left(\sqrt{q^2+M_3^2}\sqrt{p^2+M_2^2}-pq\cos\theta\right)\simeq-\frac{\tilde{t}}{2M_2}\equiv-\frac{t-M_2^2-M_3^2}{2M_2}, \tag{3.3a}$$

$$e^L\cdot k_3=\frac{1}{M_2}\left(p\sqrt{q^2+M_3^2}-q\sqrt{p^2+M_2^2}\cos\theta\right)\simeq-\frac{\tilde{t}'}{2M_2}\equiv-\frac{t+M_2^2-M_3^2}{2M_2}, \tag{3.3b}$$

$$e^T\cdot k_3=-q\sin\phi\simeq-\sqrt{-t}. \tag{3.3c}$$

Before we proceed to calculate the fermionic string scatterings for the general mass levels in the RR, we first use a simple example of bosonic string scattering [29] to illustrate a subtle difference between scatterings in the GR and RR. In the mass level $M^2_{2(B)}=4$ $(M^2_{1(B)}=M^2_{3(B)}=M^2_{4(B)}=-2)$, one of the (conformal invariant) high-energy amplitudes in the RR is for the state $(\alpha^{PLT}_{-1}+\alpha^{(L}_{-1}\alpha^{T)}_{-2})|0\rangle$. This can be seen from the first line of Eq. (2.12). For simplicity and for illustration here, we will only calculate amplitude corresponding to the state $\alpha^T_{-1}\alpha^L_{-2}|0\rangle$. We stress that in order to recover the conformal invariance, one needs to calculate the amplitude corresponding to the state $(\alpha^{PLT}_{-1}+\alpha^{(L}_{-1}\alpha^{T)}_{-2})|0\rangle$. For the general mass levels, see the discussion on the paragraph after Eq. (3.7) below. The $s-t$ channel of this amplitude (the $t-u$ channel amplitudes can be similarly discussed) can be calculated to be [29] (we use A to represent RR amplitudes and T to represent GR amplitudes, respectively, in this paper)

$$A^{TL} = \int_0^1 dx \cdot x^{k_1 \cdot k_2}(1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x}\right) \times \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2}\right] \simeq i(\sqrt{-t})\left(-\frac{1}{2M_2}\right)\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left[-\left(\frac{1}{8}t+\frac{3}{4}\right)s^3 - \frac{1}{8}(t^2-2t)s^2 - \left(\frac{1}{4}t^2 - t - 3\right)s\right]. \tag{3.4}$$

From the above calculation, one can easily see that the term $\sim\sqrt{-t}t^2s^2$ is in the leading order in the GR, but is in the subleading order in the RR. On the other hand, the term $\sim\sqrt{-t}s^3$ is in the subleading order in the GR, but is in the leading order in the RR. This observation suggests that the high-energy string scattering amplitudes in the GR and RR contain information complementary to each other.

By Eqs. (3.3a) and (3.3b), it is important to note that

$$e^P \neq e^L \text{ in the RR.} \tag{3.5}$$

This is very different from the case of GR. In the discussion of this section and Sec. IV of this paper, we will calculate the amplitudes for the polarization e^P and e^T. For the additional e^L amplitudes, the results can be trivially modified. There is another important difference between the high-energy scattering amplitudes in the RR and in the GR. It was found [29] for the bosonic string case that the number of high-energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR. In fact, instead of states in Eq. (2.2) for the GR, a class of high-energy string states at each fixed mass level $N = \sum_{n,m} np_n + mq_m$ for the RR are [29]

$$|p_n, q_m\rangle = \prod_{n>0}(\alpha^T_{-n})^{p_n}\prod_{m>0}(\alpha^P_{-m})^{q_m}|0, k\rangle. \tag{3.6}$$

At this point, we note that there are other high-energy vertices for the RR which were not considered previously for the bosonic string case [29], namely

$$|p_n, q_m, r_l\rangle = \prod_{n>0}(\alpha^T_{-n})^{p_n}\prod_{m>0}(\alpha^P_{-m})^{q_m}\prod_{l>0}(\alpha^L_{-l})^{r_l}|0, k\rangle \tag{3.7}$$

where $N = \sum_{n,m} np_n + mq_m + lr_l$. However, for the purpose of recovering the GR ratios, the vertex in Eq. (3.6) is good enough. All the results in [29] including Kummer functions and ratios, etc. remain the same if Eq. (3.7) was used. However, in order to get the *conformal invariant* RR amplitudes, one needs to consider the most general vertex in Eq. (3.7). The calculation is similar to the one for Eq. (3.6). For example, for the vertex in Eq. (2.12), one needs to calculate, in addition to others, the amplitude corresponding to $\alpha^{PLT}_{-1}|0\rangle \equiv \alpha^P_{-1}\alpha^L_{-1}\alpha^T_{-1}|0\rangle$ in the RR.

Now we come back to the discussion for the vertex in Eq. (3.6). It seems that both the saddle-point method and the method of decoupling of high-energy ZNS adopted in the calculation of GR do not apply to the case of RR. However a direct calculation is still manageable due to the following rules to simplify the calculation for the leading order amplitudes in the RR:

$$\alpha^T_{-n}\text{: 1 term (contraction of } ik_3 \cdot X \quad \text{with } \varepsilon_T \cdot \partial^n X), \tag{3.8}$$

$$\alpha^P_{-n}\text{: } \begin{cases} n > 1, & \text{1 term} \\ n = 1 & \text{2 terms (constraction of } ik_1 \cdot X \quad \text{and} \quad ik_3 \cdot X \quad \text{with} \quad \varepsilon_L \cdot \partial^n X). \end{cases} \tag{3.9}$$

For our purpose in this paper, we will only calculate four classes of scattering amplitudes corresponding to states in Eq. (2.13) to Eqs. (2.16) in the RR. There are much more high-energy fermionic string scattering amplitudes other than states we will consider in this paper. We stress that, in addition to high-energy scatterings of string states with polarizations orthogonal to the scattering plane considered previously in the GR [9], there are more high-energy string scattering amplitudes with more worldsheet fermionic operators $b^{P,T}_{-(n/2)}$ in the vertex.

A. Amplitude $|N, 2m, q\rangle \otimes |b^P_{-(3/2)}\rangle$

The first scattering amplitude we want to calculate corresponding to state in Eq. (2.15) is

$$A_1^{(N,2m,q)} = \langle \psi_1^{T^1} e^{-\phi_1} e^{ik_1X_1} \cdot (\partial X_2^T)^{N-2m-2q} \times (\partial X_2^L)^{2m}(\partial^2 X_2^L)^q \partial\psi_2^P e^{-\phi_2} \times e^{ik_2X_2} \cdot k_{\lambda 3}\psi_3^\lambda e^{ik_3X_3} \cdot k_{\sigma 4}\psi_4^\sigma e^{ik_4X_4}\rangle \tag{3.10}$$

where we have dropped out an overall factor. In Eq. (3.10), the first vertex is a vector state in the $(-)$ ghost picture, and the last two states are tachyons in the (0) ghost picture. The second state is a tensor in the $(-)$ ghost picture, so that the total superconformal ghost charges sum up to -2. The $s-t$ channel of the amplitude can be calculated to be

$$A_1^{(N,2m,q)} = \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q} \tag{3.11}$$

$$\cdot\left[\frac{e^P\cdot k_1}{-x}+\frac{e^P\cdot k_3}{1-x}\right]^{2m}\left[\frac{e^P\cdot k_1}{x^2}+\frac{e^P\cdot k_3}{(1-x)^2}\right]^q\cdot\frac{1}{x} \tag{3.12}$$

$$\cdot\{\langle\psi_1^{T^1}\partial\psi_2^P\rangle\langle\psi_3^\lambda\psi_4^\sigma\rangle-\langle\psi_1^{T^1}\psi_3^\lambda\rangle\langle\partial\psi_2^P\psi_4^\sigma\rangle+\langle\psi_1^{T^1}\psi_4^\sigma\rangle\langle\partial\psi_2^P\psi_3^\lambda\rangle\}k_{\lambda 3}k_{\sigma 4} \tag{3.13}$$

$$\simeq \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q} \tag{3.14}$$

$$\cdot\left[\frac{e^P\cdot k_1}{-x}+\frac{e^P\cdot k_3}{1-x}\right]^{2m}\left[\frac{e^P\cdot k_3}{(1-x)^2}\right]^q\cdot\frac{1}{x}\frac{1}{M_2}\left[-\frac{(e^T\cdot k_4)(k_2\cdot k_3)}{(1-x)^2}\right]. \tag{3.15}$$

In Eq. (3.12), $\frac{e^P\cdot k_1}{x^2}$ is of subleading order in the RR and $\frac{1}{x}$ is the ghost contribution. The second term of Eq. (3.13) vanishes due to the $SL(2,R)$ gauge fixing $x_1=0$, $x_2=x$, $x_3=1$ and $x_4=\infty$. The first term of Eq. (3.13) vanishes due to $e^{T^1}\cdot e^{P^2}=0$. The amplitude then reduces to

$$A_1^{(N,2m,q)} \simeq \frac{\tilde{t}}{2M_2}(\sqrt{-t})^{N-2m-2q+1}\left(\frac{\tilde{t}}{2M_2}\right)^q\int_0^1 dx x^{k_1\cdot k_2-1}(1-x)^{k_2\cdot k_3-N+2m-2}\cdot\sum_{j=0}^{2m}\binom{2m}{j}\left(\frac{s}{2M_2x}\right)^j\left(\frac{-\tilde{t}}{2M_2(1-x)}\right)^{2m-j}$$

$$= \frac{\tilde{t}}{2M_2}(\sqrt{-t})^{N-2m-2q+1}\left(\frac{\tilde{t}}{2M_2}\right)^{2m+q}\cdot\sum_{j=0}^{2m}\binom{2m}{j}(-1)^j\left(\frac{s}{\tilde{t}}\right)^j B(k_1\cdot k_2-j,k_2\cdot k_3-N+j-1). \tag{3.16}$$

The Beta function above can be approximated in the large s, but fixed t limit as follows

$$\begin{aligned}&B(k_1\cdot k_2-j,k_2\cdot k_3+j-N-1)\\&=B\left(1-\frac{s}{2}+N-j,-\frac{1}{2}-\frac{t}{2}+j\right)\\&=\frac{\Gamma(1-\frac{s}{2}+N-j)\Gamma(-\frac{1}{2}-\frac{t}{2}+j)}{\Gamma(\frac{u}{2}-1)}\\&\approx B\left(1-\frac{s}{2},-\frac{1}{2}-\frac{t}{2}\right)\left(1-\frac{s}{2}\right)^{N-j}\left(\frac{u}{2}-1\right)^{-N}\left(-\frac{1}{2}-\frac{t}{2}\right)_j\\&\approx B\left(1-\frac{s}{2},-\frac{1}{2}-\frac{t}{2}\right)\left(-\frac{s}{2}\right)^{-j}\left(-\frac{1}{2}-\frac{t}{2}\right)_j\end{aligned} \tag{3.17}$$

where

$$(a)_j = a(a+1)(a+2)\ldots(a+j-1) \tag{3.18}$$

is the Pochhammer symbol. The leading order amplitude in the RR can then be written as

$$A_1^{(N,2m,q)} \simeq \frac{\tilde{t}}{2M_2}B\left(1-\frac{s}{2},-\frac{1}{2}-\frac{t}{2}\right)\sqrt{-t}^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q}\cdot(\tilde{t})^{2m+q}\sum_{j=0}^{2m}\binom{2m}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j, \tag{3.19}$$

which is UV power-law behaved as expected. The summation in Eq. (3.19) can be represented by the Kummer function of the second kind U as follows,

$$\sum_{j=0}^{p}\binom{p}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j = 2^p(\tilde{t})^{-p}U\left(-p,\frac{t}{2}-p+\frac{3}{2},\frac{\tilde{t}}{2}\right). \tag{3.20}$$

Finally, the amplitudes can be written as

$$A_1^{(N,2m,q)} \simeq B\left(1-\frac{s}{2},-\frac{1}{2}-\frac{t}{2}\right)\sqrt{-t}^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q+1}\cdot 2^{2m}(\tilde{t})^{q+1}U\left(-2m,\frac{t}{2}-2m+\frac{3}{2},\frac{\tilde{t}}{2}\right). \tag{3.21}$$

In the above, U is the Kummer function of the second kind and is defined to be

$$U(a,c,x) = \frac{\pi}{\sin\pi c}\left[\frac{M(a,c,x)}{(a-c)!(c-1)!}-\frac{x^{1-c}M(a+1-c,2-c,x)}{(a-1)!(1-c)!}\right](c\neq 2,3,4\ldots) \tag{3.22}$$

where $M(a,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}\frac{x^j}{j!}$ is the Kummer function of the first kind. U and M are the two solutions of the Kummer equation

$$xy''(x)+(c-x)y'(x)-ay(x)=0. \tag{3.23}$$

It is crucial to note that $c=\frac{t}{2}-2m+\frac{3}{2}$, and is not a constant as in the usual case, so U in Eq. (3.21) is not a solution of the Kummer equation. This will make our analysis in Sec. IV more complicated.

There are some important observations for the high-energy amplitude in Eq. (3.21). First, the amplitude gives the universal power-law behavior for string states at *all* mass levels

$$A_1^{(N,2m,q)} \sim s^{\alpha(t)} \,(\text{in the RR}) \tag{3.24}$$

where

$$\alpha(t) = a_0 + \alpha' t, \qquad a_0 = \frac{1}{2} \quad \text{and} \quad \alpha' = 1/2. \tag{3.25}$$

This generalizes the high-energy behavior of the four massless vector amplitude in the RR to string states at arbitrary mass levels. Second, the amplitude gives the correct intercept $a_0 = \frac{1}{2}$ of fermionic string. Finally, the amplitude can be used to reproduce the ratios in Eqs. (2.19) calculated in the GR as we will see in Sec. IV.

B. Amplitude $|N+1, 2m+1, q\rangle \otimes |b^P_{-(1/2)}\rangle$

Note that this is the only case with odd integer $2m+1$. The scattering amplitude corresponding to state in Eq. (2.14) can be written as

$$\begin{aligned} A_2^{(N+1,2m+1,q)} = \langle &\psi_1^{T^1} e^{-\phi_1} e^{ik_1 X_1} \cdot (\partial X_2^T)^{N-2m-2q} \\ &\times (\partial X_2^L)^{2m+1} (\partial^2 X_2^L)^q \\ &\times \psi_2^P e^{-\phi_2} e^{ik_2 X_2} \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3 X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4 X_4} \rangle \end{aligned} \tag{3.26}$$

where we have dropped out an overall factor. The amplitude can be calculated to be

$$\begin{aligned} A_2^{(N+1,2m+1,q)} &= \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q} \\ &\quad \cdot \left[\frac{e^P\cdot k_1}{-x} + \frac{e^P\cdot k_3}{1-x}\right]^{2m+1}\left[\frac{e^P\cdot k_1}{x^2} + \frac{e^P\cdot k_3}{(1-x)^2}\right]^q \cdot \frac{1}{x} \\ &\quad \cdot \left\{\langle\psi_1^{T^1}\psi_2^P\rangle\langle\psi_3^\lambda\psi_4^\sigma\rangle - \langle\psi_1^{T^1}\psi_3^\lambda\rangle\langle\psi_2^P\psi_4^\sigma\rangle + \langle\psi_1^{T^1}\psi_4^\sigma\rangle\langle\psi_2^P\psi_3^\lambda\rangle\right\} k_{\lambda 3}k_{\sigma 4} \\ &= \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q}\left[\frac{e^P\cdot k_1}{-x} + \frac{e^P\cdot k_3}{1-x}\right]^{2m+1} \\ &\quad \cdot \left[\frac{e^P\cdot k_3}{(1-x)^2}\right]^q \frac{1}{x}\frac{1}{M_2}\left[(e^{T^1}\cdot k_3)(k_2\cdot k_4) - \frac{(e^{T^1}\cdot k_4)(k_2\cdot k_3)}{1-x}\right] \\ &\simeq (-1)^N[\sqrt{-t}]^{N-2m-2q+1}\left(-\frac{1}{2M_2}\right)^{2m+q+2}\tilde{t}^{2m+q+1}\sum_{j=0}^{2m+1}\binom{2m+1}{j}\left(-\frac{s}{\tilde{t}}\right)^j \\ &\quad \cdot \left[-(s+t+1)\int_0^1 dx x^{k_1\cdot k_2-j-1}(1-x)^{k_2\cdot k_3-N+j-1} + \tilde{t}\int_0^1 dx x^{k_1\cdot k_2-j-1}(1-x)^{k_2\cdot k_3-N+j-2}\right] \\ &\simeq [\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q+2}\tilde{t}^{2m+q+1}\sum_{j=0}^{2m+1}\binom{2m+1}{j}\left(-\frac{s}{\tilde{t}}\right)^j \\ &\quad \cdot [-(s+t+1)B(k_1\cdot k_2 - j, k_2\cdot k_3 - N + j) + \tilde{t}B(k_1\cdot k_2 - j, k_2\cdot k_3 - N + j - 1)]. \end{aligned} \tag{3.27}$$

We then do an approximation for beta function similar to the calculation for $A_1^{(N,2m,q)}$ and end up with

$$\begin{aligned} A_2^{(N+1,2m+1,q)} &\simeq B\left(1-\frac{s}{2}, -\frac{1}{2}-\frac{t}{2}\right)[\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q+2}\tilde{t}^{2m+q+1} \\ &\quad \cdot \sum_{j=0}^{2m+1}\binom{2m+1}{j}\left[(1+t)\left(\frac{2}{\tilde{t}}\right)^j\left(\frac{1}{2}-\frac{t}{2}\right)_j - \tilde{t}\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j\right] \\ &\simeq B\left(1-\frac{s}{2}, -\frac{1}{2}-\frac{t}{2}\right)[\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q+2}2^{2m+1}(\tilde{t})^q \\ &\quad \cdot \left[(1+t)U\left(-1-2m, \frac{t}{2}-2m-\frac{1}{2}, \frac{\tilde{t}}{2}\right) - \tilde{t}U\left(-1-2m, \frac{t}{2}-2m+\frac{1}{2}, \frac{\tilde{t}}{2}\right)\right]. \end{aligned} \tag{3.28}$$

Note that there are two terms in Eq. (3.28), and the first argument of the U function $a = -1-2m$ is odd. These differences will make the calculation of the ratios in Sec. IV more complicated. Finally, the amplitude gives the universal power-law behavior for string states at *all* mass levels with the correct intercept $a_0 = \frac{1}{2}$ of fermionic string.

C. Amplitude $|N+1, 2m, q\rangle \otimes |b^T_{-(1/2)}\rangle$

The third scattering amplitude corresponding to state in Eq. (2.13) is

$$A_3^{(N+1,2m,q)} = \langle \psi_1^{T^1} e^{-\phi_1} e^{ik_1X_1} \cdot (\partial X_2^T)^{N-2m-2q+1} (\partial X_2^L)^{2m} (\partial^2 X_2^L)^q \psi_2^T e^{-\phi_2} e^{ik_2X_2} \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4X_4} \rangle \tag{3.29}$$

where we have dropped out an overall factor. The scattering amplitude can be calculated to be

$$\begin{aligned} A_3^{(N+1,2m,q)} &= \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q+1} \\ &\quad \cdot \left[\frac{e^P\cdot k_1}{-x}+\frac{e^P\cdot k_3}{1-x}\right]^{2m}\left[\frac{e^P\cdot k_1}{x^2}+\frac{e^P\cdot k_3}{(1-x)^2}\right]^q\cdot\frac{1}{x} \\ &\quad \cdot \{\langle \psi_1^{T^1}\psi_2^T\rangle\langle\psi_3^\lambda\psi_4^\sigma\rangle - \langle\psi_1^{T^1}\psi_3^\lambda\rangle\langle\psi_2^T\psi_4^\sigma\rangle + \langle\psi_1^{T^1}\psi_4^\sigma\rangle\langle\psi_2^T\psi_3^\lambda\rangle\}k_{\lambda 3}k_{\sigma 4} \\ &= \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q+1}\left[\frac{e^P\cdot k_1}{-x}+\frac{e^P\cdot k_3}{1-x}\right]^{2m}\left[\frac{e^P\cdot k_3}{(1-x)^2}\right]^q \\ &\quad \cdot \frac{1}{x}\left[\frac{(e^{T^1}\cdot e^T)(k_3\cdot k_4)}{-x} + (e^{T^1}\cdot k_3)(e^T\cdot k_4) - \frac{(e^{T^1}\cdot k_4)(e^T\cdot k_3)}{1-x}\right] \\ &\simeq [\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q}\tilde{t}^{2m+q}\sum_{j=0}^{2m}\binom{2m}{j}\left(-\frac{s}{\tilde{t}}\right)^j \\ &\quad \cdot \left[-\frac{s}{2}\int_0^1 dx x^{k_1\cdot k_2-j-2}(1-x)^{k_2\cdot k_3-N+j-1} + t\int_0^1 dx x^{k_1\cdot k_2-j}(1-x)^{k_2\cdot k_3-N+j-2}\right] \\ &\simeq [\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q}\tilde{t}^{2m+q}\sum_{j=0}^{2m}\binom{2m}{j}\left(-\frac{s}{\tilde{t}}\right)^j \\ &\quad \cdot \left[-\frac{s}{2}B(k_1\cdot k_2-j-1, k_2\cdot k_3-N+j) + tB(k_1\cdot k_2-j+1, k_2\cdot k_3-N+j-1)\right]. \end{aligned} \tag{3.30}$$

We then do an approximation for beta function similar to the calculation for $A_1^{(N,2m,q)}$ and end up with

$$\begin{aligned} &\simeq -B\left(1-\frac{s}{2}, -\frac{1}{2}-\frac{t}{2}\right)[\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q}\tilde{t}^{2m+q} \\ &\quad \cdot \sum_{j=0}^{2m}\binom{2m}{j}\left[\frac{(1+t)}{2}\left(\frac{2}{\tilde{t}}\right)^j\left(\frac{1}{2}-\frac{t}{2}\right)_j - t\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j\right] \\ &\simeq -B\left(1-\frac{s}{2}, -\frac{1}{2}-\frac{t}{2}\right)[\sqrt{-t}]^{N-2m-2q+1}\left(\frac{1}{2M_2}\right)^{2m+q}2^{2m-1}\tilde{t}^q \\ &\quad \cdot \left[(1+t)U\left(-2m, \frac{t}{2}-2m+\frac{1}{2}, \frac{\tilde{t}}{2}\right) - 2tU\left(-2m, \frac{t}{2}-2m+\frac{3}{2}, \frac{\tilde{t}}{2}\right)\right]. \end{aligned} \tag{3.31}$$

In this case there are again two terms as in the amplitude A_2 but with an even argument $a = -2m$. Finally, the amplitude gives the universal power-law behavior for string states at *all* mass levels with the correct intercept $a_0 = \frac{1}{2}$ of fermionic string.

D. Amplitude $|N-1, 2m, q-1\rangle \otimes |b^T_{-(1/2)} b^P_{-(1/2)} b^P_{-(3/2)}\rangle$

The fourth scattering amplitude corresponding to state in Eq. (2.16) is

$$A_4^{(N-1,2m,q-1)} = \langle \psi_1^{T^1} e^{-\phi_1} e^{ik_1X_1} \cdot (\partial X_2^T)^{N-2m-2q} (\partial X_2^L)^{2m} (\partial^2 X_2^L)^{q-1} \psi_2^T \psi_2^P \partial \psi_2^P e^{-\phi_2} e^{ik_2X_2} \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4X_4} \rangle \tag{3.32}$$

where we have dropped out an overall factor. The scattering amplitude can be calculated to be

$$A_4^{(N-1,2m,q-1)} = \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q}\cdot\left[\frac{e^P\cdot k_1}{-x}+\frac{e^P\cdot k_3}{1-x}\right]^{2m}\left[\frac{e^P\cdot k_1}{x^2}+\frac{e^P\cdot k_3}{(1-x)^2}\right]^{q-1}\frac{1}{x}$$
$$\cdot\langle\psi_1^{T^1}\psi_2^T\rangle\langle\psi_2^P\psi_4^\sigma\rangle\langle\partial\psi_2^P\psi_3^\lambda\rangle k_{\lambda 3}k_{\sigma 4}$$
$$\simeq \int_0^1 dx x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\left[\frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q}\left[\frac{e^P\cdot k_1}{-x}+\frac{e^P\cdot k_3}{1-x}\right]^{2m}$$
$$\cdot\left[\frac{e^P\cdot k_3}{(1-x)^2}\right]^{q-1}\frac{1}{x}\frac{1}{M_2^2}\left[\frac{(e^{T^1}\cdot e^T)(k_2\cdot k_4)(k_2\cdot k_3)}{(1-x)^2}\right]$$
$$\simeq [\sqrt{-t}]^{N-2m-2q}\left(\frac{1}{2M_2}\right)^{2m+q+1}\tilde{t}^{2m+q}s\cdot\sum_{j=0}^{2m}\binom{2m}{j}\left(-\frac{s}{\tilde{t}}\right)^j\int_0^1 dx x^{k_1\cdot k_2-j}(1-x)^{k_2\cdot k_3-N+j-1}$$
$$\simeq [\sqrt{-t}]^{N-2m-2q}\left(\frac{1}{2M_2}\right)^{2m+q+1}\tilde{t}^{2m+q}s\cdot\sum_{j=0}^{2m}\binom{2m}{j}\left(-\frac{s}{\tilde{t}}\right)^j B(k_1\cdot k_2-j+1, k_2\cdot k_3-N+j). \tag{3.33}$$

With a similar approximation for the beta function, we get

$$A_4^{(N-1,2m,q-1)} \simeq B\left(1-\frac{s}{2},-\frac{1}{2}-\frac{t}{2}\right)[\sqrt{-t}]^{N-2m-2q}\left(\frac{1}{2M_2}\right)^{2m+q+1}\tilde{t}^{2m+q}\cdot(1+t)\sum_{j=0}^{2m}\binom{2m}{j}\left(-\frac{2}{\tilde{t}}\right)^j\left(\frac{1}{2}-\frac{t}{2}\right)_j$$
$$= B\left(1-\frac{s}{2},-\frac{1}{2}-\frac{t}{2}\right)[\sqrt{-t}]^{N-2m-2q}\left(\frac{1}{2M_2}\right)^{2m+q+1}\cdot 2^{2m}(\tilde{t})^q(1+t)U\left(-2m,\frac{t}{2}-2m+\frac{1}{2},\frac{\tilde{t}}{2}\right). \tag{3.34}$$

Again the amplitude gives the universal power-law behavior for string states at *all* mass levels with the correct intercept $a_0=\frac{1}{2}$ of fermionic string. In the next section we are going to use the four amplitudes calculated in this section to extract ratios of Eqs. (2.19) to (2.22) calculated in the fixed angle regime.

IV. REPRODUCING THE GR RATIOS IN THE RR

In the bosonic string calculation [29], we learned that the relative coefficients of the highest power t terms in the leading order amplitudes in the RR can be used to reproduce the ratios of the amplitudes in the GR for each fixed mass level. Here we present an explicit example. An explicit calculation of the high-energy string scattering amplitudes to some subleading orders in the RR for $M_2^2=4$ are

$$A_{TTT}\sim\frac{1}{8}\sqrt{-t}ts^3+\frac{3}{16}\sqrt{-t}t(t+6)s^2 + \frac{3t^3+84t^2-68t-864}{64}\sqrt{-t}s+O(1), \tag{4.1}$$

$$A_{LLT}\sim\frac{1}{64}\sqrt{-t}(t-6)s^3+\frac{3}{128}\sqrt{-t}(t^2-20t-12)s^2 + \frac{3t^3-342t^2-92t+5016+1728(-t)^{-1/2}}{512} \times\sqrt{-t}s+O(1), \tag{4.2}$$

$$A_{(LT)}\sim-\frac{1}{64}\sqrt{-t}(t+10)s^3-\frac{1}{128}\sqrt{-t}(3t^2+52t+60)s^2 - \frac{3[t^3+30t^2+76t-1080-960(-t)^{-1/2}]}{512} \times\sqrt{-t}s+O(1), \tag{4.3}$$

$$A_{[LT]}\sim-\frac{1}{64}\sqrt{-t}(t+2)s^3-\frac{3}{128}\sqrt{-t}(t+2)^2s^2 - \frac{(3t-8)(t+6)^2[1-2(-t)^{-1/2}]}{512}\sqrt{-t}s+O(1). \tag{4.4}$$

We have ignored an overall irrelevant factors in the above amplitudes. Note that the calculation of Eqs. (4.3) and (4.4) involves amplitude of the state $(\alpha_{-2}^T)(\alpha_{-1}^L)|0,k_2\rangle$ which can be shown to be of leading order in the RR [29], but is of subleading order in the GR as it is not in the form of Eq. (2.2). However, the contribution of the amplitude calculated from this state will not affect the ratios 8:1: − 1: − 1 in the RR [29]. One can now easily see that the ratios of the coefficients of the highest power of t in these *leading order* (s^3) coefficient functions $\frac{1}{8}:\frac{1}{64}:-\frac{1}{64}:-\frac{1}{64}$ in the RR agree with the ratios in the GR calculated in Eq. (2.8) as expected. Moreover, one further observation is that these ratios remain the same for the coefficients of the highest power of t in the *subleading orders* (s^2) $\frac{3}{16}:\frac{3}{128}:-\frac{3}{128}:-\frac{3}{128}$ and (s) $\frac{3}{64}:\frac{3}{512}:-\frac{3}{512}:-\frac{3}{512}$. More examples can be found in [29].

In this section, we are going to generalize the calculation to four classes of fermionic string states for arbitrary mass levels. We will first calculate the leading order results in this section and postpone the subleading order calculation to the next section. We begin with the first amplitude of Eq. (3.21).

A. Ratios for $|N, 2m, q\rangle \otimes |b^P_{-(3/2)}\rangle$

It is important to note that there are no linear relations among high-energy string scattering amplitudes, Eq. (3.21), of different string states for each fixed mass level in the RR. In other words, the ratios $A_1^{(N,2m,q)}/A_1^{(N,0,0)}$ are t-dependent functions and can be calculated to be

$$\frac{A_1^{(N,2m,q)}}{A_1^{(N,0,0)}} = \left(-\frac{1}{2M_2}\right)^{2m+q}(-)^m(\tilde{t}+2N+1)^{-m-q}(\tilde{t})^{2m+q} \cdot \sum_{j=0}^{2m}(-2m)_j\left(-N-1-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} \qquad (4.5)$$

where we have used Eq. (3.3a) to replace t by $\tilde{t}$. If the leading order coefficients in Eq. (4.5) extracted from the amplitudes in the RR are to be identified with the ratios calculated in the GR in Eq. (2.19), we need the following identity

$$\sum_{j=0}^{2m}(-2m)_j\left(-L-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} \qquad (4.6)$$

$$= 0(-\tilde{t})^0 + 0(-\tilde{t})^{-1} + \ldots + 0(-\tilde{t})^{-m+1} + \frac{(2m)!}{m!}(-\tilde{t})^{-m} + O\left\{\left(\frac{1}{\tilde{t}}\right)^{m+1}\right\} \qquad (4.7)$$

where $L = N + 1$ and is an integer. The coefficients of the terms $O\{(1/\tilde{t})^{m+1}\}$ in Eq. (4.7) are irrelevant for string amplitudes. If the identity of Eq. (4.7) obtained from superstring theory calculation is correct, this implies that the value of L affects only the subleading order terms $O\{(1/\tilde{t})^{m+1}\}$ in Eq. (4.7). We will show that this is indeed the case mathematically and numerically. In fact, we will show numerically that the identity is valid for arbitrary *real* L.

We will first show the cases of $L = 0, 1$, and then try to generalize the proof to arbitrary integers L. For $L = 1$, we rewrite the nontrivial leading term of the above summation in Eq. (4.6) as (we have replaced $\tilde{t}$ by t here for simplicity)

$$\sum_{j=0}^{2m}(-2m)_j\left(-1-\frac{t}{2}\right)\left(-\frac{t}{2}\right)_{j-1}\left(-\frac{2}{t}\right)^j\frac{1}{j!} = \sum_{j=0}^{2m}(-2m)_j\left(-1-\frac{t}{2}\right)\sum_{k=0}^{j-1}(-1)^{j-1-k}s(j-1,k)\left(-\frac{t}{2}\right)^k\left(-\frac{2}{t}\right)^j\frac{1}{j!}$$
$$\Rightarrow\left[\sum_{j=m}^{2m}(-2m)_j s(j-1,j-m)\frac{2^m}{j!} + \sum_{j=m+1}^{2m}(-2m)_j s(j-1,j-m-1)\frac{2^m}{j!}\right](-t)^{-m}$$
$$= 2^m\sum_{j=0}^{m}(-1)^{j+m}\binom{2m}{j+m}s(j+m-1,j)(-t)^{-m} + 2^m\sum_{j=1}^{m}(-1)^{j+m}\binom{2m}{j+m}s(j+m-1,j-1)(-t)^{-m} \qquad (4.8)$$

where we have used the signed Stirling number of the first kind $s(n, k)$ to expand the Pochhammer symbol. The definition of $s(n, k)$ is

$$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k. \qquad (4.9)$$

Thus the nontrivial leading order identity of Eq. (4.7) can be written as ($m \geq 0$)

$$F(m) \equiv \sum_{j=0}^{m}(-1)^j\binom{2m}{j+m}[s(j+m-1,j-1) + s(j+m-1,j)] = (2m-1)!! \qquad (4.10)$$

where we have used the convention that

$$s(m-1,-1) = \begin{cases} = 0, & \text{for } m \geq 1 \\ = 1, & \text{for } m = 0 \end{cases}, \qquad s(-1,0) = 0, \qquad (4.11)$$

and $(2m-1)!! = 0$ for $m = 0$. To apply the algorithm developed by Mkauers in 2007 [30], we need to introduce an auxiliary variable u and define

$$F(u,m) \equiv \sum_{j=0}^{m+u}(-1)^j\binom{2m+u}{j+m}[s(j+m-1,j-1) + s(j+m-1,j)] \equiv f_1(u,m) + f_2(u,m) \qquad (4.12)$$

where f_1 and f_2 are the two summations, each with one Stirling number, and $F(0, m) = F(m)$. By the algorithm, both f_1, f_2 satisfy the following recurrence relation [30]

$$-(1+2m+u)f(u,m) + (2m+u)f(u+1,m) + f(u,m+1) = 0, \qquad (4.13)$$

hence, so is F. Equation (4.13) is the most nontrivial step to prove Eq. (4.10). Now, note that

$$F(u,0)=\sum_{j=0}^{u}(-1)^j\binom{u}{j}=\begin{cases}1, & u=0\\ 1, & u>0\end{cases}. \tag{4.14}$$

Using the recurrence relation Eq. (4.13) and substituting $(u,m)=(1,0),(2,0)\cdots$, one can prove that

$$F(u,1)=0,\quad \forall\ u>0. \tag{4.15}$$

Similarly, by substituting $(u,m)=(1,1),(2,1),(3,1)\cdots$, one gets $F(u,2)=0$, $\forall\ u>0$. In general, we have

$$F(u,m)=0,\quad \forall\ u>0. \tag{4.16}$$

Finally we substitute $u=0$ in the Eq. (4.13) to obtain

$$-(1+2m)F(0,m)+2mF(1,m)+F(0,m+1)=0, \tag{4.17}$$

which implies

$$F(m+1)=(2m+1)F(m). \tag{4.18}$$

Equation (4.10) is thus proved by mathematical induction. Note that the case for $L=0$ corresponds to $f_2=0$ in the above calculation. We thus have proved the nontrivial part of Eq. (4.7) for $L=0$, 1.

For $L=1$, the vanishing of the coefficients of $(-\tilde{t})^0,(-\tilde{t})^{-1},\ldots,(-\tilde{t})^{-m+1}$ terms on the LHS of Eq. (4.7) means, for $1\le i\le m$,

$$\begin{aligned}G(m,i)\equiv\sum_{j=0}^{m+i}(-1)^{j-i}\binom{2m}{j+m-i}&[s(j+m-1-i,j)\\&+s(j+m-1-i,j-1)]\\=0.\end{aligned} \tag{4.19}$$

Note that for the case of $L=0$, the second term of Eq. (4.19) vanishes. To prove the identity Eq. (4.19), we need the recurrence relation of $G(m,i)$ [30]

$$\begin{aligned}&-2(1+m)^2(1+2m)G(m,i)+(2+7m+4m^2)G(m+1,i)\\&-2m(1+m)(1+2m)G(m+1,i+1)-mG(m+2,i)=0.\end{aligned} \tag{4.20}$$

Putting $i=0,1,2..$, and using the fact we have just proved, i.e. $G(m+1,0)=(2m+1)G(m,0)$, one can show that

$$G(m,i)=0\quad \text{for } 1\le i\le m. \tag{4.21}$$

Equation (4.7) is finally proved for the case of $L=0$, 1.

We now proceed to prove Eq. (4.7) for $L=2,3,4,\ldots$. To do so, we rewrite the nontrivial leading term of Eq. (4.6) in another form as

$$\begin{aligned}\sum_{j=0}^{2m}(-2m)_j\left(-L-\frac{t}{2}\right)_j\left(-\frac{2}{t}\right)^j\frac{1}{j!}&=\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}\sum_{l=0}^{j}\binom{j}{l}(-L)_{j-l}\left(-\frac{t}{2}\right)_l\left(-\frac{2}{t}\right)^j\\&=\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}\sum_{l=0}^{j}\binom{j}{l}(-L)_{j-l}\sum_{s=0}^{l}(-1)^{l-s}s(l,s)\left(-\frac{t}{2}\right)^s\left(-\frac{2}{t}\right)^j\\&=\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}\sum_{l=0}^{j}\binom{j}{l}(-L)_{j-l}\sum_{s=0}^{l}(-1)^{l-s}s(l,s)\left(-\frac{2}{t}\right)^{j-s}\\&\longrightarrow\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}\sum_{l=0}^{j}\binom{j}{l}(-L)_{j-l}(-1)^{l-j+m}s(l,j-m)\left(-\frac{2}{t}\right)^m\\&=\sum_{j=0}^{m}\binom{2m}{j+m}\sum_{l=0}^{j+m}\binom{j+m}{l}(-1)^{l+m}(-L)_{j+m-l}s(l,j)\left(-\frac{2}{t}\right)^m.\end{aligned} \tag{4.22}$$

Thus the nontrivial leading order identity of Eq. (4.7) can be written as

$$\begin{aligned}\mathcal{F}(m,L)\equiv\sum_{j=0}^{m}\binom{2m}{j+m}\sum_{l=0}^{j+m}\binom{j+m}{l}(-1)^{l+m}\\\times s(l,j)(-L)_{j+m-l}\\=(2m-1)!!,\end{aligned} \tag{4.23}$$

which is independent of L! We will again use mathematical induction to prove the identity. Firstly, we note that, for $L=0$ and $L=1$, $\mathcal{F}(m,L=0)=(2m-1)!!$ and $\mathcal{F}(m,L=1)=(2m-1)!!$ as have been proved previously, so Eq. (4.23) is true. Secondly, we notice that $\mathcal{F}(m,L)$ satisfies the following recurrence relation [30]

$$\begin{aligned}&2(1+m)(1+2m)\mathcal{F}(m,1-L)\\&\quad-2(1+m)(1+2m)\mathcal{F}(m,2-L)\\&\quad+(2+2m+N)\mathcal{F}(1+m,-L)\\&\quad-(3+2m+2N)\mathcal{F}(1+m,1-L)\\&\quad+(1+N)\mathcal{F}(1+m,2-L)=0.\end{aligned} \tag{4.24}$$

Equation (4.24) gives a recurrence relation for $\mathcal{F}(m, L)$ with three consecutive values of L. One thus can solve the equation and get the final solution $\mathcal{F}(m, L) = (2m-1)!!$. We thus have proved Eq. (4.23) for any integer L.

To complete the proof of Eq. (4.7), we need to show the vanishing of the coefficients of $(-\tilde{t})^0, (-\tilde{t})^{-1}, \ldots, (-\tilde{t})^{-m+1}$ terms on Eq. (4.6) for $L = 2, 3.4\ldots$ At this stage, the authors are unable to do the exact proof for this case. Instead, we give numerical calculation of Eq. (4.6) for some values of m. The results support the identity Eq. (4.6) and can be found in Appendix B. Moreover, the identity seems to be valid for arbitrary *real* values L not just integer. So we will take Eq. (4.6) as an identity in combinatorial theory predicted by string theory calculations. We thus have shown that high-energy superstring scattering amplitudes $A_1^{(N,2m,q)}$ of Eq. (3.19) in the RR can be used to extract the ratios $T_1^{(N,2m,q)}/T_1^{(N,0,0)}$ of Eq. (2.19) in the GR by using the Stirling number identities. That is

$$\lim_{t\to\infty}\frac{A_1^{(N,2m,q)}}{A_1^{(N,0,0)}} = \lim_{t\to\infty}\left(-\frac{1}{2M_2}\right)^{2m+q}2^{2m}(-t)^{m+2q} \times U\left(-2m, \frac{t}{2}-2m+\frac{3}{2}, \frac{t}{2}\right) = \left(-\frac{1}{2M_2}\right)^{q+m}\frac{(2m-1)!!}{(-M_2)^m} = \frac{T_1^{(N,2m,q)}}{T_1^{(N,0,0)}}. \quad (4.25)$$

B. Ratios for $|N+1, 2m+1, q\rangle \otimes |b^P_{-(1/2)}\rangle$

The ratios $A_2^{(N+1,2m+1,q)}/A_1^{(N,0,0)}$ can be calculated to be

$$\frac{A_2^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}} = \left(-\frac{1}{2M_2}\right)^{2m+q+1}(-\tilde{t})^m \cdot \left[(1+t)\times\sum_{j=0}^{2m+1}\binom{1+2m}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(\frac{1}{2}-\frac{t}{2}\right)_j - \tilde{t}\sum_{j=0}^{2m+1}\binom{1+2m}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j\right]. \quad (4.26)$$

The bracket in the above equation can be simplified by dropping out the subleading order terms in the calculation, and one obtains

$$\begin{aligned}
&(1+t)\sum_{j=0}^{2m+1}\binom{2m+1}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(\frac{1}{2}-\frac{t}{2}\right)_j - \tilde{t}\sum_{j=0}^{2m+1}\binom{2m+1}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j \\
&= (1+t)\sum_{j=0}^{2m+1}(-2m-1)_j\left(-N-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} - \tilde{t}\sum_{j=0}^{2m+1}(-2m-1)_j\left(-N-1-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} \\
&\approx \tilde{t}\sum_{j=0}^{2m+1}(-2m-1)_j\left(-N-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} - \tilde{t}\sum_{j=0}^{2m+1}(-2m-1)_j\left(-N-1-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} \\
&= 2(2m+1)\cdot\sum_{j=1}^{2m+1}(-2m)_{j-1}\left(-N-\frac{\tilde{t}}{2}\right)_{j-1}\frac{(-2/\tilde{t})^{j-1}}{(j-1)!} \\
&= 2(2m+1)\cdot\sum_{j=0}^{2m}(-2m)_j\left(-N-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!} \qquad (4.27)
\end{aligned}$$

where we have dropped out the subleading order terms in the second equality of the calculation. Finally, the ratios can be calculated to be

$$\begin{aligned}
\frac{A_2^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}} &= \left(-\frac{1}{2M_2}\right)^{2m+q+1}(-\tilde{t})^m\cdot\left[(1+t)\sum_{j=0}^{2m+1}\binom{1+2m}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(\frac{1}{2}-\frac{t}{2}\right)_j - \tilde{t}\sum_{j=0}^{2m+1}\binom{1+2m}{j}\left(\frac{2}{\tilde{t}}\right)^j\left(-\frac{1}{2}-\frac{t}{2}\right)_j\right] \\
&\simeq \left(-\frac{1}{2M_2}\right)^{2m+q+1}(-\tilde{t})^m 2(2m+1)\sum_{j=0}^{2m}(-2m)_j\left(-N-1-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!}. \qquad (4.28)
\end{aligned}$$

By using the identity Eq. (4.7), one can show that the leading order coefficients in Eq. (4.28) can be identified with the ratios calculated in the GR in Eq. (2.20). That is

$$\lim_{t\to\infty}\frac{A_2^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}} = \frac{T_2^{(N+1,2m+1,q)}}{T_1^{(N,0,0)}}. \qquad (4.29)$$

In the calculation for this case, it is crucial to reduce the upper limit of the summation $2m+1$ to $2m$. Otherwise, the identity Eq. (4.7) will not be applicable. It is remarkable to see that the leading order coefficients of Eq. (4.28) can be identified with ratios of Eq. (2.20) in the GR.

C. Ratios for $|N+1,2m,q\rangle \otimes |b^T_{-(1/2)}\rangle$

The ratios $A_3^{(N+1,2m,q)}/A_1^{(N,0,0)}$ can be calculated to be

$$\frac{A_3^{(N+1,2m,q)}}{A_1^{(N,0,0)}}=\frac{1}{2}\left(-\frac{1}{2M_2}\right)^{2m+q-1}(-\tilde{t})^m \times\sum_{j=0}^{2m}(-2m)_j\left(-N-1-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!}. \quad (4.30)$$

By using the identity Eq. (4.7), one can show that the leading order coefficients in Eq. (4.30) can be identified with the ratios calculated in the GR in Eq. (2.21). That is

$$\lim_{t\to\infty}\frac{A_3^{(N+1,2m,q)}}{A_1^{(N,0,0)}}=\frac{T_3^{(N+1,2m,q)}}{T_1^{(N,0,0)}}. \quad (4.31)$$

D. Ratios for $|N-1,2m,q-1\rangle \otimes |b^T_{-(1/2)}b^P_{-(1/2)}b^P_{-(3/2)}\rangle$

The ratios $A_4^{(N-1,2m,q-1)}/A_1^{(N,0,0)}$ can be calculated to be

$$\frac{A_4^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}}=\left(-\frac{1}{2M_2}\right)^{2m+q}(-\tilde{t})^m \times\sum_{j=0}^{2m}(-2m)_j\left(-N-1-\frac{\tilde{t}}{2}\right)_j\frac{(-2/\tilde{t})^j}{j!}. \quad (4.32)$$

By using the identity Eq. (4.7), one can show that the leading order coefficients in Eq. (4.32) can be identified with the ratios calculated in the GR in Eq. (2.22). That is

$$\lim_{t\to\infty}\frac{A_4^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}}=\frac{T_4^{(N+1,2m+1,q)}}{T_1^{(N,0,0)}}. \quad (4.33)$$

We thus have succeeded in extracting the ratios of high-energy superstring scattering amplitudes in the GR from the high-energy superstring scattering amplitudes in the RR. In the next section, we will study the subleading order amplitudes.

V. SUBLEADING ORDER AMPLITUDES

In this section, we calculate the next few subleading order amplitudes in the RR for the mass levels $M_2^2=2(N+1)=4,\ 6,\ 8$. Some results for the bosonic string calculation were presented in Eq. (4.7) to Eq. (4.7) in the last section. The relevant kinematic can be found in Appendix A. We will see that the ratios derived in Sec. IV persist to subleading order amplitudes in the RR. For the even mass levels with $(N+1)=\frac{M_2^2}{2}=\text{odd}$, we conjecture and give evidence that the existence of these ratios in the RR persists to all orders in the Regge expansion of all high-energy string scattering amplitudes. For the odd mass levels with $(N+1)=\frac{M_2^2}{2}=\text{even}$, the existence of these ratios will show up only in the first $\frac{N+1}{2}+1$ terms in the Regge expansion of the amplitudes. For the mass level $M_2^2=4$, there are three states for Eq. (2.13), and we obtain the subleading order expansions as follows.

$$|2,0,0\rangle|b^T_{-(1/2)}\rangle\to\left(\frac{1}{4}t^2-\frac{1}{4}t\right)s+\left(\frac{1}{4}t^3+\frac{9}{4}t^2+\frac{7}{4}t-\frac{5}{4}\right)s^0+\left(\frac{5}{2}t^3+18t^2+\frac{39}{2}t+4\right)s^{-1}+O[s^{-2}], \quad (5.1)$$

$$|2,2,0\rangle|b^T_{-(1/2)}\rangle\to\left(\frac{1}{32}t^2+\frac{1}{8}t+\frac{19}{32}\right)s+\left(\frac{1}{32}t^3+\frac{23}{32}t^2+\frac{35}{32}t-\frac{19}{32}\right)s^0+\left(\frac{3}{4}t^3-\frac{13}{4}t^2-\frac{39}{4}t-\frac{23}{4}\right)s^{-1}+O[s^{-2}], \quad (5.2)$$

$$|2,0,1\rangle|b^T_{-(1/2)}\rangle\to\left(-\frac{1}{16}t^2-\frac{1}{4}t+\frac{5}{16}\right)s+\left(-\frac{1}{16}t^3-\frac{15}{16}t^2-\frac{27}{16}t-\frac{29}{16}\right)s^0+\left(-\frac{3}{4}t^3-\frac{17}{4}t^2-\frac{45}{4}t-\frac{31}{4}\right)s^{-1}+O[s^{-2}]. \quad (5.3)$$

In order to simply the notation in the above equations, we have only shown the second state of the four-point functions in the correction functions to represent the scattering amplitudes on the left-hand side of each equation. We find that the ratios of the leading order coefficients of st^2 are $\frac{1}{4}:\frac{1}{32}:-\frac{1}{16}$, and it is easy to check that these are the same as the ratios in the fixed angle limit. Moreover, the ratios persist in the second subleading order terms s^0t^3 as $\frac{1}{4}:\frac{1}{32}:-\frac{1}{16}$. The ratios terminate to this order. We can also compare the ratios among different worldsheet fermionic states but with the same mass level $M_2^2=4$. We have the expansions:

$$|2,1,0\rangle|b^L_{-(1/2)}\rangle\to\left(\frac{1}{16}t^2-\frac{7}{16}t\right)s+\left(\frac{1}{16}t^3-\frac{29}{16}t^2-\frac{49}{16}t-\frac{35}{16}\right)s^0+\left(-\frac{7}{4}t^3-\frac{67}{4}t^2-\frac{117}{4}t-\frac{57}{4}\right)s^{-1}+O[s^{-2}], \quad (5.4)$$

$$|1,0,0\rangle|b^L_{-(3/2)}\rangle\to\left(-\frac{1}{8}t^2-\frac{5}{8}t\right)s+\left(-\frac{1}{8}t^3-\frac{17}{8}t^2-\frac{33}{8}t-\frac{25}{8}\right)s^0+\left(-\frac{7}{4}t^3-\frac{61}{4}t^2-\frac{109}{4}t-\frac{55}{4}\right)s^{-1}+O[s^{-2}], \quad (5.5)$$

$$|0,0,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle \to \left(\frac{1}{32}t^2+\frac{3}{16}t+\frac{5}{32}\right)s+\left(\frac{1}{32}t^3+\frac{15}{32}t^2+\frac{27}{32}t+\frac{13}{32}\right)s^0 + \left(\frac{1}{2}t^3+\frac{7}{2}t^2+\frac{11}{2}t+\frac{5}{2}\right)s^{-1}+O[s^{-2}]. \quad (5.6)$$

The ratios of the leading order coefficients are proportional to that of state $|2,0,0\rangle|b^T_{-(1/2)}\rangle$, and can be calculated to be

$$|2,0,0\rangle|b^T_{-(1/2)}\rangle:|2,1,0\rangle|b^L_{-(1/2)}\rangle:|1,0,0\rangle|b^L_{-(3/2)}\rangle :|0,0,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle=\frac{1}{4}:\frac{1}{16}:-\frac{1}{8}:\frac{1}{32}. \quad (5.7)$$

They again match with the ratios in the fixed angle limit. One can also find that the second subleading order ratios are the same $\frac{1}{4}:\frac{1}{16}:-\frac{1}{8}:\frac{1}{32}$. Again the ratios terminate to this order.

For the mass level $M_2^2=6$, there are three states in Eq. (2.13). We again calculate the subleading order expansions. Interestingly, in this case the ratios of the coefficients seem to be the same in all orders as can be seen in the following:

$$|3,0,0\rangle|b^T_{-(1/2)}\rangle \to \sqrt{-t}\left(\frac{1}{8}t^2-\frac{1}{8}t\right)s^2+\sqrt{-t}\left(\frac{3}{16}t^3+\frac{25}{16}t^2+\frac{25}{16}t-\frac{21}{16}\right)s + \sqrt{-t}\left(\frac{3}{64}t^4+\frac{197}{64}t^3+\frac{625}{32}t^2+\frac{743}{32}t+\frac{411}{64}\right)s^0 + O[s^{-1}], \quad (5.8)$$

$$|3,2,0\rangle|b^T_{-(1/2)}\rangle \to \sqrt{-t}\left(\frac{1}{96}t^2-\frac{1}{48}t+\frac{11}{32}\right)s^2+\sqrt{-t}\left(\frac{1}{64}t^3+\frac{13}{32}t-\frac{5}{8}\right)s + \sqrt{-t}\left(\frac{1}{256}t^4+\frac{9}{128}t^3-\frac{925}{256}t^2-\frac{729}{64}t-\frac{1481}{256}\right)s^0 + O[s^{-1}], \quad (5.9)$$

$$|3,0,1\rangle|b^T_{-(1/2)}\rangle \to \sqrt{-t}\left(-\frac{1}{16\sqrt{6}}t^2-\frac{3}{8\sqrt{6}}t+\frac{7}{16\sqrt{6}}\right)s^2 + \sqrt{-t}\left(-\frac{3}{32\sqrt{6}}t^3-\frac{3}{2\sqrt{6}}t^2-\frac{51}{16\sqrt{6}}t-\frac{19}{4\sqrt{6}}\right)s + \sqrt{-t}\left(-\frac{3}{128\sqrt{6}}t^4-\frac{111}{64\sqrt{6}}t^3-\frac{1841}{128\sqrt{6}}t^2-\frac{1209}{32\sqrt{6}}t - \frac{3573}{128\sqrt{6}}\right)s^0+O[s^{-1}]. \quad (5.10)$$

We find that the ratios of the leading order coefficients of $s^2t^{5/2}$ are $\frac{1}{8}:\frac{1}{96}:-\frac{1}{16\sqrt{6}}$, and they agree with the ratios in the fixed angle limit. The ratios of the second and the third order coefficients of $st^{7/2}$ and $s^0t^{9/2}$ are $\frac{3}{16}:\frac{1}{64}:-\frac{3}{32\sqrt{6}}$ and $\frac{3}{64}:\frac{1}{256}:-\frac{3}{128\sqrt{6}}$, respectively. We find that these two sets of ratios are the same with one another. We predict that the ratios persist to all orders in the expansions.

The expansions among different worldsheet fermionic states but with same mass level $M_2^2=6$ are

$$|3,0,0\rangle|b^L_{-(1/2)}\rangle \to \sqrt{-t}\left(\frac{1}{48}t^2-\frac{17}{48}t\right)s^2+\sqrt{-t}\left(\frac{1}{32}t^3-\frac{151}{96}t^2-\frac{295}{96}t-\frac{119}{32}\right)s + \sqrt{-t}\left(\frac{1}{128}t^4-\frac{249}{128}t^3-\frac{1317}{64}t^2-\frac{2883}{64}t-\frac{3831}{128}\right)s^0 + O[s^{-1}], \quad (5.11)$$

$$|2,0,0\rangle|b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(-\frac{1}{8\sqrt{6}}t^2-\frac{7}{8\sqrt{6}}t\right)s^2 + \sqrt{-t}\left(-\frac{3}{16\sqrt{6}}t^3-\frac{57}{16\sqrt{6}}t^2-\frac{129}{16\sqrt{6}}t-\frac{147}{16\sqrt{6}}\right)s + \sqrt{-t}\left(-\frac{3}{64\sqrt{6}}t^4-\frac{285}{64\sqrt{6}}t^3-\frac{1289}{32\sqrt{6}}t^2 - \frac{2831}{32\sqrt{6}}t-\frac{4011}{64\sqrt{6}}\right)s^0+O[s^{-1}], \quad (5.12)$$

$$|1,0,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(\frac{1}{96}t^2+\frac{1}{12}t+\frac{7}{96}\right)s^2 + \sqrt{-t}\left(\frac{1}{64}t^3+\frac{9}{32}t^2+\frac{31}{48}t+\frac{61}{96}\right)s + \sqrt{-t}\left(\frac{1}{256}t^4+\frac{77}{192}t^3+\frac{2531}{768}t^2+\frac{643}{96}t+\frac{3569}{768}\right)s^0 + O[s^{-1}]. \quad (5.13)$$

The ratios of the leading order coefficients are given by

$$|3,0,0\rangle|b^T_{-(1/2)}\rangle:|3,1,0\rangle|b^L_{-(1/2)}\rangle:|2,0,0\rangle|b^L_{-(3/2)}\rangle :|1,0,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle=\frac{1}{8}:\frac{1}{48}:-\frac{1}{8\sqrt{6}}:\frac{1}{96}. \quad (5.14)$$

We have checked that they agree with the ratios in the fixed angle limit. The second and the third subleading order ratios are $\frac{3}{16}:\frac{1}{32}:-\frac{3}{16\sqrt{6}}:\frac{1}{64}$ and $\frac{3}{64}:\frac{1}{128}:-\frac{3}{64\sqrt{6}}:\frac{1}{256}$, respectively. Again they agree with the ratios in the fixed angle limit. We expect that the ratios persist to all orders in the expansions.

For the mass level $M_2^2=8$, there are six states in Eq. (2.13)

$$\begin{aligned}|4,0,0\rangle|b^T_{-(1/2)}\rangle \rightarrow & \sqrt{-t}\Big(\frac{1}{16}t^3-\frac{1}{16}t^2\Big)s^3+\sqrt{-t}\Big(\frac{1}{8}t^4+t^3+\frac{5}{4}t^2-\frac{9}{8}t\Big)s^2\\ &+\sqrt{-t}\Big(\frac{1}{16}t^5+\frac{45}{16}t^4+\frac{139}{8}t^3+\frac{91}{4}t^2+\frac{137}{16}t-\frac{81}{16}\Big)s^1\\ &+\sqrt{-t}\Big(\frac{7}{4}t^5+\frac{193}{4}t^4+\frac{2013}{8}t^3+\frac{2899}{8}t^2+\frac{1381}{8}t+\frac{171}{8}\Big)s^0+O[s^{-1}],\end{aligned} \tag{5.15}$$

$$\begin{aligned}|4,2,0\rangle|b^T_{-(1/2)}\rangle \rightarrow & \sqrt{-t}\Big(\frac{1}{256}t^3-\frac{3}{64}t^2+\frac{51}{256}t\Big)s^3+\sqrt{-t}\Big(\frac{1}{128}t^4-\frac{49}{256}t^3+\frac{7}{256}t^2-\frac{251}{256}t+\frac{459}{256}\Big)s^2\\ &+\sqrt{-t}\Big(\frac{1}{256}t^5-\frac{63}{256}t^4-\frac{9}{2}t^3-\frac{893}{64}t^2-\frac{1465}{256}t-\frac{1797}{256}\Big)s^1\\ &+\sqrt{-t}\Big(-\frac{13}{128}t^5-\frac{1273}{128}t^4-\frac{6419}{64}t^3-\frac{15167}{64}t^2-\frac{26093}{128}t-\frac{5801}{128}\Big)s^0+O[s^{-1}],\end{aligned} \tag{5.16}$$

$$\begin{aligned}|4,4,0\rangle|b^T_{-(1/2)}\rangle \rightarrow & \sqrt{-t}\Big(\frac{3}{4096}t^3-\frac{129}{4096}t^2-\frac{979}{4096}t-\frac{2895}{4096}\Big)s^3+\sqrt{-t}\Big(\frac{3}{2048}t^4-\frac{327}{2048}t^3-\frac{5899}{2048}t^2-\frac{8917}{2048}t+\frac{2235}{512}\Big)s^2\\ &+\sqrt{-t}\Big(\frac{3}{4096}t^5-\frac{1017}{4096}t^4-\frac{23853}{2048}t^3-\frac{18573}{2048}t^2+\frac{76519}{4096}t-\frac{23133}{4096}\Big)s^1\\ &+\sqrt{-t}\Big(-\frac{123}{1024}t^5-\frac{19031}{1024}t^4-\frac{24895}{512}t^3+\frac{10657}{512}t^2+\frac{109321}{1024}t+\frac{81701}{1024}\Big)s^0+O[s^{-1}],\end{aligned} \tag{5.17}$$

$$\begin{aligned}|4,0,1\rangle|b^T_{-(1/2)}\rangle \rightarrow & \sqrt{-t}\Big(-\frac{1}{64\sqrt{2}}t^3-\frac{1}{8\sqrt{2}}t^2+\frac{9}{64\sqrt{2}}t\Big)s^3+\sqrt{-t}\Big(-\frac{1}{32\sqrt{2}}t^4-\frac{35}{64\sqrt{2}}t^3-\frac{83}{64\sqrt{2}}t^2-\frac{161}{64\sqrt{2}}t+\frac{81}{64\sqrt{2}}\Big)s^2\\ &+\sqrt{-t}\Big(-\frac{1}{64\sqrt{2}}t^5-\frac{53}{64\sqrt{2}}t^4-\frac{125}{16\sqrt{2}}t^3-\frac{181}{8\sqrt{2}}t^2-\frac{1187}{64\sqrt{2}}t-\frac{891}{64\sqrt{2}}\Big)s^1\\ &+\sqrt{-t}\Big(-\frac{13}{32\sqrt{2}}t^5-\frac{401}{32\sqrt{2}}t^4-\frac{1665}{16\sqrt{2}}t^3-\frac{4397}{16\sqrt{2}}t^2-\frac{8401}{32\sqrt{2}}t-\frac{2165}{32\sqrt{2}}\Big)s^0+O[s^{-1}],\end{aligned} \tag{5.18}$$

$$\begin{aligned}|4,0,2\rangle|b^T_{-(1/2)}\rangle \rightarrow & \sqrt{-t}\Big(\frac{1}{512}t^3+\frac{17}{512}t^2+\frac{63}{512}t-\frac{81}{512}\Big)s^3+\sqrt{-t}\Big(\frac{1}{256}t^4+\frac{27}{256}t^3+\frac{163}{256}t^2+\frac{89}{256}t+\frac{45}{16}\Big)s^2\\ &+\sqrt{-t}\Big(\frac{1}{512}t^5+\frac{53}{512}t^4+\frac{365}{256}t^3+\frac{1413}{256}t^2+\frac{5685}{512}t-\frac{2095}{512}\Big)s^1\\ &+\sqrt{-t}\Big(\frac{1}{32}t^5+\frac{29}{32}t^4+\frac{547}{32}t^3+\frac{1893}{32}t^2+90t+\frac{945}{16}\Big)s^0+O[s^{-1}],\end{aligned} \tag{5.19}$$

$$\begin{aligned}|4,2,1\rangle|b^T_{-(1/2)}\rangle \rightarrow & \sqrt{-t}\Big(-\frac{1}{1024\sqrt{2}}t^3+\frac{3}{1024\sqrt{2}}t^2+\frac{57}{1024\sqrt{2}}t-\frac{459}{1024\sqrt{2}}\Big)s^3\\ &+\sqrt{-t}\Big(-\frac{1}{512\sqrt{2}}t^4+\frac{33}{512\sqrt{2}}t^3+\frac{421}{512\sqrt{2}}t^2-\frac{309}{512\sqrt{2}}t+\frac{297}{64\sqrt{2}}\Big)s^2\\ &+\sqrt{-t}\Big(-\frac{1}{1024\sqrt{2}}t^5+\frac{159}{1024\sqrt{2}}t^4+\frac{2139}{512\sqrt{2}}t^3+\frac{3983}{512\sqrt{2}}t^2+\frac{27419}{1024\sqrt{2}}t+\frac{1043}{1024\sqrt{2}}\Big)s^1\\ &+\sqrt{-t}\Big(\frac{3}{32\sqrt{2}}t^5+\frac{1073}{128\sqrt{2}}t^4+\frac{1203}{32\sqrt{2}}t^3+\frac{11931}{64\sqrt{2}}t^2+\frac{1913}{8\sqrt{2}}t+\frac{13569}{128\sqrt{2}}\Big)s^0+O[s^{-1}].\end{aligned} \tag{5.20}$$

We find that the ratios of the leading order coefficients of s^3t^3 are $\frac{1}{16}:\frac{1}{256}:\frac{3}{4096}:-\frac{1}{64\sqrt{2}}:\frac{1}{512}:-\frac{1}{1024\sqrt{2}}$, and they agree with the ratios in the fixed angle limit. The ratios of the second and the third order coefficients of s^2t^4 and s^1t^5 are $\frac{1}{8}:\frac{1}{128}:\frac{3}{2048}:-\frac{1}{32\sqrt{2}}:\frac{1}{256}:-\frac{1}{512\sqrt{2}}$ and $\frac{1}{16}:\frac{1}{256}:\frac{3}{4096}:-\frac{1}{64\sqrt{2}}:\frac{1}{512}:-\frac{1}{1024\sqrt{2}}$, respectively. We find that the above three ratios are the same with one another. One can see that the ratios terminate at the order s^0t^5 as expected.

The expansions among different worldsheet fermionic states but with the same mass level $M_2^2 = 8$ are

$$|4,0,0\rangle|b^L_{-(1/2)}\rangle \to \sqrt{-t}\left(\frac{1}{128}t^3 - \frac{31}{128}t^2\right)s^3 + \sqrt{-t}\left(\frac{1}{64}t^4 - \frac{19}{16}t^3 - \frac{41}{16}t^2 - \frac{279}{64}t\right)s^2 + \sqrt{-t}\left(\frac{1}{128}t^5 - \frac{231}{128}t^4 - \frac{1289}{64}t^3 - \frac{3385}{64}t^2 - \frac{5527}{128}t - \frac{2511}{128}\right)s^1 + \sqrt{-t}\left(-\frac{55}{64}t^5 - \frac{2731}{64}t^4 - \frac{9975}{32}t^3 - \frac{22423}{32}t^2 - \frac{39563}{64}t - \frac{11223}{64}\right)s^0 + O[s^{-1}], \quad (5.21)$$

$$|4,2,0\rangle|b^L_{-(1/2)}\rangle \to \sqrt{-t}\left(\frac{3}{2048}t^3 - \frac{131}{1024}t^2 + \frac{759}{2048}t\right)s^3 + \sqrt{-t}\left(\frac{3}{1024}t^4 - \frac{1355}{2048}t^3 + \frac{7207}{2048}t^2 + \frac{3111}{2048}t + \frac{6831}{2048}t\right)s^2 + \sqrt{-t}\left(\frac{3}{2048}t^5 - \frac{2153}{2048}t^4 + \frac{8169}{1024}t^3 + \frac{32837}{1024}t^2 + \frac{133867}{2048}t + \frac{41631}{2048}\right)s^1 + \sqrt{-t}\left(-\frac{265}{512}t^5 + \frac{2259}{512}t^4 + \frac{37975}{256}t^3 + \frac{143935}{256}t^2 + \frac{342379}{512}t + \frac{140223}{512}\right)s^0 + O[s^{-1}], \quad (5.22)$$

$$|4,0,1\rangle|b^L_{-(1/2)}\rangle \to \sqrt{-t}\left(-\frac{1}{512\sqrt{2}}t^3 + \frac{11}{256\sqrt{2}}t^2 + \frac{279}{512\sqrt{2}}t\right)s^3 + \sqrt{-t}\left(-\frac{1}{256\sqrt{2}}t^4 + \frac{181}{512\sqrt{2}}t^3 + \frac{1687}{512\sqrt{2}}t^2 + \frac{943}{512\sqrt{2}}t + \frac{2511}{512\sqrt{2}}\right)s^2 + \sqrt{-t}\left(-\frac{1}{512\sqrt{2}}t^5 + \frac{363}{512\sqrt{2}}t^4 + \frac{2081}{256\sqrt{2}}t^3 + \frac{8293}{256\sqrt{2}}t^2 + \frac{27967}{512\sqrt{2}}t + \frac{3915}{512\sqrt{2}}\right)s^1 + \sqrt{-t}\left(\frac{51}{128\sqrt{2}}t^5 + \frac{1467}{128\sqrt{2}}t^4 + \frac{7523}{64\sqrt{2}}t^3 + \frac{25623}{64\sqrt{2}}t^2 + \frac{63223}{128\sqrt{2}}t + \frac{28679}{128\sqrt{2}}\right)s^0 + O[s^{-1}], \quad (5.23)$$

$$|4,2,1\rangle|b^L_{-(1/2)}\rangle \to \sqrt{-t}\left(-\frac{3}{8192\sqrt{2}}t^3 + \frac{235}{8192\sqrt{2}}t^2 + \frac{1599}{8192\sqrt{2}}t - \frac{6831}{8192\sqrt{2}}\right)s^3 + \sqrt{-t}\left(-\frac{3}{4096\sqrt{2}}t^4 + \frac{803}{4096\sqrt{2}}t^3 - \frac{809}{4096\sqrt{2}}t^2 - \frac{43879}{4096\sqrt{2}}t + \frac{861}{512\sqrt{2}}\right)s^2 + \sqrt{-t}\left(-\frac{3}{8192\sqrt{2}}t^5 + \frac{3029}{8192\sqrt{2}}t^4 - \frac{21333}{4096\sqrt{2}}t^3 - \frac{115237}{4096\sqrt{2}}t^2 - \frac{150707}{8192\sqrt{2}}t - \frac{326379}{8192\sqrt{2}}\right)s^1 + \sqrt{-t}\left(\frac{829}{4096\sqrt{2}}t^5 - \frac{45011}{4096\sqrt{2}}t^4 - \frac{68377}{2048\sqrt{2}}t^3 - \frac{657673}{2048\sqrt{2}}t^2 - \frac{2228619}{4096\sqrt{2}}t - \frac{807579}{4096\sqrt{2}}\right)s^0 + O[s^{-1}], \quad (5.24)$$

$$|3,0,0\rangle|b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(\frac{1}{32\sqrt{2}}t^3 + \frac{9}{32\sqrt{2}}t^2\right)s^3 + \sqrt{-t}\left(\frac{1}{16\sqrt{2}}t^4 + \frac{21}{16\sqrt{2}}t^3 + \frac{53}{16\sqrt{2}}t^2 + \frac{81}{16\sqrt{2}}t\right)s^2 + \sqrt{-t}\left(\frac{1}{32\sqrt{2}}t^5 + \frac{69}{32\sqrt{2}}t^4 + \frac{327}{16\sqrt{2}}t^3 + \frac{847}{16\sqrt{2}}t^2 + \frac{1485}{32\sqrt{2}}t + \frac{729}{32\sqrt{2}}\right)s^1 + \sqrt{-t}\left(\frac{9}{8\sqrt{2}}t^5 + \frac{315}{8\sqrt{2}}t^4 + \frac{284}{\sqrt{2}}t^3 + \frac{658}{\sqrt{2}}t^2 + \frac{4771}{8\sqrt{2}}t + \frac{1377}{8\sqrt{2}}\right)s^0 + O[s^{-1}], \quad (5.25)$$

$$|3,2,0\rangle|b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(\frac{1}{512\sqrt{2}}t^3 - \frac{11}{256\sqrt{2}}t^2 - \frac{279}{512\sqrt{2}}t\right)s^3 + \sqrt{-t}\left(\frac{1}{256\sqrt{2}}t^4 - \frac{147}{512\sqrt{2}}t^3 - \frac{2187}{512\sqrt{2}}t^2 - \frac{1477}{512\sqrt{2}}t - \frac{2511}{512\sqrt{2}}\right)s^2 + \sqrt{-t}\left(\frac{1}{512\sqrt{2}}t^5 - \frac{267}{512\sqrt{2}}t^4 - \frac{865}{64\sqrt{2}}t^3 - \frac{5919}{128\sqrt{2}}t^2 - \frac{37009}{512\sqrt{2}}t - \frac{8721}{512\sqrt{2}}\right)s^1 + \sqrt{-t}\left(-\frac{71}{256\sqrt{2}}t^5 - \frac{5143}{256\sqrt{2}}t^4 - \frac{26457}{128\sqrt{2}}t^3 - \frac{81497}{128\sqrt{2}}t^2 - \frac{184135}{256\sqrt{2}}t - \frac{75127}{256\sqrt{2}}\right)s^0 + O[s^{-1}], \quad (5.26)$$

$$|3,0,1\rangle|b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(-\frac{1}{256}t^3-\frac{9}{128}t^2-\frac{81}{256}t^2\right)s^3+\sqrt{-t}\left(-\frac{1}{128}t^4-\frac{57}{256}t^3-\frac{373}{256}t^2-\frac{279}{256}t-\frac{729}{256}\right)s^2 + \sqrt{-t}\left(-\frac{1}{256}t^5-\frac{61}{256}t^4-\frac{221}{64}t^3-\frac{465}{32}t^2-\frac{5955}{256}t-\frac{243}{256}\right)s^1 + \sqrt{-t}\left(-\frac{11}{128}t^5-\frac{447}{128}t^4-\frac{2847}{64}t^3-\frac{10587}{64}t^2-\frac{27511}{128}t-\frac{13131}{128}\right)s^0+O[s^{-1}], \tag{5.27}$$

$$|2,0,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(\frac{1}{256}t^3+\frac{5}{128}t^2+\frac{9}{256}t\right)s^3+\sqrt{-t}\left(\frac{1}{128}t^4+\frac{41}{256}t^3+\frac{109}{256}t^2+\frac{151}{256}t+\frac{81}{256}\right)s^2 + \sqrt{-t}\left(\frac{1}{256}t^5+\frac{73}{256}t^4+\frac{83}{32}t^3+\frac{409}{64}t^2+\frac{1503}{256}t+\frac{459}{256}\right)s^1 + \sqrt{-t}\left(\frac{21}{128}t^5+\frac{705}{128}t^4+\frac{2263}{64}t^3+\frac{4987}{64}t^2+\frac{9085}{128}t+\frac{2953}{128}\right)s^0+O[s^{-1}], \tag{5.28}$$

$$|2,2,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(\frac{1}{4096}t^3-\frac{41}{4096}t^2-\frac{501}{4096}t-\frac{459}{4096}\right)s^3 + \sqrt{-t}\left(\frac{1}{2048}t^4-\frac{29}{512}t^3-\frac{825}{1024}t^2-\frac{321}{512}t+\frac{249}{2048}\right)s^2 + \sqrt{-t}\left(\frac{1}{4096}t^5-\frac{371}{4096}t^4-\frac{4715}{4096}t^3-\frac{14947}{4096}t^2-\frac{39131}{4096}t-\frac{18295}{4096}\right)s^1 + \sqrt{-t}\left(-\frac{45}{1024}t^5-\frac{3473}{1024}t^4-\frac{17217}{1024}t^3-\frac{46833}{1024}t^2-\frac{97697}{1024}t-\frac{35037}{1024}\right)s^0+O[s^{-1}], \tag{5.29}$$

$$|2,0,1\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle \to \sqrt{-t}\left(-\frac{1}{1024\sqrt{2}}t^3-\frac{19}{1024\sqrt{2}}t^2-\frac{99}{1024\sqrt{2}}t-\frac{81}{1024\sqrt{2}}\right)s^3 + \sqrt{-t}\left(-\frac{1}{512\sqrt{2}}t^4-\frac{13}{256\sqrt{2}}t^3-\frac{41}{128\sqrt{2}}t^2-\frac{47}{256\sqrt{2}}t+\frac{45}{512\sqrt{2}}\right)s^2 + \sqrt{-t}\left(-\frac{1}{1024\sqrt{2}}t^5-\frac{57}{1024\sqrt{2}}t^4-\frac{405}{512\sqrt{2}}t^3-\frac{1945}{512\sqrt{2}}t^2-\frac{5573}{1024\sqrt{2}}t-\frac{2437}{1024\sqrt{2}}\right)s^1 + \sqrt{-t}\left(-\frac{3}{128\sqrt{2}}t^5-\frac{57}{64\sqrt{2}}t^4-\frac{737}{64\sqrt{2}}t^3-\frac{615}{16\sqrt{2}}t^2-\frac{5859}{128\sqrt{2}}t-\frac{1151}{64\sqrt{2}}\right)s^0+O[s^{-1}]. \tag{5.30}$$

The ratios of the leading order coefficients are given by

$$|4,0,0\rangle|b^T_{-(1/2)}\rangle:|4,0,0\rangle|b^L_{-(1/2)}\rangle:|4,2,0\rangle|b^L_{-(1/2)}\rangle:|4,0,1\rangle|b^L_{-(1/2)}\rangle:|4,2,1\rangle|b^L_{-(1/2)}\rangle :|3,0,0\rangle|b^L_{-(3/2)}\rangle:|3,2,0\rangle|b^L_{-(3/2)}\rangle:|3,0,1\rangle|b^L_{-(3/2)}\rangle:|2,0,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle :|2,2,0\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle:|2,0,1\rangle|b^T_{-(1/2)}b^L_{-(1/2)}b^L_{-(3/2)}\rangle = \frac{1}{16}:\frac{1}{128}:\frac{3}{2048}:-\frac{1}{512\sqrt{2}}:-\frac{3}{8192\sqrt{2}}:\frac{1}{32\sqrt{2}}:\frac{1}{512\sqrt{2}}:-\frac{1}{256}:\frac{1}{256}:\frac{1}{4096}:-\frac{1}{1024\sqrt{2}}. \tag{5.31}$$

They agree with the ratios in the fixed angle limit. It can be checked that the second and the third subleading order coefficients of s^2t^4 and s^1t^5 have the same ratios. The ratios terminate in the fourth subleading order s^0t^5 coefficients as expected.

VI. CONCLUSION

In this paper, we calculate high-energy massive superstring scattering amplitudes in the Regge regime (RR). We explicitly calculate four classes of high-energy Regge scattering amplitudes. As an application, we demonstrate the universal power-law behavior for all massive string scattering amplitudes in the RR. In particular, the amplitude gives the correct intercept $a_0 = \frac{1}{2}$ of fermonic string theory. These results generalize the well-known results for the case of high-energy four-point tachyon scattering amplitudes. Moreover, as in the bosonic string case considered previously [29], these amplitudes can be used to extract ratios among high-energy superstring scattering amplitudes in the fixed angle regime. The calculation relies on a set of Stirling number identities, which we are able to give only partial proofs of them. For this reason, we give a numerical "proof" of the whole identities. Hopefully, the complete mathematical proof of these identities suggested by string theory calculation can be worked out in the future.

In addition to the leading order calculation, we also study the subleading order amplitudes in the Regge regime for the first few mass levels. In particular, we conjecture and give evidence that the existence of the GR ratios in the RR persists to all orders in the Regge expansion of all string amplitudes for the even mass level with $(N+1)=\frac{M_2^2}{2}=$ odd. For the odd mass levels with $(N+1)=\frac{M_2^2}{2}=$ even, the existence of the GR ratios shows up only in the first $\frac{N+1}{2}+1$ terms in the Regge expansion of the amplitudes. It will be an interesting challenge to further study this subleading order effect.

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APPENDIX A: KINEMATIC RELATIONS IN THE RR

In this appendix, we list the expressions of the kinematic variables we used in the evaluation of 4-point functions in this paper. For convenience, we take the center of momentum frame and choose the momenta of particles 1 and 2 to be along the X^1-direction. The high-energy scattering plane is defined to be on the X^1-X^2 plane. (See Fig. 1.)

The momenta of the four particles are

$$k_1=\left(+\sqrt{p^2+M_1^2},-p,0\right), \tag{A1}$$

$$k_2=\left(+\sqrt{p^2+M_2^2},+p,0\right), \tag{A2}$$

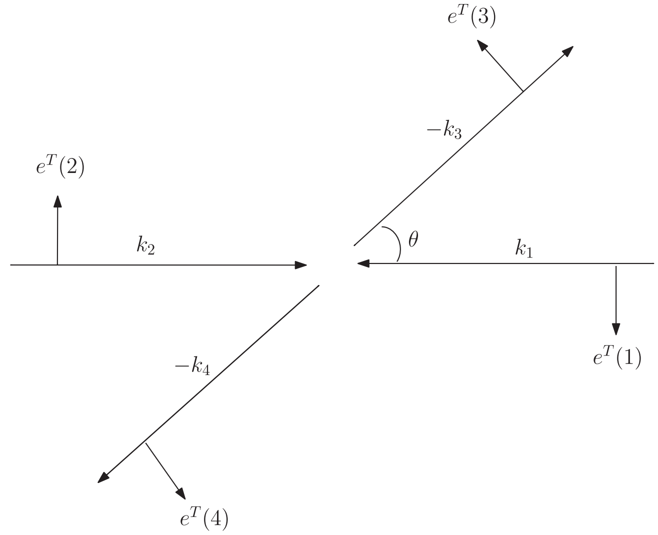

FIG. 1. Kinematic variables in the center of mass frame.

$$k_3=\left(-\sqrt{q^2+M_3^2},-q\cos\theta,-q\sin\theta\right), \tag{A3}$$

$$k_4=\left(-\sqrt{q^2+M_4^2},+q\cos\theta,+q\sin\theta\right) \tag{A4}$$

where $p\equiv|\vec{p}|$, $q\equiv|\vec{q}|$ and $k_i^2=-M_i^2$. In the calculation of the string scattering amplitudes, we use the following formulas

$$\begin{aligned}-k_1\cdot k_2&=\sqrt{p^2+M_1^2}\cdot\sqrt{p^2+M_2^2}+p^2\\&=\frac{1}{2}(s-M_1^2-M_2^2),\end{aligned} \tag{A5}$$

$$\begin{aligned}-k_2\cdot k_3&=-\sqrt{p^2+M_2^2}\cdot\sqrt{q^2+M_3^2}+pq\cos\theta\\&=\frac{1}{2}(t-M_2^2-M_3^2),\end{aligned} \tag{A6}$$

$$\begin{aligned}-k_1\cdot k_3&=-\sqrt{p^2+M_1^2}\cdot\sqrt{q^2+M_3^2}-pq\cos\theta\\&=\frac{1}{2}(u-M_1^2-M_3^2)\end{aligned} \tag{A7}$$

where the Mandelstam variables are defined as usual with

$$s+t+u=\sum_i M_i^2=2N-1. \tag{A8}$$

The center of mass energy E is defined as

$$E = \frac{1}{2}\left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2}\right) = \frac{1}{2}\left(\sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2}\right). \tag{A9}$$

We define the polarizations of the string state on the scattering plane as

$$e^P = \frac{1}{M_2}\left(\sqrt{p^2 + M_2^2}, p, 0\right), \tag{A10}$$

$$e^L = \frac{1}{M_2}\left(p, \sqrt{p^2 + M_2^2}, 0\right), \tag{A11}$$

$$e^T = (0, 0, 1). \tag{A12}$$

The projections of the momenta on the scattering plane can be calculated as (here we only list the ones we need for our calculations)

$$e^P \cdot k_1 = -\frac{1}{M_2}\left(\sqrt{p^2 + M_1^2}\sqrt{p^2 + M_2^2} + p^2\right), \tag{A13}$$

$$e^L \cdot k_1 = -\frac{p}{M_2}\left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2}\right), \tag{A14}$$

$$e^T \cdot k_1 = 0 \tag{A15}$$

and

$$e^P \cdot k_3 = \frac{1}{M_2}\left(\sqrt{q^2 + M_3^2}\sqrt{p^2 + M_2^2} - pq\cos\theta\right), \tag{A16}$$

$$e^L \cdot k_3 = \frac{1}{M_2}\left(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}\cos\theta\right), \tag{A17}$$

$$e^T \cdot k_3 = -q\sin\theta. \tag{A18}$$

We now expand the kinematic relations to the subleading orders in the RR. We first express all kinematic variables in terms of s and t, and then expand all relevant quantities in s:

$$E_1 = \frac{s - (M_2^2 + 2)}{2\sqrt{2}}, \tag{A19}$$

$$E_2 = \frac{s + (M_2^2 + 2)}{2\sqrt{2}}, \tag{A20}$$

$$|\mathbf{k_2}| = \sqrt{E_1^2 + 2}, \qquad |\mathbf{K_3}| = \sqrt{\frac{s}{4} + 2}; \tag{A21}$$

$$e_P \cdot k_1 = -\frac{1}{2M_2}s + \left(-\frac{1}{M_2} + \frac{M_2}{2}\right), \qquad \text{(exact)} \tag{A22}$$

$$\begin{aligned} e_L \cdot k_1 = &-\frac{1}{2M_2}s + \left(-\frac{1}{M_2} + \frac{M_2}{2}\right) - 2M_2 s^{-1} \\ &- 2M_2(M_2^2 - 2)s^{-2} - 2m_2(M_2^4 - 6M_2^2 + 4)s^{-3} \\ &- 2M_2(M_2^6 - 12M_2^4 + 24M_2^2 - 8)s^{-4} + O(s^{-5}), \end{aligned} \tag{A23}$$

$$e_T \cdot k_1 = 0. \tag{A24}$$

A key step is to express the scattering angle θ in terms of s and t. This can be achieved by solving

$$t = -\left(-\left(E_2 - \frac{\sqrt{s}}{2}\right)^2 + (|\mathbf{k_2}| - |\mathbf{k_3}|\cos\theta)^2 + |\mathbf{k}_3|^2\sin^2\theta\right) \tag{A25}$$

to obtain

$$\theta = \arccos\left(\frac{s + 2t - M_2^2 + 6}{\sqrt{s+8}\sqrt{\frac{(s+2)^2 - 2(s-2)M_2^2 + M_2^4}{s}}}\right). \qquad \text{(exact)} \tag{A26}$$

One can then calculate the following expansions which we used in the subleading order calculation in Sec. V

$$e_P \cdot k_3 = \frac{1}{M_2}\left(E_2\frac{\sqrt{s}}{2} - |\mathbf{k_2}||\mathbf{k_3}|\cos\theta\right) = -\frac{t + 2 - M_2^2}{2M_2}, \tag{A27}$$

$$\begin{aligned} e_L \cdot k_3 = &\frac{1}{M_2}\left(k_2\frac{\sqrt{2}}{2} - E_2 k_3\cos\theta\right) \\ = &-\frac{t + 2 + M_2^2}{2M_2} - M_2 t s^{-1} - M_2[-4(t+1) \\ &+ M_2^2(t-2)]s^{-2} - M_2[4(4+3t) - 12tM_2^2 \\ &+ (t-4)M_2^4]s^{-3} - M_2[-16(3+2t) + 24(2+3t)M_2^2 \\ &- 24(-1+t)M_2^4 + (-6+t)M_2^6]s^{-4} + O(s^{-5}), \end{aligned} \tag{A28}$$

$$
\begin{aligned}
e_T \cdot k_3 &= -|\mathbf{k_3}|\sin\theta \\
&= -\sqrt{-t} - \frac{1}{2}\sqrt{-t}(2 + t + M_2^2)s^{-1} - \frac{1}{8\sqrt{-t}}[32 + 52t + 20t^2 + t^3 + (32 + 20t - 6t^2)M_2^2 + (8 - 3t)M_2^4]s^{-2} \\
&\quad + \frac{1}{16\sqrt{-t}}[320 + 456t + 188t^2 + 22t^3 + t^4 - (-224 + 36t + 132t^2 + 5t^3)M_2^2. + (-16 - 122t + 15t^2)M_2^4 \\
&\quad + (-24 + 5t)M_2^6]s^{-3} + \frac{1}{128(-t)^{3/2}}[1024 + 12032t + 16080t^2 + 7520t^3 + 1432t^4 + 136t^5 + 5t^6 \\
&\quad - 4(-512 - 896t + 2232t^2 + 1844t^3 + 170t^4 + 7t^5)M_2^2 + 2(768 - 2240t - 2372t^2 + 1172t^3 + 35t^4)M_2^4 \\
&\quad - 4(-128 + 288t - 450t^2 + 35t^3)M_2^6 + (64 + 240t - 35t^2)M_2^8]s^{-4} + O(s^{-5}). \qquad \text{(A29)}
\end{aligned}
$$

APPENDIX B: NUMERICAL IDENTITIES

In this appendix, we give a numerical proof of the identity

$$\sum_{j=0}^{2m}(-2m)_j\left(-L - \frac{\tilde{t}}{2}\right)_j \frac{(-2/\tilde{t})^j}{j!} \qquad \text{(B1)}$$

$$= 0(-\tilde{t})^0 + 0(-\tilde{t})^{-1} + \ldots + 0(-\tilde{t})^{-m+1} + \frac{(2m)!}{m!}(-\tilde{t})^{-m} + O\left\{\left(\frac{1}{\tilde{t}}\right)^{m+1}\right\}, \qquad \text{(B2)}$$

which we used intensively in Sec. IV to rederive the ratios among high-energy scattering amplitudes in the fixed angle regime from the Regge scattering amplitudes. The nontrivial identity of Eq. (B2) has been proved for arbitrary integers L by using Stirling number identities. However, the "0 identities" were exactly proved only for $L = 0$, 1. We conjecture that all identities in Eq. (B2) are valid for arbitrary *real* L. We have done the numerical proof of the identity for m up to $m = 10$. Here we give only results of $m = 3$ and 4

$$\sum_{j=0}^{6}(-2m)_j\left(a - \frac{\tilde{t}}{2}\right)_j \frac{(-2/\tilde{t})^j}{j!} = \frac{120}{(-\tilde{t})^3} + \frac{720a^2 + 2640a + 2080}{(-\tilde{t})^4} + \frac{480a^4 + 4160a^3 + 12000a^2 + 12928a + 3840}{(-\tilde{t})^5} + \frac{64a^6 + 960a^5 + 5440a^4 + 14400a^3 + 17536a^2 + 7680a}{(-\tilde{t})^6}, \qquad \text{(B3)}$$

$$\sum_{j=0}^{8}(-2m)_j\left(a - \frac{\tilde{t}}{2}\right)_j \frac{(-2/\tilde{t})^j}{j!} = \frac{1680}{(-\tilde{t})^4} + \frac{13440a^2 + 67200a + 76160}{(-\tilde{t})^5} + \frac{13440a^4 + 152320a^3 + 595840a^2 + 930048a + 467712}{(-\tilde{t})^6} \qquad \text{(B4)}$$

$$+ \frac{3584a^6 + 68096a^5 + 501760a^4 + 1802752a^3 + 3236352a^2 + 2608128a + 645120}{(-\tilde{t})^7} + \frac{256a^8 + 7168a^7 + 82432a^6 + 501760a^5 + 1732864a^4 + 3361792a^3 + 3345408a^2 + 1290240a}{(-\tilde{t})^8} \qquad \text{(B5)}$$

where $a = -L$. We can see that a shows up only in the subleading order terms as expected. For $m = 5$, the nontrivial leading order term is $\frac{30240}{(-\tilde{t})^5}$ as expected. For $m = 10$, the nontrivial leading order term is $\frac{670442572800}{(-\tilde{t})^{10}}$ as expected.

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