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Noncommutativity in weakly curved background by canonical methods

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Using the canonical method, we investigate the Dp-brane world-volume noncommutativity in a weakly curved background. The term ''weakly curved'' means that, in the leading order, the source of nonflatness is an infinitesimally small Kalb-Ramond field $B_{\mu\nu}$, linear in coordinate, while the Ricci tensor does not contribute, being an infinitesimal of the second order. On the solution of boundary conditions, we find a simple expression for the space-time coordinates in terms of the effective coordinates and momenta. This basic relation helped us to prove that noncommutativity appears only on the world sheet boundary. The noncommutativity parameter has a standard form, but with the infinitesimally small and coordinate-dependent antisymmetric tensor $B_{\mu\nu}$. This result coincides with that obtained on the group manifolds in the limit of the large level n of the current algebra. After quantization, the algebra of the functions on the Dp-brane world volume is represented with the Kontsevich star product instead of the Moyal one in the flat background.

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I. INTRODUCTION

Quantization of the open-string ending on Dp branes has been studied in many papers [1–12]. In the presence of the Kalb-Ramond antisymmetric tensor field $B_{\mu\nu}$, the Dp brane becomes a noncommutative manifold.

In the simplest case, all background fields—the metric tensor $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, and the dilaton field Φ—are constant. Geometrically, it corresponds to the embedding of a flat Dp brane into a flat background. In that case, the dilaton field does not give any contribution, and the quadratic action represents two-dimensional free-field theory. The constant $B_{\mu\nu}$ field does not affect the dynamics in the world sheet interior. It contributes only to its boundary and it is a source of noncommutativity. Several methods have been used to investigate this case: the operator product expansion of the open-string vertex operator [2,3], the mode expansion of the classical solution [4], the methods of conformal field theory [5], and the canonical quantization for constrained systems [6,8].

In Refs. [7,8], the inclusion of a dilaton field, linear in space-time coordinates, has been investigated. Because only the gradient of the dilaton field appears in space-time field equations, this case technically behaves similarly to that with a constant background. The dilaton field induces a commutative Dp-brane coordinate in the direction of the dilaton gradient $\partial_\mu\Phi$. For some particular relation between background fields, when $\partial_\mu\Phi$ is a lightlike vector with respect to the open- or closed-string metrics, the local gauge symmetries appear. They turn some Neumann boundary conditions into Dirichlet ones and decrease the number of Dp-brane dimensions [8].

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In Refs. [9], the noncommutative properties of the Dp-brane world volume embedded in the space-time of Type IIB superstring theory have been investigated. Similarly as in the bosonic theory, the presence of σ-antisymmetric fields leads to noncommutativity of the supercoordinates. In the case of Type IIB theory, this supermultiplet beside $B_{\mu\nu}$ from the NS-NS (Naveu-Schwarz) sector contains the difference of two gravitons, $\psi^{\alpha}_{-\mu}$ from the NS-R (Ramond) sector and the symmetric part of the bispinor $F^{\alpha\beta}$ from the R-R sector.

In all previous investigations, the target space was assumed to be flat. In the present paper, we investigate the deformation of the Dp-brane world volume in a curved background. We choose a background such that the metric tensor $G_{\mu\nu}$ is constant, the antisymmetric tensor $B_{\mu\nu}$ is linear in coordinate, and its field strength $B_{\mu\nu\rho}$ is a nonvanishing parameter [3,10]. This choice is in accordance with the space-time equations of motion, obtained from the world sheet conformal invariance, as far as we can neglect the Ricci tensor. So, we demand that $B_{\mu\nu\rho}$ is an infinitesimal parameter and we work in the leading order in $B_{\mu\nu\rho}$ throughout the whole paper. The Ricci tensor is thus neglected as an infinitesimal of the second order. We call this choice the *weakly curved background*. Physically, this case corresponds to the embedding of a curved Dp brane into a curved background.

The open string with the nonvanishing field strength of the Kalb-Ramond field has been investigated in Refs. [10,11]. The correlation functions have been computed on the disk, and, therefrom, the Kontsevich product has been extracted. The considerations in Ref. [10] have been restricted to the weakly curved background, while those of Ref. [11] have been restricted to the first order in the derivatives of the background fields.

In [13], the same problem has been considered using the canonical method and some approximations based on low-energy limits. The main result is the new type of noncommutativity relation, where the noncommutativity parameter depends not only on coordinates but also on momenta. In the Lagrangian formulation, it means that it depends on the coordinates' time derivatives. This form of parameter has not been observed by the path-integral method, in Refs. [10,11]. So, the results obtained in the treatment of the same physical problem with a different formalism (in Refs. [10,11] using the path-integral method and in Ref. [13] using the canonical method) are not the same but differ essentially.

Therefore, to be able to better elucidate the evident discrepancies of these results, we developed a systematic canonical approach in which these ambiguities could be solved properly and the relation between Ref. [13] and Refs. [10,11] could be clarified. First, does the momentum-dependent term exist? Second, if this term exists, under which conditions does it disappear, like in Refs. [10,11]? Third, might there exist some new momentum-dependent terms, missed in [13] as a consequence of the low-energy limit assumption?

In the present paper, the problem of the open string in the weakly curved background is treated using canonical methods. The approach applied to the constant background fields [6,8] is generalized to the case of the curved one. The boundary conditions are treated as canonical constraints. Using the Dirac requirement (that the time derivatives of the primary constraints are also constraints) and Lagrangian equations of motion, we obtain the infinite set of constraints in the Lagrangian form.

Following the line of Refs. [8] using the Taylor expansion, we represent this infinite set of constraints at a point ($\sigma = 0$ and $\sigma = \pi$) with two σ-dependent constraints, even and odd under world sheet parity transformation (Ω: $\sigma \to -\sigma$). It is remarkable that these constraints can be expressed in compact form, in terms of coordinates, their first σ and τ derivatives, and their integrals.

At this point and thereafter, we switched from the Lagrangian to the Hamiltonian method. We checked the validity of the procedure by calculating the Poisson brackets between the Hamiltonian and the constraints, reaching the conclusion that these are, in fact, Hamiltonian constraints and that they form a complete set of constraints.

All constraints except the zero modes [14] are of the second class, and we solve them explicitly. On this solution, the original canonical variables can be expressed in terms of the effective ones. Imposing 2π periodicity, the constraints at $\sigma = \pi$ can be expressed in terms of those at $\sigma = 0$. We separately solve the symmetric and antisymmetric parts of the constraints and express Ω-odd variables in terms of Ω-even ones. So, the constraints appear as particular orbifold conditions, reducing the initial phase space to the Ω-even and 2π-periodic ones.

The transition from the initial phase space to the effective phase space on the orbifold requires some comment about the corresponding canonical brackets. We make a transition to the effective phase space with the variables q^μ and p_μ, 2π-periodic and symmetric under the transformation $\sigma \to -\sigma$, with $\sigma \in [-\pi, \pi]$, in two steps.

In the initial phase space, with the variables x^μ and π_μ, we use the standard Poisson brackets with $\sigma \in [0, \pi]$. Because the basic effective canonical variables q^μ and p_μ ($\bar{q}^\mu$ and $\bar{p}_\mu$) are not arbitrary functions but contain only the even (odd) powers in σ, their brackets do not close on the standard δ function on the interval $[0, \pi]$, but on the symmetric (antisymmetric) δ function on the interval $[-\pi, \pi]$ times 2 (see Appendix B).

We also impose the boundary condition $\Gamma_\mu = 0$. Instead of using the Dirac brackets associated with the second-class constraints Γ_μ, we solve the constraints $\Gamma_\mu = 0$ and then use the equivalent star brackets between the variables restricted to the constrained subspace.

The initial coordinates x^μ depend both on the effective coordinates q^μ and their canonically conjugated momenta p_μ. This fact is a source of noncommutativity. The coefficient in front of the momenta p_μ is not a constant, as in the case of the flat background, but depends on the effective coordinates q^μ. Because of this, the noncommutative parameter will also depend on q^μ. This fact is the source of nonassociativity.

We want to stress that, even in the curved background, after nontrivial calculations, it turns out that only the end points of the string are noncommutative, while the interior of the string commutes. At the world sheet's boundary, the Ω-odd parts of the coordinates vanish ($\bar{q}(0) = 0$ and $\bar{q}(\pi) = 0$) and, consequently, the effective coordinate is equal to the original one. So, we can say that the noncommutative parameter depends on the original variables x^μ. Formally, it has the same form as in the flat background, but now the Kalb-Ramond field is infinitesimal and linear in coordinate.

II. OPEN-STRING PROPAGATION IN A WEAKLY CURVED BACKGROUND

Let us consider the open bosonic string in the nontrivial background defined by the space-time fields: the metric $G_{\mu\nu}$ and the Kalb-Ramond antisymmetric tensor $B_{\mu\nu}$. The propagation is described by the action [15,16]

$$S = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta}G_{\mu\nu}(x) + \epsilon^{\alpha\beta}B_{\mu\nu}(x)\right]\partial_\alpha x^\mu \partial_\beta x^\nu$$

$$(\varepsilon^{01} = -1), \tag{2.1}$$

where integration goes over the two-dimensional world sheet Σ with coordinates ξ^α, $\alpha = 0, 1$. By $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$, we denote the coordinates of the D-dimensional space-time. Throughout the paper, we will use the notation $\xi^0 = \tau$, $\xi^1 = \sigma$ and $\dot{x} = \frac{\partial x}{\partial \tau}$, $x' = \frac{\partial x}{\partial \sigma}$.

In order to preserve the quantum world sheet conformal invariance, the β functions for both background fields must vanish as necessary conditions for the consistency of the theory. To the lowest order in the string slope parameter α', they have the form [15]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4}B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma} = 0, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} = 0. \tag{2.3}$$

Here, $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and the covariant derivative with respect to the space-time metric.

In fact, to fulfill conformal invariance, it is necessary to introduce an additional background field, the dilaton field Φ and the corresponding β function. Only derivatives of the dilaton field give a contribution to all β functions, so that the space-time equations of motion (2.2) and (2.3) are correct under the assumption $\Phi = \text{const}$.

It is an enormous task to make further progress with arbitrary background fields. Instead, we can employ a particular solution of the space-time field equations. We want to have the solution which admits a curved background, but to be technically as simple as possible.

It is clear that the nonzero Ricci tensor $R_{\mu\nu}$ implies a nontrivial $B_{\mu\nu\rho}$. Following [3,10], we choose the field strength of the Kalb-Ramond field to be constant ($B_{\mu\nu\rho} = \text{const}$) and infinitesimally small. This solves Eq. (2.3), and we can neglect the curvature $R_{\mu\nu}$ in (2.2) as an infinitesimal of the second order. Consequently, in the leading order, the solution of the space-time equations of motion produces the following background fields:

$$G_{\mu\nu} = \text{const}, \qquad B_{\mu\nu}[x] = \tfrac{1}{3}B_{\mu\nu\rho}x^\rho, \tag{2.4}$$

where the parameter $B_{\mu\nu\rho}$ is constant and infinitesimally small. Through the paper, we will work up to its first order. So, the chosen background is "weakly curved" as a consequence of the infinitesimally small Kalb-Ramond field $B_{\mu\nu}$, while the contribution of the Ricci curvature $R_{\mu\nu}$ can be neglected.

In the case of open string, the minimal action principle produces the equation of motion

$$\ddot{x}^\mu = x''^\mu - 2B^\mu_{\nu\rho}\dot{x}^\nu x'^\rho \tag{2.5}$$

and the boundary conditions on string end points

$$\gamma^\mu_0|_{\sigma=0,\pi} = 0, \tag{2.6}$$

where we have introduced the variable

$$\gamma^\mu_0 = x'^\mu - 2(G^{-1}B)^\mu{}_\nu\dot{x}^\nu. \tag{2.7}$$

Note that the linear background field $B_{\mu\nu}$ contributes to the equation of motion through its field strength. This is an essential difference from the case of the constant $B_{\mu\nu}$, when it does not appear in the equation of motion ($B_{\mu\nu\rho} = 0$), and the second term in the action (2.1) is topological.

III. LAGRANGIAN CONSISTENCY CONDITION

We are going to treat the boundary conditions (2.6) as the constraints. Because they must be conserved in time, their time derivative produces the new constraints, for which we again require time conservation. For technical reasons, instead of applying the Dirac consistency procedure, we will use the analogous Lagrangian consistency procedure.

A. Infinite set of constraints

Starting with the boundary condition γ^μ_0 as a constraint, with the help of the equation of motion, we obtain the infinite set of constraints at the string end points

$$\gamma^\mu_n|_{\sigma=0,\pi} = 0, \qquad \gamma^\mu_{n+1} \equiv \dot{\gamma}^\mu_n \qquad (n \geq 0). \tag{3.1}$$

In order to find the explicit form of these constraints, we introduce the following functions:

$$\begin{aligned}
\gamma^\mu &= \gamma^\mu_0 = x'^\mu - 2(G^{-1}B)^\mu{}_\nu\dot{x}^\nu, \\
\tilde{\gamma}^\mu &= \dot{x}^\mu - 2(G^{-1}B)^\mu{}_\nu x'^\nu, \\
Q^{\alpha\beta}_n &= \dot{x}^{(n)\alpha}x^{(n+1)\beta}, \\
R^{\alpha\beta}_n &= x^{(n+2)\alpha}x^{(n+1)\beta} + \dot{x}^{(n)\alpha}\dot{x}^{(n+1)\beta},
\end{aligned} \tag{3.2}$$

where $x^{(n)\alpha} = \frac{\partial^n}{\partial\sigma^n}x^\alpha$. On the equation of motion (2.5), their time derivatives in the leading order are

$$\begin{aligned}
\dot{\gamma}^\mu &= \tilde{\gamma}'^\mu, & \dot{\tilde{\gamma}}^\mu &= \gamma'^\mu - \tfrac{2}{3}B^\mu_{\alpha\beta}Q^{\alpha\beta}_0, \\
\dot{Q}^{\alpha\beta}_n &= R^{\alpha\beta}_n, & \dot{R}^{\alpha\beta}_n &= Q''^{\alpha\beta}_n - 4Q^{\alpha\beta}_{n+1}.
\end{aligned} \tag{3.3}$$

Therefore, their second time derivatives are closed on the same set of functions:

$$\begin{aligned}
\ddot{\gamma}^\mu &= \gamma''^\mu - \tfrac{2}{3}B^\mu_{\alpha\beta}Q'^{\alpha\beta}_0, & \ddot{\tilde{\gamma}}^\mu &= \tilde{\gamma}''^\mu - \tfrac{2}{3}B^\mu_{\alpha\beta}R^{\alpha\beta}_0, \\
\ddot{Q}^{\alpha\beta}_n &= Q''^{\alpha\beta}_n - 4Q^{\alpha\beta}_{n+1}, & \ddot{R}^{\alpha\beta}_n &= R''^{\alpha\beta}_n - 4R^{\alpha\beta}_{n+1}.
\end{aligned} \tag{3.4}$$

It is clear that the constraints with even indices, γ^μ_{2n}, depend on γ^μ and $Q^{\alpha\beta}$, and the ones with odd indices, γ^μ_{2n+1}, depend on $\tilde{\gamma}^\mu$ and $R^{\alpha\beta}$. Moreover, notice that every term in γ^μ_n has exactly $n+1$ derivatives over τ and σ. So, the expression of γ^μ_n should have the form

$$\begin{aligned}
\gamma^\mu_{2n} &= \gamma^{(2n)\mu} - \frac{2}{3}B^\mu_{\alpha\beta}\sum_{k=0}^{n-1}\alpha^k_{2n}Q^{(2n-2k-1)\alpha\beta}_k \qquad (n \geq 1), \\
\gamma^\mu_{2n+1} &= \tilde{\gamma}^{(2n+1)\mu} - \frac{2}{3}B^\mu_{\alpha\beta}\sum_{k=0}^{n-1}\beta^k_{2n}R^{(2n-2k-1)\alpha\beta}_k \qquad (n \geq 1),
\end{aligned} \tag{3.5}$$

with unknown constants α^k_{2n} and β^k_{2n}. We have already seen that $\gamma^\mu_0 = \gamma^\mu$ and $\gamma^\mu_1 = \tilde{\gamma}'^\mu$. From the definition $\gamma^\mu_{2n+2} = \ddot{\gamma}^\mu_{2n}$, we obtain the recurrence relation

$$\alpha_{2n+2}^0 = \alpha_{2n}^0 + 1,$$
$$\alpha_{2n+2}^k = \alpha_{2n}^k - 4\alpha_{2n}^{k-1} \qquad (k = 1, \cdots, n-1),$$
$$\alpha_{2n+2}^n = -4\alpha_{2n}^{n-1}, \tag{3.6}$$

with the solution

$$\alpha_{2n}^k = (-4)^k \binom{n}{k+1} \qquad (k = 0, \cdots, n-1). \tag{3.7}$$

Using $\gamma_{2n+1}^\mu = \dot{\gamma}_{2n}^\mu$, we conclude that $\beta_{2n}^k = \alpha_{2n}^k = (-4)^k \binom{n}{k+1}$.

B. Compact form of the constraints at $\sigma = 0$

We obtained the explicit form of the infinite set of constraints. Let us now multiply every constraint $\gamma_n^\mu|_{\sigma=0}$ with the appropriate power of σ and sum separately odd and even powers in σ. In this way, we gathered the infinite set of conditions into only two σ-dependent ones:

$$\Gamma_S^\mu(\sigma) = 0, \qquad \Gamma_A^\mu(\sigma) = 0, \tag{3.8}$$

with

$$\begin{aligned}\Gamma_S^\mu(\sigma) &\equiv \sum_{n=0}^\infty \frac{\sigma^{2n}}{(2n)!} \gamma_{2n}^\mu \Big|_{\sigma=0} \\ &= \gamma_S^\mu(\sigma) - \frac{2}{3} B^\mu{}_{\alpha\beta} \sum_{k=0}^\infty (\Gamma^Q)_k^{\alpha\beta}(\sigma), \\ \Gamma_A^\mu(\sigma) &\equiv \sum_{n=0}^\infty \frac{\sigma^{2n+1}}{(2n+1)!} \gamma_{2n+1}^\mu \Big|_{\sigma=0} \\ &= \tilde{\gamma}_A^\mu(\sigma) - \frac{2}{3} B^\mu{}_{\alpha\beta} \sum_{k=0}^\infty (\Gamma^R)_k^{\alpha\beta}(\sigma),\end{aligned} \tag{3.9}$$

where we introduced the symmetric part of γ^μ and the antisymmetric part of $\tilde{\gamma}^\mu$, defined in (3.2),

$$\begin{aligned}\gamma_S^\mu(\sigma) &\equiv \sum_{n=0}^\infty \frac{\sigma^{2n}}{(2n)!} \gamma^{(2n)\mu} \Big|_{\sigma=0}, \\ \tilde{\gamma}_A^\mu(\sigma) &\equiv \sum_{n=0}^\infty \frac{\sigma^{2n+1}}{(2n+1)!} \tilde{\gamma}^{(2n+1)\mu} \Big|_{\sigma=0}\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(\Gamma^Q)_k^{\alpha\beta}(\sigma) &\equiv \sum_{n=k+1}^\infty (-4)^k \binom{n}{k+1} \frac{\sigma^{2n}}{(2n)!} Q_k^{(2n-2k-1)\alpha\beta} \Big|_{\sigma=0}, \\ (\Gamma^R)_k^{\alpha\beta}(\sigma) &\equiv \sum_{n=k+1}^\infty (-4)^k \binom{n}{k+1} \\ &\quad \times \frac{\sigma^{2n+1}}{(2n+1)!} R_k^{(2n-2k-1)\alpha\beta} \Big|_{\sigma=0}.\end{aligned} \tag{3.11}$$

These sums can be represented in the integral form (see Appendix C 1)

$$\begin{aligned}(\Gamma^Q)_k{}^{\alpha\beta}(\sigma) = &\frac{(-1)^k \sigma}{2(k+1)!} \int_0^\sigma d\sigma_1^2 \\ &\times \int_0^{\sigma_1} d\sigma_2^2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k^2 (Q_A)_k^{\alpha\beta}(\sigma_k),\end{aligned} \tag{3.12}$$

in terms of the antisymmetric part of $Q_k^{\alpha\beta}$,

$$(Q_A)_k^{\alpha\beta}(\sigma) \equiv \sum_{n=0}^\infty \frac{\sigma^{2n+1}}{(2n+1)!} Q_k^{(2n+1)\alpha\beta} \Big|_0 = [\dot{x}^{(k)\alpha} x^{(k+1)\beta}]_A. \tag{3.13}$$

Notice that

$$(\Gamma^R)_k'^{\alpha\beta}(\sigma) = (\Gamma^Q)_k{}^{\alpha\beta}(\sigma)|_{Q\mapsto R}, \tag{3.14}$$

so that, by analogy, we can write

$$\begin{aligned}(\Gamma^R)_k{}^{\alpha\beta}(\sigma) = &\frac{(-1)^k}{4(k+1)!} \int_0^\sigma d\sigma_0{}^2 \\ &\times \int_0^{\sigma_0} d\sigma_1^2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k{}^2 (R_A)_k^{\alpha\beta}(\sigma_k),\end{aligned} \tag{3.15}$$

in terms of the antisymmetric part of $R_k^{\alpha\beta}$,

$$\begin{aligned}(R_A)_k^{\alpha\beta}(\sigma) &\equiv \sum_{n=0}^\infty \frac{\sigma^{2n+1}}{(2n+1)!} R_k^{(2n+1)\alpha\beta} \Big|_0 \\ &= [x^{(n+2)\alpha} x^{(n+1)\beta} + \dot{x}^{(n)\alpha} \dot{x}^{(n+1)\beta}]_A.\end{aligned} \tag{3.16}$$

Consequently, we can express $\Gamma_S^\mu(\sigma)$, defined in (3.9), in terms of the symmetric part of γ^μ and the antisymmetric parts of $\tilde{\gamma}^\mu$, $Q_k^{\alpha\beta}$, and $R_k^{\alpha\beta}$, defined in (3.2).

In order to separate the symmetric and antisymmetric parts under σ parity, we introduce the new variables

$$\begin{aligned}q^\mu(\sigma) &= \sum_{n=0}^\infty \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu} \Big|_{\sigma=0}, \\ \bar{q}^\mu(\sigma) &= \sum_{n=0}^\infty \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu} \Big|_{\sigma=0},\end{aligned} \tag{3.17}$$

which we will call open-string variables.

In terms of the new variables, we have

$$\begin{aligned}\gamma_S^\mu &= \bar{q}'^\mu - \tfrac{2}{3} B^\mu_{\nu\rho}(\dot{q}^\nu q^\rho + \dot{\bar{q}}^\nu \bar{q}^\rho), \\ \tilde{\gamma}_A^\mu &= \dot{\bar{q}}^\mu - \tfrac{2}{3} B^\mu_{\nu\rho}(q'^\nu q^\rho + \bar{q}'^\nu \bar{q}^\rho), \\ (Q_A)_k^{\alpha\beta} &= \dot{q}^{(k)\alpha} q^{(k+1)\beta} + \dot{\bar{q}}^{(k)\alpha} \bar{q}^{(k+1)\beta}, \\ (R_A)_k^{\alpha\beta} &= q^{(k+2)\alpha} q^{(k+1)\beta} + \bar{q}^{(k+2)\alpha} \bar{q}^{(k+1)\beta} \\ &\quad + \dot{q}^{(k)\alpha} \dot{q}^{(k+1)\beta} + \dot{\bar{q}}^{(k)\alpha} \dot{\bar{q}}^{(k+1)\beta}.\end{aligned} \tag{3.18}$$

Using two previous equations, we can rewrite the last terms in (3.9) as

$$\sum_{k=0}^{\infty}(\Gamma^Q)_k^{\alpha\beta} = h^{\alpha\beta}[\dot{q}, q] + h^{\alpha\beta}[\dot{\bar{q}}, \bar{q}],$$
$$\sum_{k=0}^{\infty}(\Gamma^R)_k^{\alpha\beta} = \int_0^\sigma d\sigma_0\{h^{\alpha\beta}[q'', q] + h^{\alpha\beta}[\bar{q}'', \bar{q}] + h^{\alpha\beta}[\dot{q}, \dot{q}] + h^{\alpha\beta}[\dot{\bar{q}}, \dot{\bar{q}}]\}(\sigma_0), \tag{3.19}$$

where we introduced the function $h^{\alpha\beta}[a, b]$,

$$h^{\alpha\beta}[a,b](\sigma) = \frac{\sigma}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\int_0^\sigma d\sigma_1^2\cdots\int_0^{\sigma_{k-1}} d\sigma_k^2 a^{(k)\alpha}(\sigma_k) \times b^{(k+1)\beta}(\sigma_k). \tag{3.20}$$

Using (E2) and (E3), we rewrite the constraints (3.9) as

$$\Gamma_S^\mu(\sigma) = \bar{q}'^\mu - \tfrac{2}{3}B^\mu_{\nu\rho}[\dot{q}^\nu q^\rho + \tfrac{1}{2}\dot{Q}^\nu q'^\rho + \tfrac{3}{2}\dot{\bar{q}}^\nu\bar{q}^\rho],$$
$$\Gamma_A^\mu(\sigma) = \dot{\bar{q}}^\mu - \tfrac{2}{3}B^\mu_{\nu\rho}[q'^\nu q^\rho + \tfrac{1}{2}\dot{Q}^\nu \dot{q}^\rho + \tfrac{3}{2}\bar{q}'^\nu\bar{q}^\rho]. \tag{3.21}$$

IV. CANONICAL FORM OF THE CONSTRAINTS AT $\sigma = 0$

Now, we are ready to make the transition from the Lagrangian to the Hamiltonian approach. Let us first introduce the canonical momenta corresponding to the coordinates x^μ,

$$\pi_\mu = \kappa(G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu), \tag{4.1}$$

and the canonical Hamiltonian,

$$H_C = \int_0^\pi d\sigma\left[\frac{1}{2\kappa}(G^{-1})^{\mu\nu}\pi_\mu\pi_\nu + \frac{\kappa}{2}G_{\mu\nu}x'^\mu x'^\nu + \frac{2}{3}B^\mu_{\nu\rho}\pi_\mu x'^\nu x^\rho\right]. \tag{4.2}$$

Similarly, as in (3.17), we introduce new, open-string momenta

$$p_\mu(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^{2n}}{(2n)!}\pi_\mu^{(2n)}\Big|_{\sigma=0},$$
$$\bar{p}_\mu(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^{2n+1}}{(2n+1)!}\pi_\mu^{(2n+1)}\Big|_{\sigma=0} \tag{4.3}$$

and rewrite the constraints (3.21) in a canonical form

$$\Gamma_S^\mu(\sigma) = \bar{q}'^\mu + \theta^{\mu\nu}(q)p_\nu + \frac{1}{2}\theta'^{\mu\nu}(q)P_\nu + \frac{3}{2}\theta^{\mu\nu}(\bar{q})\bar{p}_\nu,$$
$$\kappa\Gamma_A^\mu(\sigma) = (G^{-1})^{\mu\nu}\bar{p}_\nu + \frac{\kappa^2}{2}\theta^\mu{}_\nu(\bar{q})\bar{q}'^\nu + \frac{1}{2}\theta^{\mu\nu}(p)P_\nu, \tag{4.4}$$

where

$$\theta^{\mu\nu}[q(\sigma)] \equiv -\frac{2}{3\kappa}B^{\mu\nu}_\rho q^\rho(\sigma) = -\frac{2}{\kappa}(G^{-1})^{\mu\alpha}B_{\alpha\beta}[q(\sigma)](G^{-1})^{\beta\nu}, \tag{4.5}$$
$$P_\mu(\sigma) \equiv \int_0^\sigma d\eta p_\mu(\eta).$$

Note that, from the standard Poisson brackets

$$\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu\delta(\sigma - \bar{\sigma}), \tag{4.6}$$

we have two nontrivial relations for the Ω-even and odd subspaces

$$\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = 2\delta^\mu{}_\nu\delta_S(\sigma, \bar{\sigma}),$$
$$\{\bar{q}^\mu(\sigma), \bar{p}_\nu(\bar{\sigma})\} = 2\delta^\mu{}_\nu\delta_A(\sigma, \bar{\sigma}), \tag{4.7}$$

where δ_S and δ_A are defined in (A3). Because Γ_S^μ and Γ_A^μ, as the symmetric and antisymmetric functions, are independent, it is enough to consider the constraint $\Gamma^\mu = -\kappa(\Gamma_S^\mu - \Gamma_A^\mu)$. It weakly commutes with the Hamiltonian

$$\{H_c, \Gamma^\mu(\sigma)\} = \Gamma'^\mu(\sigma), \tag{4.8}$$

and, therefore, there are no more constraints. We can calculate the Poisson brackets, up to the linear term in the small parameter $B^{\mu\nu\rho}$, as

$$\{\Gamma^\mu(\sigma), \Gamma^\nu(\bar{\sigma})\} = -\kappa(G^{-1})^{\mu\nu}\delta'(\sigma - \bar{\sigma}) - B^{\mu\nu\rho}[\bar{p}_\rho(\sigma) - \kappa G_{\rho\tau}\bar{q}'^\tau(\sigma)]\delta(\sigma - \bar{\sigma}) \approx -\kappa(G^{-1})^{\mu\nu}\delta'(\sigma - \bar{\sigma}). \tag{4.9}$$

The sign $\approx$ is a weak equality which, in the canonical approach, means equality on the constraints. In this particular case, with the help of (4.4), from $\Gamma_S^\mu \approx 0$ and $\Gamma_A^\mu \approx 0$, it follows that $\bar{q}^\mu$ and $\bar{p}_\nu$ are proportional to $B^{\mu\nu\rho}$, so that the term in front of $\delta(\sigma - \bar{\sigma})$ in (4.9) is an infinitesimal of the second order. Therefore, we conclude that Γ^μ and, consequently, Γ_S^μ and Γ_A^μ are the second-class constraints. We will look for their solution in Sec. VI.

There is a slight improvement of the above conclusion. The Poisson brackets between the constraints Γ^μ are closed on $\delta'(\sigma - \bar{\sigma})$ and not on the $\delta(\sigma - \bar{\sigma})$ function. So, the zero mode of $\Gamma^\mu(\sigma)$,

$$\Gamma_0^\mu = \int_0^\pi d\sigma\Gamma^\mu(\sigma), \tag{4.10}$$

is the first-class constraint, because $\{\Gamma_0^\mu, \Gamma^\nu(\sigma)\} = 0$. Consequently, it is a generator of gauge symmetry with a constant parameter. We will use this fact at the end of Sec. VI to gauge away the center of mass of the coordinate.

V. CONSTRAINTS AT $\sigma = \pi$

In order to derive constraints at the other string end point $\sigma = \pi$, we will multiply every constraint $\gamma_n^\mu|_{\sigma=\pi}$ with the appropriate power of $\sigma - \pi$ and sum separately odd and even powers in $\sigma - \pi$. We obtain two new σ-dependent constraints:

$$\bar{\Gamma}^{\mu}_{S}(\sigma) = 0, \qquad \bar{\Gamma}^{\mu}_{A}(\sigma) = 0, \tag{5.1}$$

where

$$\begin{aligned} \bar{\Gamma}^{\mu}_{S}(\sigma) &= \sum_{n=0}^{\infty} \frac{(\sigma-\pi)^{2n}}{(2n)!} \gamma^{\mu}_{2n} \Big|_{\sigma=\pi} \\ &= \bar{\gamma}^{\mu}_{S}(\sigma) - \frac{2}{3} B^{\mu}{}_{\alpha\beta} \sum_{k=0}^{\infty} (\bar{\Gamma}^{Q})^{\alpha\beta}_{k}(\sigma), \\ \bar{\Gamma}^{\mu}_{A}(\sigma) &= \sum_{n=0}^{\infty} \frac{(\sigma-\pi)^{2n+1}}{(2n+1)!} \gamma^{\mu}_{2n+1} \Big|_{\sigma=\pi} \\ &= \bar{\bar{\gamma}}^{\mu}_{A}(\sigma) - \frac{2}{3} B^{\mu}{}_{\alpha\beta} \sum_{k=0}^{\infty} (\bar{\Gamma}^{R})^{\alpha\beta}_{k}(\sigma) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \bar{\gamma}^{\mu}_{S}(\sigma) &= \sum_{n=0}^{\infty} \frac{(\sigma-\pi)^{2n}}{(2n)!} \gamma^{(2n)\mu} \Big|_{\sigma=\pi}, \\ \bar{\bar{\gamma}}^{\mu}_{A}(\sigma) &= \sum_{n=0}^{\infty} \frac{(\sigma-\pi)^{2n+1}}{(2n+1)!} \tilde{\gamma}^{(2n+1)\mu} \Big|_{\sigma=\pi}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} (\bar{\Gamma}^{Q})^{\alpha\beta}_{k}(\sigma) &= \sum_{n=k+1}^{\infty} (-4)^{k} \binom{n}{k+1} \\ &\quad \times \frac{(\sigma-\pi)^{2n}}{(2n)!} Q_{k}^{(2n-2k-1)\alpha\beta} \Big|_{\sigma=\pi}, \\ (\bar{\Gamma}^{R})^{\alpha\beta}_{k}(\sigma) &= \sum_{n=k+1}^{\infty} (-4)^{k} \binom{n}{k+1} \\ &\quad \times \frac{(\sigma-\pi)^{2n+1}}{(2n+1)!} R_{k}^{(2n-2k-1)\alpha\beta} \Big|_{\sigma=\pi}. \end{aligned} \tag{5.4}$$

The $(\bar{\Gamma}^{Q})^{\alpha\beta}_{k}(\sigma)$ can be written in the integral form (Appendix C 2)

$$\begin{aligned} (\bar{\Gamma}^{Q})^{\alpha\beta}_{k}(\sigma) &= \frac{\sigma-\pi}{2(k+1)!} \int_{\sigma}^{\pi} d(\sigma_1-\pi)^2 \\ &\quad \times \int_{\sigma_1}^{\pi} d(\sigma_2-\pi)^2 \cdots \int_{\sigma_{k-1}}^{\pi} d(\sigma_k-\pi)^2 \\ &\quad \times (\bar{Q}_A)^{\alpha\beta}_{k}(\sigma_k), \end{aligned} \tag{5.5}$$

with

$$(\bar{Q}_A)^{\alpha\beta}_{k}(\sigma) \equiv \sum_{n=0}^{\infty} \frac{(\sigma-\pi)^{2n+1}}{(2n+1)!} Q_{k}{}^{(2n+1)\alpha\beta} \Big|_{\sigma=\pi}. \tag{5.6}$$

In analogy with (3.17) and (4.3), we introduce symmetric and antisymmetric variables in the neighborhood of $\sigma = \pi$,

$$\begin{aligned} \tilde{q}^{\mu}(\sigma) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu} \Big|_{\sigma=\pi}, \\ \tilde{\bar{q}}^{\mu}(\sigma) &= -\sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu} \Big|_{\sigma=\pi}, \\ \tilde{p}_{\mu}(\sigma) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} \pi^{(2n)}_{\mu} \Big|_{\sigma=\pi}, \\ \tilde{\bar{p}}_{\mu}(\sigma) &= -\sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} \pi^{(2n+1)}_{\mu} \Big|_{\sigma=\pi}. \end{aligned} \tag{5.7}$$

In the canonical form, in terms of the new variables, Eq. (5.5) can be rewritten as

$$\begin{aligned} &(\bar{\Gamma}^{Q})_{k}{}^{\alpha\beta}(\sigma) \\ &= \frac{1}{\kappa} (G^{-1})^{\alpha\gamma} \frac{\pi-\sigma}{2(k+1)!} (-1)^{k} \cdot \int_{0}^{\pi-\sigma} d\eta_1{}^2 \cdots \int_{0}^{\eta_{k-1}} d\eta_k{}^2 \\ &\quad \times [\tilde{p}^{(k)}_{\gamma} \tilde{q}^{(k+1)\beta} + \tilde{\bar{p}}^{(k)}_{\gamma} \tilde{\bar{q}}^{(k+1)\beta}](\eta_k), \end{aligned} \tag{5.8}$$

where we introduced $\eta_k = \pi - \sigma_k$. As before, we can observe that

$$(\bar{\Gamma}^{R})'^{\alpha\beta}_{k}(\sigma) = (\bar{\Gamma}^{Q})_{k}{}^{\alpha\beta}(\sigma)|_{Q\mapsto R} \tag{5.9}$$

stands. Finally, we can write the explicit form of the σ-dependent constraints in $\sigma = \pi$:

$$\begin{aligned} \bar{\Gamma}^{\mu}_{S}(\sigma) &= \Big\{ -\tilde{\bar{q}}'^{\mu} + \theta^{\mu\nu}[\tilde{q}]\tilde{p}_{\nu} + \frac{1}{2}\theta'^{\mu\nu}[\tilde{q}]\tilde{P}_{\nu} \\ &\quad + \frac{3}{2}\theta^{\mu\nu}[\tilde{\bar{q}}]\tilde{\bar{p}}_{\nu} \Big\} (\pi-\sigma), \\ \kappa\bar{\Gamma}^{\mu}_{A}(\sigma) &= \Big\{ (G^{-1})^{\mu\nu}\tilde{\bar{p}}_{\nu} - \frac{\kappa^2}{2}\theta^{\mu}{}_{\nu}[\tilde{\bar{q}}]\tilde{\bar{q}}'^{\nu} - \frac{1}{2}\theta^{\mu\nu}[\tilde{p}]\tilde{P}_{\nu} \Big\} \\ &\quad \times (\pi-\sigma), \end{aligned} \tag{5.10}$$

where all variables on the right-hand side depend on $\pi - \sigma$. Comparing (4.4) with (5.10), we find the relations

$$\begin{aligned} \Gamma^{\mu}_{S}[q, p, \bar{q}, \bar{p}, q', \bar{q}', P](\sigma) &= \bar{\Gamma}^{\mu}_{S}[\tilde{q}, \tilde{p}, \tilde{\bar{q}}, \tilde{\bar{p}}, -\tilde{q}', -\tilde{\bar{q}}', -\tilde{P}] \\ &\quad \times (\pi-\sigma), \\ \Gamma^{\mu}_{A}[p, \bar{q}, \bar{p}, \bar{q}', P](\sigma) &= \Gamma^{\mu}_{A}[\tilde{p}, \tilde{\bar{q}}, \tilde{\bar{p}}, -\tilde{\bar{q}}', -\tilde{P}](\pi-\sigma). \end{aligned} \tag{5.11}$$

Note that, for all variables, we have $z(\sigma) = \tilde{z}(\pi - \sigma)$, where $z = \{q, p, \bar{q}, \bar{p}\}$. For the corresponding σ derivatives and σ integrals, there is an additional minus sign (e.g., $q'^{\mu}(\sigma) = -\tilde{q}'^{\mu}(\pi - \sigma)$, $P_{\mu}(\sigma) = -P_{\mu}(\pi - \sigma)$), which is equivalent to the above relations.

With the help of (3.17), (4.3), and (5.7), we can conclude that, if we demand 2π periodicity of the original coordinates and momenta,

$$x(\sigma) = x(\sigma + 2\pi), \qquad \pi(\sigma) = \pi(\sigma + 2\pi). \tag{5.12}$$

By solving the σ-dependent constraints at $\sigma = 0$, we solve the σ-dependent constraints at $\sigma = \pi$, also.

VI. NONCOMMUTATIVITY ON THE STRING END POINTS

Instead of constructing the Dirac brackets, we are going to solve the second-class constraints $\Gamma_S^\mu(\sigma) = 0$ and $\Gamma_A^\mu(\sigma) = 0$ explicitly. Up to the linear term in the infinitesimal parameter $B_{\mu\nu\rho}$, we obtain

$$\bar{q}^\mu(\sigma) = -\int_0^\sigma d\sigma_0 \Big(\theta^{\mu\nu}[q]p_\nu + \tfrac{1}{2}\theta'^{\mu\nu}[q]P_\nu\Big)(\sigma_0),$$
$$\bar{p}_\mu(\sigma) = -\tfrac{1}{2}\theta_\mu{}^\nu[p(\sigma)]P_\nu(\sigma). \tag{6.1}$$

Notice that both $\bar{q}^\mu$ and $\bar{p}_\mu$ are proportional to $B_{\mu\nu\rho}$. Therefore, we neglected $\theta^{\mu\nu}(\bar{q})\bar{p}_\nu$ in Γ_S^μ and $\theta^\mu{}_\nu(\bar{q})\bar{q}'^\nu$ in Γ_A^μ, because they are of a higher order in $B_{\mu\nu\rho}$. By solving the constraints, we obtained the expressions for the antisymmetric variables $\bar{q}^\mu$ and $\bar{p}_\mu$, in terms of the symmetric ones, q^μ and p_μ. So, we can express the original variables in terms of the new ones:

$$x^\mu(\sigma) = q^\mu(\sigma) - \int_0^\sigma d\sigma_0 \Big(\theta^{\mu\nu}[q]p_\nu + \tfrac{1}{2}\theta'^{\mu\nu}[q]P_\nu\Big) \times (\sigma_0),$$
$$\pi_\mu(\sigma) = p_\mu(\sigma) - \tfrac{1}{2}\theta_\mu{}^\nu[p(\sigma)]P_\nu(\sigma). \tag{6.2}$$

Let us stress that, from the moment we solved the constraints, the open-string variables q^μ and p_ν became the fundamental quantities, while the closed-string variables x^μ and π_μ became derived ones. So, the phase space of the effective theory (obtained on the solution of the boundary condition) is a subspace containing only the even powers in σ, with the canonical variables q^μ and p_μ and the star brackets (see Appendix B)

$$^\star\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = 2\delta^\mu_\nu \delta_S(\sigma, \bar{\sigma}). \tag{6.3}$$

Since the coordinates x^μ depend both on effective coordinates q^μ and effective momenta p_μ, they are noncommutative.

Using these relations, we can calculate the star brackets between the composed variables x^μ. The second term in the first relation in (6.2) is infinitesimal, and, therefore, only the star brackets between the first and second terms give the nontrivial contribution

$$^\star\{q^\mu(\sigma), \bar{q}^\nu(\bar{\sigma})\} = 2\theta^{\mu\nu}[q(\sigma)]\theta_S(\bar{\sigma}, \sigma) + \int_0^{\bar{\sigma}} d\sigma_0 \theta'^{\mu\nu}[q(\sigma_0)]\theta_S(\sigma_0, \sigma). \tag{6.4}$$

Substituting this result into the expression for $^\star\{x^\mu(\sigma), x^\nu(\bar{\sigma})\}$ and using the properties of the θ function (see Appendix A), we get

$$^\star\{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = \{\theta^{\mu\nu}[q(\sigma)] + \theta^{\mu\nu}[q(\bar{\sigma})]\}\theta(\sigma + \bar{\sigma}). \tag{6.5}$$

Notice that the term with $\theta(\sigma - \bar{\sigma})$ disappears.

If we separate a center-of-mass variable $x^\mu_{\rm cm} = \frac{1}{\pi}\int_0^\pi d\sigma x^\mu(\sigma)$, we can write

$$x^\mu(\sigma) = X^\mu(\sigma) + x^\mu_{\rm cm} \tag{6.6}$$

and obtain

$$^\star\{X^\mu(\sigma), X^\nu(\bar{\sigma})\} = \theta^{\mu\nu}[q(\sigma)]\begin{cases} -1 & \sigma, \bar{\sigma} = 0 \\ 1 & \sigma, \bar{\sigma} = \pi \\ 0 & \text{otherwise.}\end{cases} \tag{6.7}$$

So, the interior of the string is commutative, and only the string end points are noncommutative. The noncommutative parameter $\theta^{\mu\nu}$ now depends on the effective coordinates q^μ. Because $\bar{q}(0) = 0$ and $\bar{q}(\pi) = 0$, we can rewrite the right-hand side of (6.7) in terms of x^μ, instead of in terms of q^μ. In order to close the algebra on the same variables X^μ using the gauge symmetry, generated by the zero mode of the constraints Γ_0^μ (4.10), we can gauge away $x^\mu_{\rm cm}$. Therefore, the final form of the noncommutativity relation takes the form

$$^\star\{X^\mu(\sigma), X^\nu(\bar{\sigma})\} = \theta^{\mu\nu}[X(\sigma)]\begin{cases} -1 & \sigma, \bar{\sigma} = 0 \\ 1 & \sigma, \bar{\sigma} = \pi \\ 0 & \text{otherwise.}\end{cases} \tag{6.8}$$

Formally, the noncommutative parameter $\theta^{\mu\nu}$, defined in (4.5), has the same structure as in the flat case. But, in the curved background, it is infinitesimally small and linear in coordinate, as well as the Kalb-Ramond field $B_{\mu\nu}$.

VII. CANONICAL QUANTIZATION AND THE KONTSEVICH STAR PRODUCT

In the quantization procedure, we associate a corresponding operator with every variable, and the star brackets are replaced by the commutator. It follows from (6.8) that the noncommutativity appears only on the string end points. The noncommutative parameters at $\sigma = 0$ and $\sigma = \pi$ differ only in sign. So, it is enough to consider the $\sigma = 0$ case

$$[\hat{X}^\mu, \hat{X}^\nu] = -i\theta^{\mu\nu}(\hat{X}), \tag{7.1}$$

where, from now on, we use the notation $X^\mu \equiv X^\mu(\sigma = 0)$.

We are interested in the algebra of the functions defined on the Dp-brane world volume. We will show that it is deformed because the Dp brane propagates in a background with the nontrivial Kalb-Ramond field $B_{\mu\nu}$. In order to uniquely assign an operator $\hat{f}(\hat{X})$ to any function $f(X)$, we introduce the Weyl prescription procedure

$$f(X) = \frac{1}{(2\pi)^D}\int d^D k \tilde{f}(k) e^{-ikX} \Rightarrow$$
$$\hat{f}(\hat{X}) = \frac{1}{(2\pi)^D}\int d^D k \tilde{f}(k) e^{-ik\hat{X}}. \tag{7.2}$$

If we have two functions and two associated operators $f \longrightarrow \hat{f}$ and $g \longrightarrow \hat{g}$, we define the star product demanding the prescription $f \star g \longrightarrow \hat{f}\,\hat{g}$. Because of the X dependence of the $\theta^{\mu\nu}$, the $\star$ is the Kontsevich product [17] which, up to the second order in $\theta^{\mu\nu}$, is equal to

$$f \star g = fg + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \partial_\nu g - \frac{1}{8}\theta^{\mu\nu}\theta^{\rho\sigma}\partial_\mu\partial_\rho f \partial_\nu \partial_\sigma g - \frac{1}{12}\theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}(\partial_\rho\partial_\mu f \partial_\nu g - \partial_\mu f \partial_\rho \partial_\nu g) + \mathcal{O}(\theta^3). \tag{7.3}$$

It can be shown that

$$(f \star g)\star h - f \star (g \star h) = \tfrac{1}{6}[\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\mu\rho} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}]\partial_\mu f \partial_\nu g \partial_\rho h + \mathcal{O}(\theta^3). \tag{7.4}$$

If we denote the inverse of $\theta^{\mu\nu}$ by $\theta_{\rho\sigma}$ ($\theta^{\mu\nu}\theta_{\nu\rho} = \delta^\mu_\rho$), we can rewrite (7.4) as

$$(f \star g)\star h - f \star (g \star h) = \tfrac{1}{6}\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma}\theta_{\alpha\beta\gamma}\partial_\mu f \partial_\nu g \partial_\rho h, \tag{7.5}$$

where

$$\theta_{\mu\nu\rho} = \partial_\mu\theta_{\nu\rho} + \partial_\nu\theta_{\rho\mu} + \partial_\rho\theta_{\mu\nu} \tag{7.6}$$

is a nonassociativity parameter.

In our case, the noncommutativity parameter

$$\theta^{\mu\nu}(\hat{X}) \equiv -\frac{2}{3\kappa}B^{\mu\nu}{}_\rho \hat{X}^\rho \tag{7.7}$$

is infinitesimal, and the Kontsevich product is associative, because the right-hand side of (7.4) is of the second order in the small parameter $B_{\mu\nu\rho}$.

VIII. CONCLUSIONS AND DISCUSSIONS

In the present paper, we investigated the geometry of Dp branes in a curved background. We chose the simplest possible case of an infinitesimally curved background, where the Kalb-Ramond field is infinitesimally small and linear in coordinate. In such a case, we avoided working with a nonconstant metric tensor, because, as a consequence of the space-time field equations, the Ricci tensor is an infinitesimal of the second order and can be neglected in the leading order.

In obtaining the Poisson brackets between the original coordinates x^μ, it was useful to express them in terms of the effective coordinates q^μ and the corresponding canonical momenta p_μ. On the other hand, Lagrangian formalism is more appropriate for working with an infinite set of constraints. So, we used an "adopted canonical approach."

We treated the boundary conditions as constraints. The basic technical problem was the derivation of the Dirac consistency conditions. Instead to commute the constraints with the Hamiltonian in order to obtain new constraints, we found it more appropriate to use the Lagrangian approach. With the help of the Lagrangian equations of motion, we obtained the time derivatives of the primary constraints in leading orders. According to the Dirac requirement, they were constraints, also. Therefore, by further application of this procedure, we obtained the infinite set of constraints in the Lagrangian form.

Following the procedure of Refs. [8], we substituted an infinite set of constraints at string end points, with two sets of σ-dependent constraints using the Taylor expansion. We found it convenient to separate sums with even and odd powers of σ, because Ω-symmetric and antisymmetric functions are independent. Note that these constraints are infinite sums, bilinear in coordinate, with one τ derivative and an arbitrary degree of σ derivatives. The main formulas were derived in Appendixes C, D, and E.

This stage was a good point for the transition from the Lagrangian to the Hamiltonian method. We expressed the τ derivative of coordinates ($\dot{x}^\mu$), in terms of the momenta π_μ and the σ derivative (x'^μ), and obtained constraints in the Hamiltonian form. Then, we were in a position to check the validity of our procedure. Because the Ω-even and odd parts of the constraints are independent, it was useful to consider their difference as the single constraint. The Poisson brackets between the Hamiltonian and this constraint are just the σ derivative of the constraint. It means that it weakly commutes with the Hamiltonian. First, this proved that the expression obtained from the boundary conditions with the help of the Lagrangian consistency procedure is really the Hamiltonian constraint. Second, we concluded that there were no more constraints, and the consistency procedure was completed. So, we showed the equivalence with the standard Dirac consistency procedure by rewriting the constraints in the canonical form.

The Poisson brackets between the constraints in the leading order are closed on the metric tensor times the σ derivative of the δ function. The metric tensor is regular ($\det G_{\mu\nu} \neq 0$), and, consequently, all constraints except the zero modes [14] are of the second class. There are two possibilities to deal with second-class constraints. The usual approach is to find the Dirac brackets, but, in our particular case, it is simpler to solve them explicitly. As a consequence of the σ derivative of the δ function, the zero modes of the constraints are of the first class. They are generators of global symmetry, which we used to gauge away the center of mass of the coordinate.

The simple solution (6.2) of the original closed-string coordinates x^μ, in terms of the effective open-string coordinates q^μ and the momenta p_ν, defines the noncommutative product and its properties. The noncommutativity parameter $\theta^{\mu\nu}$, (4.5), formally has the same form as in the flat background, but with the Kalb-Ramond field (2.4), linear in coordinate. Taking into account the fact that, at the string boundaries, $\bar{q}(0) = 0$ and $\bar{q}(\pi) = 0$ and gauging away the center-of-mass coordinate, we can rewrite explicitly the noncommutative relations in terms of the original variables

$$\begin{aligned} {}^{\star}\{X^{\mu}(0), X^{\nu}(0)\} &= -\theta^{\mu\nu}[X(0)] = f^{\mu\nu}{}_{\rho}X^{\rho}(0), \\ {}^{\star}\{X^{\mu}(\pi), X^{\nu}(\pi)\} &= \theta^{\mu\nu}[X(\pi)] = -f^{\mu\nu}{}_{\rho}X^{\rho}(\pi), \end{aligned} \tag{8.1}$$

where $f^{\mu\nu}{}_{\rho} = \frac{2}{3\kappa}B^{\mu\nu}{}_{\rho}$.

In Refs. [10,11], the noncommutative product is defined only on the world sheet boundary using the path-integral method. In fact, the explicit expression of the star product has been extracted from the correlation functions computed on the disk.

On the other hand, we used the canonical approach and explicitly solved the boundary conditions. We want to stress that only the space-time coordinates of the string end points are noncommutative. For any points of the string interior, the commutation relations are standard. This is obvious in a decoupling limit ($\alpha' \to \sqrt{\varepsilon}\alpha'$, $G_{\mu\nu} \to \varepsilon G_{\mu\nu}$, $\varepsilon \to 0$), when all degrees of freedom in the string interior can be gauged away [18]. We could also expect such a result, without a decoupling limit, but for the constant $B_{\mu\nu}$ field, because it does not appear in the equations of motion and does not affect the string interior [6,8]. In our case, this is a nontrivial result, because the coordinate-dependent $B_{\mu\nu}$ field contributes to the equations of motion and affects the string interior. Furthermore, even at the first glance, one sees that the constraints cannot be imposed only on the string end points. In fact, γ_0^{μ}, which are defined only on the boundary, are just prime constraints. In order to obtain all the constraints, one must apply the Dirac consistency procedure, which leads to the full set of constraints $\Gamma^{\mu}(\sigma)$, which is nontrivial at the string interior. So, even in the case when the term with the Kalb-Ramond field in the action is not topological, it is possible to restrict noncommutativity only to the world sheet boundary.

Let us now discuss the relation between our case of branes in a weakly curved background and branes on a group manifold [2,3,12]. Note that the strings moving on the group manifold are described by the Wess-Zumino-Novikov-Witten (WZNW) model. Owing to the conformal symmetries of the WZNW model, the space-time Eqs. (2.2) and (2.3) are automatically satisfied.

Strings moving on the three-sphere S^3 of radius R are described by the WZNW model, with the group $SU(2)$ at level n. As a consequence of the Dirac quantization condition, it follows that the radius of the three-sphere is quantized, $R^2 = \alpha' n$, where the integer n is also the level of the corresponding current algebra. In the limit of large n, the group manifold becomes more and more flat, and the three-sphere approaches the flat three-space. So, in the language of the group manifold, the large level n corresponds to the weakly curved background of the present paper. In that sense, our result (8.1) corresponds to Eq. (4.6) of Ref. [3], and the structure constants $f^{\mu\nu}{}_{\rho}$ are proportional to the field strength of the Kalb-Ramond field $B^{\mu\nu}{}_{\rho}$.

As mentioned in Ref. [3], these relations have been obtained as an extension of the flat-background expressions and "naively applied" to the curved background. In the present paper, we derive Eq. (4.6) of Ref. [3] and prove that it is correct. Our derivation is not restricted to the case of the $SU(2)$ group.

After quantization, using the Weyl normal ordering prescription, we showed that the product of the operators, defined on the Dp-brane world volume, is isomorphic to the Kontsevich product of the ordinary functions. In the case of the weakly curved background, the Kontsevich product turns to an associative one, because the nonassociative term is an infinitesimal of the second order.

Let us discuss one additional possibility in our approach. Instead of using the composed variables X^{μ} with the noncommutativity relation (7.1) and the Weyl normal ordering prescription (7.2), we can treat q^{μ} and p_{μ} as fundamental variables. Then, we can define normal ordering :: for the operators $\hat{q}^{\mu}$ and $\hat{p}_{\mu}$ and, to any function $f(x)$, according to (6.2), assign the operator

$$f(x) \to :\hat{f}\left\{\hat{q}^{\mu} - \int_0^{\sigma} d\sigma_0\left[\theta^{\mu\nu}(\hat{q})\hat{p}_{\nu} + \frac{1}{2}\theta'^{\mu\nu}(\hat{q})\hat{P}_{\nu}\right]\right\}:. \tag{8.2}$$

Consequently, we can introduce the new star product, specifying new normal ordering and using Eq. (8.2) and the commutation relation (4.7). The new star product is defined along the whole string and not only on the string end points. We find this approach more fundamental but, in the particular case of the world sheet boundary, it produces the same Kontsevich star product. We will discuss this new definition of the star product elsewhere.

In the present paper, in order to simplify calculations, we neglected the constant part $b_{\mu\nu}$ of the linear Kalb-Ramond field $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^{\rho}$, introduced in Eq. (2.4). In that case, the new, momentum-dependent term of Ref. [13] goes to zero. This resolves the second ambiguity from the Introduction, that the result of Refs. [10,11] is valid for $b_{\mu\nu} = 0$.

From where does the momentum-dependent term appear? It can come from the Poisson brackets between the momentum-dependent terms of the relation (6.2), the basic expression of the initial coordinates x^{μ} in terms of the effective canonical variables q^{μ} and p_{μ}. In the present paper (for $b_{\mu\nu} = 0$), this part is an infinitesimal of the second order, so we neglect it. In the case of $b_{\mu\nu} \neq 0$, this part produces a nontrivial momentum-dependent result, because the expression $\theta^{\mu\nu}(q)$ acquires the constant finite term $\theta_0^{\mu\nu} \neq 0$.

In our next paper, Ref. [19], we apply the same canonical method for the case $b_{\mu\nu} \neq 0$ and obtain a momentum-dependent noncommutativity parameter. Besides the standard expression of Refs. [10,11], it contains the term of Ref. [13] and some other momentum-dependent terms. This result will resolve all the ambiguities mentioned in the Introduction and it represents a complete expression of the noncommutative parameter of the weakly curved background.

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APPENDIX A: 2π-PERIODIC FUNCTIONS

In this Appendix, we will introduce the Fourier expansion of the ordinary, symmetric, and antisymmetric delta and step functions. In addition, we define I_k functions as k integrals of the symmetric θ functions and investigate their properties.

1. Step and delta functions

The Fourier series of the 2π-periodic δ function and the θ step function has the forms

$$\delta(\sigma) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n\geq 1}\cos n\sigma \qquad (\sigma \in [0, 2\pi]), \tag{A1}$$

$$\theta(\sigma) = \frac{1}{2\pi}\left(\sigma + 2\sum_{n\geq 1}\frac{1}{n}\sin n\sigma\right), \tag{A2}$$

where, by definition, $\theta(\sigma) = \int_0^\sigma d\sigma_0 \delta(\sigma_0)$. Let us define the delta and step functions, symmetric and antisymmetric, under σ parity:

$$\begin{aligned} \delta_S(\sigma, \bar\sigma) &= \tfrac{1}{2}[\delta(\sigma - \bar\sigma) + \delta(\sigma + \bar\sigma)], \\ \delta_A(\sigma, \bar\sigma) &= \tfrac{1}{2}[\delta(\sigma - \bar\sigma) - \delta(\sigma + \bar\sigma)], \\ \theta_S(\sigma, \bar\sigma) &= \tfrac{1}{2}[\theta(\sigma - \bar\sigma) + \theta(\sigma + \bar\sigma)], \\ \theta_A(\sigma, \bar\sigma) &= \tfrac{1}{2}[\theta(\sigma - \bar\sigma) - \theta(\sigma + \bar\sigma)]. \end{aligned} \tag{A3}$$

Using (A1) and (A2), we can rewrite them in the form

$$\begin{aligned} \delta_S(\sigma, \bar\sigma) &= \frac{1}{2\pi}\left[1 + 2\sum_{n\geq 1}\cos n\sigma \cos n\bar\sigma\right], \\ \delta_A(\sigma, \bar\sigma) &= \frac{1}{\pi}\sum_{n\geq 1}\sin n\sigma \sin n\bar\sigma, \\ \theta_S(\sigma, \bar\sigma) &= \frac{1}{2\pi}\left[\sigma + 2\sum_{n\geq 1}\frac{1}{n}\sin n\sigma \cos n\bar\sigma\right], \\ \theta_A(\sigma, \bar\sigma) &= -\frac{1}{2\pi}\left[\bar\sigma + 2\sum_{n\geq 1}\frac{1}{n}\cos n\sigma \sin n\bar\sigma\right]. \end{aligned} \tag{A4}$$

These functions satisfy the following properties:

$$\begin{aligned} \delta_S(\sigma, \bar\sigma) &= \delta_S(\bar\sigma, \sigma), & \delta_A(\sigma, \bar\sigma) &= \delta_A(\bar\sigma, \sigma), \\ \delta_S(\sigma, -\bar\sigma) &= \delta_S(\sigma, \bar\sigma), & \delta_A(\sigma, -\bar\sigma) &= -\delta_A(\sigma, \bar\sigma), \\ \theta_S(\bar\sigma, \sigma) &= -\theta_A(\sigma, \bar\sigma), & \partial_\sigma\theta_S(\sigma, \bar\sigma) &= \delta_S(\sigma, \bar\sigma), \\ \partial_\sigma\theta_A(\sigma, \bar\sigma) &= \delta_A(\sigma, \bar\sigma), & \partial_{\bar\sigma}\theta_S(\sigma, \bar\sigma) &= -\delta_A(\sigma, \bar\sigma). \end{aligned} \tag{A.5}$$

We will use the relations

$$\begin{aligned} &\int_0^\sigma d\sigma_1 f(\sigma_1)\delta(\sigma_1 - \bar\sigma) = f(\bar\sigma)[\theta(\sigma - \bar\sigma) + \theta(\bar\sigma)], \\ &\int_0^\sigma d\sigma_1 f_S(\sigma_1)\delta_S(\sigma_1, \bar\sigma) = f_S(\bar\sigma)\theta_S(\sigma, \bar\sigma), \\ &\int_0^\sigma d\sigma_1 f_A(\sigma_1)\delta_A(\sigma_1, \bar\sigma) = f_A(\bar\sigma)\theta_S(\sigma, \bar\sigma), \end{aligned} \tag{A6}$$

where f_S and f_A are symmetric and antisymmetric functions under σ parity, $f_S(-\sigma) = f_S(\sigma)$, and $f_A(-\sigma) = -f_A(\sigma)$. Using the fact that

$$\theta(\sigma) = \begin{cases} 0 & \sigma = 0 \\ 1/2 & 0 < \sigma < 2\pi \\ 1 & \sigma = 2\pi \end{cases} \quad (\sigma \in [0, 2\pi]), \tag{A7}$$

we obtain

$$\theta_S(\sigma, \bar\sigma) = \begin{cases} 0 & \sigma = \bar\sigma = 0 \\ 1/2 & \sigma = \bar\sigma = \pi \\ 1/4 & \sigma = \bar\sigma \neq 0, \pi \\ 1/2 & \sigma > \bar\sigma \\ 0 & \sigma < \bar\sigma \end{cases} \qquad (\sigma, \bar\sigma \in [0, \pi]), \tag{A8}$$

which will be useful in the derivation of the properties of the I_k functions.

2. Integrals of the symmetric θ functions

We define two variable functions $I_k(\sigma, \bar\sigma)$, $(\sigma, \bar\sigma \in [0, \pi])$ as multiple integrals of the symmetric step function

$$I_0(\sigma, \bar{\sigma}) = \theta_S(\sigma, \bar{\sigma}),$$
$$I_k(\sigma, \bar{\sigma}) = \int_0^{\sigma} d\sigma_1{}^2 \int_0^{\sigma_1} d\sigma_2{}^2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k^2 \theta_S(\sigma_k, \bar{\sigma}) \quad (k \geq 1). \tag{A9}$$

They have the following properties:

$$\partial_\sigma I_k(\sigma, \bar{\sigma}) = 2\sigma I_{k-1}(\sigma, \bar{\sigma}),$$
$$\partial_{\bar{\sigma}} I_k(\sigma, \bar{\sigma}) = -2\bar{\sigma} I_{k-1}(\sigma, \bar{\sigma}) \qquad (k \geq 1). \tag{A10}$$

Using (A8) and the mathematical induction, it can be shown that

$$I_k(\sigma, \bar{\sigma}) = \begin{cases} \frac{1}{2k!}(\sigma^2 - \bar{\sigma}^2)^k & \sigma > \bar{\sigma} \\ 0 & \sigma \leq \bar{\sigma} \end{cases} \quad (k \geq 1). \tag{A11}$$

In the derivation of the summation formula in Appendix D, we will need an expression for the k-th derivative ($k \leq n$) of the I_n function over the second variable. The expression $\partial_{\bar{\sigma}}^k I_n(\sigma, \bar{\sigma})$ is a polynomial of the $(2n - k)$-th order in $\bar{\sigma}$, so it can be written in the form

$$\partial_{\bar{\sigma}}^k I_n(\sigma, \bar{\sigma}) = \sum_{q=0}^{[k/2]} a_q^k \bar{\sigma}^{k-2q} I_{n-k+q}(\sigma, \bar{\sigma}). \tag{A12}$$

Using the mathematical induction, we obtain the recursion relation for coefficients a_q^k, with the solution

$$a_0^k = (-2)^k \qquad (k \geq 0),$$
$$a_q^k = (-2)^{k-q} \binom{k}{2q} (2q - 1)!! \qquad (k \geq 2q). \tag{A13}$$

APPENDIX B: INDUCED BRACKETS IN THE REDUCED PHASE SPACE

The solution of the constraints $\Gamma_\mu(\sigma) = 0$ and $\bar{\Gamma}_\mu(\sigma) = 0$ reduces the phase space, leaving only half of the degrees of freedom. Let us clarify the relation between the brackets associated with the initial and the reduced phase spaces. We will distinguish two nontrivial steps. In Appendix B 1, we will take into account the symmetries of the basic canonical variables under σ parity Ω, and, in Appendix B 2, we will impose the second-class constraints Γ_μ.

1. The phase space reduced by Ω-even and Ω-odd projections

First, we need the expression for the brackets between the basic canonical variables q^μ and p_μ ($\bar{q}^\mu$ and $\bar{p}_\mu$) in the interval $[0, \pi]$. Note that they are not closed on the standard δ function, because they are not arbitrary functions on that interval, but they contain only even (odd) powers of σ. The easiest way to impose this restriction is just an extension to the domain $[-\pi, \pi]$, when they become symmetric (antisymmetric) functions under $\sigma \to -\sigma$. Then, we have

$$\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = \delta_\nu^\mu \delta_S(\sigma, \bar{\sigma}),$$
$$\{\bar{q}^\mu(\sigma), \bar{p}_\nu(\bar{\sigma})\} = \delta_\nu^\mu \delta_A(\sigma, \bar{\sigma}), \tag{B1}$$

with $\sigma, \bar{\sigma} \in [-\pi, \pi]$, where, by definition,

$$\int_{-\pi}^{\pi} d\bar{\sigma} q^\mu(\bar{\sigma}) \delta_S(\bar{\sigma}, \sigma) = q^\mu(\sigma),$$
$$\int_{-\pi}^{\pi} d\bar{\sigma} \bar{q}^\mu(\bar{\sigma}) \delta_S(\bar{\sigma}, \sigma) = \bar{q}^\mu(\sigma). \tag{B2}$$

Separating the integration domain in two parts, from $-\pi$ to 0 and from 0 to π, and changing the integration variable in the first part $\bar{\sigma} \to -\bar{\sigma}$, we obtain

$$2\int_0^{\pi} d\bar{\sigma} q^\mu(\bar{\sigma}) \delta_S(\bar{\sigma}, \sigma) = q^\mu(\sigma),$$
$$2\int_0^{\pi} d\bar{\sigma} \bar{q}^\mu(\bar{\sigma}) \delta_S(\bar{\sigma}, \sigma) = \bar{q}^\mu(\sigma). \tag{B3}$$

So, the unit functions on the interval $[0, \pi]$ for functions with only an even or odd power in σ are $2\delta_S(\bar{\sigma}, \sigma)$ and $2\delta_A(\bar{\sigma}, \sigma)$, respectively. Therefore, the brackets which we are looking for have a form

$$\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = 2\delta_\nu^\mu \delta_S(\sigma, \bar{\sigma}),$$
$$\{\bar{q}^\mu(\sigma), \bar{p}_\nu(\bar{\sigma})\} = 2\delta_\nu^\mu \delta_A(\sigma, \bar{\sigma}), \qquad \sigma, \bar{\sigma} \in [0, \pi]. \tag{B4}$$

In the initial phase space, with the canonical variables $x^\mu(\sigma)$, $\pi_\mu(\sigma)$, and $\sigma \in [0, \pi]$, the standard Poisson brackets (4.6) are valid. Applying relations (B4), we obtain Poisson brackets (4.7).

2. The phase space reduced by the constraint $\Gamma_\mu = 0$

On the solution of the boundary conditions, we obtain the reduced phase space with 2π-periodic canonical variables $q^\mu(\sigma)$ and $p_\mu(\sigma)$, defined in (3.17) and (4.3). For arbitrary functions $F(x, \pi)$ and $G(x, \pi)$, defined on the initial phase space, we introduce their restrictions on the reduced phase space, as a value on the solution of the boundary conditions

$$f(q, p) = F(x, \pi)|_{\Gamma_\mu = 0}, \qquad g(q, p) = G(x, \pi)|_{\Gamma_\mu = 0}. \tag{B5}$$

As was shown in Sec. 2.3.2 of Ref. [20], the Poisson brackets in the effective phase space are, in fact, the Dirac brackets in the initial phase space associated with the second-class constraints $\Gamma_\mu = 0$,

$$^\star\{f, g\} = \{F, G\}_{\text{Dirac}}|_{\Gamma_\mu = 0}. \tag{B6}$$

To distinguish the new brackets from those of the initial phase space, we denoted them by a star. Applying the first relation (B4) to the star brackets (B6), we obtain (6.3).

Finally, we should check the 2π-periodicity conditions (5.12). For the σ-symmetric functions ($q^\mu(\sigma)$ and $p_\mu(\sigma)$ and their algebraic combinations), they are automatically satisfied. The σ-antisymmetric functions (the σ derivative and σ integral of the symmetric functions) must vanish both at $\sigma = 0$ and $\sigma = \pi$ because of the antisymmetry and 2π periodicity, respectively.

APPENDIX C: INTEGRAL FORM OF $(\Gamma^Q)^{\alpha\beta}$ AND $(\bar{\Gamma}^Q)^{\alpha\beta}$

1. The case $\sigma = 0$

We will show that $(\Gamma^Q)^{\alpha\beta}$, defined in (3.11), is equal to

$$(\Gamma^Q)_k{}^{\alpha\beta}(\sigma) = \frac{(-1)^k\sigma}{2(k+1)!}\int_0^\sigma d\sigma_1^2 \times \int_0^{\sigma_1} d\sigma_2^2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k^2 (Q_A)_k^{\alpha\beta}(\sigma_k). \tag{C1}$$

In order to prove the above relation, it is useful to define the auxiliary variable as

$$(\gamma^Q)_{kq}^{\alpha\beta}(\sigma) = \sum_{n=k+1}^{\infty} \frac{\sigma^{2n-2q-1}}{(2n-2q-1)!}\frac{(n-q-1)!}{(n-k-1)!} \times Q_k^{\alpha\beta(2n-2k-1)}\Big|_{\sigma=0} \qquad (q = 0, 1, \cdots, k) \tag{C2}$$

and rewrite $(\Gamma^Q)_k^{\alpha\beta}$ as

$$(\Gamma^Q)_k^{\alpha\beta}(\sigma) = (-4)^k \frac{\sigma}{2(k+1)!}(\gamma^Q)_{k0}^{\alpha\beta}(\sigma). \tag{C3}$$

Observing that $(\gamma^Q)_{kq}^{\alpha\beta}$ satisfies

$$(\gamma^Q)_{kq}^{\prime\alpha\beta}(\sigma) = \frac{\sigma}{2}(\gamma^Q)_{kq+1}^{\alpha\beta}(\sigma), \qquad \Rightarrow (\gamma^Q)_{kq}^{\alpha\beta}(\sigma) = \frac{1}{4}\int_0^\sigma d\sigma_1^2(\gamma^Q)_{kq+1}^{\alpha\beta}(\sigma_1) \tag{C4}$$

and using the fact that

$$(\gamma^Q)_{kk}^{\alpha\beta} = (Q_A)_k^{\alpha\beta}(\sigma) \equiv \sum_{n=0}^{\infty}\frac{\sigma^{2n+1}}{(2n+1)!}Q_k^{(2n+1)\alpha\beta}\Big|_0 = [\dot{x}^{(k)\alpha}x^{(k+1)\beta}]_A \tag{C5}$$

is the antisymmetric part of $Q_k^{\alpha\beta}$, we obtain (C1).

2. The case $\sigma = \pi$

A similar integral form can be obtained for $(\bar{\Gamma}^Q)^{\alpha\beta}$, defined in (5.4). As before, it is useful to define the auxiliary variable as

$$(\bar{\gamma}^Q)_{kq}^{\alpha\beta}(\sigma) = \sum_{n=k+1}^{\infty}\frac{(\sigma-\pi)^{2n-2q-1}}{(2n-2q-1)!}\frac{(n-q-1)!}{(n-k-1)!} \times Q_k^{\alpha\beta(2n-2k-1)}\Big|_{\sigma=\pi} \qquad (q = 0, 1, \cdots, k) \tag{C6}$$

and rewrite $(\bar{\Gamma}^Q)_k^{\alpha\beta}$ as

$$(\bar{\Gamma}^Q)_k^{\alpha\beta}(\sigma) = (-4)^k\frac{\sigma-\pi}{2(k+1)!}(\bar{\gamma}^Q)_{k0}^{\alpha\beta}(\sigma). \tag{C7}$$

Observing that $(\bar{\gamma}^Q)_{kq}^{\alpha\beta}$ satisfies

$$(\bar{\gamma}^Q)_{kq}^{\prime\alpha\beta}(\sigma) = \frac{\sigma-\pi}{2}(\bar{\gamma}^Q)_{kq+1}^{\alpha\beta}(\sigma) \Rightarrow (\bar{\gamma}^Q)_{kq}^{\alpha\beta}(\sigma) = \int_\sigma^\pi d\sigma_1\frac{\pi-\sigma_1}{2}(\bar{\gamma}^Q)_{kq+1}^{\alpha\beta}(\sigma_1), \tag{C8}$$

we obtain

$$(\bar{\Gamma}^Q)_k{}^{\alpha\beta}(\sigma) = \frac{\sigma-\pi}{2(k+1)!}\int_\sigma^\pi d(\sigma_1-\pi)^2 \times \int_{\sigma_1}^\pi d(\sigma_2-\pi)^2\cdots\int_{\sigma_{k-1}}^\pi d(\sigma_k-\pi)^2 \times (\bar{Q}_A)_k^{\alpha\beta}(\sigma_k), \tag{C9}$$

with

$$(\bar{\gamma}^Q)_{kk}^{\alpha\beta}(\sigma) = (\bar{Q}_A)_k^{\alpha\beta}(\sigma) \equiv \sum_{n=0}^{\infty}\frac{(\sigma-\pi)^{2n+1}}{(2n+1)!}Q_k{}^{(2n+1)\alpha\beta}\Big|_{\sigma=\pi}. \tag{C10}$$

APPENDIX D: SUMMATION FORMULA

In this Appendix, we will derive the relation

$$S^\rho(x|\sigma,\bar{\sigma}) \equiv \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\frac{\partial^k}{\partial\bar{\sigma}^k}[\bar{\sigma}x^{(k+1)\rho}(\bar{\sigma})I_k(\sigma,\bar{\sigma})] = \frac{1}{2}\theta_S(\sigma,\bar{\sigma})[x^\rho(\sigma)+x^\rho(-\sigma)-2x^\rho(-\bar{\sigma})], \tag{D1}$$

which we will use in Appendix E.

Using the Leibniz rule and Eq. (A12), the sum in the above expression can be rewritten as an expansion in the I_k functions,

$$S^\rho(x|\sigma,\bar{\sigma}) = \sum_{l=0}^{\infty}C_l^\rho(\bar{\sigma})I_l(\sigma,\bar{\sigma}), \tag{D2}$$

with the coefficients

$$C_l^\rho(\bar{\sigma}) = \sum_{q=0}^{l}\sum_{k=2l-q}^{\infty}\frac{1}{(k+1)!}\binom{k}{q} \times [\bar{\sigma}x^{\rho(k+1)}(\bar{\sigma})]^{(q)}a_{l-q}^{k-q}\bar{\sigma}^{k-2l+q}. \tag{D3}$$

For $l=0$, we obtain

$$C_0^\rho = \tfrac{1}{2}[x^\rho(\bar\sigma) - x^\rho(-\bar\sigma)]. \tag{D4}$$

With the help of (A13), we can rewrite the coefficients (D3) in the form

$$C_l^\rho(\bar\sigma) = \sum_{m=0}^{\infty} K_{lm}\frac{(-2\bar\sigma)^m}{m!}x^{(m+2l)\rho} \qquad (l\ge 1), \tag{D5}$$

where

$$K_{l0} = \frac{(-1)^l}{l!}(2l+1)R_{2l+1,l},$$
$$K_{lm} = \frac{(-1)^l}{l!}\frac{(2l+m)}{2}R_{2l+m,l} \qquad (m\ge 1), \tag{D6}$$

and $R_{m,n}$ are defined in Appendix D 1. With the help of (D17), we can rewrite (D5) as

$$C_l^\rho(\bar\sigma) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-2\bar\sigma)^m}{m!}\frac{(m+l-1)!}{(m+2l-1)!}x^{(m+2l)\rho}(\bar\sigma) \quad (l\ge 1). \tag{D7}$$

Let us now define the auxiliary function of two variables as

$$\bar C_l^\rho(\eta,\bar\sigma) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-2\eta)^m}{m!}\frac{(m+l-1)!}{(m+2l-1)!}x^{(m+2l)\rho}(\bar\sigma) \quad (l\ge 1). \tag{D8}$$

Obviously, $C_l^\rho(\bar\sigma) = \bar C_l^\rho(\bar\sigma,\bar\sigma)$. Let us define

$$\bar C_l^{k\rho}(\eta_k,\bar\sigma) = \int_0^{\eta_k} d\eta_{k-1}\bar C_l^{k-1\rho}(\eta_{k-1},\bar\sigma) \quad (1\le k\le l-1),$$
$$\bar C_l^{0\rho}(\eta_k,\bar\sigma) = \bar C_l^\rho(\eta_k,\bar\sigma). \tag{D9}$$

We can show that $\bar C_l^{l-1\rho}$ is equal to

$$\bar C_l^{l-1\rho}(\eta,\bar\sigma) = -\frac{1}{(4\eta)^l}\Big[x^{l\rho}(\bar\sigma-2\eta) - \sum_{n=0}^{2l-2}\frac{(-2\eta)^n}{n!}x^{(n+1)\rho}(\bar\sigma)\Big], \tag{D10}$$

and, using

$$\bar C_l^\rho(\eta,\bar\sigma) = \partial_\eta^{\,l-1}\bar C_l^{l-1\rho}(\eta,\bar\sigma), \tag{D11}$$

we obtain

$$\bar C_l^\rho(\eta,\bar\sigma) = \frac{(-1)^l}{2}\sum_{n=0}^{l-1}\frac{(2l-n-2)!}{n!(l-n-1)!}(2\eta)^{-(2l-n-1)} \times[x^{(n+1)\rho}(\bar\sigma-2\eta) - (-1)^n x^{(n+1)\rho}(\bar\sigma)] \tag{D12}$$

and, finally,

$$C_l^\rho(\bar\sigma) = \frac{(-1)^l}{2}\sum_{n=0}^{l-1}\frac{(2l-n-2)!}{n!(l-n-1)!}(2\bar\sigma)^{-(2l-n-1)} \times[x^{(n+1)\rho}(-\bar\sigma) - (-1)^n x^{(n+1)\rho}(\bar\sigma)]. \tag{D13}$$

Substituting (D13) and (A11) into expression (D2) without the first term

$$S_1^\rho(x|\sigma,\bar\sigma) = \sum_{l=1}^{\infty} C_l^\rho(\bar\sigma)I_l(\sigma,\bar\sigma), \tag{D14}$$

after straightforward calculation for $\sigma > \bar\sigma$, we obtain

$$S_1^\rho(x|\sigma,\bar\sigma) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-2\bar\sigma)^{n+1}}{n!}[(-1)^{n+1}x^{(n+1)\rho}(-\bar\sigma) + x^{(n+1)\rho}(\bar\sigma)]\tilde S_n(y),$$
$$y = \frac{1}{4}\Big[1-\Big(\frac{\sigma}{\bar\sigma}\Big)^2\Big], \tag{D15}$$

where $\tilde S_n(y)$ is defined in (D21). Substituting (D25) into (D15), we obtain

$$S_1^\rho(x|\sigma,\bar\sigma) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\sigma-\bar\sigma)^{n+1}[x^{(n+1)\rho}(\bar\sigma) + (-1)^{n+1}x^{(n+1)\rho}(-\bar\sigma)]$$
$$= \frac{1}{4}[x^\rho(\sigma) + x^\rho(-\sigma) - x^\rho(\bar\sigma) - x^\rho(-\bar\sigma)] \quad (\sigma>\bar\sigma). \tag{D16}$$

Using the properties of θ functions and Eq. (D4), we obtain (D1).

1. Coefficients $R_{m,n}$

In this Appendix, we will prove the relation

$$R_{m,n} \equiv \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{m-k} = (-1)^n\frac{n!(m-n-1)!}{m!} \quad (m>n), \tag{D17}$$

used in (D6). Let us introduce the auxiliary function

$$f_{mn}(\alpha) \equiv \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{m-k}\alpha^{m-k} \qquad (m>n), \tag{D18}$$

which has the properties

$$f_{mn}(0) = 0, \qquad f_{mn}(1) = R_{m,n}. \tag{D19}$$

Differentiating f_{mn} over α, we obtain

$$f'_{mn}(\alpha) = (-1)^n\alpha^{m-n-1}(1-\alpha)^n. \tag{D20}$$

Integrating this relation on the interval (0,1), using (D19) and the properties of the gamma function, we obtain (D17).

2. Functions $\tilde{S}_n$

Let us define the function

$$\tilde{S}_n(y) = \sum_{k=0}^{\infty} \frac{(2k+n)!}{k!(k+n+1)!} y^{k+n+1} \quad (n \geq 0). \tag{D21}$$

It satisfies the recurrence relation

$$\partial_y \tilde{S}_{n+1}(y) = 2y\partial_y \tilde{S}_n(y) - (n+1)\tilde{S}_n(y), \tag{D22}$$

which, after the change of variables $y = \frac{1-\alpha^2}{4}$, becomes

$$\partial_\alpha \tilde{S}_{n+1}(\alpha) = \frac{1}{2}(1-\alpha^2)\partial_\alpha \tilde{S}_n(\alpha) + \frac{\alpha}{2}(n+1)\tilde{S}_n(\alpha). \tag{D23}$$

It is easy to check that the expression

$$\tilde{S}_n(\alpha) = \frac{(1-\alpha)^{n+1}}{2^{n+1}(n+1)} \tag{D24}$$

is a solution of the above equation. Recalling that $\alpha = \frac{\sigma}{\bar{\sigma}}$, we have

$$\tilde{S}_n(\sigma, \bar{\sigma}) = \frac{1}{n+1} \frac{(\sigma - \bar{\sigma})^{n+1}}{(-2\bar{\sigma})^{n+1}}. \tag{D25}$$

APPENDIX E: EXPRESSION FOR $h^{\alpha\beta}$ WHICH TURNS CONSTRAINTS TO THE COMPACT FORM

Let us derive the compact expression for the functions $h^{\alpha\beta}$, defined in (3.20), as

$$h^{\alpha\beta}(a,b)(\sigma) = \frac{\sigma}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \int_0^\sigma d\sigma_1^2 \cdots \int_0^{\sigma_{k-1}} d\sigma_k^2 a^{(k)\alpha}(\sigma_k) \times b^{(k+1)\beta}(\sigma_k). \tag{E1}$$

The result is different when both variables a and b are σ-symmetric,

$$h^{\alpha\beta}(a,b)(\sigma) = \frac{1}{2} A^\alpha(\sigma) b'^\beta(\sigma), \qquad A^\alpha(\sigma) \equiv \int_0^\sigma d\eta a^\alpha(\eta), \tag{E2}$$

and σ-antisymmetric,

$$h^{\alpha\beta}(\bar{a}, \bar{b})(\sigma) = \tfrac{1}{2}\bar{a}^\alpha(\sigma)\bar{b}^\beta(\sigma). \tag{E3}$$

We will prove (E2) by substituting $a^{(k)\alpha}(\sigma_k)$, written as

$$a^{(k)\alpha}(\sigma_k) = 2\int_0^\pi d\eta a^\alpha(\eta) \frac{\partial^k}{\partial \sigma_k^k} \delta_S(\eta, \sigma_k), \tag{E4}$$

into the expression for $h^{\alpha\beta}(a,b)(\sigma)$. Integrating over σ_k, we obtain

$$h^{\alpha\beta}(a,b)(\sigma) = \frac{\sigma}{2} a^\alpha b'^\beta + \int_0^\pi d\eta a^\alpha(\eta) \times \sum_{k=1}^{\infty} \frac{2\sigma}{(k+1)!} \partial_\eta^k [\eta b^{(k+1)\beta}(\eta) I_{k-1}(\sigma, \eta)], \tag{E5}$$

where I_k is defined in (A9). Note that the sum in the last expression is $\partial_\sigma S_1^\beta(a|\sigma, \eta)$, where S_1^β is defined in (D14). Therefore, using (D1) for $x \to a$, we obtain (E2). In the case when both $\bar{a}$ and $\bar{b}$ are σ-antisymmetric, observing that

$$\bar{a}^{(k)}(\sigma_k) = \int_0^{\sigma_k} d\eta \bar{a}^{(k+1)}(\eta), \qquad \bar{b}^{(k+1)}(\sigma_k) = (\bar{b}')^{(k)}(\sigma_k), \tag{E6}$$

with the help of (E2), we obtain

$$h^{\alpha\beta}(\bar{a}, \bar{b})(\sigma) = h^{\beta\alpha}\left(\bar{b}', \int \bar{a}\right) = \frac{1}{2}\bar{a}^\alpha \bar{b}^\beta. \tag{E7}$$

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