An AdS_3 dual for minimal model CFTs

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We propose a duality between the 2d $\mathcal{W}_N$ minimal models in the large N 't Hooft limit, and a family of higher spin theories on AdS_3. The 2d conformal field theories (CFTs) can be described as Wess-Zumino-Witten coset models, and include, for $N = 2$, the usual Virasoro unitary series. The dual bulk theory contains, in addition to the massless higher spin fields, two complex scalars (of equal mass). The mass is directly related to the 't Hooft coupling constant of the dual CFT. We give convincing evidence that the spectra of the two theories match precisely for all values of the 't Hooft coupling. We also show that the renormalization group flows in the 2d CFT agree exactly with the usual AdS/CFT prediction of the gravity theory. Our proposal is in many ways analogous to the Klebanov-Polyakov conjecture for an AdS_4 dual for the singlet sector of large N vector models.

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I. INTRODUCTION

Two-dimensional conformal field theories are probably the best understood amongst all quantum field theories. The local conformal symmetry described by the Virasoro algebra is in many cases powerful enough to lead to a complete determination of the operator spectrum, as well as to explicit formulas for the correlation functions. These theories thus give concrete instances of nontrivial fixed points of the renormalization group, many of which have a realization in statistical mechanical systems.

In higher dimensional conformal field theories (CFTs), without the luxury of the local Virasoro symmetry, we have had to resort to other techniques to learn about nontrivial fixed points. One of the fruitful approaches has been to consider theories in which one has very many interacting degrees of freedom, the so-called large N limit. For example, for vector models in $2 + 1$ dimensions with $\mathcal{O}(N)$ number of fields, one can infer the existence of nontrivial fixed points in the large N limit. In fact, in this limit, the fixed points are perturbatively accessible, and one can compute, in a systematic $\frac{1}{N}$ expansion, anomalous dimensions and correlation functions. Thus the O(N) vector model exhibits the analogue of the Wilson-Fisher fixed point without having to resort to methods such as the ϵ expansion.

While it has always been surmised that the large N limit is some kind of mean field like description, it was not until the advent of the AdS/CFT duality that one could make this idea precise (at least for gauge theories). This duality gives a classical description of the leading large N behavior. The unexpected feature was that this was in terms of a higher dimensional theory which typically involves gravity in an asymptotically AdS spacetime. In the case of matrix valued fields with $\mathcal{O}(N^2)$ degrees of freedom, the relevant description is believed to be in terms of a classical string theory. If one then takes the further limit of ultrastrong coupling ($\lambda \gg 1$), the classical (super-)string theory reduces to Einstein (super-)gravity. This idea has had tremendous success in the last decade or so, and its repercussions are now even being felt in domains once far removed from string theory.

The connection of the large N limit to gravity, however, remains very mysterious, and our current understanding is very much tied to the origins of the duality in D-branes and string theory (with all its additional baggage of supersymmetry and so on). One would like to have examples which are shorn of any unnecessary ingredients, and which give an idea of how this connection comes about. Such "distilled" versions of the gauge-gravity duality are also interesting from the point of view of applications to realistic systems which often do not involve supersymmetry, for example. Moreover, since an Einstein gravity dual forces us into the regime of very strong coupling one would need to move away from this limit to describe systems with a coupling of order one. Generically, this would require us to be in a stringy regime with a large number of operators of finite anomalous dimensions. The technical complications of quantizing strings in (asymptotically) AdS spacetimes prevents us from studying this regime easily.

An interesting *via media* is afforded by the so-called higher spin theories in AdS spacetimes [1]. These are theories containing (generically) an infinite number of massless interacting fields with spin $s \geq 2$ (see [2] for an introduction). It has been suggested by several people [3–6] that these theories might be relevant for the description of (a sector of) the weak coupling limit of large N gauge theories. However, a striking and concrete conjecture was made in 2002 by Klebanov and Polyakov [7] who

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suggested that a particular higher spin theory on AdS_4 might be exactly dual to the singlet sector of the interacting (as well as the free) O(N) vector model in 2 + 1 dimensions at large N. This is interesting for several of the reasons discussed above. The O(N) model has a close relation to various statistical mechanical systems. The interacting fixed point is nontrivial and yet not strongly coupled. Finally, it is a concrete duality which goes beyond the Einstein gravity limit and yet does not involve an entire stringy spectrum of operators. Recent calculations have provided nontrivial, interesting evidence for this conjecture, see in particular [8–10].

The aim of this paper is to propose another duality of this nature. In fact, we shall return to the well-understood class of 2d CFTs and look for signatures of a higher dimensional classical gravity like description in a suitable large N limit. This will give rise to a controlled environment in which to study the puzzle of the emergence of a gravity dual.

The large N limit of various field theories in two and higher dimensions has been much studied. For some reason, however, this limit does not appear to have been much explored in the context of 2d CFTs (see however [11–13]), perhaps because they are solvable by other means. In this paper we study a family of minimal model CFTs which are given by coset WZW models

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}, \tag{1.1}$$

where the denominator is the diagonal $\mathfrak{su}(N)$ subalgebra, and the subscripts refer to the level of the current algebra. This family of CFTs includes in the special case of $N = 2$ the usual coset description of the unitary minimal models ($c < 1$) of the Virasoro algebra [14]. Though the generalization of these theories to arbitrary N has been less studied compared to the $N = 2$ case, several important facts about them are known. In particular, the spectrum of primaries and the fusion rules follow directly from those of the Wess-Zumino-Witten (WZW) models, and the characters can in principle be deduced. More interestingly, these theories are known [15] to possess a higher spin $\mathcal{W}_N$ symmetry [16–18] (for a review see [19]), and the different minimal models (for finite, fixed N and different values of k) are related to one another by an integrable renormalization group (RG) flow. More details about the $\mathcal{W}_N$ minimal models are explained in Sec. II.

Here, we will look at these theories in the large N limit. Specifically, we will define a 't Hooft limit (see also [13]) in which we take

$$N, k \to \infty; \qquad 0 \le \lambda \equiv \frac{N}{k+N} \le 1 \quad \text{fixed.} \tag{1.2}$$

It is interesting that the limit appears to be well defined and nontrivial. In particular, these theories behave like vector models, since their central charge equals $c_N(\lambda) \simeq N(1 - \lambda^2)$ and hence scales as N. The discrete set of CFTs coalesce into a line labeled by the 't Hooft coupling λ, where $\lambda = 0$ behaves like a free theory (of N complex fermions), while $\lambda = 1$ is some sort of "strong" coupling region. Notice that the coupling always remains of order one—an indication of the absence of a dual Einstein gravity regime. Furthermore, the spectrum of primaries simplifies remarkably in the 't Hooft limit, in that the dependence of the conformal dimensions on the coupling λ becomes essentially linear. One of the nice features of this model compared to the O(N) vector model is the existence of the additional continuous parameter λ, which makes it closer to the supersymmetric gauge theories in higher dimensions. The details of the 't Hooft limit are explained in Sec. III.

In Sec. IV we describe the higher spin theories on AdS_3 which are dual to these large N CFTs. As was mentioned before, the CFTs have a higher spin $\mathcal{W}_N$ symmetry, and so it is natural that the bulk theory also possesses such a symmetry. In fact, it was recently pointed out in [20,21] that higher spin theories in AdS_3 possess, at the classical level, an asymptotic symmetry group which is indeed a two-dimensional $\mathcal{W}$-algebra. This generalizes the observation of Brown-Henneaux for asymptotic Virasoro symmetries in Einstein gravity on AdS_3 [22]. Here we will consider a theory of higher spins, which contains, in addition to a large N tower of massless higher spin gauge fields, two complex scalar fields (of equal mass). It is known that scalar fields can appear as additional matter fields in these AdS_3 theories (precisely in pairs of equal mass). However, their mass cannot be arbitrary since it is related to a parameter Δ of the algebra which plays the role analogous to α' [23–25].[1] We will see that this parameter (and hence the mass) is indeed mapped to the 't Hooft coupling (1.2) of the CFT via

$$\Delta = 1 - \lambda \qquad \Rightarrow \qquad M^2 = -(1 - \lambda^2). \tag{1.3}$$

In Sec. V we provide support for this conjecture. The first piece of evidence consists in matching the spectrum of the CFT with that of the bulk theory. In fact, a 1-loop computation for the higher spin gauge fields in AdS_3 [26] had already revealed a match with the vacuum character of the $\mathcal{W}_N$ algebra. In the specific higher spin theory being considered here one has additional scalar fields in the bulk, as well as additional primary fields in the CFT. We find highly nontrivial evidence that the two match for *all* values of the coupling λ. This requires the spectrum of dimensions in the CFT to take a special form which it obligingly does, but only in the large N 't Hooft limit.

The next piece of evidence consists of relating the behavior of the CFTs under the RG flow with that in AdS_3. The lowest nontrivial primary operator $\mathcal{O}$ has conformal dimension $h_- = \bar{h}_- = \frac{1}{2}(1 - \lambda)$ in the 't Hooft limit. The "double trace" operator $\mathcal{O}\mathcal{O}^\dagger$ is thus relevant,

[1]We thank Misha Vasiliev for this remark.

and it is known to be the operator that induces the RG flow to the nearby minimal model. Furthermore, one knows that it flows in the IR to an irrelevant operator of the form $\mathcal{O}'\mathcal{O}'^{\dagger}$, where $\mathcal{O}'$ has dimensions (in the 't Hooft limit) $h_+ = \bar{h}_+ = \frac{1}{2}(1+\lambda)$. Note that

$$\Delta_+ = (h_+ + \bar{h}_+) = 2 - \Delta_- = 2 - (h_- + \bar{h}_-). \quad (1.4)$$

This precisely corresponds to what we have learnt from AdS/CFT. In fact, one of the bulk complex scalar fields ϕ that one has to add to the higher spin theory is precisely dual to $\mathcal{O}$, while the other, ϕ', is dual to $\mathcal{O}'$. Though both fields in AdS_3 have the same mass, there is a choice in how they are quantized [27,28]. In fact, we have to quantize them in opposite ways such that they correspond to the two different dimensions $\Delta_\pm$ for the operators dual to them. As was argued on general grounds in [29] (see also [30]), the RG flow takes one from the quantization corresponding to Δ_- in the UV to that for Δ_+ in the IR. In other words, the operator corresponding to ϕ, namely $\mathcal{O}$, must flow in the IR to the operator corresponding to ϕ', namely $\mathcal{O}'$. This is thus in perfect agreement with the CFT result we mentioned earlier.

In Sec. VI we outline a heuristic way to "derive" this duality. This uses the fact that the bulk description of the higher spin fields is a Chern-Simons theory [31,32]. One might therefore imagine the boundary theory to be a WZW theory. In fact, there is a very specific set of boundary conditions associated with requiring the spacetime to be asymptotically AdS_3—this is, for example, clearly explained in [33], see also [20,21]. From the point of view of the WZW theory, these boundary conditions lead to a specific gauging (Hamiltonian reduction) which goes by the name of (classical) Drinfeld-Sokolov (DS) reduction. The bulk description in terms of the higher spin theories is, of course, classical and we do not have a quantum definition of the theory. What we propose is that the quantum version of the above classical DS reduction would define the quantum theory. This quantum theory is believed to be equivalent to the above coset model, provided that the levels are suitably identified [17,34–37]. In order to apply this line of reasoning to our situation the main open problem is to understand how to describe the scalar fields in this formulation. This approach should hopefully lead to a detailed understanding of the emergence of gravity and higher spin diffeomorphisms in AdS_3.

Finally, Sec. VII contains concluding remarks. We have sequestered various details of the CFT and higher spin theories into three appendices.

II. A FAMILY OF MINIMAL MODEL CONFORMAL FIELD THEORIES

In this section we describe the 2d CFTs which will be the key players on the field theory side. Here we outline some of their important properties. In the next section we will take the 't Hooft large N limit of these models.

A. The minimal $\mathcal{W}_N$ models

The CFTs we are interested are the so-called $\mathcal{W}_N$ minimal models. They are most easily described in terms of a coset [15]

$$\frac{\mathfrak{g}_k \oplus \mathfrak{g}_1}{\mathfrak{g}_{k+1}}, \quad (2.1)$$

where, in order to obtain $\mathcal{W}_N$, we consider $\mathfrak{g} = \mathfrak{su}(N)$. The central charge of the coset equals

$$c = \dim(\mathfrak{g})\left[\frac{k}{k+h^\vee} + \frac{1}{1+h^\vee} - \frac{k+1}{k+1+h^\vee}\right], \quad (2.2)$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{su}(N)$ we have $h^\vee = N$, and the central charge becomes

$$c_N(p) = (N-1)\left[1 - \frac{N(N+1)}{p(p+1)}\right] \leq (N-1), \quad (2.3)$$

where we have introduced the parameter $p = k + N \geq (N+1)$ that will sometimes be useful. Note that for $N = 2$ this is just the familiar unitary series of the Virasoro minimal models that can be described by the above Goddard-Kent-Olive construction with $\mathfrak{g} = \mathfrak{su}(2)$ [14].

For the smallest value $p = N+1$, (i.e. $k = 1$), we have a theory with central charge $c = \frac{2(N-1)}{N+2}$ which has an alternative realization in terms of a theory of $\mathbb{Z}_N$ parafermions [38]. The other extreme case corresponds to $p \to \infty$ (taking $k \to \infty$ while keeping N finite), where $c = (N-1)$, and the symmetry algebra is equivalent to the Casimir algebra of the $\mathfrak{su}(N)$ affine algebra at level $k = 1$ [15,18]. The Casimir algebra consists of all $\mathfrak{su}(N)$ singlets in the affine vacuum representation. Since the affine algebra is at level one, it can be realized in terms of $(N-1)$ free bosons.

B. The minimal model representations

The actual coset theory does not just involve the vacuum representation of the coset algebra (2.1). The other states of the theory fall into highest weight representations of the coset algebra. These are labeled by $(\rho, \mu; \nu)$, where ρ is a highest weight representation (hwr) of $\mathfrak{g}_k$, μ is a hwr of $\mathfrak{g}_1$, and ν is a hwr of $\mathfrak{g}_{k+1}$.[2] Only those combinations are allowed where ν appears in the decomposition of $(\rho \oplus \mu)$ under the action of $\mathfrak{g}_{k+1}$. The relevant selection rule is simply

$$\rho + \mu - \nu \in \Lambda_R(\mathfrak{g}), \quad (2.4)$$

where $\Lambda_R(\mathfrak{g})$ is the root lattice of $\mathfrak{g}$. In addition, there are field identifications: the two triplets

$$(\rho, \mu; \nu) \cong (A\rho, A\mu; A\nu), \quad (2.5)$$

[2]It is important to note though that the states in the coset do *not* transform under any nontrivial representations of $\mathfrak{su}(N)$.

define the same highest weight representation of the coset algebra, provided that A is an outer automorphism of the affine algebra corresponding to $\mathfrak{g}_l$, with $l = k$, $l = 1$ and $l = k + 1$, respectively. For $\mathfrak{g} = \mathfrak{su}(N)$, the group of outer automorphisms is $\mathbb{Z}_N$, generated by the cyclic rotation of the affine Dynkin labels, i.e. the map

$$[\lambda_0; \lambda_1, \ldots, \lambda_{N-1}] \mapsto [\lambda_1; \lambda_2, \ldots, \lambda_{N-1}, \lambda_0], \qquad (2.6)$$

where the first entry is the affine Dynkin label. In this notation, the allowed highest weight representations of $\mathfrak{su}(N)$ at level k are labeled by

$$P_k^+(\mathfrak{su}(N)) = \left\{[\lambda_0; \lambda_1, \ldots, \lambda_{N-1}]: \lambda_j \in \mathbb{N}_0, \quad \sum_{j=0}^{N-1} \lambda_j = k\right\}. \qquad (2.7)$$

Note that the field identification (2.5) does not have any fixed points since $\mathbb{Z}_N$ acts transitively on the highest weight representations of $\mathfrak{su}(N)$ at level $k = 1$.

C. Conformal weights

It is easy to see that for *any* choice of highest weight representations $(\rho; \nu)$, there always exists a unique $\mu \in P_1^+(\mathfrak{su}(N))$, such that $\rho + \mu - \nu \in \Lambda_R(\mathfrak{su}(N))$. Thus we may label the highest weight representations of the coset algebra in terms of unconstrained pairs $(\rho; \nu)$. These pairs are still subject to the field identifications

$$(\rho; \nu) \cong (A\rho; A\nu). \qquad (2.8)$$

The conformal weight of the corresponding highest weight representation equals then

$$h(\rho; \nu) = \frac{C_N(\rho)}{N+k} + \frac{C_N(\mu)}{N+1} - \frac{C_N(\nu)}{N+k+1} + n, \qquad (2.9)$$

where $C_N(\sigma)$ is the eigenvalue of the quadratic Casimir operator of $\mathfrak{g} = \mathfrak{su}(N)$—hence the N-dependence—in the representation σ, see Appendix B 1 for our normalization convention. Here the representation $\mu \in P_1^+(\mathfrak{su}(N))$ is uniquely determined by the condition that $\rho + \mu - \nu \in \Lambda_R(\mathfrak{su}(N))$. Furthermore, n is a non-negative integer, describing the "height" (i.e. the conformal weight above the ground state) at which the $\mathfrak{g}_{k+1}$ primary ν appears in the representation $(\rho \oplus \mu)$. Unfortunately, an explicit formula for n is not available, but it is not difficult to work out n for simple examples. Alternatively, one may use the Drinfeld-Sokolov description of these models (that is briefly reviewed in Appendix B). In that language the highest weight representations are labeled by $(\Lambda^+, \Lambda^-) \cong (\rho; \nu)$, and the conformal weights equal

$$h(\Lambda^+, \Lambda^-) = \frac{1}{2p(p+1)}(|(p+1)(\Lambda^+ + \hat{\rho}) - p(\Lambda^- + \hat{\rho})|^2 - \hat{\rho}^2), \qquad (2.10)$$

where $\hat{\rho}$ is the Weyl vector of $\mathfrak{su}(N)$. For $N = 2$ (the Virasoro minimal models), (2.10) reduces to the familiar formula

$$h(r, s) = \frac{(r(p+1) - sp)^2 - 1}{4p(p+1)} = h(p - r, p + 1 - s) \qquad (2.11)$$

with $1 \le r \le p - 1$, $1 \le s \le p$. Here we have identified $\Lambda^+ = \frac{(r-1)}{\sqrt{2}}$ and $\Lambda^- = \frac{(s-1)}{\sqrt{2}}$.

In the following, the primary where $\nu = [1, 0^{N-1}] = \mathrm{f}$ is the fundamental representation[3] with $\rho = [0^{N-1}] = 0$ the trivial representation will play an important role. Then (2.9) gives—in this case $\mu = \mathrm{f}$ with $n = 0$

$$h(0; \mathrm{f}) = \frac{C_N(\mathrm{f})}{N+1} - \frac{C_N(\mathrm{f})}{N+k+1} = \frac{(N-1)}{2N}\left(1 - \frac{N+1}{N+k+1}\right), \qquad (2.12)$$

where we have used that $C_N(\mathrm{f}) = \frac{1}{2}(\Lambda_\mathrm{f}, \Lambda_\mathrm{f} + 2\hat{\rho}) = \frac{N^2-1}{2N}$. On the other hand, for the coset representation with $\rho = \mathrm{f}$ and $\nu = 0$, μ is the antifundamental representation, $\mu = \bar{\mathrm{f}}$, and we get (again with $n = 0$)

$$h(\mathrm{f}; 0) = \frac{C_N(\mathrm{f})}{N+k} + \frac{C_N(\mathrm{f})}{N+1} = \frac{(N-1)}{2N}\left(1 + \frac{N+1}{N+k}\right). \qquad (2.13)$$

An example with $n = 1$ arises for the case where $\rho = 0$ and $\nu = \mathrm{adj}$, the adjoint representation. Then $\mu = 0$ but $n = 1$, and we obtain

$$h(0; \mathrm{adj}) = 1 - \frac{C_N(\mathrm{adj})}{N+k+1} = 1 - \frac{N}{N+k+1}, \qquad (2.14)$$

where we have used that $C_N(\mathrm{adj}) = h^\vee = N$. Finally, the representation with $\rho = \mathrm{adj}$ and $\nu = 0$ also has $\mu = 0$ and $n = 1$, and the conformal weight is

$$h(\mathrm{adj}; 0) = 1 + \frac{C_N(\mathrm{adj})}{N+k} = 1 + \frac{N}{N+k}. \qquad (2.15)$$

D. Fusion rules and characters

The fusion rules of the coset theory follow directly from the mother and daughter theory. Indeed, in terms of the triplets $(\rho, \mu; \nu)$ the fusion rules are simply

$$\mathcal{N}_{(\rho_1,\mu_1;\nu_1)(\rho_2,\mu_2;\nu_2)}{}^{(\rho_3,\mu_3;\nu_3)} = \mathcal{N}^{(k)}_{\rho_1\rho_2}{}^{\rho_3}\, \mathcal{N}^{(1)}_{\mu_1\mu_2}{}^{\mu_3}\, \mathcal{N}^{(k+1)}_{\nu_1\nu_2}{}^{\nu_3}, \qquad (2.16)$$

[3]Note that the representation of the affine $\mathfrak{su}(N)$ algebra has N entries as in (2.6). Below we will mostly drop the affine Dynkin label, and use a description in terms of the usual $(N-1)$ Dynkin labels for representations of $\mathfrak{su}(N)$.

where the fusion rules on the right-hand side are those of $\mathfrak{g}_k$, $\mathfrak{g}_1$ and $\mathfrak{g}_{k+1}$, respectively. Note that the fusion rules are invariant under the field identification (2.5). Since the fusion rules of the level one factor are just a permutation matrix, we can also directly give the fusion rules for the representatives $(\rho;\nu)$ as

$$\mathcal{N}_{(\rho_1;\nu_1)(\rho_2;\nu_2)}{}^{(\rho_3;\nu_3)} = \mathcal{N}^{(k)}_{\rho_1\rho_2}{}^{\rho_3}\mathcal{N}^{(k+1)}_{\nu_1\nu_2}{}^{\nu_3}. \tag{2.17}$$

Closed form expressions for the characters of the minimal $\mathcal{W}_N$ highest weight representations are known in terms of branching functions, see, for example, Eq. (7.51) of [19]. However, these expressions are often difficult to evaluate explicitly. In the following we shall mainly be interested in the large k limit of these models, in which case the low-lying terms of the characters simplify. In particular, the vacuum character becomes in this limit

$$\chi_{(0;0)}(q) = q^{-(c_N/24)}\Big(\prod_{s=2}^{N}\prod_{n=s}^{\infty}\frac{1}{(1-q)^n} + \mathcal{O}(q^{k+1})\Big), \tag{2.18}$$

since for $k\to\infty$ the character is that of the Casimir algebra, see Eq. (7.18) of [19]. For finite k the corrections to this formula are a consequence of the null-vectors of the $\mathfrak{g}_k$ and $\mathfrak{g}_{k+1}$ factors in (2.1). For the case of the vacuum representation with $\rho=\nu=0$, these appear first at height $h=k+1$.

For the other characters there is a similar formula in terms of branching functions of the affine level one representation to the horizontal (finite-dimensional) Lie algebra. However, as far as we are aware, no simple explicit formulas for these branching functions are known.[4] We have worked out the first few branching rules for some small representations in Appendix C, and from that we can conclude that

$$\chi_{(0;\mathrm{f})}(q) = q^{h(0;\mathrm{f})}(1+q+2q^2+4q^3+\cdots) \tag{2.19}$$

$$\chi_{(0;\mathrm{adj})}(q) = q^{h(0;\mathrm{adj})}(1+2q+4q^2+\cdots) \tag{2.20}$$

$$\chi_{(0;[0,1,0^{N-3}])}(q) = q^{h(0;[0,1,0^{N-3}])}(1+q+3q^2+\cdots) \tag{2.21}$$

$$\begin{aligned}\chi_{(0;[2,0^{N-2}])}(q) &= q^{h(0;[0,1,0^{N-3}])}(q+q^2+\cdots)\\ &= q^{h(0;[2,0^{N-2}])}(1+q+\cdots).\end{aligned} \tag{2.22}$$

These formulas will play an important role below.

E. The RG flows

For fixed (finite) N the models with different values of k (or p) are related to one another by an RG flow. This is most familiar for the Virasoro minimal models, for which the perturbing field in the UV is the (1,3) field with $h_{1,3}=\frac{p-1}{p+1}$ [39].[5] In the above conventions this field corresponds to $(\rho;\nu)=(0;\mathrm{adj})$, which has indeed $h(0;\mathrm{adj})=\frac{p-1}{p+1}$, see (2.14). The RG flow that is induced by this relevant perturbation connects the p-th unitary minimal model in the UV, to the $(p-1)$st in the IR. In the IR, the perturbing (1,3) field of the UV theory has become irrelevant. Indeed, it can be identified with the (3,1) field of the $(p-1)$'st minimal model [39]. The latter field has conformal dimension $h_{3,1}=\frac{p+1}{p-1}$ in the $(p-1)$'st minimal model, and hence can be identified with the $(\rho;\nu)=(\mathrm{adj};0)$ field in that theory, see (2.15).

Similarly, the (1,2)-field can be identified with $(\rho;\nu)=(0;\mathrm{f})$. In the IR it flows to the (2,1)-field of the $(p-1)$'st minimal model [40]. The latter field can be identified with the $(\rho;\nu)=(\mathrm{f};0)$ field in that theory. Note that this is compatible with the above since we have the fusion rules (for $p\geq 4$)

$$\begin{aligned}(1,2)\otimes(1,2) &= (1,1)\oplus(1,3)\\ (2,1)\otimes(2,1) &= (1,1)\oplus(3,1).\end{aligned} \tag{2.23}$$

Thus the normal-ordered product of the (1,2) field with itself is the (1,3) field, and similarly for the (2,1) and (3,1) field.

The generalization to $N>2$ is believed to follow a similar pattern, see for example [41]. The relevant field $(0;\mathrm{adj})$ of the p'th $\mathcal{W}_N$ minimal model induces an RG flow, whose IR fixed point is the $(p-1)$'st $\mathcal{W}_N$ minimal model. In the IR the perturbing field becomes irrelevant, and is to be identified with the $(\mathrm{adj};0)$ field of the $(p-1)$'st model, i.e.

$$(0;\mathrm{adj})_p \quad \overset{\text{RG}-\text{flow by }(0;\text{adj})}{\longrightarrow} \quad (\mathrm{adj};0)_{p-1}. \tag{2.24}$$

The analogue of the (1,2) field for $N>2$ is slightly more subtle. For $N>2$, charge conjugation of SU(N) is nontrivial, and there are therefore two fields that play that role. Indeed, the analogues of the fusion rules (2.23) are now

$$\begin{aligned}(0;\mathrm{f})\otimes(0;\bar{\mathrm{f}}) &= (0;0)\oplus(0;\mathrm{adj})\\ (\mathrm{f};0)\otimes(\bar{\mathrm{f}};0) &= (0;0)\oplus(\mathrm{adj};0),\end{aligned} \tag{2.25}$$

where $\bar{\mathrm{f}}$ denotes the antifundamental representation of $\mathfrak{su}(N)$. Note that the conformal dimension of the $(0;\bar{\mathrm{f}})$ field obviously equals that of $(0;\mathrm{f})$, and similarly, $h(\bar{\mathrm{f}};0)=h(\mathrm{f};0)$. The analogue of the RG-flow for the (1,2) field is then

$$(0;\mathrm{f})_p \quad \overset{\text{RG}-\text{flow by }(0;\text{adj})}{\longrightarrow} \quad (\mathrm{f};0)_{p-1} \tag{2.26}$$

[4]We thank Terry Gannon for discussions about this point.

[5]Here, and in the following, we mean by (1,3) the field whose left- and right-moving conformal dimension is $h=\bar{h}=h(1,3)$.

$$(0;\bar{\mathrm{f}})_p \quad \overset{\mathrm{RG-flow\ by\ (0;adj)}}{\longrightarrow} \quad (\bar{\mathrm{f}};0)_{p-1}. \tag{2.27}$$

As we shall see, these RG flows have a very nice interpretation in the bulk theory, following the general analysis of [29]; see also [7].

III. THE LARGE N 'T HOOFT LIMIT

With all of these preparations in place, we can now explain the large N limit we shall be considering. If we take $N \to \infty$ for constant k, then it follows from (2.3) that $c_N \simeq 2k + \mathcal{O}(N^{-1})$, remembering that $p = k + N$ [11,12]. In this limit, however, many important fields will have vanishing conformal dimension. For example, this will be the case for $(0;\mathrm{f})$ and $(0;\mathrm{adj})$, see (2.12) and (2.14).

It is therefore more interesting to consider the 't Hooft like limit (see also [13]), where we take both N, $k \to \infty$, but keep the (renormalized) 't Hooft coupling

$$\lambda = \frac{N}{k+N} < 1 \tag{3.1}$$

fixed. In this limit we get a family of CFTs with an effectively continuous central charge

$$c_N(\lambda) \simeq N(1-\lambda^2) < N. \tag{3.2}$$

Note that the central charge scales as N. In this sense, these theories behave like vector like models (whose degrees of freedom scale as N), rather than like gauge theory models (where the number of degrees of freedom scales as N^2).

Note that the "free case", $\lambda = 0$, corresponds to first taking $k \to \infty$, before taking $N \to \infty$. At finite N, the limit $k \to \infty$ leads to a theory with $c = N - 1$. In the large N limit, we then expect this theory to have a description in terms of a singlet sector of N free complex fermions.[6] This would be closely analogous to the free vector model considered in [7]. In our context, the "singlet sector" condition arises automatically as a consequence of the coset construction, and does not have to be added in by hand.

Next we turn to the conformal weights. It follows from (2.10) that they become in this limit

$$h(\Lambda^+,\Lambda^-) \simeq \frac{1}{2}(\Lambda^+ - \Lambda^-)^2 + \frac{1}{N+k}(\Lambda^+ - \Lambda^-, \hat{\rho}). \tag{3.3}$$

Note that the second term is typically at least of the same order, since $\hat{\rho}^2 = \frac{N(N^2-1)}{12}$. For example, for the fields discussed above, we find in this 't Hooft limit

$$h(0;\mathrm{f}) = \frac{1}{2}(1-\lambda), \qquad h(\mathrm{f};0) = \frac{1}{2}(1+\lambda), \tag{3.4}$$

as well as

$$h(0;\mathrm{adj}) = 1-\lambda, \qquad h(\mathrm{adj};0) = 1+\lambda. \tag{3.5}$$

Obviously, this also agrees with the formulas obtained from the coset description, Eqs. (2.12), (2.13), (2.14), and (2.15).

We should stress that there are ambiguities in how to define the large N limit, and that we have implicitly made a choice in the above. For example, for $N = 2$, there exist at least two different (natural) $k \to \infty$ limits of the unitary minimal models that have been considered in the literature [42,43]. They lead to quite different limit theories: the spectrum of [42] is continuous, and the resulting theory seems to be similar to Liouville theory (see also [44,45]), while the spectrum of [43] is discrete. Both are believed to lead to consistent correlation functions, and thus both seem to define viable large k limits.

While these limits have only been analyzed for $N = 2$, it is not difficult to see how their respective analogues would differ in our case. In order to explain this, let us consider the representations of the form $(R;R)$, whose conformal dimension equals

$$\begin{aligned} h(R;R) &= C_N(R)\left(\frac{1}{N+k} - \frac{1}{N+k+1}\right) \\ &= \frac{C_N(R)}{(N+k)(N+k+1)}. \end{aligned} \tag{3.6}$$

Since the Casimir $C_N(R)$ is of order $\mathcal{O}(N)$ (for representations with a finite number of boxes in the Young tableau, the coefficient is one half times the number of boxes $B(R)$, see (B11)), the conformal weight then behaves in the large N limit as

$$h(R;R) = \frac{B(R)}{2} \times \frac{\lambda^2}{N}, \tag{3.7}$$

where $B(R)$ is an integer. In the 't Hooft limit, both N and k become large, and hence representations R_N with an arbitrarily large number of boxes $B(R_N)$ are allowed. There are now essentially two possibilities we can consider: we can either define the fields of the limit theory to be those associated to a family of representations R_N with fixed $B(R_N)$, and then take $N \to \infty$—in this case the conformal weight will approach $h(R_N;R_N) \to 0$. Or, we consider fields, where, as we take $N \to \infty$, we also take $B(R_N) \to \infty$, keeping only their ratio fixed. The latter prescription leads to a continuous spectrum (and is the analogue of the proposal of [42]), while the former leads to a discrete spectrum as in [43]. As will become clear below, the dual of the bulk gravity theory we are about to discuss corresponds to the second option, i.e. to a limit theory with a discrete spectrum. Indeed, the fields associated to the gravity dual are those that appear in finite tensor powers of the fundamental (and antifundamental) representation, and therefore $B(R_N)$ will not grow with N.

[6] For large N, we may ignore the difference between $\hat{su}(N)_1$ and $\hat{u}(N)_1$. The latter theory has a description in terms of N complex free fermions.

We should also note that $h = 0$ is not the only limit point; for example, for the representations of the form $(R \otimes \mathrm{f}; R)$ the conformal dimension behaves as

$$h(R \otimes \mathrm{f}; R) \simeq \frac{1}{2} + \frac{B(R)}{2} \times \frac{\lambda^2}{N}, \tag{3.8}$$

etc. Finally, we note that excitations of order $\frac{1}{N}$ are typically seen in symmetric orbifold CFTs arising from fractionalized momentum; the above behavior could therefore be indicative of some string theory interpretation of our higher spin theory.

IV. THE HIGHER SPIN AdS_3 DUAL

Now we want to switch gears and describe the dual gravity theory for the above large N family of 2d CFTs; this will turn out to be a higher spin theory.

Higher spin field theories in three dimensions are relatively more tractable than their higher dimensional counterparts. First, the massless higher spin fields themselves do not contain any propagating degrees of freedom (see e.g. [26] for a recent discussion). Second, one can (classically) truncate consistently to a finite number of them [46]. For instance, one can have theories in which one has massless fields of spin $s = 2, 3, \ldots, N$ only, for any $N \geq 2$. Third, there exists a Chern-Simons formulation of the classical action for these theories [47,48]. For theories with a maximal spin N, the Chern-Simons gauge group is $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ (in Lorentzian signature) or $SL(N, \mathbb{C})$ in Euclidean signature. Thus the interacting theory of the higher spin fields can be expressed relatively compactly compared to the higher dimensional cases.

To be a bit more specific, the higher spin gauge fields can be expressed in terms of generalized vielbein and connection variables (generalizing the familiar case of gravity)

$$e_\mu^{a_1 \cdots a_{s-1}}, \qquad \omega_\mu^{a_1 \cdots a_{s-1}}, \tag{4.1}$$

where s is the spin of the gauge field. In a theory with maximal spin N, all these variables for the fields with $s = 2, 3, \ldots, N$ can be packaged together into two $SL(N, \mathbb{R})$ (or one $SL(N, \mathbb{C})$, depending on the signature) gauge fields. This reflects the fact that all these fields are part of one single multiplet under the higher spin symmetry. The action is then given by

$$S = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\tilde{A}], \tag{4.2}$$

where

$$S_{\mathrm{CS}}[A] = \frac{k_{\mathrm{CS}}}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{4.3}$$

The Chern-Simons level k_{CS} (to be distinguished from the k that appeared in the previous section) is related to the AdS radius by the classical relation

$$k_{\mathrm{CS}} = \frac{\ell}{4G_N}. \tag{4.4}$$

In [21] (see also [20]) it was argued, using the above Chern-Simons formulation, that the theory with maximal spin N has an asymptotic $\mathcal{W}_N$ symmetry algebra. Already at the classical level one sees a centrally extended algebra with a central charge whose value was determined to be the same as the Brown-Henneaux result for Einstein gravity on AdS_3

$$c = \frac{3\ell}{2G_N}. \tag{4.5}$$

In Appendix A we summarize, for completeness, the salient features of the framelike formulation and its relation to the more conventional Fronsdal description in terms of symmetric tensor fields of higher rank.

So far we have only discussed pure higher spin theories. In three dimensions one can also have, in addition to the higher spin fields, *separate* matter multiplets (for a survey of these matters see [25]). While in higher dimensions the matter fields always lie in the same multiplet as the higher spin fields, in three dimensions the matter multiplet is distinct and contains *only* scalar and/or fermion fields. Moreover, the fields in the matter multiplet can be massive since they are not in the same representation as the gauge fields. The mass is related to a deformation parameter[7] of the higher spin algebra as

$$M^2 = \Delta(\Delta - 2). \tag{4.6}$$

Typically the matter multiplet contains four scalars, two with mass (4.6), and two with $M^2 = \Delta(\Delta + 2)$. These scalars can additionally transform under a global symmetry group. However, it is consistent to truncate this multiplet[8] to just the two scalars of mass (4.6), and this is what will be relevant for the following. The interacting theory of these scalars with the higher spin fields was constructed in [23,24]. Finally, we should mention that for generic Δ, it is no longer possible to truncate the massless fields to a maximal spin. Thus once we have added such fields (as we are about to do), we have to take the $N \to \infty$ limit.

We can now describe the higher spin theory we are interested in. It contains, in addition to the higher spin gauge fields, a matter multiplet containing two complex scalar fields of the same mass (4.6). We will take M^2 to lie in the window

$$-1 \leq M^2 \leq 0. \tag{4.7}$$

As is by now familiar from various AdS/CFT applications this implies that there are two alternative conformally

[7]The deformation parameter is sometimes denoted by ν in the literature on higher spin theories [25]. We suggestively call it Δ here since it has exactly the same relation to the mass as the conformal dimension of the boundary operator.

[8]We thank Misha Vasiliev for discussions about this point.

invariant quantizations (which we denote by ($\pm$)) of these scalar fields. These correspond to the two different roots $\Delta_\pm$ of (4.6) determining the asymptotic fall-off behavior. We shall take one of the scalars, which we call ϕ, in the $(-)$-quantization and thus corresponding to $\Delta = \Delta_-$. The other scalar, ϕ', will be taken in the $(+)$-quantization corresponding to $\Delta = \Delta_+$. We will denote this particular one parameter family of theories by HS(M^2).

Our proposal can now be stated as follows. The $\mathcal{W}_N$ minimal model CFT with 't Hooft coupling λ is dual, in the large N 't Hooft limit, to the HS(M^2) theory with the identification

$$\Delta_- = 1 - \lambda, \qquad \Delta_+ = 1 + \lambda. \tag{4.8}$$

Note that both scalars have the same mass which is given by

$$M^2 = -(1 - \lambda^2). \tag{4.9}$$

Before we begin to discuss this proposal further, let us note that both the CFT and the higher spin theory have the same $\mathcal{W}_\infty$ symmetry. It then makes sense to identify the central charges; this leads to

$$c_{\text{bulk}} = \frac{3\ell}{2G_N} = c_N(\lambda) = N(1 - \lambda^2). \tag{4.10}$$

The bulk theory is only well-defined in the large N limit (since we can only add massive scalar fields in this limit). Note that large N means that G_N is small $\sim \frac{1}{N}$ (in units where $\ell = 1$); thus the large N limit is indeed the semiclassical limit, where one can trust the bulk description. For finite N, we may view the CFT (in its full $\frac{1}{N}$ expansion) as the quantum definition of the higher spin theory.

In the next section we will present some nontrivial checks of the proposal at leading order in N. We shall also give a heuristic derivation of some parts of the duality in Sec. VI.

V. CHECKS OF THE PROPOSAL

In this section we shall subject the above proposal to essentially two consistency checks. First we shall explain in quite some detail (see Sec. V A) that the spectrum of the two theories agrees. More specifically, we shall study the quantum 1-loop partition function of the higher spin theory, and see how it reproduces the full CFT spectrum in the 't Hooft limit. This is quite a detailed consistency check, and it probes much of the structure of the CFT. The second consistency check concerns the RG-flow for which we observe a beautiful matching with the bulk analysis (Sec. V B).

A. The spectrum

In this section we want to calculate the 1-loop partition function of the higher spin theory and compare it to the full CFT spectrum. There are basically two parts to this calculation. For the higher spin fields, the 1-loop determinant was computed recently in [26] using the heat kernel techniques of [49]. For $N \to \infty$ the answer is

$$Z^{(1)}_{\text{HS}} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}. \tag{5.1}$$

The higher spin theory HS(M^2) we are interested in also contains two complex scalar fields, one corresponding to $\Delta = \Delta_+$ and one with $\Delta = \Delta_-$, see (4.8). The 1-loop contribution from each complex scalar field is [50] (see also [49])

$$Z^{(1)}_{\text{scalar}} = \prod_{l=0,l'=0}^{\infty}\frac{1}{(1-q^{h+l}\bar{q}^{h+l'})^2}, \tag{5.2}$$

where $h = \frac{\Delta}{2}$. Thus defining

$$h_\pm = \frac{1}{2}(1\pm\lambda) = \frac{1}{2}(1\pm\sqrt{1+M^2}), \tag{5.3}$$

the total 1-loop partition function is

$$Z^{(1)}_{\text{total}} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}\times\prod_{l_1=0,l_1'=0}^{\infty}\frac{1}{(1-q^{h_-+l_1}\bar{q}^{h_-+l_1'})^2} \times\prod_{l_2=0,l_2'=0}^{\infty}\frac{1}{(1-q^{h_++l_2}\bar{q}^{h_++l_2'})^2}. \tag{5.4}$$

Our claim is that this partition function *agrees* with the full CFT partition function of the $\mathcal{W}_N$ model in the 't Hooft limit!

We have so far not managed to find an analytic proof of this statement, but we shall give below what we regard to be highly nontrivial evidence in favor of this claim. Before we begin with the detailed checks, we should first explain intuitively why this could be true.

The first factor coming from $Z^{(1)}_{\text{HS}}$ can be identified with a (generic) vacuum character of the $\mathcal{W}_\infty$-algebra [26]. In our case, the character of the vacuum representation of the coset CFT is *not* generic since we consider the limit of rational theories at finite k. However, as was explained in Sec. II D, see, in particular, Eq. (2.18), the null vectors only modify the answer at height $k+1$, and thus this modification does not play any role in the 't Hooft limit. We therefore conclude that the contribution from the higher spin gauge fields—the first factor of $Z^{(1)}_{\text{total}}$—reproduces precisely the vacuum character from the CFT perspective.

The full CFT has obviously many additional states; indeed, the coset representations are labeled by the pairs $(R_1; R_2)$, and the full spectrum (at finite k and N) will include all such sectors. However, given the structure of the fusion rules, all states of the CFT can be obtained by taking successive fusion products of the generating fields

$$(0;\text{f}), \qquad (0;\bar{\text{f}}) \qquad \text{and} \qquad (\text{f};0), \qquad (\bar{\text{f}};0), \tag{5.5}$$

where f and $\bar{\text{f}}$ are the fundamental and antifundamental representation of $\mathfrak{su}(N)$. Note that in the large N limit, the two sectors corresponding to $(0;\text{f})$ and $(0;\bar{\text{f}})$ (and similarly for $(\text{f};0)$ and $(\bar{\text{f}};0)$) effectively decouple; at finite N, the field $(0;\bar{\text{f}})$ obviously appears in the $(N-1)$-fold fusion of $(0;\text{f})$ with itself, but in the 't Hooft limit we have to include both separately.

Now the key observation is that the conformal dimension of the first two fields in (5.5) is $h = h_-$, while that of the second two fields is $h = h_+$, see Eq. (3.4). This suggests the identification

$$\prod_{l_1=0,l_1'=0}^{\infty} \frac{1}{(1-q^{h_-+l_1}\bar{q}^{h_-+l_1'})^2} \quad \longleftrightarrow \quad (0;\text{f})^{\otimes s_1} \otimes (0;\bar{\text{f}})^{\otimes s_2}, \tag{5.6}$$

i.e. that the product on the left gives the contributions of the fusion products involving multiple copies of $(0;\text{f})$ and $(0;\bar{\text{f}})$. Similarly, the other term should be identified with

$$\prod_{l_2=0,l_2'=0}^{\infty} \frac{1}{(1-q^{h_++l_2}\bar{q}^{h_++l_2'})^2} \quad \longleftrightarrow \quad (\text{f};0)^{\otimes r_1} \otimes (\bar{\text{f}};0)^{\otimes r_2}. \tag{5.7}$$

Putting all factors together then accounts for the full spectrum of the CFT. In the following we want to check this proposal in more detail. We shall consider the different pieces in turn.

1. *The fusion powers of* $(0;\text{f})$

The simplest consistency check is to consider the square root of (5.6), and confirm that it reproduces the states that appear in the fusion powers of $(0;\text{f})$, say. (Obviously, the analysis is identical for the fusion powers of $(0;\bar{\text{f}})$.) Expanding out the first few terms in (5.2) with $h = h_-$ leads to

$$\begin{aligned} Z^{(1)} = {}& q^h\bar{q}^h(1+q+q^2+q^3+\cdots)(1+\bar{q}+\bar{q}^2+\bar{q}^3 \\ &+\cdots) + q^{2h}\bar{q}^{2h}(1+q+2q^2+\cdots)(1+\bar{q} \\ &+2\bar{q}^2+\cdots) + q^{2h+1}\bar{q}^{2h+1}(1+q+\cdots) \\ &\times(1+\bar{q}+\cdots)+\cdots. \end{aligned} \tag{5.8}$$

In order to identify this with $\mathcal{W}_N$ characters, we also have to multiply the expression with the 1-loop determinant coming from the higher spin fields, $Z_{\text{HS}}^{(1)}$ (5.1). Then the low-lying terms of $Z^{(1)} \cdot Z_{\text{HS}}^{(1)}$ look like the sum of three representations with conformal dimensions $h_1 = h$, $h_2 = 2h$ and $h_3 = 2h+1$, whose characters are

$$\chi_{h_1}(q) = q^h(1+q+2q^2+4q^3+\cdots) \tag{5.9}$$

$$\chi_{h_2}(q) = q^{2h}(1+q+3q^2+\cdots) \tag{5.10}$$

$$\chi_{h_3}(q) = q^{2h+1}(1+q+\cdots), \tag{5.11}$$

respectively. Since $h = h_- = \frac{1}{2}(1-\lambda) = h(0;\text{f})$, these characters agree then precisely, in the 't Hooft limit, with the characters for the representation $(0;\text{f})$, see (2.19), the representation $(0;[0,1,0^{N-3}])$, see (2.21), and the representation $(0;[2,0^{N-2}])$, see (2.22), respectively. Here we have used that the conformal dimension of these fields, in the 't Hooft limit, are

$$\begin{aligned} h(0;[0,1,0^{N-3}]) &= \frac{(N-2)(N+1)}{N}\left(\frac{1}{N+1} - \frac{1}{N+k+1}\right) \\ &\simeq 1-\lambda = 2h(0;\text{f}), \end{aligned} \tag{5.12}$$

as well as

$$\begin{aligned} h(0;[2,0^{N-2}]) &= \frac{2(N+k)}{N+k+1}\frac{N-1}{N} - \frac{N-1}{N+k+1} \\ &\simeq 2-\lambda = 2h(0;\text{f})+1. \end{aligned} \tag{5.13}$$

Note that these two representations are precisely the representations that appear in the fusion of $(0;\text{f})$ with itself,

$$(0;\text{f}) \otimes (0;\text{f}) = (0;[0,1,0^{N-3}]) \oplus (0;[2,0^{N-2}]), \tag{5.14}$$

in accordance with the fact that the terms that are proportional to second powers of h correspond to two-particle states in the bulk.

2. *Higher orders*

We would expect that this pattern continues for higher powers of q and $\bar{q}$. While we have not yet attempted to prove this in general, there is one simple consistency check we have performed. Since h has a nontrivial λ-dependence, the above can only work out if the λ-dependence is additive under taking tensor products. It follows from (3.3) that the λ-dependent term is proportional to $(\Lambda^+ - \Lambda^-, \hat{\rho})$. For representations that have a finite number of boxes in the Young tableau, the argument in (B11) then implies that in the large N limit

$$\frac{1}{N+k}(\Lambda^+ - \Lambda^-, \hat{\rho}) \simeq \frac{\lambda}{2}(B(\Lambda^+) - B(\Lambda^-)), \tag{5.15}$$

where $B(\Lambda^\pm)$ is the number of boxes in the Young tableau of $\Lambda^\pm$. For representations that appear in finite tensor powers of the fundamental, the number of boxes is conserved under taking tensor products (for N sufficiently large), and since the fusion rules do not mix Λ^+ and Λ^- (that contribute with opposite sign), the statement follows.

3. *Fusion products of* $(0;\text{f})$ *and* $(0;\bar{\text{f}})$

It is clear that the above analysis works identically for the other factor in (5.6), the one associated to fusion products of $(0;\bar{\text{f}})$. However, in order to check (5.6), we also have to verify that the fusion products involving both $(0;\text{f})$ and $(0;\bar{\text{f}})$ work out. The leading "mixed" term arises by taking terms proportional to $q^{h+l}\bar{q}^{h+l'}$ from both factors in (5.6); it is easy to see that their total contribution is precisely

$$\frac{q^{2h_-}\bar{q}^{2h_-}}{(1-q)^2(1-\bar{q})^2}. \tag{5.16}$$

Taking into account the $\mathcal{W}$-descendants, this then implies that the character of the corresponding CFT representation should be (in the 't Hooft limit)

$$\chi_{(0;\mathrm{adj})}(q) = q^{1-\lambda}\frac{1}{(1-q)^2}\prod_{s=2}^{N}\prod_{n=s}^{\infty}\frac{1}{(1-q^n)} = q^{1-\lambda}(1+2q+4q^2+\cdots). \tag{5.17}$$

Because these states are single-particle in each factor, they should arise from the tensor product (2.25), and hence transform in the (0; adj) representation of the coset algebra—the other representation that appears in this fusion product is the identity representation that is already accounted for by $Z^{(1)}_{\mathrm{HS}}$. This works out precisely (to the order to which we have done the calculation), because (5.17) agrees exactly with the character of (0; adj), see Eq. (2.20).

4. *Fusion products of* (f; 0) *and* ($\bar{\mathrm{f}}$; 0)

It is fairly straightforward to see that the analysis works essentially identically for the terms in (5.7). The main difference is that we now have to determine the leading behavior of the characters of the representations (R; 0) with R being in turn $R=\mathrm{f}, \bar{\mathrm{f}}$, adj, *etc.* It is not difficult to show that the leading behavior of the character of (R; 0) is in fact the same as that for (0; $\bar{R}$). For concreteness, let us concentrate on the case when $R=\mathrm{f}$. For (f; 0), i.e. $\rho=\mathrm{f}$ and $\nu=0$, we have $\mu=\bar{f}$. Then the leading behavior of the character is described by the branching function where we count the multiplicities with which the $\bar{\mathrm{f}}$-representation appears in the level $k=1$ representation based on $\mu=\bar{\mathrm{f}}$, since we have to look at those representations of the level $k=1$ factor that lead to the trivial representation when tensored with the ground state representation $\rho=\mathrm{f}$. However, this branching function is precisely what gives the leading part of the (0; $\bar{\mathrm{f}}$) character. The other cases work essentially identically. Thus we conclude that the contribution from the left-hand-side of (5.7) accounts for the tensor products of (f; 0) and ($\bar{\mathrm{f}}$; 0).

5. *Fusion product of* (f; 0) *and* (0; $\bar{\mathrm{f}}$)

Finally we also have to look at the terms that involve both contributions from (5.6) and (5.7). By the same argument as that leading up to (5.16) it is clear that the leading term of the gravity calculation is

$$\frac{q^{h_++h_-}\bar{q}^{h_++h_-}}{(1-q)^2(1-\bar{q})^2}. \tag{5.18}$$

Since $h_+ + h_- = 1$, and taking into account the $\mathcal{W}$-descendants, this then implies that the character of the corresponding CFT representation should be (in the 't Hooft limit)

$$\chi(q) = q^1\frac{1}{(1-q)^2}\prod_{s=2}^{N}\prod_{n=s}^{\infty}\frac{1}{(1-q^n)} = q^1(1+2q+4q^2+\cdots). \tag{5.19}$$

Let us first consider the case $(\mathrm{f};0)\otimes(0;\bar{\mathrm{f}}) = (\bar{\mathrm{f}};\mathrm{f})$, in which case we should expect (5.19) to agree with the character of ($\bar{\mathrm{f}}$; f). For $\rho=\bar{\mathrm{f}}$, $\nu=\mathrm{f}$, the level $k=1$ representation is $\mu=[0,1,0^{N-3}]$. In this level $k=1$ affine representation we then have to look for those representations S of the finite-dimesional $\mathfrak{su}(N)$ algebra that have the property that

$$\bar{\mathrm{f}}\otimes S \supset \mathrm{f}. \tag{5.20}$$

Among the representations S that appear in the affine level $k=1$ representation of $\mu=[0,1,0^{N-3}]$, the only ones that satisfy (5.20) are

$$S=[0,1,0^{N-3}] \quad \text{and} \quad S=[2,0^{N-2}]. \tag{5.21}$$

Thus the leading behavior of the character $\chi_{(\bar{\mathrm{f}},\mathrm{f})}$ equals the sum of the branching functions of the level $k=1$ representation $\mu=[0,1,0^{N-3}]$ into $[0,1,0^{N-3}]$ and $[2,0^{N-2}]$, respectively. This can be read off from (C10)–(C12), and at least the first terms we have worked out reproduce precisely (5.19). Note that the conformal dimension of the primary of the ($\bar{\mathrm{f}}$, f) representation is

$$h(\bar{\mathrm{f}};\mathrm{f}) = \frac{C_N(\bar{\mathrm{f}})}{N+k} + \frac{C_N([0,1,0^{N-3}])}{N+1} - \frac{C_N(\mathrm{f})}{N+k+1} = \frac{N^2-1}{2N}\frac{1}{(N+k)(N+k+1)} + \frac{1}{N+1}\frac{(N-2)(N+1)}{N} \simeq 1 \tag{5.22}$$

in the 't Hooft limit, thus accounting also correctly for the q^1 leading power.

6. *Fusion product of* (f; 0) *and* (0; f)

For the case where we consider instead the fusion product

$$(\mathrm{f};0)\otimes(0;\mathrm{f}) = (\mathrm{f};\mathrm{f}) \tag{5.23}$$

the gravity calculation is identical. However, now the CFT character is different. For (f; f), we have $\mu=0$, and we have to look for the multiplicities with which $S=[0^{N-1}]$ and $S=[1,0^{N-3},1]$ appear in the decomposition of the level $k=1$ vacuum representation. Actually, by the argument leading to (5.17), the latter contribution corresponds precisely to (5.19), and thus we have

$$\chi_{(\mathrm{f};\mathrm{f})}(q) = \chi_{(0;0)}(q) + q^1(1+2q+4q^2+\cdots). \tag{5.24}$$

The fact that the limit character decomposes in this manner suggests that the underlying representation (f; f) becomes reducible in this limit. Actually, this phenomenon is familiar from the $k \rightarrow \infty$ limit of the $N = 2$ unitary minimal models, see [43] [Remark 4.1.7]. What it means is that in the limit $k \rightarrow \infty$, the representation (f; f) contains a "null-vector" that generates the subrepresentation corresponding to the second sum in (5.24). A natural way to deal with these additional null-vectors was proposed in [43] (see also [51]), where it was referred to as "scaling up the additional null-vectors." It amounts to rescaling the states in such a way that *only* the descendants of the null-vectors, i.e. the second sum in (5.24) survives in the limit.[9] This is precisely what the gravity calculation also seems to require. We therefore find again perfect agreement between the CFT and the gravity calculation.

B. The RG flow

As mentioned in Sec. II E, the minimal models have an RG flow relating two nearby theories (labeled by p and $p-1$). The operator responsible for the flow is the least relevant operator (0; adj). In the 't Hooft limit we saw in (3.5) that its dimension becomes $h = 1 - \lambda$. Combining with the similar operator for the right mover we have a relevant operator for the full CFT. In the 't Hooft limit, the RG flow going from p to $p-1$ changes the 't Hooft coupling as

$$\delta\lambda = \frac{\lambda^2}{N}. \tag{5.25}$$

Though the 't Hooft coupling only changes infinitesimally in the large N limit, there is nevertheless an order one change in the central charge

$$\delta c = -2\lambda^3. \tag{5.26}$$

This indicates that one should be able to see a reflection of this RG flow in the bulk as well.

As we have seen above, see (2.25), the field (0; adj) is actually the normal ordered product of the two fields (0; f) and $(0; \bar{\mathrm{f}})$. Writing $\mathcal{O} = (0; \mathrm{f})$, and using that $(0; \bar{\mathrm{f}})$ is the conjugate of (0; f), the full perturbation is of the form

$$S_{\mathrm{pert}} = g \int d^2z \mathcal{O}\mathcal{O}^\dagger, \tag{5.27}$$

where both $\mathcal{O}$ and $\mathcal{O}^\dagger$ have conformal dimension $\frac{1}{2}(1-\lambda, 1-\lambda)$ (see (3.4)). Thus the perturbation is indeed by a "double trace" operator [52]. As is familiar from general AdS/CFT considerations, see in particular [29], it corresponds to a flow between two different bulk theories. The scalar field corresponding to $\mathcal{O}$ (with dimension Δ_-) is quantized in the $(-)$-quantization in the UV. Under the RG flow, it flows to a theory with the $(+)$-quantization in the IR where it corresponds to an irrelevant operator $\mathcal{O}'$ with dimension Δ_+. Here $\Delta_\pm$ are the two roots of the equation $M^2 = \Delta(\Delta - 2)$ for the mass of the bulk scalar field.

This now ties in perfectly with what we know of the corresponding RG flow in the 2d CFT. As was explained in Sec. II E, the RG flow takes the operator (0; adj) to (adj; 0), and indeed (0; f) to (f; 0) (as well as $(0; \bar{\mathrm{f}})$ to $(\bar{\mathrm{f}}; 0)$). Translated into the above language it follows that the operator $\mathcal{O}'$ can be identified with (f; 0) (and similarly for the conjugate field). This operator has conformal dimension $\frac{1}{2}(1+\lambda, 1+\lambda)$, see (3.4). In particular, we see that it is dual to the scalar field in the $(+)$-quantization, as expected. Note that this statement only holds in the 't Hooft limit, where we have the relation $h(0; \mathrm{f}) + h(\mathrm{f}; 0) = 1$ (and similarly $h(\bar{\mathrm{f}}; 0) + h(0; \bar{\mathrm{f}}) = 1$).

Thus the two complex scalar fields in the bulk with mass $M^2 = -(1-\lambda^2)$, where one is in the $(+)$-quantization and the other in the $(-)$-quantization, fit exactly with what one expects from general consideration of RG flows in AdS/CFT. Furthermore, the picture ties in perfectly with what is known about the flow in the dual CFT.

VI. TOWARDS A DERIVATION

Finally, we want to sketch a possible way in which one can at least heuristically establish the relation between the bulk theory of higher spins in AdS_3, and the dual coset models studied in Secs. II and III. Our starting point will be the Chern-Simons description of the higher spin theory which was mentioned in Sec. IV. Let us consider, for definiteness, the Lorentzian signature theory with gauge group $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$. As in Sec. IV we denote the corresponding gauge fields by A and $\tilde{A}$. We will be interested in taking the large N limit eventually (to consistently couple with matter) but for now we will take N to be finite for simplicity.

In describing the bulk gravity (or higher spin theories) in a Chern-Simons formulation it is absolutely crucial to specify the boundary conditions properly. Since there are no propagating modes in the bulk, all the dynamics essentially arises from the boundary conditions. For the case of pure gravity on AdS_3 (corresponding to $N = 2$) this has been carefully studied over the years, and there is a reasonably straightforward generalization for any value of N [20,21]. We will mainly follow the very clear presentation by [33], and refer to this paper as well as [20,21] for more details as well as references to the original literature.

We will work in coordinates where the global AdS_3 metric reads as

$$ds^2 = \ell^2\left(1 + \frac{r^2}{\ell^2}\right)dt^2 - \left(1 + \frac{r^2}{\ell^2}\right)^{-1}dr^2 - r^2 d\phi^2. \tag{6.1}$$

The boundary is a cylinder parametrized by t, ϕ or more naturally $w = t - \phi$, $\tilde{w} = t + \phi$. To have a well-defined variational principle for the Chern-Simons action (4.2), we need to either add a boundary term, or specify suitable boundary conditions for the gauge fields. The natural

[9]In addition, there will be an overall infinite normalization factor, reflecting the volume divergence of the gravity calculation (that has been dropped in these 1-loop calculations).

choice of boundary conditions on the boundary cylinder, which obviates the need for an additional boundary term is (see [21,33])

$$A_{\bar{w}} = 0, \qquad \tilde{A}_w = 0. \tag{6.2}$$

In other words, the gauge fields $A, \tilde{A}$ have only left-moving and right-moving components at the boundary, respectively. It also suggests (using Eqs. (A5) and (A6)) that the gauge fields are effectively in SU(N) at the boundary [33].

At this stage one is tempted to view the boundary dynamics as that of two chiral WZW theories with gauge group SU(N). However, the above boundary conditions are not complete, since they do not guarantee that the geometry is asymptotically AdS_3, i.e. they do not yet include the analogue of the Brown-Henneaux boundary conditions. In the case of pure gravity the relevant boundary condition for the gauge fields in the Chern-Simons formulation was first worked out in [53]. This was recently generalized to any N in [21]. Roughly speaking, the additional condition removes all components of the gauge field, except for the lowest-spin components in the decomposition of the algebra with respect to some principal $\mathfrak{sl}(2, \mathbb{R})$ embedding, see Eq. (54) of [33] and Sec. 4.2 of [21]. As a consequence the WZW model is gauged, and the resulting theory describes the (classical) Drinfeld-Sokolov reduction [21]. This is what is responsible for reducing the affine Kac-Moody algebra to a $\mathcal{W}_N$-algebra, and by this route the classical $\mathcal{W}$-algebra for the asymptotic symmetry generators was established [20,21].

At the quantum level, we propose that the analogous statement should involve the *quantum* Drinfeld-Sokolov reduction of the affine $\mathfrak{su}(N)$ algebra. The quantum mechanical treatment of the DS reduction is more involved (and quite different) from the classical reduction. In particular, it necessitates the introduction of ghosts to take care of the constraints (gauging). A full BRST analysis was carried out in [37], and we summarize some of the results in Appendix B. What is important for us is the observation [17,19,34,35,37] that the CFT at the quantum level is equivalent precisely to the coset theory

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}. \tag{6.3}$$

It is important to note here that the level k of the coset theory is not the same as that of the quantum DS reduced theory (and therefore of the original Chern-Simons theory), but rather that given in (B2).

Obviously, the discussion so far only involves the higher spin degrees of freedom. Our proposal also suggests that we have to add two massive complex scalar fields (of the same mass) to this theory in order to identify it with the full dual CFT. It would be very interesting to understand these scalar fields from the point of view of the Chern-Simons theory. This could then also open the way to a more conceptual understanding of the duality.

VII. FINAL REMARKS

In this paper we have made a proposal for a duality between a family of higher spin theories on AdS_3, and a 't Hooft like limit of a family of 2d CFTs. More specifically, we have argued that the 't Hooft limit of the $\mathcal{W}_N$ minimal models is dual to the higher spin theory on AdS_3, where one adds to the massless higher spin fields two massive complex scalars. Unlike the massless higher spin fields, these massive scalars actually have propagating degrees of freedom. There is a free parameter on either side, namely, the 't Hooft coupling λ of the 2d CFT, and the mass parameter M of the massive scalars, and they are directly related to one another, see Eq. (4.9). We have checked this proposal by matching the spectra in quite some detail. We have also shown that the RG flow of the 2d CFT, relating the different minimal models to one another, agrees very nicely with the gravity description.

Our proposal is in some sense the natural 3d analogue of the 4d higher spin conjecture of Klebanov & Polyakov [7]. Indeed, at the free point, $\lambda = 0$, the 2d CFT is described by some sort of singlet sector of free fermions transforming in the fundamental and antifundamental representation, and is thus the natural lower dimensional analogue of the O(N) vector model of Klebanov and Polyakov. However, unlike their case where the duality is only defined for two special CFTs, we actually have a full 1-parameter family of conformal fixed points for which we can identify the dual higher spin theory. Furthermore, our 2d CFTs are limits of consistent $\mathcal{W}_N$ minimal models, and there is no need for a projection to a singlet sector. Indeed, the coset construction seems to take care of this automatically.

The family of theories we consider interpolate between the free theory at $\lambda = 0$, and the $\lambda = 1$ theory. The latter, from the point of view of the bulk theory, corresponds to the higher spin algebra with massless scalars, $\Delta = M^2 = 0$. Indeed, nonzero Δ plays the role of α' corrections as already mentioned in the introduction. It would be good to see in detail how the interpolation between these two limits via the general interacting theories takes place. From the field theory point of view it would be nice to understand the interacting theories from a Landau-Ginzburg picture as in the case of $N = 2$.[10] We note that the RG flow between the $\mathcal{W}_N$ minimal models is integrable [41]. Thus, one should be able to study the bulk interpretation of the massive two dimensional field theories which are in some sense deformed Gross-Neveu models.

The $\mathcal{W}_N$ minimal models that appear on the CFT side are nonsupersymmetric conformal field theories that are believed to describe the multicritical behavior of $\mathbb{Z}_N$

[10]We thank Shiraz Minwalla for a stimulating discussion on this topic.

symmetric statistical systems [16]. Our proposed duality could therefore also lead to new insights into such statistical systems.

There are a number of open problems that deserve further study. First of all, it would be good to establish the matching of the spectra to all orders, i.e. to complete the analysis of Sec. V. On the bulk side, it is important to write down the interactions involving the massive scalar fields in the Chern-Simons formulation which appears not to have been done in the higher spin literature. This would also be necessary in order to flesh out the arguments of Sec. VI and thus understand the underlying mechanism of the duality. The close connection between the $\mathcal{W}_\infty$ algebra and the algebra of area-preserving diffeomorphisms of 2-surfaces [11] could be important in this context. One could also compute correlation functions in the CFT (in the planar limit) and compare with the bulk computation using, for instance, the techniques of [5,8,9]. As far as we are aware, the correlation functions for these $\mathcal{W}_N$ models have not yet been worked out for general N, and it would be important to understand their large N limit. It is conceivable that the bulk description might give an alternative, less tedious way of determining them (at least, in the planar limit), thus making the bulk description useful from a practical point of view in 2d CFTs! There are also some natural generalizations one can envisage, in particular, to cases involving fermions and/or supersymmetry, as well as to cosets of O(N) and Sp(N), rather than SU(N).

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APPENDIX A: HIGHER SPIN THEORIES ON AdS_3

Let us first recall some basic features of massless higher spin theories at the noninteracting level [54,55] (see for example [2,56] for reviews and more references).

The massless spin s fields in three dimensions are completely symmetric tensors $\varphi_{\mu_1\mu_2\cdots\mu_s}$ subject to a double trace constraint

$$\varphi_{\mu_5\cdots\mu_s\alpha\lambda}{}^{\alpha\lambda} = 0. \tag{A1}$$

This constraint only makes sense if $s \geq 4$. In, addition we have a gauge invariance leading to the identification of field configurations

$$\varphi_{\mu_1\mu_2\cdots\mu_s} \sim \varphi_{\mu_1\mu_2\cdots\mu_s} + \nabla_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}. \tag{A2}$$

The gauge parameter $\xi_{\mu_2\cdots\mu_s}$ is a symmetric tensor of rank $(s-1)$ which is, in addition, traceless, *i.e.* $\xi^{\lambda}_{\mu_3\cdots\mu_s\lambda} = 0$. This last constraint only makes sense for $s \geq 3$.

In higher dimensional AdS spacetimes we need to have an infinite tower of these fields to obtain classically consistent interacting theories. It is a special property of AdS_3 [46] that, for every $N \geq 2$, we can have consistent (again classical) truncations to theories which have a spectrum containing a single massless field for each spin $s = 2, \ldots, N$.

While it is possible to write down an action for these theories in terms of the Fronsdal fields given above, it is much simpler to recast it in terms of Chern-Simons gauge fields [47]. This generalizes the observation of [31,32] for the case of pure gravity, i.e. $N = 2$, for which the Einstein-Hilbert action can be reexpressed in terms of an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (or $SL(2,\mathbb{C})$ in Euclidean signature) Chern-Simons theory. In the case of maximal spin N the higher spin theory has an $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ (or $SL(N,\mathbb{C})$ in Euclidean signature) Chern-Simons description. Thus the theory is labeled by two parameters, N and the AdS radius ℓ.

Many of the properties of the Chern-Simons description have been reviewed and studied in detail in [21]. We just summarize a few of the central points. One describes the higher spin fields in terms of generalized vielbein and connection variables

$$e_\mu^{a_1\cdots a_{s-1}} \qquad \omega_\mu^{a_1\cdots a_{s-1}}, \tag{A3}$$

and relates these to the Fronsdal fields (about an AdS background) via the relation (at the linearized level)

$$\varphi_{\mu_1\mu_2\cdots\mu_s} = \frac{1}{s}\bar{e}^{a_1}_{(\mu_1}\cdots\bar{e}^{a_{s-1}}_{\mu_{s-1}}e_{\mu_s)a_1\cdots a_{s-1}}. \tag{A4}$$

Here $\bar{e}^a_\mu$ is the (usual) vielbein for the background AdS_3 metric. In addition, to the diffeomorphism invariance, the generalized vielbeins and connections transform under local ‘‘frame rotations’’ which are parametrized by a gauge parameter $\Lambda^{ba_1\cdots a_{s-1}}$.

For the Chern-Simons formulation, one considers the combinations

$$j_\mu^{a_1\cdots a_{s-1}} = \left(\omega + \frac{e}{\ell}\right)_\mu^{a_1\cdots a_{s-1}}, \qquad \tilde{j}_\mu^{a_1\cdots a_{s-1}} = \left(\omega - \frac{e}{\ell}\right)_\mu^{a_1\cdots a_{s-1}}, \tag{A5}$$

and then defines the gauge potentials as

$$A = (j^a_\mu T_a + \cdots + j^{a_1\cdots a_{N-1}}_\mu T_{a_1\cdots a_{N-1}})dx^\mu$$
$$\tilde{A} = (\tilde{j}^a_\mu T_a + \cdots + \tilde{j}^{a_1\cdots a_{N-1}}_\mu T_{a_1\cdots a_{N-1}})dx^\mu. \tag{A6}$$

Here the T_a are generators of $SL(2, \mathbb{R})$, while

$$T_{a_1\cdots a_{s-1}} \sim T_{(a_1} \cdots T_{a_{s-1})}. \tag{A7}$$

We can thus view the A, $\tilde{A}$ as $SL(N, \mathbb{R})$ gauge fields. The action of these higher spin fields is given by (4.2) together with (4.3).

Note that in this description in terms of gauge fields, the "physical" Fronsdal fields are actually singlets under the diagonal $SL(N, \mathbb{R})$ gauge group. This generalizes the well-known fact that the metric field is a singlet under local Lorentz frame rotations (which are the diagonal $SL(2, \mathbb{R})$ transformations). However, all observables in the bulk higher spin theory should be singlets under the $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ (or $SL(N, \mathbb{C})$) gauge transformations.

While (A4) gives the relation between the Fronsdal fields and the frame fields (gauge fields) at the linearized level, there is a generalization to the full nonlinear theory as well—see Sec. 4.3 of [21]. It was observed there that the Fronsdal fields (for the case of $N = 3$) are simply expressed in terms of the Casimir generators of $SL(3, \mathbb{R})$. This is expected to generalize to the case of arbitrary N [21]. This also fits in with the present proposal in which the vacuum sector of the CFT, which contains the Casimir algebra of $SU(N)$, describes the pure higher spin field excitations.

APPENDIX B: THE DRINFELD-SOKOLOV DESCRIPTION

In the Drinfeld-Sokolov description of the $\mathcal{W}_N$ theories one starts with some WZW model, and then reduces the theory by imposing suitable constraints, see e.g. [19] for a review. These constrained WZW models also give a description of Toda theories [36] that were known to be closely related to the $\mathcal{W}_N$ models [17]. In the case of interest to us, the WZW model is $SU(N)$ at level k_{DS}, and in the quantum version the resulting theory has central charge

$$c_N(k_{DS}) = (N-1)\left[1 - N(N+1)\frac{(k_{DS}+N-1)^2}{(k_{DS}+N)}\right]. \tag{B1}$$

For large k_{DS} the central charge goes as $c_N(k_{DS}) \simeq -k_{DS}N(N^2-1)$; for $N = 2$ this reduces to the relation $c_2(k_{DS}) \simeq -6k_{DS}$ of [33], where it is also argued that k_{DS} should be chosen to be negative. In order to relate the Drinfeld-Sokolov construction to the coset construction described in section II, we have to identify

$$\frac{1}{p} = \frac{1}{k+N} = \frac{1}{k_{DS}+N} - 1. \tag{B2}$$

Then (B1) agrees with (2.2).

In the Drinfeld-Sokolov description the highest weight representations are labeled by $(\Lambda^+, \Lambda^-) \cong (\rho; \nu)$—we are using the notation of [19]—and the conformal weight equals (see Eqs. (6.74) and (7.53) of [19])

$$h(\Lambda^+, \Lambda^-) = \frac{c_N}{24} - \frac{(N-1)}{24} + \frac{1}{2p(p+1)}\Big|(p+1)(\Lambda^+ + \hat{\rho}) - p(\Lambda^- + \hat{\rho})\Big|^2, \tag{B3}$$

where the central charge c_N is given by (B1), with the relation between p, k and k_{DS} being determined by (B2). Furthermore, $\hat{\rho}$ is the Weyl vector of $\mathfrak{su}(N)$, i.e. one half the sum of all positive roots, whose square equals $\hat{\rho}^2 = \frac{N(N^2-1)}{12}$. This then leads to (2.10).

1. Examples and Casimirs

It is not difficult to check that the two formulas (2.9) and (2.10) actually agree in simple cases. For example, both give $h = 0$ for the vacuum representation with $(\rho; \nu) = (0; 0)$ (and $n = 0$). A more interesting case arises if either ρ $(= \Lambda^+)$ or ν $(= \Lambda^-)$ vanish. In that case, (2.10) becomes

$$h(0, \Lambda^-) = \frac{1}{2(p+1)}(p(\Lambda^-)^2 - 2(\Lambda^-, \hat{\rho})) \tag{B4}$$

or

$$h(\Lambda^+, 0) = \frac{1}{2p}((p+1)(\Lambda^+)^2 + 2(\Lambda^+, \hat{\rho})). \tag{B5}$$

Specializing further to the case that $\Lambda^\pm$ equals the fundamental (f) or adjoint (adj) representation, we then obtain

$$h(0, \mathrm{f}) = \frac{(N-1)}{2N}\left(1 - \frac{N+1}{p+1}\right), \qquad h(\mathrm{f}, 0) = \frac{(N-1)}{2N}\left(1 + \frac{N+1}{p}\right), \tag{B6}$$

and

$$h(0, \mathrm{adj}) = 1 - \frac{N}{p+1}, \qquad h(\mathrm{adj}, 0) = 1 + \frac{N}{p}. \tag{B7}$$

Here we have used that the inverse Cartan matrix $C^{-1}_{ij} = C^{-1}_{ji}$ for $\mathfrak{su}(N)$ equals

$$C^{-1}_{ij} = \frac{i(N-j)}{N} \quad \text{for} \quad (i \le j), \qquad \text{and} \quad \sum_{i=1}^{N-1} C^{-1}_{ij} = \frac{j}{2}(N-j), \tag{B8}$$

from which it follows that

$$(\Lambda_{\mathrm{f}})^2 = \frac{(N-1)}{N}, \qquad (\Lambda_{\mathrm{f}}, \hat{\rho}) = \frac{(N-1)}{2}, \quad \text{and} \quad (\Lambda_{\mathrm{adj}})^2 = 2, \qquad (\Lambda_{\mathrm{adj}}, \hat{\rho}) = (N-1). \tag{B9}$$

In our conventions the quadratic Casimir is defined to be

$$C_N(\Lambda) = \frac{1}{2}[(\Lambda, \Lambda) + 2(\Lambda, \hat{\rho})] = \sum_{i<j} \Lambda_i \Lambda_j \frac{i(N-j)}{N} + \frac{1}{2}\sum_{j=1}^{N-1} \Lambda_j^2 \frac{j(N-j)}{N} + \sum_{j=1}^{N-1} \Lambda_j \frac{j}{2}(N-j), \tag{B10}$$

where Λ_j are the Dynkin labels of the weight Λ. For weights that appear in finite powers of the fundamental representation—these are the weights with a finite number of boxes in the Young tableau—the leading term in the large N limit is

$$C_N(\Lambda) \simeq \frac{N}{2}\sum_{j=1}^{N-1} j\Lambda_j = \frac{N}{2} B(\Lambda), \tag{B11}$$

where $B(\Lambda)$ denotes the number of boxes in the Young tableau of Λ.

APPENDIX C: BRANCHING FUNCTIONS

In order to determine the low-lying terms of the coset characters in the large k limit, we have to determine the decomposition of level $k = 1$ affine representations in terms of representations of the horizontal (zero-mode) algebra. Since the zero modes commute with L_0, we can do this separately level by level. This is to say, we decompose the affine level $k = 1$ representation ν in terms of L_0 eigenspaces as

$$\mathcal{H}_\nu = \bigoplus_{n=0}^{\infty} \mathcal{H}_\nu^{(n)}, \tag{C1}$$

and then decompose each $\mathcal{H}_\nu^{(n)}$ under the action of the zero modes. We have performed this analysis for the first few values of n and some small representations (assuming that N is sufficiently large—for the following $N \geq 5$ will suffice).[11] Explicitly we find

$$\mathcal{H}_{[0^{N-1}]}^{(n=0)} = [0^{N-1}] \tag{C2}$$

$$\mathcal{H}_{[0^{N-1}]}^{(n=1)} = [1, 0^{N-3}, 1] \tag{C3}$$

$$\mathcal{H}_{[0^{N-1}]}^{(n=2)} = [0, 1, 0^{N-5}, 1, 0] \oplus 2 \cdot [1, 0^{N-3}, 1] \oplus [0^{N-1}] \tag{C4}$$

$$\mathcal{H}_{[0^{N-1}]}^{(n=3)} = [2, 0^{N-4}, 1, 0] \oplus [0, 1, 0^{N-4}, 2] \oplus 2 \cdot [0, 1, 0^{N-5}, 1, 0] \oplus 4 \cdot [1, 0^{N-3}, 1] \oplus 2 \cdot [0^{N-1}], \tag{C5}$$

$$\mathcal{H}_{\rm f}^{(n=0)} = [1, 0^{N-2}] \tag{C6}$$

$$\mathcal{H}_{\rm f}^{(n=1)} = [0, 1, 0^{N-4}, 1] \oplus [1, 0^{N-2}] \tag{C7}$$

$$\mathcal{H}_{\rm f}^{(n=2)} = [2, 0^{N-3}, 1] \oplus [0^{N-3}, 2, 0] \oplus 2 \cdot [0, 1, 0^{N-4}, 1] \oplus 2 \cdot [1, 0^{N-2}] \tag{C8}$$

$$\mathcal{H}_{\rm f}^{(n=3)} = [1, 1, 1, 0^{N-4}] \oplus [0^{N-3}, 1, 2] \oplus 2 \cdot [2, 0^{N-3}, 1] \oplus [0^{N-3}, 2, 0] \oplus 5 \cdot [0, 1, 0^{N-4}, 1] \oplus 4 \cdot [1, 0^{N-2}], \tag{C9}$$

and

$$\mathcal{H}_{[0,1,0^{N-3}]}^{(n=0)} = [0, 1, 0^{N-3}] \tag{C10}$$

$$\mathcal{H}_{[0,1,0^{N-3}]}^{(n=1)} = [2, 0^{N-2}] \oplus [0^{N-3}, 1, 1] \oplus [0, 1, 0^{N-3}] \tag{C11}$$

$$\mathcal{H}_{[0,1,0^{N-3}]}^{(n=2)} = [1, 1, 0^{N-4}, 1] \oplus [2, 0^{N-2}] \oplus 2 \cdot [0^{N-3}, 1, 1] \oplus 3 \cdot [0, 1, 0^{N-3}]. \tag{C12}$$

[11] We thank Roberto Volpato for helping us check these identities.

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