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Constants of motion for constrained Hamiltonian systems: A particle around a charged rotating black hole

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We discuss constants of motion of a particle under an external field in a curved spacetime, taking into account the Hamiltonian constraint, which arises from the reparametrization invariance of the particle orbit. As the necessary and sufficient condition for the existence of a constant of motion, we obtain a set of equations with a hierarchical structure, which is understood as a generalization of the Killing tensor equation. It is also a generalization of the conventional argument in that it includes the case when the conservation condition holds only on the constraint surface in the phase space. In that case, it is shown that the constant of motion is associated with a conformal Killing tensor. We apply the hierarchical equations and find constants of motion in the case of a charged particle in an electromagnetic field in black hole spacetimes. We also demonstrate that gravitational and electromagnetic fields exist in which a charged particle has a constant of motion associated with a conformal Killing tensor.

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I. INTRODUCTION

Black holes are of great importance in modern physics. In astrophysics, black holes are thought to be the central engines of active galactic nuclei. The gravitational and electromagnetic fields of black holes play crucial roles for the production of the energy actually observed. On the other hand, higher-dimensional black holes, in recent years, have been given much attention in the context of unified theories of interactions. An important task at present is to reveal their properties because the existence of the extra dimensions could be verified by observations of the phenomena concerned with black holes.

A test particle is an important probe of black hole spacetimes because the study of the motions of a test particle provides an important insight into the physical properties of black holes. In the case of a charged test particle, the motion provides information on both the gravitational and electromagnetic fields. The constants of motion, i.e., the conserved quantities along a particle trajectory, are useful for the analysis of the motion of the particle. In particular, one can conclude that the system is integrable by finding a sufficient number of constants of motion that commute with each other under the Poisson bracket.

In the case of a particle without charge when the trajectory is a geodesic on a curved spacetime, the existence of a one-parameter group of isometries generated by a Killing vector implies the existence of a constant of motion that is linear in the momentum. The constants of motion that are nonlinear in the momentum arise from Killing tensors. For example, in the Kerr spacetime, there exists a constant of motion quadratic in the momentum [1] that arises from a Killing tensor of rank 2 [2]. Furthermore, the existence of a rank-2 Killing tensor was recently shown in the Kerr-Newman-Unti-Tamburino-de Sitter black holes in any dimensionality [3–8].

A constant of motion that is quadratic in the momentum for a charged particle in the Kerr-Newman spacetime was also found [1] by the method of the Hamilton-Jacobi equation. It was shown that the constant of motion is also related to a rank-2 Killing tensor in this case [9,10], and a set of coupled equations is obtained that should be satisfied by the constants of motion for the charged particle [10] (see also [11] for recent works).

The motion of a test particle in a curved spacetime is described by a world line with arbitrary parametrization. This reparametrization invariance gives rise to the Hamiltonian constraint. In this paper, we discuss a generalization of the conservation condition for the system of a particle that is subject to external fields in the Hamiltonian formalism. We consider the conservation condition with the constraint taken into account. Namely, we require that the conservation equation hold under the constraint condition. As a result, we obtain a generalized set of equations, which is the necessary and sufficient condition for the existence of a constant of motion for a particle in an external field. The equations have hierarchical structure, and the topmost equation in the hierarchy is the conformal Killing tensor equation.

As applications, we first consider systems of a charged particle in electromagnetic fields around black holes, namely, a test Maxwell field on the Kerr spacetime, the Kerr-Newman spacetime, and a five-dimensional charged rotating black hole. In these cases, the Hamiltonian constraint does not play any role in finding constants of motion since the metrics admit rank-2 Killing tensors. In the final

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example, we demonstrate that a constant of motion can exist that is related to a conservation condition holding only on the constraint surface. The example is constructed by conformal transformation of a spacetime with a test Maxwell field.

This paper is organized as follows. In the following section, we review the relation between the constants of motion of a free particle and geometrical quantities by using the Hamiltonian formalism. In Sec. III, we formulate the condition for the existence of a constant of motion for a particle in an external field when there is a constraint condition. We obtain a set of equations with a hierarchical structure as a result. The equations are applied to systems of a charged particle in Sec. IV. Finally, Sec. V is devoted to a summary.

II. CONSERVED QUANTITIES OF A FREE PARTICLE IN THE HAMILTONIAN FORMALISM

In this section, we review the relation between geometrical quantities of a curved spacetime $(\mathcal{M}, g_{\mu\nu})$ and conserved quantities of a free particle in $(\mathcal{M}, g_{\mu\nu})$. It is well known that the solutions of the Killing equation, Killing fields, give conserved quantities along the trajectory of the particle, which is a geodesic. We shall derive the relation by using the Hamiltonian formalism. The relation will be generalized in the following section.

Let H be the Hamiltonian of a free particle given by

$$H = \frac{1}{2m}(g^{\mu\nu} p_\mu p_\nu + m^2), \tag{1}$$

where m is the mass and p_μ is the canonical momentum of the particle. The Hamilton equation for (1) leads to the geodesic equation. Let F be a dynamical quantity of the free particle represented by a function on the phase space with coordinates (x^μ, p_ν). In the Hamiltonian formalism, if F is a constant of motion, i.e., a conserved quantity along the orbit of the particle, it commutes with H under the Poisson bracket,

$$\frac{dF}{d\tau} = \{F, H\}_{\mathrm{PB}} := \frac{\partial F}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} - \frac{\partial H}{\partial x^\mu}\frac{\partial F}{\partial p_\mu} = 0, \tag{2}$$

where the bracket with PB denotes the Poisson bracket, and τ is the proper time of the particle.

Here, we assume that F is written in the form

$$F(x^\mu, p_\mu) = \xi^\mu p_\mu, \tag{3}$$

where ξ^μ is a vector field on $\mathcal{M}$. For F in the form of (3), Eq. (2) takes the form

$$\{F, H\}_{\mathrm{PB}} = \frac{1}{m}\xi^{\mu;\nu} p_\mu p_\nu = 0, \tag{4}$$

where the semicolon denotes the covariant derivative. Hence, we have the *Killing equation*

$$\xi^{\mu;\nu} = 0. \tag{5}$$

A solution ξ^μ of the equation, a *Killing vector*, gives a constant of motion F.

We can generalize the Killing equation to a higher-rank tensor equation. Let us assume that F has the form of

$$F = \overset{(k)}{K}{}^{\mu_1\cdots\mu_k} p_{\mu_1}\cdots p_{\mu_k}, \tag{6}$$

where $\overset{(k)}{K}$ denotes a completely symmetric tensor field of rank k. Then Eq. (2) yields the Killing tensor equation for the rank-k tensor

$$\overset{(k)}{K}{}^{(\mu_1\cdots\mu_k;\mu_{k+1})} = 0. \tag{7}$$

That is, Killing tensors, which are geometrical quantities, are related with constants of motion.

III. FORMULATION OF GENERALIZED KILLING EQUATIONS

In this section, we generalize the Killing tensor equation derived in the previous section to include the case when the conservation condition holds only on the constraint surface in the phase space.

Let us first derive the Hamiltonian for a free particle. The geodesic is obtained from the variational principle with the action being the arc length of a curve connecting two points,

$$S = -m\int_C \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda. \tag{8}$$

The action S has the reparametrization invariance $\lambda \longrightarrow \lambda'$. Introducing a Lagrange multiplier $N(\lambda)$, we can construct an equivalent action in the quadratic form,

$$S = \int_C \left(\frac{m}{2} g_{\mu\nu}\frac{dx^\mu}{Nd\lambda}\frac{dx^\nu}{Nd\lambda} - \frac{m}{2}\right) Nd\lambda. \tag{9}$$

By the standard procedure of moving from the Lagrange formalism to the Hamilton formalism, we can derive the Hamiltonian,

$$H = \frac{N}{2m}(g^{\mu\nu} p_\mu p_\nu + m^2). \tag{10}$$

Variation by N yields the mass shell condition or the Hamiltonian constraint,

$$\mathcal{H}(x^\mu, p_\mu) = g^{\mu\nu} p_\mu p_\nu + m^2 \approx 0. \tag{11}$$

The last equality, *weak equality*, defines a constraint surface in the phase space where the actual particle motion is confined on the hypersurface. The action (1) is restricted to be used with an affine parameter, while an arbitrary parameter is allowed in (9).

Let us generalize the Killing tensor equation, taking into account the case when the conservation condition holds only on the constraint surface. We begin with a Hamiltonian slightly more general than (10),

$$H = \frac{N}{2m}(g^{\mu\nu}p_\mu p_\nu + B^\rho p_\rho + V), \tag{12}$$

to describe a system of a particle in an external field, where B^ρ and V are a vector field and a scalar field, respectively, on $\mathcal{M}$. By setting the differentiation of H by N equal to zero, we get the Hamiltonian constraint equation

$$\mathcal{H} = g^{\mu\nu}p_\mu p_\nu + B^\rho p_\rho + V \approx 0. \tag{13}$$

We shall examine the conservation condition for the function $F(x^\mu, p_\mu)$ on the constrained system. Since the particle motion is realized only on the constraint surface, then it suffices that F commutes with H only on the constraint surface, i.e.,

$$\{F, H\}_{\rm PB} = \frac{N}{2m}\{F, \mathcal{H}\}_{\rm PB} \approx 0. \tag{14}$$

This is equivalent to

$$\{F, \mathcal{H}\}_{\rm PB} + \phi\mathcal{H} = 0, \tag{15}$$

where ϕ is an arbitrary function on the phase space.

We assume that F and ϕ are expanded in the form,

$$F = \sum_k \overset{(k)}{K}{}^{\mu_1\cdots\mu_k} p_{\mu_1}\cdots p_{\mu_k} =: \sum_k \overset{(k)}{K}\cdot p^k, \tag{16}$$

$$\phi = \sum_l \overset{(l)}{\lambda}{}^{\mu_1\cdots\mu_l} p_{\mu_1}\cdots p_{\mu_l} =: \sum_l \overset{(l)}{\lambda}\cdot p^l, \tag{17}$$

where $\overset{(k)}{K}{}^{\mu_1\cdots\mu_k}$ and $\overset{(l)}{\lambda}{}^{\mu_1\cdots\mu_l}$ are symmetric tensor fields of rank k and rank l, respectively, on $\mathcal{M}$. The right-hand sides of (16) and (17) are abbreviations of contraction with p's. Substituting these expressions into (15), we have

$$\sum_k \Big(-[\overset{(k-1)}{K}, \overset{(2)}{g}]_{\rm S} - [\overset{(k)}{K}, \overset{(1)}{B}]_{\rm S} - [\overset{(k+1)}{K}, \overset{(0)}{V}]_{\rm S} + \overset{(k-2)}{\lambda}\overset{\rm S}{\otimes}\overset{(2)}{g} + \overset{(k-1)}{\lambda}\overset{\rm S}{\otimes}\overset{(1)}{B} + \overset{(k)}{\lambda}\overset{(0)}{V}\Big)\cdot p^k = 0, \tag{18}$$

where we understand that $\overset{(k)}{K} = 0$ and $\overset{(k)}{\lambda} = 0$ for $k < 0$, and $\overset{\rm S}{\otimes}$ denotes the symmetric tensor product. The bracket with the subscript S is the Schouten bracket [12], which is the map from symmetric tensor fields $\overset{(k)}{X}$ and $\overset{(l)}{Y}$ of rank k and rank l, respectively, to a rank-$(k + l - 1)$ symmetric tensor field $[\overset{(k)}{X}, \overset{(l)}{Y}]_{\rm S}$ defined by

$$[\overset{(k)}{X}, \overset{(l)}{Y}]_{\rm S}\cdot p^{k+l-1} = -\{\overset{(k)}{X}\cdot p^k, \overset{(l)}{Y}\cdot p^l\}_{\rm PB}. \tag{19}$$

Since (18) should be satisfied for any $p^k = p_{\mu_1}p_{\mu_2}\cdots p_{\mu_k}$, then the each coefficient of p^k vanishes, i.e.,

$$[\overset{(k-1)}{K}, \overset{(2)}{g}]_{\rm S} + [\overset{(k)}{K}, \overset{(1)}{B}]_{\rm S} + [\overset{(k+1)}{K}, \overset{(0)}{V}]_{\rm S} - \overset{(k-2)}{\lambda}\overset{\rm S}{\otimes}\overset{(2)}{g} - \overset{(k-1)}{\lambda}\overset{\rm S}{\otimes}\overset{(1)}{B} - \overset{(k)}{\lambda}\overset{(0)}{V} = 0. \tag{20}$$

Let us call the set of equations (20) the *Killing hierarchy* because it is a generalization of the Killing equation.

When the highest rank of the hierarchy is N, namely, when $\overset{(l+1)}{K} = 0$ and $\overset{(l)}{\lambda} = 0$ for $l \geq N$, the Killing hierarchy reads

$$-[\overset{(N)}{K}, \overset{(2)}{g}]_{\rm S} + \overset{(N-1)}{\lambda}\overset{\rm S}{\otimes}\overset{(2)}{g} = 0, \tag{21}$$

$$-[\overset{(N-1)}{K}, \overset{(2)}{g}]_{\rm S} - [\overset{(N)}{K}, \overset{(1)}{B}]_{\rm S} + \overset{(N-2)}{\lambda}\overset{\rm S}{\otimes}\overset{(2)}{g} + \overset{(N-1)}{\lambda}\overset{\rm S}{\otimes}\overset{(1)}{B} = 0, \tag{22}$$

$$-[\overset{(k-1)}{K}, \overset{(2)}{g}]_{\rm S} - [\overset{(k)}{K}, \overset{(1)}{B}]_{\rm S} - [\overset{(k+1)}{K}, \overset{(0)}{V}]_{\rm S} + \overset{(k-2)}{\lambda}\overset{\rm S}{\otimes}\overset{(2)}{g} + \overset{(k-1)}{\lambda}\overset{\rm S}{\otimes}\overset{(1)}{B} + \overset{(k)}{\lambda}\overset{(0)}{V} = 0, \qquad 2 \leq k \leq N-1, \tag{23}$$

$$-[\overset{(0)}{K}, \overset{(2)}{g}]_{\rm S} - [\overset{(1)}{K}, \overset{(1)}{B}]_{\rm S} - [\overset{(2)}{K}, \overset{(0)}{V}]_{\rm S} + \overset{(0)}{\lambda}\overset{(1)}{B} + \overset{(1)}{\lambda}\overset{(0)}{V} = 0, \tag{24}$$

$$-[\overset{(0)}{K}, \overset{(1)}{B}]_{\rm S} - [\overset{(1)}{K}, \overset{(0)}{V}]_{\rm S} + \overset{(0)}{\lambda}\overset{(0)}{V} = 0, \tag{25}$$

where $\overset{(2)}{g}\cdot p^2$ is assumed to be nonvanishing. The structure of the Killing hierarchy tells us that we should solve them from the highest-rank equation (21). Since the highest-rank equation is the conformal Killing equation, the existence of a conformal Killing tensor is necessary for a constant of motion to exist. If the Killing hierarchy admits a nontrivial solution, then there exists a constant of motion associated with a conformal Killing tensor.

In the case of a free particle, i.e., $\overset{(1)}{B} = 0$ and $\overset{(0)}{V} = \text{const}$, the Killing hierarchy reduces to a set of decoupled conformal Killing equations if the particle is massless, $\overset{(0)}{V} = 0$, and to a set of decoupled Killing equations if the particle is massive, $\overset{(0)}{V} = m^2$. This fact is shown in Appendix A.

In the case of $\overset{(k)}{\lambda} = 0$ for all k the Killing hierarchy reduces to

$$-[\overset{(N)}{K},\overset{(2)}{g}]_S=0,$$
$$-[\overset{(N-1)}{K},\overset{(2)}{g}]_S-[\overset{(N)}{K},\overset{(1)}{B}]_S=0,$$
$$-[\overset{(k-1)}{K},\overset{(2)}{g}]_S-[\overset{(k)}{K},\overset{(1)}{B}]_S-[\overset{(k+1)}{K},\overset{(0)}{V}]_S=0,\quad 2\le k\le N-1,$$
$$-[\overset{(0)}{K},\overset{(2)}{g}]_S-[\overset{(1)}{K},\overset{(1)}{B}]_S-[\overset{(2)}{K},\overset{(0)}{V}]_S=0,$$
$$-[\overset{(0)}{K},\overset{(1)}{B}]_S-[\overset{(1)}{K},\overset{(0)}{V}]_S=0. \tag{26}$$

These equations were obtained by Sommers [10] and van Holten [11]. Several applications are found in [13].

IV. KILLING HIERARCHY FOR A CHARGED PARTICLE

In this section, we apply the Killing hierarchy to the systems of an electrically charged particle subject to an external electromagnetic field as an important application of our formalism. We consider the Hamiltonian of a charged particle in the form

$$H=\frac{N}{2m}[g^{\mu\nu}(p_\mu-qA_\mu)(p_\nu-qA_\nu)+m^2], \tag{27}$$

where m and q are the mass and the electric charge of the particle, respectively, and A_μ denotes the gauge potential. Substituting

$$B^\mu=-2qA^\mu,\qquad V=q^2A_\mu A^\mu+m^2 \tag{28}$$

in (21)–(25), we obtain the Killing hierarchy for a charged particle:

$$-[\overset{(N)}{K},g]_S+\overset{(N-1)}{\lambda}\overset{S}{\otimes}g=0, \tag{29}$$

$$-[\overset{(N-1)}{K},g]_S+2q[\overset{(N)}{K},A]_S+\overset{(N-2)}{\lambda}\overset{S}{\otimes}g-2q\overset{(N-1)}{\lambda}\overset{S}{\otimes}A=0, \tag{30}$$

$$-[\overset{(k-1)}{K},g]_S+2q[\overset{(k)}{K},A]_S-q^2[\overset{(k+1)}{K},A^2]_S \tag{31}$$

$$+\overset{(k-2)}{\lambda}\overset{S}{\otimes}g-2q(\overset{\lambda}{k-1})\overset{S}{\otimes}A+\overset{(k)}{\lambda}(q^2A^2+m^2)=0,$$
$$2\le k\le N-1, \tag{32}$$

$$-[\overset{(0)}{K},g]_S+2q[\overset{(1)}{K},A]_S-q^2[\overset{(2)}{K},A^2]_S-2q\overset{(0)}{\lambda}A$$
$$+\overset{(1)}{\lambda}(q^2A^2+m^2)=0, \tag{33}$$

$$2q[\overset{(0)}{K},A]_S-q^2[\overset{(1)}{K},A^2]_S+\overset{(0)}{\lambda}(q^2A^2+m^2)=0, \tag{34}$$

where A^2 denotes the squared norm of A^μ. Summing up all equations contracted by A's, we have

$$m^2(q^{N-1}\overset{(N-1)}{\lambda}\cdot A^{N-1}+q^{N-2}\overset{(N-2)}{\lambda}\cdot A^{N-2}+\cdots$$
$$+q\overset{(1)}{\lambda}\cdot A^1+\overset{(0)}{\lambda})=0. \tag{35}$$

One can also derive (35) by setting $p_\mu=qA_\mu$ in (15) after calculating the Poisson bracket.

We note that the existence of constants of motion linear in momenta for a massive particle requires the existence of a Killing vector. This is so because, when $N=1$ and $m\neq 0$, Eq. (35) reduce to $\overset{(0)}{\lambda}=0$, so that the highest-rank equation (29) becomes the Killing vector equation.

In what follows, we first consider three spacetimes with an electromagnetic field: test electromagnetic fields called the Wald solutions on the Kerr background, the four-dimensional Kerr-Newman black hole, and the five-dimensional charged rotating black hole. Since these systems admit a rank-2 Killing tensor $\overset{(2)}{K}$, we consider rank-2 solutions of the Killing hierarchy with $\overset{(l)}{\lambda}=0$,

$$-[\overset{(2)}{K},g]_S=0, \tag{36}$$

$$-[\overset{(1)}{K},g]_S+2q[\overset{(2)}{K},A]_S=0, \tag{37}$$

$$-[\overset{(0)}{K},g]_S+2q[\overset{(1)}{K},A]_S-q^2[\overset{(2)}{K},A^2]_S=0. \tag{38}$$

Next, in the final subsection, we demonstrate the existence of a constant of motion of a charged particle associated with a conformal Killing tensor. We construct a four-dimensional spacetime that admits a nontrivial conformal Killing tensor by a conformal transformation of the Minkowski spacetime. Making use of the conformal invariance of the Maxwell theory in four dimensions, we construct a solution of the Maxwell field on the spacetime. We show that there exists a suitable conformal transformation such that the charged particle system has a constant of motion associated with the conformal Killing tensor, so that the conservation equation holds only on the constraint surface in the phase space.

A. Wald solutions on Kerr geometry

Let us consider an electromagnetic field on the Kerr geometry. If a Ricci flat metric admits a Killing vector, the Killing vector solves the vacuum Maxwell equation as a test gauge 4-potential in the Lorentz gauge. This is called the Wald solution [14].

The Kerr metric is given by

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{39}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{40}$$

$$\Delta = r^2 + a^2 - 2Mr. \tag{41}$$

The spacetime admits two commuting Killing vectors $\xi = \partial_t$ and $\psi = \partial_\phi$, where a and M are the rotation and mass parameters, respectively. On the spacetime,

$$A^\mu = c_1\xi^\mu + c_2\psi^\mu \tag{42}$$

is a solution of the vacuum Maxwell equations, where c_1 and c_2 are arbitrary constants. The solution (42) has no magnetic charge. In the case $c_1 = 2ac_2$, it has no electric charge either. Furthermore, the Kerr metric also admits the Killing tensor

$$K^{\mu\nu} = 2\Sigma l^{(\mu}n^{\nu)} + r^2 g^{\mu\nu}, \tag{43}$$

which commutes with both of ξ and ψ, where

$$l^\mu = \frac{r^2+a^2}{\Delta}\xi^\mu + \frac{a}{\Delta}\psi^\mu + (\partial_r)^\mu, \quad n^\mu = \frac{r^2+a^2}{2\Sigma}\xi^\mu + \frac{a}{2\Sigma}\psi^\mu - \frac{\Delta}{2\Sigma}(\partial_r)^\mu. \tag{44}$$

Let us solve the Killing hierarchy (36). The Killing tensor (43), solves the highest-order equation of the hierarchy (36). We thus set

$$\overset{(2)}{K}{}^{\mu\nu} = K^{\mu\nu}. \tag{45}$$

Since $\overset{(2)}{K}$ commutes with A, which is a linear combination of ξ and ψ, the second equation (37) reduces to the Killing vector equation. Thus, we have

$$\overset{(1)}{K}{}^\mu = \alpha\xi^\mu + \beta\psi^\mu, \tag{46}$$

where α and β are arbitrary constants. Then the last equation (38) can be written as

$$\overset{(0)}{K}{}_{,\mu} = q^2 K_\mu{}^\nu A^2{}_{,\nu}. \tag{47}$$

By inspecting the integrability condition of the partial differential Eq. (47), we find that this equation is integrable only when $c_2 = 0$, i.e., $A = \xi$. In the case, the solution is given by

$$\overset{(0)}{K} = q^2 K_{\mu\nu}\xi^\mu\xi^\nu. \tag{48}$$

Thus, we have found a constant of motion of a charged particle associated with a rank-2 Killing tensor, $F = (\alpha\xi^\mu + \beta\psi^\mu)p_\mu + K_{\mu\nu}(p_\mu p_\nu + q^2\xi^\mu\xi^\nu)$.

We conclude that, in the case $A = \xi$, the system has independent Poisson-commuting constants of motion,

$$\xi^\mu p_\mu, \qquad \psi^\mu p_\mu, \qquad \text{and} \qquad K_{\mu\nu}u^\mu u^\nu, \tag{49}$$

where $u^\mu := \frac{1}{m}(p^\mu - qA^\mu)$ is the four velocity of the particle. In (49), we used the fact that $K_\mu u^\nu u^\mu$ is a linear combination of $\xi^\mu p_\mu$ and $\psi^\mu p_\mu$, and $K^{\mu\nu}(p_\mu p_\nu + q^2\xi_\mu\xi_\nu)$. We remark that no constant of motion associated with the rank-2 Killing tensor exists in the electrically neutral case $c_1 = 2ac_2$.

B. Kerr-Newman black holes

The Kerr-Newman spacetime is the exact solution of electrically charged rotating black hole in the Einstein-Maxwell system. The spacetime metric is given by (39) and (40) with

$$\Delta = r^2 + a^2 + e^2 - 2Mr, \tag{50}$$

instead of (41), and the electromagnetic 4-potential is given by

$$A = -\frac{er}{\Sigma}(dt - a\sin^2\theta d\phi), \tag{51}$$

where e is electric charge of the black hole. As the Kerr metric, the Kerr-Newman metric admits two Killing vectors $\xi = \partial_t$ and $\psi = \partial_r$ and the Killing tensor $K^{\mu\nu}$ in (43), where Δ in l and n is now given by (50).

In a manner similar to that in the previous section, we can find the solution to the set of Eqs. (36)–(38),

$$\overset{(2)}{K}{}_{\mu\nu} = K_{\mu\nu}, \tag{52}$$

$$\overset{(1)}{K}{}_\mu = -2qK^\nu{}_\mu A_\nu + \alpha\xi_\mu + \beta\psi_\mu, \tag{53}$$

$$\overset{(0)}{K} = q^2 K^{\mu\nu}A_\mu A_\nu, \tag{54}$$

where α and β are arbitrary constants. Then the constant of motion associated with the Killing tensor is given by

$$F = K^{\mu\nu}p_\mu p_\nu + (-2qK^{\mu\nu}A_\nu + \alpha\xi^\mu + \beta\psi^\mu)p_\mu + q^2K^{\mu\nu}A_\mu A_\nu = K_{\mu\nu}u^\mu u^\nu + \alpha\xi^\mu p_\mu + \beta\psi^\mu p_\mu. \tag{55}$$

As the Wald solution $A = \xi$ on the Kerr metric discussed in the previous section, the Kerr-Newman metric admits the independent constants of motion, $\xi^\mu p_\mu$, $\psi^\mu p_\mu$, and $K_{\mu\nu}u^\mu u^\nu$, for a charged particle. These two examples have common properties, i.e., both are rotating black holes with an electric monopole and a magnetic field falling off toward infinity.

The constant of motion for a charged particle associated with the Killing tensor is referred to as Carter's constant, which was first obtained by the Hamilton-Jacobi method in Ref. [1].

C. Five-dimensional charged black holes

We demonstrate that our formalism is applicable to a charged particle moving around a five-dimensional charged black hole. Here, we consider the five-dimensional charged rotating black hole with the following metric and the electromagnetic 5-potential [15],

$$ds^2 = -\frac{(\rho^2 dt + 2e\nu)dt}{\rho^2} + \frac{2e\nu\omega}{\rho^2} + \frac{f}{\rho^4}(dt-\omega)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + (r^2+b^2)\cos^2\theta d\psi^2, \tag{56}$$

$$A = \frac{\sqrt{3}e}{2\rho^2}(dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi), \tag{57}$$

where

$$S = a^2\cos^2\theta + b^2\sin^2\theta, \quad \rho^2 = r^2 + S, \quad f = 2M\rho^2 - e^2, \tag{58}$$

$$\begin{aligned} \nu &= b\sin^2\theta d\phi + a\cos^2\theta d\psi, \\ \omega &= a\sin^2\theta d\phi + b\cos^2\theta d\psi, \end{aligned} \tag{59}$$

$$\Delta_r = \frac{(r^2+a^2)(r^2+b^2) + e^2 + 2abe}{r^2} - 2M. \tag{60}$$

The black hole is characterized by mass parameter M, charge parameter e, and two spin parameters a and b. The metric is an exact solution in the five-dimensional Einstein-Maxwell-Chern-Simons theory. The metric admits three Killing vectors ∂_t, ∂_ϕ, and ∂_ψ, and an irreducible Killing tensor [16]

$$K^{\mu\nu} = -Sg^{\mu\nu} - S(\partial_t)^\mu(\partial_t)^\nu + \frac{1}{\sin^2\theta}(\partial_\phi)^\mu(\partial_\phi)^\nu + \frac{1}{\cos^2\theta}(\partial_\psi)^\mu(\partial_\psi)^\nu + (\partial_\theta)^\mu(\partial_\theta)^\nu. \tag{61}$$

These Killing vectors and tensor commute with each other. Hence, we can discuss the existence of the constants of motion of a charged particle associated with the three Killing vectors and the Killing tensor.

Let us consider the Killing hierarchy (36)–(38), with the rank-2 Killing tensor $\overset{(2)}{K}{}^{\mu\nu} = K^{\mu\nu}$, which solves (36). We try to find the solution of the second equations (37) of the form

$$\overset{(1)}{K}{}^{\mu;\nu} + \overset{(1)}{K}{}^{\nu;\mu} = B^{\mu\nu}, \tag{62}$$

where $B^{\mu\nu} = 2q(A^\lambda K^{\mu\nu}{}_{,\lambda} - 2K^{\lambda(\mu}A^{\nu)}{}_{,\lambda})$. Since t, ϕ, and ψ are the Killing coordinates, Eq. (62) has a simple form, which is shown in Appendix B.

To find solutions of (37), we assume that $\overset{(1)}{K}$ depends only on r and θ as $\overset{(2)}{K}$ does. It turns out that

$$\overset{(1)}{K}{}^r = 0, \qquad \overset{(1)}{K}{}^\theta = 0 \tag{63}$$

from the explicit form of (62), and the other components of $\overset{(1)}{K}$ satisfy the following equations:

$$\begin{aligned} &g^{rr}\overset{(1)}{K}{}^t{}_{,r} = B^{tr}, \quad g^{\theta\theta}\overset{(1)}{K}{}^t{}_{,\theta} = B^{t\theta}, \quad g^{rr}\overset{(1)}{K}{}^\phi{}_{,r} = B^{r\phi}, \\ &g^{\theta\theta}\overset{(1)}{K}{}^\phi{}_{,\theta} = B^{\theta\phi}, \quad g^{rr}\overset{(1)}{K}{}^\psi{}_{,r} = B^{r\psi}, \quad g^{\theta\theta}\overset{(1)}{K}{}^\psi{}_{,\theta} = B^{\theta\psi}. \end{aligned} \tag{64}$$

After some calculations, we can find an explicit solution

$$\overset{(1)}{K}{}^\mu = 2qSA^\mu + \alpha(\partial_t)^\mu + \beta(\partial_\phi)^\mu + \gamma(\partial_\psi)^\mu, \tag{65}$$

where α, β, and γ are arbitrary constants.

Let us solve (38). It can be rewritten as

$$\overset{(0)}{K}{}_{,t} = \overset{(0)}{K}{}_{,\phi} = \overset{(0)}{K}{}_{,\psi} = 0, \tag{66}$$

$$g^{rr}\overset{(0)}{K}{}_{,r} = -\frac{3q^2e^2S}{2r^3\Delta_r\rho^6}[(r^4 - (ab+e)^2)S + r^4((a^2+b^2) + 2(r^2 - M))], \tag{67}$$

$$g^{\theta\theta}\overset{(0)}{K}{}_{,\theta} = -\frac{3q^2e^2r^2}{4\Delta_r\rho^6}(a+b)(a-b)\sin 2\theta. \tag{68}$$

We can easily integrate these equations to get

$$\overset{(0)}{K} = -q^2SA^\mu A_\mu. \tag{69}$$

As a result, we obtain a constant of motion associated with the Killing tensor,

$$F = K^{\mu\nu}p_\mu p_\nu + 2qSA^\mu p_\mu - q^2SA^\mu A_\nu \tag{70}$$

for a charged particle moving in the five-dimensional charged rotating black holes in addition to the momentum components p_t, p_ϕ and p_ψ. These four constants of motions Poisson commute mutually.

D. Constant of motion associated with a conformal Killing tensor

In this subsection, we demonstrate that the conservation equation can hold only on the constraint surface. In such a case, the constant of motion is associated with a conformal Killing tensor. We present an example for this case, but unfortunately we have not found one as a solution of the Einstein equation.

Let $(\mathcal{M}^4, \bar{g})$ be the Minkowski spacetime. Consider the metric $\bar{g}$ spanned by the polar coordinates as

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{71}$$

The Minkowski spacetime admits a Killing tensor

$$K^{\mu\nu} = (\partial_\theta)^\mu(\partial_\theta)^\nu + \frac{1}{\sin^2\theta}(\partial_\phi)^\mu(\partial_\phi)^\nu. \tag{72}$$

As a solution of the test electromagnetic field on the flat background, we take $\bar{A} = \partial_\phi$. This is the special case of the Wald solution discussed in Sec. IV A, with $M = 0$ and $a = 0$. As was mentioned there, there is no constant of motion associated with the Killing tensor $K^{\mu\nu}$ for a charged particle.

Let $(\mathcal{M}^4, g)$ be a spacetime with $g_{\mu\nu} = e^\Phi \bar{g}_{\mu\nu}$ where Φ is a function on $\mathcal{M}^4$. The spacetime $(\mathcal{M}^4, g)$ is conformally flat. Since the Maxwell theory in a four-dimensional spacetime has conformal invariance, the gauge 4-potential $A_\mu := \bar{A}_\mu$ solves the Maxwell equations on $(\mathcal{M}^4, g)$. The tensor $K^{\mu\nu}$ given by (72), satisfying

$$-[K, g]_S = 2(K\partial\Phi)\overset{S}{\otimes} g \tag{73}$$

is a conformal Killing tensor on $(\mathcal{M}^4, g)$, where $K\partial$ denotes the derivative operator $K^{\mu\nu}\partial_\nu$. We show that a constant of motion associated with the conformal Killing tensor for a class of the conformal factor can exist .

We shall try to find a solution of the Killing hierarchy, which starts with a rank-2 conformal Killing tensor that is not a Killing tensor. Namely, we shall solve (29)–(34) with (35), i.e.,

$$-[\overset{(2)}{K}, g]_S + \overset{(1)}{\lambda} \overset{S}{\otimes} g = 0, \tag{74}$$

$$-[\overset{(1)}{K}, g]_S + 2q[\overset{(2)}{K}, A]_S + \overset{(0)}{\lambda} g - 2q\overset{(1)}{\lambda} \overset{S}{\otimes} A = 0, \tag{75}$$

$$\begin{aligned} -[\overset{(0)}{K}, g]_S + 2q[\overset{(1)}{K}, A]_S - q^2[\overset{(2)}{K}, A^2]_S \\ - 2q\overset{(0)}{\lambda} A + \overset{(1)}{\lambda}(q^2A^2 + m^2) = 0, \end{aligned} \tag{76}$$

with

$$m^2(\overset{(0)}{\lambda} + q\overset{(1)}{\lambda} \cdot \overset{(1)}{A}) = 0, \tag{77}$$

where $\overset{(1)}{\lambda}$ is not identically zero, and the gauge potential A^μ is given by $A^\mu = e^{-\Phi}(\partial_\phi)^\mu$.

The tensor $\overset{(2)}{K}{}^{\mu\nu} = K^{\mu\nu}$ given by (72) solves (74) with

$$\overset{(1)}{\lambda} = -2(K\partial\Phi). \tag{78}$$

From the algebraic condition (77), we have

$$\overset{(0)}{\lambda} = -qA \cdot \overset{(1)}{\lambda} = 2qe^{-\Phi}r^2\partial_\phi\Phi. \tag{79}$$

We assume that the function Φ is in the form $\Phi = \Phi(t, r, \theta)$, so that we have $\overset{(0)}{\lambda} = 0$ and $[K, A]_S = 0$. Then the second equation (75) reduces to the Killing vector equation

$$[\overset{(1)}{K}, g]_S = 0. \tag{80}$$

For simplicity, we choose the trivial solution $\overset{(1)}{K} = 0$. The remaining equation (76) is given by

$$[\overset{(0)}{K}, g]_S = -2q^2e^{-\Phi}K\partial\bar{g}_{\phi\phi} - 2m^2(K\partial\Phi), \tag{81}$$

or explicitly,

$$\partial_t\overset{(0)}{K} = \partial_r\overset{(0)}{K} = \partial_\phi\overset{(0)}{K} = 0, \tag{82}$$

$$\partial_\theta\overset{(0)}{K} = \bar{g}_{\theta\theta}(q^2\partial_\theta\bar{g}_{\phi\phi} + m^2e^\Phi\partial_\theta\Phi). \tag{83}$$

A constant of motion exists if the function Φ is chosen such that the partial differential Eqs. (82) and (83) for $\overset{(0)}{K}$ are integrable.

As the simplest case, we consider that the right-hand side of (83) vanishes, namely,

$$m^2\partial_\theta e^\Phi + 2q^2r^2\sin\theta\cos\theta = 0. \tag{84}$$

In this case, Eqs. (82) and (83) admit a trivial solution $\overset{(0)}{K} = 0$. We can easily integrate (84) to obtain

$$e^\Phi = \frac{q^2}{m^2}r^2\cos^2\theta + f(t, r), \tag{85}$$

where $f(t, r)$ is an arbitrary positive function.

Therefore, if we choose (85) as the conformal factor, then the quadratic quantity $F = K \cdot p^2$ is a constant of motion of a charged particle associated with the conformal Killing tensor. The quantity is conserved only on the constraint surface in the phase space.

V. SUMMARY

In this paper, we have discussed constants of motion for a test particle in a curved spacetime. For the particle that is subjected to an external field, we have obtained the condition for the existence of the constant of motion in the form of coupled equations with a hierarchical structure. There, we have taken into consideration the Hamiltonian constraint condition for the particle, which arises from the reparametrization invariance of particle's world line. The equation at the top of the hierarchy is the conformal Killing tensor equation. Then, the existence of constant of motion requires that the metric admits a conformal Killing tensor. If the Killing hierarchy has a nontrivial solution, a constant of motion associated with the conformal Killing tensor exists.

As applications of the formalism, we have considered systems of a charged particle in Maxwell's fields on black holes. In the case of a charged particle in the Kerr-Newman black holes, we have rediscovered a constant of motion quadratic in the canonical momenta, which has been found via the Hamilton-Jacobi method [1]. In the case of a charged particle in the electromagnetic field without an electric charge constructed by Wald's method on the Kerr black holes, we have shown that the Killing hierarchy is not integrable. The nonexistence of constant of motion does not depend on the choice of coordinate, in contrast to the fact that the discovery of constant of motion was due to suitable coordinates in the Hamilton-Jacobi method. We have found a new constant of motion as a solution of the hierarchical equations for a charged particle around a five-dimensional charged rotating black hole, which is a solution for the Einstein-Maxwell-Chern-Simons theory. Since these metrics admit Killing tensor of rank 2, the constants of motion in these examples are associated with the Killing tensors.

As the final example in this paper, we have considered Maxwell's field on an artificial conformal flat spacetime. For a charged particle in this field, we constructed a constant of motion that is associated with rank-2 conformal Killing tensor, i.e., the conservation equation holds only on the Hamiltonian constraint surface in the phase space. It would be interesting to find the constants of motion for a particle moving in a solution of the Einstein-Maxwell system. The extension to wider classes of interactions is an important future work.

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APPENDIX A: KILLING HIERARCHY FOR A FREE PARTICLE

Let us consider the system of a free particle in the framework of the Killing hierarchy. The explicit form of the Hamiltonian is given by (10), and the constraint equation is given by (11). Then the Killing hierarchy (20) reads

$$-[\overset{(k-1)}{K}, g]_{\rm S} + \overset{(k-2)}{\lambda} \overset{\rm S}{\otimes} g + m^2 \overset{(k)}{\lambda} = 0, \qquad k \geq 2, \\ -[\overset{(0)}{K}, g]_{\rm S} + m^2 \overset{(1)}{\lambda} = 0, \qquad m^2 \overset{(0)}{\lambda} = 0. \tag{A1}$$

If the particle is massless, i.e. $m = 0$, then the constraint (11) becomes

$$\mathcal{H} = g^{\mu\nu} p_\mu p_\nu \approx 0, \tag{A2}$$

and the Killing hierarchy becomes

$$-[\overset{(k-1)}{K}, g]_{\rm S} + \overset{(k-2)}{\lambda} \overset{\rm S}{\otimes} g = 0, \qquad k \geq 2, \qquad -[\overset{(0)}{K}, g]_{\rm S} = 0. \tag{A3}$$

All the equations for $\overset{(k)}{K}$ $(k \geq 1)$ become decoupled conformal Killing equations. Therefore, a nontrivial conformal Killing tensor with nonvanishing $\overset{(l)}{\lambda}$ gives a constant of motion conserved only on the constraint surface (A2).

If the particle is massive, i.e., $m \neq 0$, we consider symmetric tensors $\overset{(k)}{\tilde{K}}$, which satisfy the linear differential equations

$$-[\overset{(k)}{\tilde{K}}, g]_{\rm S} + m^2 \overset{(k+1)}{\lambda} = 0. \tag{A4}$$

The solutions of the linear differential equations (A4) have the form

$$\overset{(k)}{\tilde{K}} = \overset{(k)}{\tilde{K}}_{\rm H} + \overset{(k)}{\tilde{K}}_{\rm I}, \tag{A5}$$

where the homogeneous part $\overset{(k)}{\tilde{K}}_{\rm H}$ is a solution of the Killing equation

$$[\overset{(k)}{\tilde{K}}_{\rm H}, g]_{\rm S} = 0, \tag{A6}$$

and $\overset{(k)}{\tilde{K}}_{\rm I}$ is an inhomogeneous solution satisfying the original equations (A4),

$$-[\overset{(k)}{\tilde{K}}_{\rm I}, g]_{\rm S} + m^2 \overset{(k+1)}{\lambda} = 0. \tag{A7}$$

Using $\tilde{K}_{\rm H}$ and $\tilde{K}_{\rm I}$, we can construct the solution of (A1) as

$$\overset{(k)}{K} = \overset{(k)}{\tilde{K}}_{\rm H} + \overset{(k)}{\tilde{K}}_{\rm I} + \frac{1}{m^2} \overset{(k-2)}{\tilde{K}}_{\rm I} \overset{\rm S}{\otimes} g. \tag{A8}$$

For this solution, the conserved quantity has the form

$$\begin{aligned} F &= \sum_k \overset{(k)}{K} \cdot p^k \\ &= \sum_k (\overset{(k)}{\tilde{K}}_{\rm H} \cdot p^k + \overset{(k)}{\tilde{K}}_{\rm I} \cdot p^k) + \sum_k \left(\frac{1}{m^2} \overset{(k-2)}{\tilde{K}}_{\rm I} \overset{\rm S}{\otimes} g \right) \cdot p^k \\ &= \sum_k \overset{(k)}{\tilde{K}}_{\rm H} \cdot p^k + \frac{1}{m^2} \sum_k \overset{(k)}{\tilde{K}}_{\rm I} \cdot p^k (g \cdot p^2 + m^2) \\ &\approx \sum_k \overset{(k)}{\tilde{K}}_{\rm H} \cdot p^k. \end{aligned} \tag{A9}$$

Therefore, the homogeneous solutions, that is, solutions for the Killing equations, contribute to the conserved quantity. The constant of motion for a massive free particle requires the existence of the Killing tensor.

APPENDIX B: EQUATIONS IN THE CASE OF FIVE-DIMENSIONAL BLACK HOLES

As a supplement to Sec. IV C, we give the explicit form of Eqs. (62) in terms of the components:

$$(t,r):\ g^{rr}\overset{(1)}{K}{}^{t}{}_{,r}+g^{tt}\overset{(1)}{K}{}^{r}{}_{,t}+g^{\phi t}\overset{(1)}{K}{}^{r}{}_{,\phi}+g^{\psi t}\overset{(1)}{K}{}^{r}{}_{,\psi}=B^{tr}, \tag{B1}$$

$$(t,\theta):\ g^{\theta\theta}\overset{(1)}{K}{}^{t}{}_{,\theta}+g^{tt}\overset{(1)}{K}{}^{\theta}{}_{,t}+g^{\phi t}\overset{(1)}{K}{}^{\theta}{}_{,\phi}+g^{\psi t}\overset{(1)}{K}{}^{\theta}{}_{,\psi}=B^{t\theta}, \tag{B2}$$

$$(r,\phi):\ g^{t\phi}\overset{(1)}{K}{}^{r}{}_{,t}+g^{\phi\phi}\overset{(1)}{K}{}^{r}{}_{,\phi}+g^{\psi\phi}\overset{(1)}{K}{}^{r}{}_{,\psi}+g^{rr}\overset{(1)}{K}{}^{\phi}{}_{,r}=B^{r\phi}, \tag{B3}$$

$$(r,\psi):\ g^{t\psi}\overset{(1)}{K}{}^{r}{}_{,t}+g^{\phi\psi}\overset{(1)}{K}{}^{r}{}_{,\phi}+g^{\psi\psi}\overset{(1)}{K}{}^{r}{}_{,\psi}+g^{rr}\overset{(1)}{K}{}^{\psi}{}_{,r}=B^{r\psi}, \tag{B4}$$

$$(\theta,\phi):\ g^{t\phi}\overset{(1)}{K}{}^{\theta}{}_{,t}+g^{\phi\phi}\overset{(1)}{K}{}^{\theta}{}_{,\phi}+g^{\psi\phi}\overset{(1)}{K}{}^{\theta}{}_{,\psi}+g^{\theta\theta}\overset{(1)}{K}{}^{\phi}{}_{,\theta}=B^{\theta\phi}, \tag{B5}$$

$$(\theta,\psi):\ g^{t\psi}\overset{(1)}{K}{}^{\theta}{}_{,t}+g^{\phi\psi}\overset{(1)}{K}{}^{\theta}{}_{,\phi}+g^{\psi\psi}\overset{(1)}{K}{}^{\theta}{}_{,\psi}+g^{\theta\theta}\overset{(1)}{K}{}^{\psi}{}_{,\theta}=B^{\theta\psi}, \tag{B6}$$

where components of $B^{\mu\nu}$ are given explicitly by

$$B^{t\theta}=\frac{\sqrt{3}qe}{\Delta_r\rho^6}[(r^2+a^2)(r^2+b^2)+abe]\times(a+b)(a-b)\sin 2\theta, \tag{B7}$$

$$B^{\theta\phi}=\frac{\sqrt{3}qe}{\Delta_r\rho^6}[b(ab+e)+ar^2](a+b)(a-b)\sin 2\theta, \tag{B8}$$

$$B^{\theta\psi}=\frac{\sqrt{3}qe}{\Delta_r\rho^6}[a(ab+e)+br^2](a+b)(a-b)\sin 2\theta, \tag{B9}$$

$$B^{tr}=\frac{2\sqrt{3}qeS}{r^3\Delta_r\rho^6}[[(r^2+a^2)(r^2+b^2)+abe]\times[r^2\Delta_r+\rho^2(a^2+b^2+2(r^2-M))]-r^2\rho^2\Delta_r(a^2+b^2+2r^2)], \tag{B10}$$

$$B^{r\phi}=\frac{2\sqrt{3}qeS}{r^3\Delta_r\rho^6}[(b(e+ab)+ar^2)\times[r^2\Delta_r+\rho^2(a^2+b^2+2(r^2-M))]-ar^2\Delta_r\rho^2], \tag{B11}$$

$$B^{r\psi}=\frac{2\sqrt{3}qeS}{r^3\Delta_r\rho^6}[(a(e+ab)+br^2)\times[r^2\Delta_r+\rho^2(a^2+b^2+2(r^2-M))]-br^2\Delta_r\rho^2]. \tag{B12}$$

The other components of the equation are trivial.

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