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Abelian symmetries in the two-Higgs-doublet model with fermions

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We classify all possible implementations of an Abelian symmetry in the two-Higgs-doublet model with fermions. We identify those symmetries which are consistent with nonvanishing quark masses and a Cabibbo-Kobayashi-Maskawa quark-mixing matrix (CKM), which is not block-diagonal. Our analysis takes us from a plethora of possibilities down to 246 relevant cases, requiring only 34 distinct matrix forms. We show that applying Z_n with $n \geq 4$ to the scalar sector leads to a continuous $U(1)$ symmetry in the whole Lagrangian. Finally, we address the possibilities of spontaneous CP violation and of natural suppression of the flavor-changing neutral currents. We explain why our work is relevant even for non-Abelian symmetries.

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I. INTRODUCTION

The least known aspect of the electroweak interactions is its scalar sector. In the standard model (SM) there is only one Higgs but, although this is an economical choice, there is no fundamental reason for nature to adopt it. Ultimately, the number of Higgs fields, like the number of fermion families before it, must be assessed experimentally. Partly for this reason, there has been a great interest in multi-Higgs models. This is also due to the fact that many interesting new effects arise, such as the presence of charged scalars, the possibility for CP violation in the scalar sector, and the possibility for spontaneous CP violation, to name a few.

One problem with multi-Higgs models is that they involve many more parameters than needed in the SM. This problem can be tamed by invoking discrete symmetries. A complete classification of the impact of discrete and continuous symmetries in the scalar sector of the two-Higgs-doublet model (THDM) has been discussed in the literature [1,2], and some incursions exist into theories with more than two Higgs doublets [3,4]. There are also several articles discussing specific implementations of discrete symmetries in both the scalar and fermion sectors, but no complete classification exists. This is the problem we tackle here.

This article is organized as follows: In Sec. II we introduce our notation and show the impact that a choice of Abelian symmetries in the scalar and fermion sectors has on the Yukawa matrices. *A priori* there are 3^{18} possibilities. In Sec. III we show how simple experimental considerations, such as the absence of massless quarks and the non-block-diagonal nature of the CKM matrix can be used to curtail this number down to 246. Up to permutations, these involve only 34 forms of Yukawa matrices, which we show explicitly. Since any finite discrete group has an Abelian subgroup, our classification is important even for those considering non-Abelian family symmetries. We present two important results in Sec. IV. Our classification is then used to address two questions: whether one can have spontaneous CP violation, in Sec. V; and whether one can relate the flavor changing neutral current interactions with the CKM matrix, in Sec. VI. We draw our conclusions in Sec. VII.

II. NOTATION

A. The Lagrangian

Let us consider a $SU(2) \otimes U(1)$ gauge theory with two hypercharge-one Higgs doublets, denoted by Φ_a, where $a = 1, 2$. The scalar potential may be written as

$$-\mathcal{L}_{\rm H} = Y_{ab}(\Phi_a^\dagger \Phi_b) + \frac{1}{2} Z_{ab,cd}(\Phi_a^\dagger \Phi_b)(\Phi_c^\dagger \Phi_d), \quad (1)$$

where Hermiticity implies

$$Y_{ab} = Y_{ba}^*, \qquad Z_{ab,cd} \equiv Z_{cd,ab} = Z_{ba,dc}^*. \quad (2)$$

Minimization of this potential leads to the vacuum expectation values (vevs) $\langle \Phi_a \rangle = v_a$.

The theory contains also 3 families of left-handed quark doublets (q_L), right-handed down-type quarks (n_R), and right-handed up-type quarks (p_R). For the most part, we will ignore the leptonic sector, since the analysis would be similar. The Yukawa Lagrangian may be written as

$$\mathcal{L}_{\rm Y} = -\bar{q}_L[(\Gamma_1 \Phi_1 + \Gamma_2 \Phi_2) n_R + (\Delta_1 \tilde{\Phi}_1 + \Delta_2 \tilde{\Phi}_2) p_R] + \text{H.c.}, \quad (3)$$

where $\tilde{\Phi}_k \equiv i\tau_2 \Phi_k^*$, and q_L, n_R, and p_R are 3-vectors in flavor space. The 3×3 matrices Γ_k, Δ_k, contain the complex Yukawa couplings to the right-handed down-type quarks and up-type quarks, respectively.

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B. Basis transformations

The Lagrangian can be rewritten in terms of new fields obtained from the original ones by simple basis transformations

$$\begin{aligned} \Phi_a \to \Phi'_a = U_{ab}\Phi_b, \qquad & q_L \to q'_L = U_L q_L, \\ n_R \to n'_R = U_{nR} n_R, \qquad & p_R \to p'_R = U_{pR} p_R, \end{aligned} \tag{4}$$

where $U \in U(2)$ is a 2×2 unitary matrix, while $\{U_L, U_{nR}, U_{pR}\} \in U(3)$ are 3×3 unitary matrices. Under these unitary basis transformations, the gauge-kinetic terms are unchanged, but the coefficients Y_{ab} and $Z_{ab,cd}$ are transformed as

$$Y_{ab} \to Y'_{ab} = U_{a\alpha} Y_{\alpha\beta} U^*_{b\beta}, \tag{5}$$

$$Z_{ab,cd} \to Z'_{ab,cd} = U_{a\alpha} U_{c\gamma} Z_{\alpha\beta,\gamma\delta} U^*_{b\beta} U^*_{d\delta}, \tag{6}$$

while the Yukawa matrices change as

$$\begin{aligned} \Gamma_a \to \Gamma'_a = U_L \Gamma_\alpha U^\dagger_{nR} (U^\dagger)_{\alpha a} \\ \Delta_a \to \Delta'_a = U_L \Delta_\alpha U^\dagger_{pR} (U^\top)_{\alpha a}. \end{aligned} \tag{7}$$

Notice that we have kept the notation of showing explicitly the indices in scalar space, while using matrix formulation for the quark flavor spaces. The basis transformations may be utilized in order to absorb some of the degrees of freedom of Y, Z, Γ, and/or Δ, which implies that not all parameters in the Lagrangian have physical significance.

C. Symmetries in the THDM

We will now assume that the Lagrangian is invariant under the symmetry

$$\begin{aligned} \Phi_a \to \Phi^S_a = S_{ab}\Phi_b, \qquad & q_L \to q^S_L = S_L q_L, \\ n_R \to n^S_R = S_{nR} n_R, \qquad & p_R \to p^S_R = S_{pR} p_R, \end{aligned} \tag{8}$$

where $S \in U(2)$, while $\{S_L, S_{nR}, S_{pR}\} \in U(3)$. As a result of this symmetry,

$$Y_{ab} = S_{a\alpha} Y_{\alpha\beta} S^*_{b\beta}, \tag{9}$$

$$Z_{ab,cd} = S_{a\alpha} S_{c\gamma} Z_{\alpha\beta,\gamma\delta} S^*_{b\beta} S^*_{d\delta}, \tag{10}$$

$$\Gamma_a = S_L \Gamma_\alpha S^\dagger_{nR} (S^\dagger)_{\alpha a}, \tag{11}$$

$$\Delta_a = S_L \Delta_\alpha S^\dagger_{pR} (S^\top)_{\alpha a}. \tag{12}$$

Under the basis transformation of Eq. (4), the specific form of the symmetry in Eq. (8) is altered as

$$S' = U S U^\dagger, \tag{13}$$

$$S'_L = U_L S_L U^\dagger_L, \tag{14}$$

$$S'_{nR} = U_{nR} S_{nR} U^\dagger_{nR}, \tag{15}$$

$$S'_{pR} = U_{pR} S_{pR} U^\dagger_{pR}. \tag{16}$$

Suppose that one has chosen to apply the symmetry $\{S, S_L, S_{nR}, S_{pR}\}$ in some basis. By a judicious choice of $\{U, U_L, U_{nR}, U_{pR}\}$ one may bring the symmetry into the form

$$S = \mathrm{diag}\{e^{i\theta_1}, e^{i\theta_2}\}, \tag{17}$$

$$S_L = \mathrm{diag}\{e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}\}, \tag{18}$$

$$S_{nR} = \mathrm{diag}\{e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}\}, \tag{19}$$

$$S_{pR} = \mathrm{diag}\{e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}\}. \tag{20}$$

What about global phases? Clearly, an overall phase change has no effect on the symmetry. For example, taking $U = e^{i\theta}\mathbb{1}_2$, leaves $S' = S$. However, it is easy to see from Eqs. (9)–(12) that the symmetry

$$\tilde{S} = e^{i\tilde{\theta}} S, \quad \tilde{S}_L = e^{i\tilde{\alpha}} S_L, \quad \tilde{S}_{nR} = e^{i\tilde{\beta}} S_{nR}, \quad \tilde{S}_{pR} = e^{i\tilde{\gamma}} S_{pR}, \tag{21}$$

imposes the same restrictions on the Lagrangian as the symmetry $\{S, S_L, S_{nR}, S_{pR}\}$, as long as

$$e^{i(\tilde{\beta} - \tilde{\alpha} - \tilde{\theta})} = 1 \quad \text{and} \quad e^{i(\tilde{\gamma} - \tilde{\alpha} + \tilde{\theta})} = 1. \tag{22}$$

This can be used to bring Eqs. (17)–(20) into the form

$$S = \mathrm{diag}\{1, e^{i\theta}\}, \tag{23}$$

$$S_L = \mathrm{diag}\{e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}\}, \quad \text{with } \alpha_1 = 0, \tag{24}$$

$$S_{nR} = \mathrm{diag}\{e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}\}, \tag{25}$$

$$S_{pR} = \mathrm{diag}\{e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}\}. \tag{26}$$

For $\theta = \pi$, $S = \mathrm{diag}(1, -1)$ leads to the usual Z_2 Higgs potential. Any other value of $0 < \theta < 2\pi$, leads to the full $U(1)$ symmetric Higgs potential. For example, with $\theta = 2\pi/3$, $S^3 = \mathbb{1}_2$, and a Z_3 symmetry is imposed on the scalar fields. Nevertheless, because the scalar potential only has quadratic and quartic terms, the resulting Higgs potential has the full $U(1)$ Peccei-Quinn symmetry [3]. If this symmetry is broken spontaneously by the vacuum, we will have massless particles. As a result, great care must be taken when imposing what may look like discrete symmetries in multi-Higgs models. Substituting Eqs. (17)–(20) in Eqs. (11) and (12), we find

$$(\Gamma_a)_{ij} = e^{i(\alpha_i - \beta_j - \theta_a)} (\Gamma_a)_{ij}, \tag{27}$$

$$(\Delta_a)_{ij} = e^{i(\alpha_i - \gamma_j + \theta_a)} (\Delta_a)_{ij}, \tag{28}$$

where *no sum* over i and j is intended on the right-hand sides. For the simplified form in Eq. (23) we set $\theta_1 = 0$ and $\theta_2 = \theta$. Furthermore, we will always take $\theta \neq 0 \pmod{2\pi}$, since we are only interested in symmetries which *do transform the scalar fields*. It will prove useful to keep α_1 explicitly, bearing in mind that it can be set equal to zero without loss of generality. These equations constitute our starting point for what follows.

D. Preliminary constraints on the Yukawa matrices

We will concentrate first on the down-type Yukawa matrices Γ_a. Given a symmetry written in the form of Eqs. (23)–(26) we conclude from Eq. (27) that

(i) $(\Gamma_1)_{ij}$ can take any value if $\theta_{ij} = 0$;
(ii) $(\Gamma_1)_{ij} = 0$ if $\theta_{ij} \neq 0$;
(iii) $(\Gamma_2)_{ij}$ can take any value if $\theta_{ij} = \theta$; $(\Gamma_2)_{ij} = 0$ if $\theta_{ij} \neq \theta$;

where we have defined

$$\theta_{ij} = \alpha_i - \beta_j. \tag{29}$$

We conclude that, for a matrix S characterized by a given $\theta \neq 0$, there are only three possibilities:

(i) $\theta_{ij} = 0 \Rightarrow (\Gamma_1)_{ij} =$ any and $(\Gamma_2)_{ij} = 0$;
(ii) $\theta_{ij} = \theta \Rightarrow (\Gamma_1)_{ij} = 0$ and $(\Gamma_2)_{ij} =$ any;
(iii) $\theta_{ij} \neq 0, \theta \Rightarrow (\Gamma_1)_{ij} = 0 = (\Gamma_2)_{ij}$.

All conditions on θ_{ij} are mod(2π). Noticing that only five θ_{ij} are independent, we will take these to be θ_{11}, θ_{12}, θ_{13}, θ_{21}, and θ_{31}. Then,

$$\begin{aligned} \theta_{22} &= \theta_{21} + \theta_{12} - \theta_{11}, & \theta_{23} &= \theta_{21} + \theta_{13} - \theta_{11}, \\ \theta_{32} &= \theta_{31} + \theta_{12} - \theta_{11}, & \theta_{33} &= \theta_{31} + \theta_{13} - \theta_{11}. \end{aligned} \tag{30}$$

For each $\theta \neq 0$, we must only consider five θ_{ij}. The possibilities $\theta_{ij} = 0$ and $\theta_{ij} = \theta$ are simple to enumerate. Unfortunately, the impact of $\theta_{ij} \neq 0$, θ depends on the exact value of θ_{ij}. Thus, there are far more than the 3^5 possibilities one might naively expect. For example, choosing $\{\theta_{11}, \theta_{12}, \theta_{21}\} = \{7\theta, 2\theta, 2\theta\}$ and $\theta = \sqrt{2}\pi$, we conclude that the (1, 1), (1, 2), and (2, 1) entries of Γ_1 and Γ_2 matrices vanish, as do the (2, 2) entries. In contrast, choosing $\{\theta_{11}, \theta_{12}, \theta_{21}\} = \{4\theta, 2\theta, 2\theta\}$ and $\theta = \sqrt{2}\pi$ we conclude that the (1, 1), (1, 2), and (2, 1) entries of Γ_1 and Γ_2 matrices vanish, but the (2, 2) entry of Γ_1 need not vanish.[1]

Some possibilities are trivially inconsistent with experiment. For example, choosing $\{\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{31}\} = \{0, \theta, \theta, \theta, \theta\}$, then the matrix

$$\Theta = \{\theta_{ij}\} \tag{31}$$

becomes

$$\Theta = \begin{bmatrix} 0 & \theta & \theta \\ \theta & 2\theta & 2\theta \\ \theta & 2\theta & 2\theta \end{bmatrix}. \tag{32}$$

For $\theta \neq 0$ (identity operation) and $\theta \neq \pi$ (usual Z_2 symmetry), we are lead to Yukawa matrices of the form

$$\Gamma_1 = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{bmatrix}. \tag{33}$$

Upon spontaneous electroweak symmetry breaking, the down-type quark mass matrix will arise from the bidiagonalization of

$$v_1\Gamma_1 + v_2\Gamma_2 = \begin{bmatrix} v_1 a_{11} & v_2 b_{12} & v_2 b_{13} \\ v_2 b_{21} & 0 & 0 \\ v_2 b_{31} & 0 & 0 \end{bmatrix}, \tag{34}$$

whose determinant is zero. As a result, this model would lead to one massless quark, which is ruled out by experiment. Notice that choosing $\{\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{31}\} = \{\theta, 0, 0, 0, 0\}$ would lead to Yukawa matrices of the form

$$\Gamma_1 = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{35}$$

This is the same as Eq. (33), with the substitution $\Phi_1 \leftrightarrow \Phi_2$. Said otherwise, these possibilities represent the same model. The interchange $\Phi_1 \leftrightarrow \Phi_2$ cuts down the number of distinct models by almost a factor of 2.

An old model by Lavoura [5] had

$$\begin{aligned} S &= \mathrm{diag}\{1, -1\}, & S_L &= \mathrm{diag}\{1, 1, 1\}, \\ S_{nR} &= \mathrm{diag}\{1, 1, -1\}, & S_{pR} &= \mathrm{diag}\{1, 1, 1\}. \end{aligned} \tag{36}$$

Thus

$$\Theta = \begin{bmatrix} 0 & 0 & \theta \\ 0 & 0 & \theta \\ 0 & 0 & \theta \end{bmatrix}, \tag{37}$$

leading to Yukawa matrices of the form

$$\Gamma_1 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}. \tag{38}$$

A model where

$$\Theta = \begin{bmatrix} 0 & \theta & 0 \\ 0 & \theta & 0 \\ 0 & \theta & 0 \end{bmatrix}, \tag{39}$$

will be indistinguishable from Lavoura's model, as will a model where the θs move to the first column. Such permutations will further cut down the number of distinct models.

[1]Notice that the freedom to choose $\alpha_1 = 0$ does not reduce the number of possibilities.

For the up-type sector we define

$$\bar{\theta}_{ij} = \alpha_i - \gamma_j. \tag{40}$$

As before, for a matrix S characterized by a given $\theta \neq 0$, there are only three possibilities:

(1) $\bar{\theta}_{ij} = 0 \Rightarrow (\Delta_1)_{ij} = \text{any}$ and $(\Delta_2)_{ij} = 0$;

(2) $\bar{\theta}_{ij} = -\theta \Rightarrow (\Delta_1)_{ij} = 0$ and $(\Delta_2)_{ij} = \text{any}$;

(3) $\bar{\theta}_{ij} \neq 0, -\theta \Rightarrow (\Delta_1)_{ij} = 0 = (\Delta_2)_{ij}$.

All conditions on $\bar{\theta}_{ij}$ are mod(2π). Clearly we can choose independently $\bar{\theta}_{11}$, $\bar{\theta}_{12}$, and $\bar{\theta}_{13}$, and then

$$\begin{aligned} \bar{\theta}_{21} &= \theta_{21} - \theta_{11} + \bar{\theta}_{11} & \bar{\theta}_{22} &= \theta_{21} - \theta_{11} + \bar{\theta}_{12}, \\ \bar{\theta}_{23} &= \theta_{21} - \theta_{11} + \bar{\theta}_{13}, & \bar{\theta}_{31} &= \theta_{31} - \theta_{11} + \bar{\theta}_{11} \\ \bar{\theta}_{32} &= \theta_{31} - \theta_{11} + \bar{\theta}_{12}, & \bar{\theta}_{33} &= \theta_{31} - \theta_{11} + \bar{\theta}_{13}. \end{aligned} \tag{41}$$

There are 9 entries in the down-type Yukawa matrices. For each there are only three possibilities (the entry exists in Γ_1 but not in Γ_2; the entry exists in Γ_2 but not in Γ_1; the entry does not exist in either). The same occurs for the up-type Yukawa matrices. As a result, we would have potentially 3^{18} possibilities. But, as we have illustrated above, interchange and permutations help cut this number down. More importantly, many of the models entail massless quarks, a diagonal CKM matrix, or other inconsistencies with experiment. These are ruled out. This is what we turn to next.

III. MODEL CLASSIFICATION

A. The left-space

The left-handed space (where the left-handed quark doublets live) is rather constrained because it affects the down-type quark mass matrix, the up-type quark mass matrix, and also the CKM matrix. The quark mass matrices are obtained by bi-diagonalizing the matrices

$$\Gamma \equiv v_1 \Gamma_1 + v_2 \Gamma_2, \tag{42}$$

$$\Delta \equiv v_1^* \Delta_1 + v_2^* \Delta_2, . \tag{43}$$

whose two indices live in different spaces. But both indices of the Hermitian matrices

$$\begin{aligned} H_d &\equiv \Gamma\Gamma^\dagger \\ &= |v_1|^2 \Gamma_1 \Gamma_1^\dagger + |v_2|^2 \Gamma_2 \Gamma_2^\dagger + v_1 v_2^* \Gamma_1 \Gamma_2^\dagger + v_1^* v_2 \Gamma_2 \Gamma_1^\dagger \end{aligned} \tag{44}$$

$$\begin{aligned} H_u &\equiv \Delta\Delta^\dagger \\ &= |v_1|^2 \Delta_1 \Delta_1^\dagger + |v_2|^2 \Delta_2 \Delta_2^\dagger + v_1^* v_2 \Delta_1 \Delta_2^\dagger + v_1 v_2^* \Delta_2 \Delta_1^\dagger \end{aligned} \tag{45}$$

live on the left-space. These matrices can be diagonalized through unitary matrices V_{dL} and V_{uL} as

$$V_{dL} H_d V_{dL}^\dagger = D_d^2 = \text{diag}\{m_d^2, m_s^2, m_b^2\}, \tag{46}$$

$$V_{uL} H_u V_{uL}^\dagger = D_u^2 = \text{diag}\{m_u^2, m_c^2, m_t^2\}, \tag{47}$$

where $V = V_{uL} V_{dL}^\dagger$ is the CKM matrix.

We may now see the impact of the symmetry on the left-space and how it affects the quark masses and mixings. We start from Eq. (11) in the form

$$\Gamma_1 = S_L \Gamma_1 S_{nR}^\dagger, \qquad \Gamma_2 = S_L \Gamma_2 S_{nR}^\dagger e^{-i\theta}, \tag{48}$$

which, using the simplified form of S_L in Eq. (18), we can combine into

$$\begin{aligned} \Gamma_1 \Gamma_1^\dagger &= S_L \Gamma_1 \Gamma_1^\dagger S_L^\dagger \\ &= \begin{bmatrix} A_{11} & A_{12} e^{i\alpha_{12}} & A_{13} e^{-i\alpha_{31}} \\ A_{21} e^{-i\alpha_{12}} & A_{22} & A_{23} e^{i\alpha_{23}} \\ A_{31} e^{i\alpha_{31}} & A_{32} e^{-i\alpha_{23}} & A_{33} \end{bmatrix}, \end{aligned} \tag{49}$$

$$\begin{aligned} \Gamma_2 \Gamma_2^\dagger &= S_L \Gamma_2 \Gamma_2^\dagger S_L^\dagger \\ &= \begin{bmatrix} B_{11} & B_{12} e^{i\alpha_{12}} & B_{13} e^{-i\alpha_{31}} \\ B_{21} e^{-i\alpha_{12}} & B_{22} & B_{23} e^{i\alpha_{23}} \\ B_{31} e^{i\alpha_{31}} & B_{32} e^{-i\alpha_{23}} & B_{33} \end{bmatrix}, \end{aligned} \tag{50}$$

$$\begin{aligned} \Gamma_1 \Gamma_2^\dagger &= S_L \Gamma_1 \Gamma_2^\dagger S_L^\dagger e^{i\theta} \\ &= \begin{bmatrix} C_{11} e^{i\theta} & C_{12} e^{i(\alpha_{12}+\theta)} & C_{13} e^{-i(\alpha_{31}-\theta)} \\ C_{21} e^{-i(\alpha_{12}-\theta)} & C_{22} e^{i\theta} & C_{23} e^{i(\alpha_{23}+\theta)} \\ C_{31} e^{i(\alpha_{31}+\theta)} & C_{32} e^{-i(\alpha_{23}-\theta)} & C_{33} e^{i\theta} \end{bmatrix}, \end{aligned} \tag{51}$$

$$\begin{aligned} \Gamma_2 \Gamma_1^\dagger &= S_L \Gamma_2 \Gamma_1^\dagger S_L^\dagger e^{-i\theta} \\ &= \begin{bmatrix} D_{11} e^{-i\theta} & D_{12} e^{i(\alpha_{12}-\theta)} & D_{13} e^{-i(\alpha_{31}+\theta)} \\ D_{21} e^{-i(\alpha_{12}+\theta)} & D_{22} e^{-i\theta} & D_{23} e^{i(\alpha_{23}-\theta)} \\ D_{31} e^{i(\alpha_{31}-\theta)} & D_{32} e^{-i(\alpha_{23}+\theta)} & D_{33} e^{-i\theta} \end{bmatrix}. \end{aligned} \tag{52}$$

In the previous four equations, $A = \Gamma_1 \Gamma_1^\dagger$, $B = \Gamma_2 \Gamma_2^\dagger$, $C = \Gamma_1 \Gamma_2^\dagger$, and $D = \Gamma_2 \Gamma_1^\dagger$, respectively. We have defined

$$\alpha_{12} = \alpha_1 - \alpha_2, \qquad \alpha_{23} = \alpha_2 - \alpha_3, \qquad \alpha_{31} = \alpha_3 - \alpha_1, \tag{53}$$

which satisfy

$$\alpha_{12} + \alpha_{23} + \alpha_{31} = 0. \tag{54}$$

It is easy to see that the up-type Yukawa matrices satisfy identical equations, with $\theta \to -\theta$.

We define the set

$$\begin{aligned} \mathcal{J} &= \{x\colon x = 0(\text{mod}2\pi) \vee x = \theta(\text{mod}2\pi) \vee x \\ &= -\theta(\text{mod}2\pi)\}. \end{aligned} \tag{55}$$

If $\alpha_{12}, \alpha_{23}, \alpha_{31} \notin \mathcal{J}$, then the matrices $\Gamma_1\Gamma_1^\dagger$, $\Gamma_2\Gamma_2^\dagger$, $\Delta_1\Delta_1^\dagger$, and $\Delta_2\Delta_2^\dagger$ are diagonal, while all 12 and 21 combinations vanish. As a result, H_d and H_u are diagonal and the CKM matrix V is unity. This is ruled out by experiment.

As a result, at least one α_{ij} must belong to $\mathcal{J}$. Let us imagine that $\alpha_{12} \in \mathcal{J}$, while α_{23}, $\alpha_{31} \notin \mathcal{J}$. In that case, H_d and H_u are block-diagonal, so are the matrices V_{dL} and V_{uL}, and so is the CKM matrix V. This is also ruled out by experiment. We are left with the cases where

(1) one α_{ij} is not in $\mathcal{J}$, while the two others are in $\mathcal{J}$;
(2) all α_{ij} belong to $\mathcal{J}$. Next we study these cases in detail.

B. Odd one out

We look at the case where only one α_{ij} is not in $\mathcal{J}$. Let us take this to be $\alpha_{12} \notin \mathcal{J}$, α_{23}, $\alpha_{31} \in \mathcal{J}$. It is easy to see that the only possibilities that satisfy this requirement are $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{2\theta, -\theta, -\theta\}$ and $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{-2\theta, \theta, \theta\}$. The second possibility arises from the first through the interchange $\alpha_1 \leftrightarrow \alpha_2$. These symmetries act on the left of the Yukawa matrices and, thus, we go from one to the other by simply interchanging the first two rows of the corresponding Yukawa matrices. Similarly, the relevant cases where $\alpha_{23} \notin \mathcal{J}$, α_{31}, $\alpha_{13} \notin \mathcal{J}$, and $\alpha_{31} \notin \mathcal{J}$, α_{12}, $\alpha_{23} \in \mathcal{J}$ are related to the case shown here by mere permutations among the rows of the respective Yukawa matrices. As a result, we show only the case $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{2\theta, -\theta, -\theta\}$. Using Eqs. (29), we obtain $\theta_{21} = \theta_{11} - 2\theta$, $\theta_{31} = \theta_{11} - \theta$. From Eqs. (30) we get

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} - 2\theta & \theta_{12} - 2\theta & \theta_{13} - 2\theta \\ \theta_{11} - \theta & \theta_{12} - \theta & \theta_{13} - \theta \end{bmatrix}. \quad (56)$$

The entries of this matrix which equal $0 (\mathrm{mod} 2\pi)$ lead to corresponding entries in Γ_1; those which equal $\theta (\mathrm{mod} 2\pi)$ lead to corresponding entries in Γ_2; all others lead to vanishing entries in Γ_1, Γ_2, and, thus, in Γ. Recall that Γ cannot have a row of zeros nor a column of zeros; otherwise there would be a massless quark. This is a very powerful constraint. Let us consider the columns first. Since there must be at least one entry on each column, we conclude that $\theta_{1j} \in \{0, \theta, 2\theta, 3\theta\} (\mathrm{mod} 2\pi)$. This would seem to lead to 4^3 possibilities. However, if $\theta_{11} = \theta_{12} = \theta_{13}$, then there would be a (forbidden) row of zeros. The reason for this is that we are considering the case where $2\theta = \alpha_{12} \notin \mathcal{J}$, implying that $\theta \neq z_1 \pi$ and $\theta \neq z_2 2\pi/3$ with z_1 and z_2 integers—keeping the interval $[0, 2\pi]$, $\theta \notin \{0, 2\pi/3, \pi, 4\pi/3\}$. This means that $-\theta$, $\pm 2\theta$, and 3θ can never equal $0 (\mathrm{mod} 2\pi)$, nor can they equal $\theta (\mathrm{mod} 2\pi)$. Consider, for example, the possibility that $\theta_{11} = \theta_{12} = \theta_{13} = 3\theta$. Then, Θ would have 3θ on the first row, θ on the second row, and 2θ on the last row. Because 3θ and 2θ cannot equal 0 nor $\theta (\mathrm{mod} 2\pi)$, this would imply that the first and last rows of Γ_1, Γ_2, and Γ vanish, leading to massless quarks. Also, possibilities where two θ_{ij} are equal to 0 or to 3θ lead to a 2×2 block of zeros in Γ (implying massless quarks) and are, thus, excluded. There remain only eight independent forms for the Γ_i matrices ($\theta \notin \{0, 2\pi/3, \pi, 4\pi/3\}$):[2]

(i) $\theta_{11} = \theta_{12} = \theta, \theta_{13} = 2\theta \ (\mathrm{mod} 2\pi)$

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ & & \\ & & \mathrm{x} \end{bmatrix},$$
$$\theta \neq 2\pi/3, \pi, 4\pi/3; \quad (57)$$

(ii) $\theta_{11} = \theta_{12} = \theta, \theta_{13} = 3\theta \ (\mathrm{mod} 2\pi)$

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \\ \mathrm{x} & \mathrm{x} & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \\ & & \end{bmatrix},$$
$$\theta \neq 2\pi/3, \pi, 4\pi/3; \quad (58)$$

(iii) $\theta_{11} = \theta_{12} = 2\theta, \theta_{13} = 0 \ (\mathrm{mod} 2\pi)$

$$\Gamma_1 = \begin{bmatrix} & & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & & \\ \mathrm{x} & \mathrm{x} & \end{bmatrix},$$
$$\theta \neq 2\pi/3, \pi, 4\pi/3; \quad (59)$$

(iv) $\theta_{11} = \theta_{12} = 2\theta, \theta_{13} = \theta \ (\mathrm{mod} 2\pi)$

$$\Gamma_1 = \begin{bmatrix} & & \\ \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & & \\ \mathrm{x} & \mathrm{x} & \end{bmatrix},$$
$$\theta \neq 2\pi/3, \pi, 4\pi/3; \quad (60)$$

(v) $\theta_{11} = 0, \theta_{12} = 2\theta, \theta_{13} = \theta \ (\mathrm{mod} 2\pi)$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & & \\ & \mathrm{x} & \end{bmatrix},$$
$$\theta \neq 2\pi/3, \pi, 4\pi/3; \quad (61)$$

[2]Equations (57)–(64) are invariant under the symmetry for all θ, but they are only the most general forms consistent with the symmetry for those symmetries where $\theta \neq 2\pi/3, \pi, 4\pi/3$. See Sec. IV B for details.

(vi) $\theta_{11} = 0$, $\theta_{12} = 3\theta$, $\theta_{13} = \theta$ (mod2π)

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ & & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & \mathrm{x} & \\ & & \end{bmatrix}, \qquad \theta \neq 2\pi/3, \pi, 4\pi/3; \tag{62}$$

(vii) $\theta_{11} = 0$, $\theta_{12} = 2\theta$, $\theta_{13} = 3\theta$ (mod2π)

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ & \mathrm{x} & \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & & \mathrm{x} \\ & \mathrm{x} & \end{bmatrix}, \qquad \theta \neq 2\pi/3, \pi, 4\pi/3; \tag{63}$$

(viii) $\theta_{11} = \theta$, $\theta_{12} = 2\theta$, $\theta_{13} = 3\theta$ (mod2π)

$$\Gamma_1 = \begin{bmatrix} & & \\ & \mathrm{x} & \\ \mathrm{x} & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & & \\ & & \mathrm{x} \\ & \mathrm{x} & \end{bmatrix}, \qquad \theta \neq 2\pi/3, \pi, 4\pi/3. \tag{64}$$

The x denotes an allowed complex entry; vacant positions mean that the entry is zero. All other allowed cases with $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{2\theta, -\theta, -\theta\}$ are related to these by permutations among the columns. This corresponds to a mere renaming of the down-type right-handed fields $\{n_{R1}, n_{R2}, n_{R3}\}$, having no physical significance. As explained above, all permutations of the rows correspond to physically allowed cases other than $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{2\theta, -\theta, -\theta\}$. As a result, all column and row permutations of the matrices in Eqs. (57)–(64) correspond to physically allowed models; permutations on columns have no physical effect; permutations on rows also have no physical effect but must be performed simultaneously on the down-type matrices Γ and on the up-type matrices Δ. For example, a specific THDM with Z_4 was proposed in [6] in the context of nearest neighbor interaction matrices, corresponding to our Eq. (61) for both the up-type and down-type quarks (with rows 1 and 2 interchanged).

C. All in $\mathcal{J}$

We now turn to the cases where α_{12}, α_{23}, $\alpha_{31} \in \mathcal{J}$. This means that each α_{ij} can only take the values 0, θ, or $-\theta$ (mod2π). There would seem to be 3^3 possibilities. But Eq. (54) allows us to exclude a few. For example, taking $-\alpha_{12} = \alpha_{23} = \alpha_{31} = \theta(\mathrm{mod}2\pi)$ into Eq. (54) would mean that $\theta = 0(\mathrm{mod}2\pi)$, a case we are not considering since it corresponds to unconstrained scalar fields: $\Phi_1 \to \Phi_1$, $\Phi_2 \to \Phi_2$. There are some cases which are possible only for specific values of θ. Postponing those for subsections III C 4 and III C 5, we are left with the following cases: (i) $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, 0, 0\}$; (ii) $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, -\theta, \theta\}$ (interchanging rows on the Yukawa matrices for this case one reaches the cases $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{-\theta, \theta, 0\}$) and $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{\theta, 0, -\theta\}$); and (iii) $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, \theta, -\theta\}$ (interchanging rows on the Yukawa matrices for this case one reaches the cases $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{\theta, -\theta, 0\}$) and $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{-\theta, 0, \theta\}$).

1. $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, 0, 0\}$ *and any* θ

In this case, $\alpha_1 = \alpha_2 = \alpha_3$ and $\theta_{11} = \theta_{21} = \theta_{31}$, leading to

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} & \theta_{12} & \theta_{13} \end{bmatrix}. \tag{65}$$

Because a column of zeros in both Γ_1 and Γ_2 would lead to massless quarks, we must have $\theta_{1j} \in \{0, \theta\}$. There are 2^3 possibilities; each column must exist in either Γ_1 or Γ_2. Ignoring cases which differ only by permutation of the columns, we are left with the following structures:

(i) All θ_{1j} equal 0

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}, \qquad \text{any } \theta; \tag{66}$$

(ii) Two θ_{1j} equal 0

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & & \mathrm{x} \\ & & \mathrm{x} \end{bmatrix}, \qquad \text{any } \theta; \tag{67}$$

(iii) One θ_{1j} equals 0

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ \mathrm{x} & & \\ \mathrm{x} & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \mathrm{x} \\ & \mathrm{x} & \mathrm{x} \\ & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \text{any } \theta. \tag{68}$$

This is the same as Eq. (67), with the interchange $\Phi_1 \leftrightarrow \Phi_2$.

(iv) No θ_{1j} equals 0

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix}, \quad \text{any } \theta. \tag{69}$$

This is the same as Eq. (66), with the interchange $\Phi_1 \leftrightarrow \Phi_2$.

2. $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, -\theta, \theta\}$ *and any* θ

Here[3] $\theta_{21} = \theta_{11}$, $\theta_{31} = \theta_{11} + \theta$, and

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} + \theta & \theta_{12} + \theta & \theta_{13} + \theta \end{bmatrix}, \tag{70}$$

implying that $\theta_{1j} \in \{0, \theta, -\theta\}$. Ignoring cases which differ only by permutation of the columns, we are left with the following structures:

(i) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, 0\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & & \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix}, \quad \text{any } \theta; \tag{71}$$

(ii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, \theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \end{bmatrix}, \quad \theta \neq \pi; \tag{72}$$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \end{bmatrix}, \quad \theta = \pi. \tag{73}$$

The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{\theta, 0, 0\}$ and $\{0, \theta, 0\}$ are obtained from these through column permutations.

(iii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, \theta, \theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ \mathrm{x} & & \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \mathrm{x} \\ & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & & \end{bmatrix}, \quad \theta \neq \pi; \tag{74}$$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ \mathrm{x} & & \\ & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \mathrm{x} \\ & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & & \end{bmatrix}, \quad \theta = \pi. \tag{75}$$

The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{\theta, \theta, 0\}$ and $\{\theta, 0, \theta\}$ are obtained from these through column permutations.

(iv) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, -\theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & & \\ \mathrm{x} & \mathrm{x} & \end{bmatrix}, \quad \theta \neq \pi. \tag{76}$$

Setting $\theta = \pi$ we reobtain Eq. (73). The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{0, -\theta, 0\}$ and $\{-\theta, 0, 0\}$ are obtained from these through column permutations.

(v) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, \theta, -\theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ \mathrm{x} & & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \\ & \mathrm{x} & \\ \mathrm{x} & & \end{bmatrix}, \quad \theta \neq \pi. \tag{77}$$

Setting $\theta = \pi$ we reobtain Eq. (75). The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{0, -\theta, \theta\}$, $\{\theta, -\theta, 0\}$, $\{\theta, 0, -\theta\}$, $\{-\theta, 0, \theta\}$, and $\{-\theta, \theta, 0\}$ are obtained from these through column permutations.

(vi) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{\theta, \theta, -\theta\}$

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ & & \end{bmatrix}, \quad \theta \neq \pi; \tag{78}$$

[3]Recall that the cases $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{-\theta, \theta, 0\}$) and $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{\theta, 0, -\theta\}$ are obtained from this through permutations on the rows of the Yukawa matrices.

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ & & \end{bmatrix}, \quad \theta = \pi. \tag{79}$$

The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{\theta, -\theta, \theta\}$, and $\{-\theta, \theta, \theta\}$ are obtained from these through column permutations.

For those wishing to check that all possibilities have been considered, we refer to the footnote.[4]

3. $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, \theta, -\theta\}$ *and any* θ

Here[5] $\theta_{21} = \theta_{11}$, $\theta_{31} = \theta_{11} - \theta$, and

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} - \theta & \theta_{12} - \theta & \theta_{13} - \theta \end{bmatrix}, \tag{80}$$

implying that $\theta_{1j} \in \{0, \theta, 2\theta\}$.

Ignoring cases which differ only by permutation of the columns, we are left with the following structures:

(i) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, \theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \mathrm{x} \\ & & \mathrm{x} \\ & & \end{bmatrix}, \quad \theta \neq \pi; \tag{81}$$

Performing $\Phi_1 \leftrightarrow \Phi_2$ and exchanging the first and third columns on Eq. (81) we obtain Eq. (74). Setting $\theta = \pi$ in this case would lead directly to Eq. (73). The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{0, \theta, 0\}$ and $\{\theta, 0, 0\}$ are obtained from these through column permutations.

(ii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, \theta, \theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ \mathrm{x} & & \\ & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \mathrm{x} \\ & \mathrm{x} & \mathrm{x} \\ & & \end{bmatrix}, \quad \theta \neq \pi; \tag{82}$$

Performing $\Phi_1 \leftrightarrow \Phi_2$ and exchanging the first and third columns on Eq. (82) we obtain Eq. (72). Setting $\theta = \pi$ in this case would lead directly to Eq. (75). The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{\theta, 0, \theta\}$ and $\{\theta, \theta, 0\}$ are obtained from these through column permutations.

(iii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, 2\theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ \mathrm{x} & \mathrm{x} & \\ & & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & & \\ & & \mathrm{x} \end{bmatrix}, \quad \theta \neq \pi; \tag{83}$$

Performing $\Phi_1 \leftrightarrow \Phi_2$ on Eq. (83) we obtain Eq. (78). Setting $\theta = \pi$ in this case would lead directly to the special case of $\theta = \pi$ in Eq. (71). The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{0, 2\theta, 0\}$ and $\{2\theta, 0, 0\}$ are obtained from these through column permutations.

(iv) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, \theta, 2\theta\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ \mathrm{x} & & \\ & \mathrm{x} & \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \\ & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \quad \theta \neq \pi. \tag{84}$$

Performing $\Phi_1 \leftrightarrow \Phi_2$ and exchanging the first and second columns on Eq. (84) we obtain Eq. (77). Setting $\theta = \pi$ in this case would lead to Eq. (73), after interchanging the second and third columns. The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{0, 2\theta, \theta\}$, $\{\theta, 2\theta, 0\}$, $\{\theta, 0, 2\theta\}$, $\{2\theta, 0, \theta\}$, and $\{2\theta, \theta, 0\}$ are obtained from these through column permutations.

(v) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{\theta, \theta, \theta\}$

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ & & \end{bmatrix}, \quad \text{any } \theta. \tag{85}$$

Performing $\Phi_1 \leftrightarrow \Phi_2$ on Eq. (85) we obtain Eq. (71). Notice that the special case of $\theta = \pi$ had already shown up in Eq. (79).

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[4]We have also checked that

(i) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{0, -\theta, -\theta\}$, $\{-\theta, 0, -\theta\}$, and $\{-\theta, -\theta, 0\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (75), if $\theta = \pi$;

(ii) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{\theta, \theta, \theta\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (79), if $\theta = \pi$;

(iii) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{\theta, -\theta, -\theta\}$, $\{-\theta, \theta, -\theta\}$, and $\{-\theta, -\theta, \theta\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (79), if $\theta = \pi$; The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{-\theta, -\theta, -\theta\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (79), if $\theta = \pi$.

[5]Recall that the cases $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{\theta, -\theta, 0\}$) and $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{-\theta, 0, \theta\}$ are obtained from this through permutations on the rows of the Yukawa matrices.

(vi) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{\theta, \theta, 2\theta\}$

$$\Gamma_1 = \begin{bmatrix} & & \\ & & \\ \times & \times & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} \times & \times & \\ \times & \times & \\ & & \times \end{bmatrix}, \quad \theta \neq \pi; \tag{86}$$

Performing $\Phi_1 \leftrightarrow \Phi_2$ on Eq. (86) we obtain Eq. (76). Setting $\theta = \pi$ in this case would lead to Eq. (75), after interchanging the first and third columns. The cases with $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal to $\{\theta, 2\theta, \theta\}$ and $\{2\theta, \theta, \theta\}$ are obtained from these through column permutations.

For those wishing to check that all possibilities have been considered, we refer to the footnote.[6]

4. *Special cases with $\theta = \pi$*

We continue to explore the cases where each α_{ij} can only take the values 0, θ, or $-\theta$ (mod2π). Certain cases are only valid for $\theta = \pi$. For example, consider $\alpha_{12} = 0(\text{mod}2\pi)$ and $\alpha_{23} = \alpha_{31} = \theta(\text{mod}2\pi)$. Taking $\theta \in [0, 2\pi]$, this can only happen for $\theta = \pi$, due to Eq. (54). This forces us to consider the case $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{0, \pi, \pi\}$. The cases $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{\pi, 0, \pi\}$ and $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\} = \{\pi, \pi, 0\}$ are obtained from this by permuting the rows on the respective Yukawa matrices. In this case, $\theta_{21} = \theta_{11}$, $\theta_{31} = \theta_{11} + \pi$, and

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} + \pi & \theta_{12} + \pi & \theta_{13} + \pi \end{bmatrix}, \tag{87}$$

implying that $\theta_{1j} \in \{0, \pi\}$. There are 2^3 such cases, all of which lead to a matrix Γ where all entries may be non-vanishing.[7] We continue to ignore cases which differ only by permutation of the columns. It is easy to see that we have already considered all possible structures. Indeed, when all θ_{1j} equal π, we recover Eq. (71); when two θ_{1j} equal π, we recover Eq. (73); when only one θ_{1j} equals π, we recover Eq. (75); and when no θ_{1j} equals π, we recover Eq. (79).

5. *Special cases with $\theta = 2\pi/3$*

We now turn to the last two cases where each α_{ij} can only take the values 0, θ, or $-\theta$ (mod2π). Because of Eq. (54), we can have $\alpha_{12} = \alpha_{23} = \alpha_{31} = \pm\theta$ if and only if $\theta = 2\pi/3$. The case $\alpha_{12} = \alpha_{23} = \alpha_{31} = -2\pi/3$ (or, which is the same, $4\pi/3$) is obtained by exchanging any two rows of the Yukawa matrices. We choose the case $\alpha_{12} = \alpha_{23} = \alpha_{31} = 2\pi/3$, implying that $\theta_{21} = \theta_{11} - 2\pi/3$, $\theta_{31} = \theta_{11} + 2\pi/3$, and

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} - 2\pi/3 & \theta_{12} - 2\pi/3 & \theta_{13} - 2\pi/3 \\ \theta_{11} + 2\pi/3 & \theta_{12} + 2\pi/3 & \theta_{13} + 2\pi/3 \end{bmatrix}, \tag{88}$$

implying that $\theta_{1j} \in \{0, 2\pi/3, -2\pi/3\}$. Recall that $\theta_{11} = \theta_{12} = \theta_{13}$ is excluded because it would lead to massless quarks.

Ignoring cases which differ only by permutation of the columns, we are left with the following structures:

(i) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, 2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} \times & \times & \times \\ & & \times \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & & \times \\ & & \\ \times & \times & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{89}$$

(ii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} \times & \times & \\ & & \\ & & \times \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & & \\ & & \times \\ \times & \times & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{90}$$

(iii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 2\pi/3, 2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} \times & & \\ & \times & \times \\ & & \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} & \times & \times \\ & & \\ \times & & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{91}$$

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[6] We have also checked that

(i) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{0, 0, 0\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (71), if $\theta = \pi$;

(ii) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{0, 2\theta, 2\theta\}$, $\{2\theta, 0, 2\theta\}$, and $\{2\theta, 2\theta, 0\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (71), if $\theta = \pi$;

(iii) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{\theta, 2\theta, 2\theta\}$, $\{2\theta, \theta, 2\theta\}$, and $\{2\theta, 2\theta, \theta\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (73), if $\theta = \pi$;

(iv) The cases where $\{\theta_{11}, \theta_{12}, \theta_{13}\}$ equal $\{2\theta, 2\theta, 2\theta\}$ lead to vanishing quark masses, if $\theta \neq \pi$, and to Eq. (71), if $\theta = \pi$.

[7] Of course, some entry may be zero by accident. The point is that this value is not required by a symmetry of this type and, as such, it is not invariant under the renormalization group equations.

(iv) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & \mathrm{x} & \\ & & \mathrm{x} \\ \mathrm{x} & & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{92}$$

(v) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, -2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} \mathrm{x} & & \\ & & \\ & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} & & \\ & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{93}$$

(vi) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{2\pi/3, 2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} & & \\ \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & \mathrm{x} & \\ & & \mathrm{x} \\ & & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{94}$$

(vii) $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{2\pi/3, -2\pi/3, -2\pi/3\}$

$$\Gamma_1 = \begin{bmatrix} & & \\ \mathrm{x} & & \\ & \mathrm{x} & \mathrm{x} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \mathrm{x} & & \\ & \mathrm{x} & \mathrm{x} \\ & & \end{bmatrix}, \quad \theta = 2\pi/3; \tag{95}$$

Care must be exercised when comparing these matrices with those shown previously. Consider, for example, Eq. (89). $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, 2\pi/3\}$, with $\{\theta_{21}, \theta_{31}\} = \{\theta_{11} - 2\pi/3, \theta_{11} + 2\pi/3\} = \{-2\pi/3, 2\pi/3\}$. We might worry about Eq. (72), where one can also choose $\{\theta_{11}, \theta_{12}, \theta_{13}\} = \{0, 0, 2\pi/3\}$. However, there, $\{\theta_{21}, \theta_{31}\} = \{\theta_{11}, \theta_{11} + 2\pi/3\} = \{0, 2\pi/3\}$.

D. Yukawa matrices for up-type quarks

So far, we have only shown the Yukawa matrices for the down-type quarks. We will now show that it is trivial to get the Yukawa matrices for the up-type quarks from those for the down-type quarks. Let us start from some specific transformation of the left-handed fields, characterized by α_{12} and α_{31}. From Eqs. (30) and (41) we get $\theta_{21} = \theta_{11} - \alpha_{12}$, $\theta_{31} = \theta_{11} + \alpha_{31}$, so that

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{11} - \alpha_{12} & \theta_{12} - \alpha_{12} & \theta_{13} - \alpha_{12} \\ \theta_{11} + \alpha_{31} & \theta_{12} + \alpha_{31} & \theta_{13} + \alpha_{31} \end{bmatrix}, \tag{96}$$

$$\bar{\Theta} = \begin{bmatrix} \bar{\theta}_{11} & \bar{\theta}_{12} & \bar{\theta}_{13} \\ \bar{\theta}_{11} - \alpha_{12} & \bar{\theta}_{12} - \alpha_{12} & \bar{\theta}_{13} - \alpha_{12} \\ \bar{\theta}_{11} + \alpha_{31} & \bar{\theta}_{12} + \alpha_{31} & \bar{\theta}_{13} + \alpha_{31} \end{bmatrix}. \tag{97}$$

Each entry on the column j of Θ is of the form $\theta_{1j} + b$. We then followed the procedure

$$\begin{aligned} \theta_{1j} + b = 0(\mathrm{mod} 2\pi) &\Rightarrow \text{entry is in } \Gamma_1, \\ \theta_{1j} + b = \theta(\mathrm{mod} 2\pi) &\Rightarrow \text{entry is in } \Gamma_2. \end{aligned} \tag{98}$$

Let us call $\bar{\theta}_{1j} = \theta_{1j} - \theta$. Then, if $\theta_{1j} + b = 0(\theta)$, we find $\bar{\theta}_{1j} + b = -\theta(0)$, meaning that this is an entry in Δ_2 (Δ_1). Thus

$$\begin{aligned} \bar{\theta}_{1j} + b = (\theta_{1j} - \theta) + b = -\theta(\mathrm{mod} 2\pi) &\Rightarrow \text{entry is in } \Delta_2, \\ \bar{\theta}_{1j} + b = (\theta_{1j} - \theta) + b = 0(\mathrm{mod} 2\pi) &\Rightarrow \text{entry is in } \Delta_1. \end{aligned} \tag{99}$$

The argument goes both ways, so we can find all cases for the up-type Yukawa matrices Δ by starting from all cases for the down-type Yukawa matrices Γ and performing the following procedure:

(i) $\theta_{1j} \to \bar{\theta}_{1j} = \theta_{1j} - \theta$;
(ii) $\Gamma_1 \to \Delta_2$;
(iii) $\Gamma_2 \to \Delta_1$.

Of course, one can shuffle differently the columns of $\{\Gamma_1, \Gamma_2\}$ and $\{\Delta_2, \Delta_1\}$, since they live on different right-handed spaces.

E. Counting the number of models

The only purpose of our parameter counting is to show the enormous amount of cases which have been killed by the simple requirements that there be no massless quarks and that the CKM matrix not be block-diagonal. As pointed out at the end of Sec. II, there are potentially $3^{18} = 387.420.489$ different models. Notice that this number does not include permutations that lead to the same form for the Yukawa matrices. But, it does include permutations which, although leading to different forms of the Yukawa matrices, have no impact on the physical observables. This same procedure must be followed when we count the number of distinct forms of the Yukawa matrices based on the analysis of the previous sections.

The forms shown in Sec. III B correspond to $6_L \times (3 + 3 + 3 + 3 + 6 + 6 + 6 + 6)_{nR} \times (3 + 3 + 3 + 3 + 6 + 6 + 6 + 6)_{pR} = 7776$. The subindices L, nR, and pR correspond to the permutations of rows, down-type columns, and up-type columns (respectively), that lead to the same physics. But, as in the 3^{18} possibilities above, the counting has been performed so that no two structures look the same. The numbers in $(3+3+3+3+6+6+6+6)_{nR}$ correspond to the number of possibilities in Eqs. (57)–(64), respectively.

To be specific, let us look at Eq. (57). Exchanging the first and second column leaves the form invariant. This is counted as one structure. However, exchanging the third and first columns leads to a new structure. So does an exchange between the third and second column. There are thus three possibilities. This explains the first "3" in $(3+3+3+3+6+6+6+6)_{nR}$. The rest of the counting procedure follows the same lines.

The forms shown in Sec. III C 1 correspond to $1_L \times (1+3+3+1)_{nR} \times (1+3+3+1)_{pR} = 64$. The forms shown in Sec. III C 2 with $\theta \neq \pi$ correspond to $3_L \times (1+3+3+6+3+3)_{nR} \times (1+3+3+6+3+3)_{pR} = 1083$. The forms shown in Sec. III C 2 with $\theta = \pi$ correspond to $3_L \times (3+3+3)_{nR} \times (3+3+3)_{pR} = 243$. The forms shown in Sec. III C 3 correspond to $3_L \times (1+3+3+6+3+3)_{nR} \times (1+3+3+6+3+3)_{pR} = 1083$. Finally, forms shown in Sec. III C 5 correspond to $6_L \times (3+3+3+6+3+3+3)_{nR} \times (3+3+3+6+3+3+3)_{pR} = 3456$. There are thus 13 705 distinct surviving possibilities.

This may seem like a large number, but notice that we have eliminated 387 406 784 *a priori* conceivable Yukawa structures. The simple requirements of quarks with nonzero mass and a CKM matrix which is not block-diagonal provides a drastic reduction in the number of possibilities. Said otherwise, the huge majority of Yukawa matrices consistent with Abelian symmetries do not survive simple experimental constraints. We should also point out that any two structures which differ only by permutations of the rows (simultaneously in Γ and Δ), and/or by permutations of the columns of Γ, and/or by permutations of the columns of Δ give exactly the same physics. Permutations aside, we are left with the $8+4+9+6+7=34$ possibilities for the down-type Yukawa matrices shown in Eqs. (57)–(64), (66)–(69), (71)–(79), (81)–(86), and (89)–(95), with similar structures for the up-type Yukawa matrices. Combining appropriately, we get $8\times 8 + 4\times 4 + 9\times 9 + 6\times 6 + 7\times 7 = 246$ overall models. Those that differ only by $\Phi_1 \leftrightarrow \Phi_2$ will lead to the same physics. Of those, a few can be further excluded because they do not yield any *CP* violation. The possibility of spontaneous *CP* violation will be addressed in Sec. V.

IV. TWO IMPORTANT RESULTS

A. Most discrete symmetries have the same impact

We have considered a symmetry in the scalar sector $S = \mathrm{diag}\{1, e^{i\theta}\}$. Of course, if the Lagrangian is invariant under S, it is invariant under any power of S. In this way, if $\theta = 2\pi/n$, then the Z_n group is generated. If $\theta \neq 2\pi/n$, then one generates a discrete, but infinite, group. For simplicity we will refer to the Z_n groups in what follows.

We now turn to an important result from our previous analysis. We know that choosing $\theta = 2\pi/3$ or $\theta = 2\pi/5$ leads to the same Higgs potential. Indeed, any $\theta \neq 0, \pi$ leads to the same Higgs potential as the continuous $U(1)$ Peccei-Quinn symmetry [3]. From this point of view, applying any Z_n ($n \geq 3$), or even $U(1)$ is the same. With the results presented in the previous section, we see that this is no longer the case when the fermions are added. As shown here, the symmetry Z_3 allows Yukawa structures not allowed for other Z_n. Remarkably, all Z_n with $n \geq 4$ have the same impact on the full Lagrangian, even when fermions are introduced.

B. Most discrete symmetries imply an accidental continuous symmetry

The notation $\theta \neq 2\pi/3, \pi, 4\pi/3$ used in Eqs. (57)–(64) means that the form of the matrices shown is the most general consistent with values of θ which differ from $2\pi/3$, π, and $4\pi/3$. But one should notice that the form of the matrices shown is left invariant even if $\theta = 2\pi/3, \pi, 4\pi/3$. The point is that, in general, for those special values of θ these matrix forms are *not the most general* consistent with the symmetries. For example, Eq. (57) is not the most general matrix consistent with $\theta_{11} = \theta_{12} = \theta$, $\theta_{13} = 2\theta$ (mod2π) when $\theta = \pi$. That form is shown in Eq. (75). But one can see that, indeed, Eq. (57) is a particular case of Eq. (75). So, Eqs. (57)–(64) are invariant under the symmetry for all θ, but they are only the most general forms consistent with the symmetry for those symmetries where $\theta \neq 2\pi/3, \pi, 4\pi/3$. The dedicated reader can check this explicitly by comparing these forms with the forms presented for the special cases $\theta = \pi$ and $\theta = 2\pi/3$.

This has a very important consequence. A matrix form which is invariant under the symmetry for some value of $\theta \neq 2\pi/3, \pi, 4\pi/3$ will be invariant under the symmetry for all values of θ, meaning that the Yukawa sector will be invariant under $U(1)$. Since this is also true for the Higgs potential, we conclude that, for the cases in Sec. III B: (i) Imposing Z_2 on the scalars does not imply a larger symmetry, neither in the Higgs sector, nor in the Yukawa sector; (ii) Imposing Z_3 on the scalars implies a continuous symmetry in the Higgs sector, but not in the Yukawa sector; (iii) Imposing Z_n, $n \geq 4$ on the scalars implies a continuous symmetry, both in the Higgs sector and in the Yukawa sector.

The other cases can be analyzed in a similar fashion. For the cases in Sec. III C 1: (i) Imposing Z_2 on the scalars implies a continuous symmetry in Yukawa the sector, but not in the Higgs sector; (ii) Imposing Z_n, $n \geq 3$ on the scalars implies a continuous symmetry, both in the Higgs sector and in the Yukawa sector. For the cases in Secs. III C 2 and III C 3: (i) Imposing Z_2 on the scalars does not imply a larger symmetry, neither in the Higgs sector, nor in the Yukawa sector; (ii) Imposing Z_n, $n \geq 3$ on the scalars implies a continuous symmetry, both in the Higgs sector and in the Yukawa sector.

V. SPONTANEOUS *CP* VIOLATION

A. Strict two-Higgs-doublet model

Let us now look at the possible vacua of a theory with only two Higgs doublets and three fermion generations, and their implications for *CP* violation at the Lagrangian level. We are interested in implementations of discrete Abelian symmetries, like Z_n, for which the scalar potential of Eq. (1) can be written as

$$\begin{aligned} V = {} & m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - m_{12}^2 (\Phi_1^\dagger \Phi_2 + \text{H.c.}) \\ & + \frac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 \\ & + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\ & + \frac{1}{2}\lambda_5 [(\Phi_1^\dagger \Phi_2)^2 + \text{H.c.}], \end{aligned} \tag{100}$$

where all the parameters are real. We have included the soft-breaking parameter m_{12}^2, taken to be real so that *CP* is not explicitly broken. For a Z_2 symmetry—$\theta = \pi$ in Eq. (23)—the λ_5 coupling is present in the potential. For Z_n, $n \geq 3$, or indeed any other value for θ different from 0 or π, the symmetry sets λ_5 to zero and the potential is indistinguishable from the Peccei-Quinn one [7]. At the minimum, the scalar fields develop vevs which we take to be given by, without loss of generality

$$\langle \Phi_1 \rangle = v_1 = u_1, \qquad \langle \Phi_2 \rangle = v_2 = u_2 + i u_3, \tag{101}$$

with all u_i real. A vacuum with $u_3 \neq 0$ *may* lead to spontaneous *CP* violation (SCPV) in the scalar sector—however, the presence of a phase in the vacuum is no guarantee of SCPV. To verify whether SCPV occurs in the scalar sector, we must calculate the basis invariant quantities of Ref. [8], which was done for all possible THDM scalar potentials in [9]. The minimization conditions are given by $\partial V / \partial u_i = 0$, from which we obtain

$$\begin{aligned} 0 = {} & [m_{11}^2 + \lambda_1 u_1^2 + (\lambda_3 + \lambda_4)(u_2^2 + u_3^2) \\ & + \lambda_5 (u_2^2 - u_3^2)] u_1 - m_{12}^2 u_2 \end{aligned} \tag{102}$$

$$0 = [m_{22}^2 + \lambda_2 (u_2^2 + u_3^2) + (\lambda_3 + \lambda_4 + \lambda_5) u_1^2] u_2 - m_{12}^2 u_1 \tag{103}$$

$$0 = [m_{22}^2 + \lambda_2 (u_2^2 + u_3^2) + (\lambda_3 + \lambda_4 - \lambda_5) u_1^2] u_3. \tag{104}$$

From these we see that solutions with $u_3 = 0$ are always possible. There are several interesting cases:

(i) $\theta = \pi$, exact Z_2 symmetry ($m_{12}^2 = 0$, $\lambda_5 \neq 0$): from Eqs. (104) and (103), any solution with $u_3 \neq 0$ automatically implies either $u_1 = 0$ or $u_2 = 0$. Both solutions lead to no SCPV in the scalar sector (see [9]).

(ii) $\theta = \pi$, soft-broken Z_2 symmetry (m_{12}^2, $\lambda_5 \neq 0$): both solutions without SCPV in the scalar sector ($u_3 = 0$) and with SCPV in the scalar sector ($u_3 \neq 0$) are possible, depending on the values of potential's parameters [10].

(iii) $\theta \neq \{0, \pi\}$, exact $U(1)$ symmetry ($m_{12}^2 = \lambda_5 = 0$): the equations above only determine the sum $u_2^2 + u_3^2$, and as such the relative phase of the vevs is arbitrary. These vacua lead to no SCPV in the scalar sector [9] and in fact generate an axion.

(iv) $\theta \neq \{0, \pi\}$, soft-broken $U(1)$ symmetry ($m_{12}^2 \neq 0$, $\lambda_5 = 0$): from Eqs. (104) and (103), we see that any solution with $u_3 \neq 0$ leads to $u_1 = 0$ which, considering Eq. (102), also implies $u_2 = 0$. Thus, no SCPV vacuum can occur in this case. Vacua with $u_3 = 0$ possess no axion.

The existence of an axion in one of the cases above is easy to understand: as was explained earlier, the imposition of a discrete symmetry with $\theta \neq \{0, \pi\}$ (for instance a Z_n symmetry with $n \geq 3$) on the scalar potential leads to an accidental Peccei-Quinn continuous $U(1)$ symmetry. Any vacuum for which both fields acquire a vev will break that symmetry and lead to a zero mass for the pseudoscalar. This corresponds in fact to the appearance of an additional Goldstone boson (other than the three usual ones arising from the breaking of the gauge symmetry). Analytically, the pseudoscalar mass is given by

$$m_A^2 = \frac{v^2}{u_1 u_2} m_{12}^2 - 2\lambda_5 v^2, \tag{105}$$

with $v^2 = u_1^2 + u_2^2$, for vacua with $u_3 = 0$.[8] From this we see that: the Z_2 potential will never lead to an axion, since $\lambda_5 \neq 0$; the exact $U(1)$ symmetry forces this mass to be zero; and the soft-broken Z_n potential again has no axion, as the pseudoscalar mass is directly proportional to the soft-breaking parameter.

The scalar vevs originate the fermion masses, but also have a contribution to *CP* breaking at the Lagrangian level, whether they are real or complex. In fact, the Jarlskog invariant, which measures *CP* violation in the weak interactions, is given by [11]

$$\begin{aligned} J &= \text{Tr}[H_u, H_d]^3 \\ &= 6i(m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2) \\ &\quad \times (m_b^2 - m_d^2)(m_s^2 - m_d^2)\text{Im}(V_{us} V_{cb} V_{ub}^* V_{cs}^*), \end{aligned} \tag{106}$$

where the matrices H_d and H_u have been defined in Eqs. (44) and (45). In the SM, since no *CP* breaking can

[8] In the case of the exact $U(1)$ symmetry an arbitrary phase between the vevs is possible, but it has no effect on the scalar masses whatsoever.

TABLE I. Possibilities of CP violation for THDM with Abelian symmetries. “Yes” means that the model’s parameters can generate a nonzero value for the Jarlskog invariant. The “$U(1)$” models are those for which one has imposed a discrete symmetry of the form of Eq. (23), with $\theta \neq 0, \pi$.

Model	Lagrangian with explicit CP breaking	CP-conserving Lagrangian
Exact Z_2	Yes	No—real vacuum or vev phase gives $J = 0$
Soft-broken Z_2	Yes	Yes
Exact $U(1)$	Yes	No—vacuum gives axion
Soft-broken $U(1)$	Yes	No—vacuum with phase impossible

arise spontaneously, it is explicitly broken with complex Yukawa couplings. In the THDM, we can study models where one has demanded that the full Lagrangian be CP invariant, such that the matrices Γ_i and Δ_i will be real, and the only possibility of producing a nonzero Jarlskog invariant will be the vevs having a relative phase. Since such a vacuum is impossible for the soft-broken $U(1)$ scalar potential, we conclude that models with an Abelian symmetry (other than Z_2) and with an explicit CP conservation are ruled out, since for them J would always be zero. Nonetheless, there is a distinction worth making: the special forms found for the matrices with $\theta = 2\pi/3$ (Z_3 symmetry), given in Sec. III E, would give a nonzero Jarlskog invariant *if a vacuum with a complex phase could be produced*; all the other Yukawa matrices we have obtained for the cases $\theta \neq \pi, 2\pi/3$ give $J = 0$ *even if a complex vacuum existed*. As such, the only models allowed are those, like the SM, where CP is explicitly broken by the Yukawa couplings.

As for the Z_2 model, the *exact* symmetry is also ruled out when CP is explicitly preserved—no phase from the vevs can originate $J \neq 0$, even for the odd case $u_2 = 0$, allowed by Eqs. (102)–(104): in that case, there is a phase of $\pi/2$ in the vaccum, but it has no bearing on J, which gives zero. In the soft-broken Z_2 model, a vacuum with a relative phase in the vevs may be obtained and it leads to CP violation, both in the scalar and the Yukawa sectors [10]. And as before, Z_2 models with explicit CP breaking are in principle perfectly viable. We summarize this analysis in Table I.

A few observations are in order:

(i) We have not considered in this analysis the so-called “inert vacua”, where either $\langle\Phi_1\rangle = 0$ or $\langle\Phi_2\rangle = 0$, possible in the case of exact symmetries (Z_2 or $U(1)$). These give an acceptable J only in the case of explicit CP breaking.

(ii) The Z_3 case is special. Let us again consider the case of explicit CP conservation. Unlike the remaining symmetries with $\theta \neq \pi, 2\pi/3$, a vacuum with complex vevs would give $J \neq 0$. Such a vacuum is impossible in the THDM, but one can conceive (like the authors of [12] did) models with two doublets and additional gauge singlets, capable of producing the desired form for the vevs [13].

B. Complex vacua and the Jarlskog invariant

The vacua of a Z_n potential may be easily altered by introducing soft-breaking terms, as discussed in the previous section, or by the inclusion of extra singlet scalars.

TABLE II. We assume that $\theta = \pi$, that all Yukawa entries are real, that the vevs have a relative complex phase, and we calculate J. The down-type Yukawas were chosen according to the equations along the first line, and the up-type Yukawas were chosen according to the equations along the first column. We denote the entries where $J = 0$, all others allow for $J \neq 0$, depending on the values of the parameters.

Equations for Yukawa matrices	(71)	(73)	(75)	(79)
(71)	0			
(73)				
(75)				
(79)				0

TABLE III. We assume that $\theta = 2\pi/3$, that all Yukawa entries are real, that the vevs have a relative complex phase, and we calculate J. The down-type Yukawas were chosen according to the equations along the first line, and the up-type Yukawas were chosen according to the equations along the first column. We denote the entries where $J = 0$, all others allow for $J \neq 0$, depending on the values of the parameters.

Equations for Yukawa matrices	(89)	(90)	(91)	(92)	(93)	(94)	(95)
(89)	0		0				
(90)		0			0		
(91)	0		0				
(92)							
(93)		0			0		
(94)						0	0
(95)						0	0

Here we discuss those cases where the introduction of singlet scalars implies a relative phase between v_1 and v_2, and we ask whether this provokes the appearance of a phase in the CKM matrix when all Yukawa couplings are real.[9]

To do this, we calculated the Jarlskog invariant of Eq. (106), assuming a relative phase between v_1 and v_2 for all the 246 models of Yukawa matrices (assumed real) which we have identified. In almost all cases $J = 0$. The only exceptions occur for $\theta = \pi$ or $\theta = 2\pi/3$. The results are presented in Tables II and III, respectively. These tables will be useful for the study of spontaneous *CP* violation in models with two scalar doublets and various scalar singlets, in the presence of Abelian symmetries.

VI. NATURAL SUPPRESSION OF FLAVOR CHANGING NEUTRAL SCALAR INTERACTION

Measurements in the mixing of neutral mesons (such as $K - \bar{K}$, $B_d - \bar{B}_d$, etc.) lead to tight constraints on flavor changing neutral scalar interactions (FCNSI). The discrete symmetry Z_2 was introduced in the scalar sector by Glashow and Weinberg [14] and, independently, by Paschos [15], precisely to preclude such FCNSI. But there are several other options to curtail FCNSI. For example, one may invoke large scalar masses, or introduce approximate flavor symmetries [16]. Perhaps more interestingly, one may relate the FCNSI with the CKM matrix. In a very nice article, Branco, Grimus, and Lavoura (BGL) used discrete Abelian symmetries in order to construct one such THDM [12], following earlier work by Lavoura [5]. The BGL model corresponds to the use of our Eq. (78) for the up-type Yukawa matrices and of our Eq. (71) for the down-type Yukawa matrices.

One may now ask the question: is there any other implementation of Abelian symmetries which leads to a relation between FCNSI and the CKM matrix? Although we have all possible implementations of Abelian symmetries, the question is difficult to answer analytically because it involves diagonalizing the mass matrices. Indeed, the quark mass basis is obtained with the basis transformation

$$d_L = V_{dL} n_L, \quad d_R = V_{dR} n_R, \quad u_L = V_{uL} p_L, \quad u_R = V_{uR} p_R, \tag{107}$$

where we have used $q_L = (n_L, p_L)^\top$. The unitary matrices V_{dL}, V_{dR}, V_{uL}, and V_{uR} are chosen such that

$$\begin{aligned} \mathrm{diag}\{m_d, m_s, m_b\} = D_d = V_{dL}[v_1\Gamma_1 + v_2\Gamma_2]V_{dR}^\dagger, \\ \mathrm{diag}\{m_u, m_c, m_t\} = D_u = V_{uL}[v_1^*\Delta_1 + v_2^*\Delta_2]V_{uR}^\dagger. \end{aligned} \tag{108}$$

The CKM matrix is $V = V_{uL}V_{dL}^\dagger$. The matrices controlling the FCNSI are

$$\begin{aligned} N_d = V_{dL}[v_2^*\Gamma_1 - v_1^*\Gamma_2]V_{dR}^\dagger, \\ N_u = V_{uL}[v_2\Delta_1 - v_1\Delta_2]V_{uR}^\dagger. \end{aligned} \tag{109}$$

Botella, Branco, and Rebelo [17] have proposed a method to identify BGL-type implementations while sidestepping the diagonalization procedure. They start from the relation [5]

$$N_d = \frac{v_2^*}{v_1} D_d - \frac{v^2}{v_1} V_{dL}\Gamma_2 V_{dR}^\dagger, \tag{110}$$

obtained by combining Eqs. (108) and (109), and using $v^2 = |v_1|^2 + |v_2|^2$. Based on this they propose the following *sufficient* conditions for BGL implementation: (i) $v_1^*\Delta_1 + v_2^*\Delta_2$ is block-diagonal; and (ii) there exists a matrix P such that (iia) $P\Gamma_2 = k\Gamma_2$ (for some number k), and (iib) $P\Gamma_1 = 0$. As they stress, the condition can be applied with an up-type/down-type quark interchange.

We start by noticing that Eqs. (108) and (109) can also be combined into

$$N_d = -\frac{v_1^*}{v_2} D_d + \frac{v^2}{v_2} V_{dL}\Gamma_1 V_{dR}^\dagger, \tag{111}$$

implying that an equally good *sufficient* conditions for BGL implementation is: (i) $v_1^*\Delta_1 + v_2^*\Delta_2$ is block-diagonal; and (ii) there exists a matrix P such that (iia) $P\Gamma_1 = k\Gamma_1$ (for some k), and (iib) $P\Gamma_2 = 0$. Again, the condition can be applied with an up-type/down-type quark interchange. The new condition is just a $\Phi_1 \leftrightarrow \Phi_2$ transformation of the previous condition, useful to us when looking for all possible BGL-type implementations.

Since we have tabled all possible matrices, we are able to see that only Eq. (78) can lead to a block-diagonal $v_1^*\Delta_1 + v_2^*\Delta_2$ for the up-type quarks. We must now check all compatible down-type Yukawa matrices, namely, Eqs. (71), (72), (74), (76), and (77), and see whether they satisfy condition ii).[10] We have checked that only for Eq. (71) can one find a matrix P consistent with the constraints (ii).

This gives a unique character to the work of Branco, Grimus, and Lavoura [12]. They have developed the *only* possible implementation of a relation between FCNSI and the CKM matrix which uses Abelian symmetries and is consistent with the *sufficient* conditions above. There are only two caveats. First, we have only checked the sufficient conditions developed by Ref. [17] and extended here. *A priori*, one can entertain the possible existence of cases which *do not* satisfy the sufficient conditions presented,

[9]Of course, the inclusion of scalar gauge singlets has no impact on the Yukawa matrices we have found in the previous sections, since singlet scalars have no coupling to the fermions.

[10]The possibility that both the up-type and down-type Yukawa matrices are given by Eq. (78) is excluded, since it would lead to a block-diagonal CKM matrix.

but where the FCNSI are indeed related to the CKM matrix. In the cases where we could perform the analysis analytically, we have found no such case. Second, in some cases condition (ii) is violated because it leads to constraints on the nonzero matrix elements of the Yukawa matrices. It could be that some non-Abelian group might lead to further zeros on the Yukawa matrices, thus evading the problem. Although possible, such a case would be difficult to construct because more zeros in the Yukawa matrices will, more often than not, lead to massless quarks or to a block-diagonal CKM matrix.

In light of our analysis, that a BGL [12] case was found by inspection in the THDM is truly remarkable.

VII. CONCLUSIONS

We have studied the restrictions on the Yukawa matrices imposed by discrete Abelian symmetries acting on the scalar and fermion sectors of the THDM. Using known experimental constraints, we have reduced the number of possible cases from 3^{18} to 246. Ignoring row and column permutations, we are left with 34 types of down-type Yukawa matrices (and the same for up-type quarks), which we table explicitly.

We have found that imposing a symmetry Z_n ($n \geq 4$) on the scalars always leads to an accidental $U(1)$ symmetry; that applying a Z_3 symmetry on the scalars leads to an accidental $U(1)$ symmetry in the scalar sector but not necessarily in the fermion sector; and that applying a Z_2 symmetry on the scalars does not lead to an accidental $U(1)$ symmetry in either sector.

We show that only Z_2 with soft-breaking in the scalar sector enables spontaneous CP violation. We also show that the proposal of Branco, Grimus and Lavoura [12] is unique, in our context, and conjecture that this uniqueness might hold even when non-Abelian symmetries are considered in the THDM.

Finally, we stress that our results have a very wide applicability in model building because all discrete non-Abelian groups have a Z_n subgroup, for some value of n. For a given non-Abelian group, pick one of its Z_n subgroups and diagonalize its generator. Applying that generator as a symmetry of the Lagrangian, one falls into one of the 34 Yukawa matrices we have shown explicitly. The action of further generators (which, of course, need not be diagonalizable in the same basis) will, in general, lead to further constraints on the Yukawa matrices. Given the low number of entries in many of our Yukawa matrices, and the likelihood of further constraints setting them to zero, the action of further generators will often lead to matrices inconsistent with experimental constraints.

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