PHYSICAL REVIEW D **83,** 065017 (2011)

Self-interacting Elko dark matter with an axis of locality

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(Received 6 October 2009; published 10 March 2011)

Here we report that Elko (for ***E****igenspinoren des* **L***adungs***k***onjugations***o***perators*) breaks Lorentz symmetry in a rather subtle and unexpected way by containing a "hidden" preferred direction. Along this preferred direction, a quantum field based on Elko enjoys locality. In the form reported here, Elko offers a mass dimension one fermionic dark matter with a quartic self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases.

DOI: 10.1103/PhysRevD.83.065017 PACS numbers: 11.10.Lm, 11.30.Cp, 11.30.Er, 95.35.+d

I. INTRODUCTION

The particle nature of dark matter is still unsettled. What we do know is that it is expected to be endowed with a self-interaction [1–4]. The indicated self-interaction would ordinarily suggest that dark matter must be some sort of scalar field. However, as shown in [5,6], the Elko (for ***E****igenspinoren des* **L***adungs***k***onjugations***o***perators*, the reason for this definition will become clear in Sec. I B) quantum field is endowed with mass-dimension one, a property that allows for unsuppressed Elko self-interaction. Further consequences of the mass dimensionality of Elko are that its possible interactions with the mass-dimension $3/2$ Dirac and Majorana fields are suppressed by one order of unification scale and that it cannot enter the standard model (SM) doublets. This, along with the fact that Elko does not carry the standard U(1) gauge invariance, renders Elko a natural dark matter candidate [5,6].

Here we report that Elko breaks Lorentz symmetry in a rather subtle and unexpected way by containing a "hidden" preferred direction. All inertial frames that move with a constant velocity along this direction are physically equivalent. Along this direction, a quantum field based on Elko enjoys locality.

Our discourse begins with a review of the SM matter fields in Sec. I A. In Sec. I B we recapitulate the known problems with the interpretation of Majorana spinors as commuting numbers, and argue that these problems evaporate under a more careful examination [5,6]. The pace is deliberately slow. The discussion is designed to provide the right setting for the taken departure. Sections II and III form the core of this communication. The discussion on the Elko dual presented in Sec. II B is a significant addition to the previous work on Elko [5,6]. The dramatically changed locality structure arises from certain phases and identification introduced in the Elko spinors at rest [see Eqs. (16a)–(16d)]. Section II C reminds the reader that Elko satisfies the Klein-Gordon, but not the Dirac, equation. The Elko spin sums are given in Sec. II D. These spin sums are needed for examining the locality structure of the Elko quantum fields and had to be reevaluated due to the mentioned changes in the Elko rest spinors [7]. These carry the seeds of the mentioned preferred direction. Section III formally introduces the Elko quantum fields. Section III A makes an argument to identify Elko with self-interacting dark matter that is endowed with an axis of locality. In the form reported here, Elko offers a mass-dimension one fermionic dark matter with self-interaction and a preferred axis of locality. The locality result crucially depends on a judicious choice of phases. The paper ends with summarizing remarks and questions in Sec. IV. Appendix A provides supplementary information.

A. The matter field underlying the SM

The matter field underlying the SM is a four-component spinor field [8] with historical origin in Dirac's celebrated 1928 paper [9]

$$\Psi(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E(\mathbf{p})}} \times [\,\underbrace{u(x;\mathbf{p},\sigma)}_{=u(\mathbf{p},\sigma)e^{-ip^\mu x_\mu}}\, a(\mathbf{p},\sigma) + \underbrace{v(x;\mathbf{p},\sigma)}_{=v(\mathbf{p},\sigma)e^{+ip^\mu x_\mu}}\, b^\dagger(\mathbf{p},\sigma)], \tag{1}$$

where σ takes the values $\pm 1/2$. The zero-momentum coefficient functions may be symbolically written as

$$u(0,1/2) = \begin{bmatrix} \uparrow \\ \uparrow \end{bmatrix}, \qquad u(0,-1/2) = \begin{bmatrix} \downarrow \\ \downarrow \end{bmatrix}, \tag{2}$$

$$v(0,1/2) = \begin{bmatrix} \downarrow \\ -\downarrow \end{bmatrix}, \qquad v(0,-1/2) = \begin{bmatrix} -\uparrow \\ \uparrow \end{bmatrix}, \tag{3}$$

where

$$\uparrow \equiv \sqrt{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \downarrow \equiv \sqrt{m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

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in the "polarization basis." In the helicity basis, these are eigenspinors of the helicity operator with a specific choice of phases. These phases are determined, e.g., by the locality condition [10].

Without any reference to the Dirac equation (see Ref. [8] for a detailed argument), the coefficient functions are determined from the condition that, under the homogeneous Lorentz transformations, the field components superimpose with other field components via spacetime-independent elements (of 4×4 matrices). These matrices must furnish a finite dimensional representation of the homogeneous Lorentz group.

The coefficient functions for arbitrary momentum are obtained by the action of the boost

$$u(\mathbf{p}, \sigma) = \kappa u(0, \sigma), \tag{5}$$

where $\kappa \equiv \kappa_r \oplus \kappa_\ell$. The explicit expressions for κ_r and κ_ℓ are given below.

The only nontrivial freedom that $\Psi(x)$ still carries is the specialization to the case where $b^\dagger(\mathbf{p}, \sigma)$ is identified with $a^\dagger(\mathbf{p}, \sigma)$. Otherwise, the Poincaré spacetime symmetries along with the symmetries of charge-conjugation, parity and time-reversal, and the demand of locality uniquely determine the field $\Psi(x)$. Seen in this light, the field coefficients $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ are eigenspinors of the $\gamma^\mu p_\mu$ operator with eigenvalues $+m$ and $-m$, respectively.

The annihilation of the field $\Psi(x)$ by the Dirac operator $(i\gamma^\mu \partial_\mu - m\mathbb{I})$ follows as a result of this structure. The Dirac equation is not assumed. Rather, it emerges as a direct consequence of the merger of quantum mechanics and Poincaré spacetime symmetries for spin $1/2$. The apparent simplicity of the Dirac field can be somewhat misleading to the uninitiated. For instance, a change in sign in the right-hand side of the expression for $v(0, -1/2)$ in Eq. (3) yields a quantum field that is nonlocal when $b^\dagger(\mathbf{p}, \sigma)$ is identified with $a^\dagger(\mathbf{p}, \sigma)$. Even though the mentioned change in phase does not destroy the locality in the original field, it does violate spacetime symmetries in a hidden way. A systematic study of such subtle loss of symmetries and locality remains largely unexplored.

For historical reasons, the field $\Psi(x)$ is known as the Dirac field, while the identification of $b^\dagger(\mathbf{p}, \sigma)$ with $a^\dagger(\mathbf{p}, \sigma)$ yields what has come to be known as the Majorana field [9,11]. The coefficient functions $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ are the usual Dirac spinors. They can be interpreted as being a direct sum of the right-handed and left-handed Weyl spinors with specific helicities and phases.

B. Majorana spinors: A critique

History clearly demarcates the introduction of the Majorana field. It was introduced in 1937 by Ettore Majorana [11]. As regards Majorana spinors, we (i.e., the authors) do not know of their historical birth.

While in the operator formalism of quantum field theory Dirac spinors are treated as commuting numbers, it is curious that Majorana spinors are treated as Grassmann variables. This is deemed necessary, due to what are considered otherwise unavoidable problems. (Consider for instance Aitchison and Hey's attempt to construct a Hamiltonian density [12].) What further adds to the problem is that, taken by itself, a Majorana spinor is nothing but a Weyl spinor in the four-component form. As shown by Ahluwalia and Grumiller [5,6], both of these problems can be circumvented. A hint toward a solution for the first problem may be found by noting that, unlike Dirac spinors, the Majorana spinors are not eigenspinors of the Dirac operator. Instead, they are eigenspinors of the square of the Dirac operator. This suggests that the problem lies not with the Majorana spinors but instead with the Lagrangian density assumed by Aitchison and Hey [12]. The latter of the two mentioned problems also has a similar solution. The usual set of Majorana spinors consists of two spinors, both of which have eigenvalue one under the operation of the charge conjugation operator. This is the self-conjugate set. However, as pointed out in Refs. [5,6], there also exists the anti self-conjugate set. Once these are added, the complete set of four spinors—the Elko (for ***E****igenspinoren des* **L***adungs****k****onjugations* ***o****peratoren*)—span the four-dimensional representation space of spin $1/2$ and come to par with the Dirac spinors.

Let us briefly review the canonical wisdom. In doing so we shall explicitly show the cost at which the above changes are implemented. Whether or not this ought to be a cost we should be willing to pay is ultimately a matter for experiment to decide. At the very least, we shall know what it is that we would reject if we were to choose to confine ourselves to the canonical wisdom.

According to the received wisdom, the Majorana spinors airse as follows. If ϕ_ℓ is a massive Weyl spinor of left-handed nature, then $\sigma_2 \phi_\ell^*$ transforms as a right-handed Weyl spinor. For this reason ([13], p. 20), we can construct a special type of four-component spinor called a Majorana spinor:

$$\psi_M = \begin{pmatrix} -\sigma_2 \phi_\ell^* \\ \phi_\ell \end{pmatrix}. \tag{6}$$

It is self-conjugate under charge conjugation. For ϕ_ℓ there are two choices: a positive helicity and a negative helicity. As such, we have two rather than four four-component spinors. Thus the folklore: the Majorana spinor is a Weyl spinor in four-component form [13]. It is self-evident and remains unquestioned in our discourse.

An immediate sign of trouble appears if one naïvely introduces a Lagrangian density $\mathcal{L}_M = \bar{\psi}_M (i\gamma^\mu \partial_\mu - m)\psi_M$. The usual route at this stage is to treat the components of the Weyl spinors as Grassmann numbers; otherwise, one encounters the often-quoted problems ([12], App. P). The Ahluwalia-Grumiller work [5,6] strongly indicates that this approach may be hiding certain fundamental properties of Majorana spinors. Or, to put it more precisely, having taken the Grassmann route, we may have overlooked a rich and fertile ground where Majorana spinors are treated as commuting number spinors. To unearth these aspects, we shall treat the massive Weyl spinors as two-component eigenspinors of the helicity operator ([14], p. 111). The fermionic statistics are implemented through the canonical field operator formalism [8,15] and not by treating them as Grassmann fields [16]. The Elko formalism was born in this spirit and attended to a widespread, but rarely spoken, discontent with abandoning Majorana spinors as commuting numbers.

A straightforward calculation now shows that, (i) under the Dirac dual, the norm $\bar{\psi}_M\psi_M$ identically vanishes (so, no Dirac mass term); and (ii) in the momentum space, ψ_M is not an eigenspinor of the $\gamma_\mu p^\mu$ operator $\gamma_\mu p^\mu \psi_M \neq \pm m\psi_M$ (and so Majorana spinors do not satisfy the Dirac equation ([12], App. P). This already suggests that constructing a mass dimension $3/2$ fermionic field in terms of Majorana spinors may not be possible [18]. The lesson to be learned is this: It is *not* sufficient that one consider the ‘‘simplest candidates for a kinematic spinor term’’ in the construction of a field equation, as found in almost [19] every text book on quantum field theory [21]. Rather, one must ensure that the associated Green’s function be proportional to the vacuum expectation value of the time-ordered product of certain field operators. This lesson, we think, has a much larger significance in that Lagrangian densities must be derived and not assumed. Neglecting this may induce all manner of pathologies. How this task is to be accomplished—at least for spin $1/2$—is one of the wider contributions of this communication.

The assertion about reduction in the degrees of freedom for Majorana spinors also faces trouble if one notes that the relevant charge conjugation operator has not one, but two, real eigenvalues: $+1$ (giving the usual self-conjugate Majorana spinors) and -1. There is no physical or mathematical reason to abandon, or ‘‘project out,’’ the latter. The sense in which the folklore still survives is that, by an appropriate similarity transformation, half of these (i.e., those corresponding to the positive eigenvalue) can be mapped to real four-component spinors, while those corresponding to the negative eigenvalue can be transformed into purely imaginary four-component spinors.

II. ELKO: DEPARTURE FROM GRASSMANN INTERPRETATION OF MAJORANA SPINORS

The interpretation of the Majorana spinors in terms of Grassmann variables is elegant. It is mathematically sound and has found widespread applications in modern quantum field theory. Yet it breaks with the tradition of field operator formalism which would have required these spinors to be commuting number coefficient functions in a field. In their work [5,6], Ahluwalia and Grumiller formulated a treatment of Majorana spinors in the operator formalism. Towards this end, they included two additional spinors to the canonical Majorana spinors, thus forming a complete set of dual helicity eigenspinors of the charge conjugation operator for spin $1/2$. In order to avoid confusion with the incomplete set, the Majorana spinors, they introduced the name Elko, which, as already mentioned, was taken from the German *Eigenspinoren des Ladungskonjugationsoperators*.

The quantum field expanded with Elko spinors is not a quantum field in the sense of Weinberg [8]. Specifically, the uniqueness of the Dirac field, modulo its specialization to the Majorana field, implies that the program we embark upon necessarily violates Lorentz symmetry. This feature, which had remained hidden in our previous discourse, we now unearth. In our opinion, this has the potential to open up an entirely new perspective on dark matter—the decision being in the hands of experiments. To a pure theoretician, the interest might be in its mathematical structure.

In this communication we confine our primary attention to spin $1/2$, but we construct Elko in such a way that the procedure immediately generalizes to all spins. This is facilitated by the use of Wigner’s time-reversal operator $\Theta = -i\sigma_2$, rather than the Pauli matrix σ_2 that appears in Ramond’s primer in the context of Majorana spinors. We shall use the phrase Elko for spinors as well as for the quantum fields constructed from them. The context shall be assumed to remove any ambiguity.

A. Construction of Elko

To construct Elko it is first necessary to introduce the charge conjugation operator. This we do as follows. Under parity, P, $\mathbf{x} \to -\mathbf{x}$; hence, the rapidity parameter $\boldsymbol{\varphi} = \varphi\hat{\mathbf{p}}$ changes sign. Thus, to implement this transformation on the boost operator, we require a matrix of the form

$$S(P) = \exp[i\vartheta]\underbrace{\begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}}_{\gamma^0}\mathcal{R}, \qquad \vartheta \in \mathbb{R}, \tag{7}$$

with $\mathbf{p} \equiv p(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, and $\mathcal{R} = \{\theta \to \pi - \theta, \phi \to \pi + \phi, p \to p\}$. If care is taken that the eigenvalues of the helicity operator change sign under P, the arguments given in Ref. [6] fix the phase $\exp[i\vartheta]$ to be i. The operator $S(P)$ now has four doubly degenerate eigenspinors, carrying opposite eigenvalues of $S(P)$—call these u and v sectors. The operator

$$\mathcal{C} = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K, \tag{8}$$

where K is the complex conjugation operator, formally interchanges the opposite parity sectors: $u \overset{\mathcal{C}}{\leftrightarrow} v$. It is apparent that $\mathcal{C}$ is the standard charge conjugation operator of Dirac. In the context of Eq. (8), Wigner’s time-reversal operator Θ is defined as $\Theta \mathbf{J} \Theta^{-1} = -\mathbf{J}^*$, where $\mathbf{J}$ are a set of rotation generators for the representation space under consideration. For spin 1/2, $\Theta[\boldsymbol{\sigma}/2]\Theta^{-1} = -[\boldsymbol{\sigma}/2]^*$. We use the realization

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

To construct Elko, let $\phi_\ell(\mathbf{p})$ be a left-handed Weyl spinor of spin 1/2. Under a Lorentz boost, it transforms as $\phi_\ell(\mathbf{p}) = \kappa_\ell \phi_\ell(\boldsymbol{\epsilon})$, with

$$\kappa_\ell = \exp\left(-\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2m}}\left(\mathbb{I} - \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\right). \tag{9}$$

The $\boldsymbol{\epsilon}$ is defined as $\mathbf{p}|_{p\to 0}$, and not as $\mathbf{p}|_{p=0}$. In the usual notation, the boost parameter $\boldsymbol{\varphi}$ is defined as

$$\cosh\varphi = \frac{E}{m} = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \sinh\varphi = \frac{p}{m} = \gamma\beta, \quad \hat{\boldsymbol{\varphi}} = \hat{\mathbf{p}}. \tag{10}$$

By $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ we denote the Pauli matrices. The symbol $\mathbb{I}$ represents an identity matrix, while in what follows $\mathbb{O}$ shall be used for a null matrix (their dimensionality shall be apparent from the context). For $\phi_\ell(\mathbf{p})$, we have two possibilities:

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \phi_\ell^\pm(\mathbf{p}) = \pm \phi_\ell^\pm(\mathbf{p}).$$

Following Ref. [6] we now note that, under a Lorentz boost, $\vartheta\Theta\phi_\ell^*(\mathbf{p})$ transforms as a right-handed Weyl spinor, $[\vartheta\Theta\phi_\ell^*(\mathbf{p})] = \kappa_r[\vartheta\Theta\phi_\ell^*(\boldsymbol{\epsilon})]$, with

$$\kappa_r = \exp\left(+\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2m}}\left(\mathbb{I} + \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\right), \tag{11}$$

where ϑ is an unspecified phase to be determined below. The helicity of $\vartheta\Theta\phi_\ell^*(\mathbf{p})$ is *opposite* to that of $\phi_\ell(\mathbf{p})$,

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}[\vartheta\Theta(\phi_\ell^\pm(\mathbf{p}))^*] = \mp[\vartheta\Theta(\phi_\ell^\pm(\mathbf{p}))^*]. \tag{12}$$

The argument that led to *two* Majorana spinors, now instead takes us to their cousins, the *four* four-component spinors with the general form

$$\chi(\mathbf{p}) = \begin{pmatrix} \vartheta\Theta\phi_\ell^*(\mathbf{p}) \\ \phi_\ell(\mathbf{p}) \end{pmatrix}. \tag{13}$$

The $\chi(\mathbf{p})$ become eigenspinors of the charge conjugation operator, Elko, with real eigenvalues if the phase ϑ is restricted to $\pm i$:

$$\mathcal{C}\chi(\mathbf{p})|_{\vartheta=\pm i} = \pm\chi(\mathbf{p})|_{\vartheta=\pm i}. \tag{14}$$

One can motivate the well-known Dirac spinors in a parallel manner; as eigenspinors of the parity operator $S(P)$. In that case, the right- and left-transforming components are necessarily endowed with the same helicity. For Elko, the right- and left-transforming components carry opposite helicity. So, whereas Dirac spinors may exist as eigenspinors of the helicity operator, the Elko cannot. This eventually is reflected in many of the results that we arrive at.

To give Elko a concrete form, we adopt the global phases so that, "at rest," the left-handed Weyl spinors take the form [22]

$$\phi_\ell^+(\boldsymbol{\epsilon}) = \sqrt{m}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}, \tag{15a}$$

$$\phi_\ell^-(\boldsymbol{\epsilon}) = \sqrt{m}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}. \tag{15b}$$

Eqs. (15a) and (15b), along with Eq. (13) and the demand of locality allow us to explicitly write the self-conjugate spinors ($\vartheta = +i$) and anti–self-conjugate spinors ($\vartheta = -i$) at rest:

$$\xi_{\{-,+\}}(\boldsymbol{\epsilon}) \equiv +\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^+(\boldsymbol{\epsilon}),\vartheta=+i} \tag{16a}$$

$$\xi_{\{+,-\}}(\boldsymbol{\epsilon}) \equiv +\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^-(\boldsymbol{\epsilon}),\vartheta=+i} \tag{16b}$$

$$\zeta_{\{-,+\}}(\boldsymbol{\epsilon}) \equiv +\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^-(\boldsymbol{\epsilon}),\vartheta=-i} \tag{16c}$$

$$\zeta_{\{+,-\}}(\boldsymbol{\epsilon}) \equiv -\chi(\boldsymbol{\epsilon})|_{\phi_\ell(\boldsymbol{\epsilon})\to\phi_\ell^+(\boldsymbol{\epsilon}),\vartheta=-i}. \tag{16d}$$

For comparison with Eqs. (2)–(4), the above in polarization basis may be written as

$$\xi_{\{-,+\}}(\boldsymbol{\epsilon}) = \begin{bmatrix} i\Downarrow \\ \Uparrow \end{bmatrix}, \qquad \xi_{\{+,-\}}(\boldsymbol{\epsilon}) = \begin{bmatrix} -i\Uparrow \\ \Downarrow \end{bmatrix}, \tag{17}$$

$$\zeta_{\{-,+\}}(\boldsymbol{\epsilon}) = \begin{bmatrix} i\Uparrow \\ \Downarrow \end{bmatrix}, \qquad \zeta_{\{+,-\}}(\boldsymbol{\epsilon}) = -\begin{bmatrix} -i\Downarrow \\ \Uparrow \end{bmatrix}. \tag{18}$$

The $\Uparrow$ and $\Downarrow$ differ from $\uparrow$ and $\downarrow$ of Eq. (4) by the phases, $e^{\pm i\phi/2}$, which even in the polarization basis prove to be essential if locality is to be preserved. In the context of Weinberg's work on the uniqueness of the Dirac field (modulo the special case of the Majorana field in the sense of Majorana's original 1937 paper [11]), a comparison with Eqs. (2) and (3) already tells us that a quantum field that fully respects Lorentz symmetries cannot be built in terms of ξ and ζ Elko spinors. The task then is to unearth this violation, and see how strong, or how weak, the said violation is.

The $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ for an arbitrary momentum are now readily obtained:

$$\xi(\mathbf{p}) = \kappa\xi(\boldsymbol{\epsilon}), \qquad \zeta(\mathbf{p}) = \kappa\zeta(\boldsymbol{\epsilon}), \qquad \kappa \equiv \kappa_r \oplus \kappa_\ell. \tag{19}$$

B. A systematic construction of Elko dual, orthonormality, and completeness

The norm of Elko under the Dirac dual $\bar{\chi}(\mathbf{p}) \equiv [\chi(\mathbf{p})]^\dagger\gamma^0$ identically vanishes. However, it is more appropriate to seek a "metric" η such that the product $[\chi_i(\mathbf{p})]^\dagger\eta\chi_j(\mathbf{p})$—with $\chi_i(p)$ as any one of the four Elko spinors—remains invariant under an arbitrary Lorentz transformation. This requirement can be readily shown to translate into the following constraints on η:

$$[J_i, \eta] = 0, \qquad \{K_i, \eta\} = 0. \tag{20}$$

Since the only property of the generators of rotations and boosts that enters the derivation of the above constraints is that $\mathbf{J}^\dagger = \mathbf{J}$ and $\mathbf{K}^\dagger = -\mathbf{K}$, the result applies to all *finite* dimensional representations of the Lorentz group. It need not be restricted to Elko alone. Seen in this light, there is no nontrivial solution for η for either the right-handed or the left-handed Weyl spinors. For $r \oplus \ell$ representation space, the most general solution is found to carry the form

$$\eta = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}. \tag{21}$$

It is now convenient to introduce the notation $\chi_1(\mathbf{p}) \equiv \xi_{\{-,+\}}(\mathbf{p})$, $\chi_2(\mathbf{p}) \equiv \xi_{\{+,-\}}(\mathbf{p})$, $\chi_3(\mathbf{p}) \equiv \zeta_{\{-,+\}}(\mathbf{p})$, and $\chi_4(\mathbf{p}) \equiv \zeta_{\{+,-\}}(\mathbf{p})$. Then 16 values of $[\chi_i(\mathbf{p})]^\dagger \eta \chi_j(\mathbf{p})$ as i and j vary from 1 to 4 are given in Table I.

To allow for the possibility of parity covariance, we set $b = a$. (This treats r and ℓ Weyl spaces on the same footing.) To make the invariant norms real, we give a and b the common value of $\pm i$, resulting in $\eta = \pm i\gamma^0$. In what follows, the choice of the signs shall be dictated by the convenience of bookkeeping.

Guided by these results, we now introduce the *Elko dual*

$$\vec{\chi}_{\{\mp,\pm\}}(\mathbf{p}) \equiv \mp i[\chi_{\{\pm,\mp\}}(\mathbf{p})]^\dagger \gamma^0. \tag{22}$$

Under the new dual, the orthonormality relations read

$$\vec{\xi}_\alpha(\mathbf{p})\xi_{\alpha'}(\mathbf{p}) = +2m\delta_{\alpha\alpha'}, \tag{23a}$$

$$\vec{\zeta}_\alpha(\mathbf{p})\zeta_{\alpha'}(\mathbf{p}) = -2m\delta_{\alpha\alpha'}, \tag{23b}$$

along with $\vec{\xi}_\alpha(\mathbf{p})\zeta_{\alpha'}(\mathbf{p}) = 0$ and $\vec{\zeta}_\alpha(\mathbf{p})\xi_{\alpha'}(\mathbf{p}) = 0$. The dual helicity index α ranges over the two possibilities: $\{+,-\}$ and $\{-,+\}$, and $-\{\pm,\mp\} \equiv \{\mp,\pm\}$. The completeness relation

$$\frac{1}{2m}\sum_\alpha [\xi_\alpha(\mathbf{p})\vec{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(\mathbf{p})\vec{\zeta}_\alpha(\mathbf{p})] = \mathbb{I} \tag{24}$$

establishes that we need to use *both* the self-conjugate as well as the anti–self-conjugate spinors to fully capture the relevant degrees of freedom.

TABLE I. The values of $[\chi_i(\mathbf{p})]^\dagger \eta \chi_j(\mathbf{p})$ evaluated using η. The i runs from 1 to 4 along the rows and j does the same across the columns.

0	$-im(a+b)$	$-im(a-b)$	0
$im(a+b)$	0	0	$-im(a-b)$
$-im(a-b)$	0	0	$im(a+b)$
0	$-im(a-b)$	$-im(a+b)$	0

C. Elko satisfies the Klein-Gordon, not Dirac, equation

Because we are going to encounter several unexpected results, we pause to examine the behavior of $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors under the action of the operator $\gamma^\mu p_\mu$. This brute force exercise serves the pedagogic purpose of countering some prejudices some readers may inevitably carry from their prior studies. Additionally, in the context of Aitchison and Hey's concern that one encounters a problem in constructing a Lagrangian density for Majorana spinors if they are not treated as Grassmann variables ([12], App. P), we provide the origin of that concern and offer a solution.

We already have explicit expressions for $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors. On these we act $\gamma^\mu p_\mu$ and find the following identities:

$$\gamma^\mu p_\mu \xi_{\{-,+\}}(\mathbf{p}) = +im\xi_{\{+,-\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_1(\mathbf{p}) = +im\chi_2(\mathbf{p}) \tag{25a}$$

$$\gamma^\mu p_\mu \xi_{\{+,-\}}(\mathbf{p}) = -im\xi_{\{-,+\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_2(\mathbf{p}) = -im\chi_1(\mathbf{p}) \tag{25b}$$

$$\gamma^\mu p_\mu \zeta_{\{-,+\}}(\mathbf{p}) = -im\zeta_{\{+,-\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_3(\mathbf{p}) = -im\chi_4(\mathbf{p}) \tag{25c}$$

$$\gamma^\mu p_\mu \zeta_{\{+,-\}}(\mathbf{p}) = +im\zeta_{\{-,+\}}(\mathbf{p}) \Leftrightarrow \gamma^\mu p_\mu \chi_4(\mathbf{p}) = +im\chi_3(\mathbf{p}). \tag{25d}$$

Applying $\gamma^\nu p_\nu$ to Eq. (25a) from the left and then using (25b) on the resulting right-hand side, and repeating the same procedure for the remaining equations, we get

$$\begin{aligned}(\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2)\xi_{\{\mp,\pm\}}(\mathbf{p}) = 0, \\ (\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2)\zeta_{\{\mp,\pm\}}(\mathbf{p}) = 0.\end{aligned} \tag{26}$$

Now using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ yields the Klein-Gordon equation (in momentum space) for the $\xi(\mathbf{p})$ and $\zeta(\mathbf{p})$ spinors. Aitchison and Hey's concern is thus overcome. The problem resides in the approach of constructing the "simplest candidates for a kinematic spinor term."

D. Elko spin sums: A preferred axis

We now look at the spin sums in Eq. (24) separately. These evaluate to

$$\sum_\alpha \xi_\alpha(\mathbf{p})\vec{\xi}_\alpha(\mathbf{p}) = m[\mathcal{G}(\mathbf{p}) + \mathbb{I}], \tag{27a}$$

$$\sum_\alpha \zeta_\alpha(\mathbf{p})\vec{\zeta}_\alpha(\mathbf{p}) = m[\mathcal{G}(\mathbf{p}) - \mathbb{I}], \tag{27b}$$

which together *define* $\mathcal{G}(\mathbf{p})$. A *direct evaluation* of the left-hand side of the above equations gives

$$\mathcal{G}(\mathbf{p}) = i\begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

For later reference, we note that $\mathcal{G}(\mathbf{p})$ is an odd function of $\mathbf{p}$:

$$\mathcal{G}(\mathbf{p}) = -\mathcal{G}(-\mathbf{p}). \tag{29}$$

But since $\mathcal{G}(\mathbf{p})$ is independent of p and θ, it is more instructive to translate the above expression into

$$\mathcal{G}(\phi) = -\mathcal{G}(\pi + \phi). \tag{30}$$

This serves to define a preferred axis, z_e [23]. Another hint for a preferred axis arises when one notes that the spinor structure of Elko does not enjoy covariance under usual local $U(1)$ transformation with phase $\exp(i\alpha(x))$. However, $U_E(1) = \exp(i\gamma^0\alpha(x))$—and not $U_M(1) = \exp(i\gamma^5\alpha(x))$ as one would have thought ([24], p. 72)—preserves various aspects of the Elko structure. Similar comments apply to the non-Abelian gauge transformations of the SM.

For a comparison with the Dirac counterpart (see App. A 1), we define $g^\mu \equiv (0, \mathbf{g})$ with $\mathbf{g} = -[1/\sin(\theta)]\partial\hat{\mathbf{p}}/\partial\phi = (\sin\phi, -\cos\phi, 0)$. Note may be taken that g^μ is a unit spacelike four-vector, $g_\mu g^\mu = -1$. Furthermore, $g_\mu p^\mu = 0$. In terms of g^μ, $\mathcal{G}(\mathbf{p})$ may be written as

$$\mathcal{G}(\mathbf{p}) = \gamma^5(\gamma_1 \sin\phi - \gamma_2\cos\phi) = \gamma^5\gamma_\mu g^\mu. \tag{31}$$

This gives Eqs. (27a) and (27b) the form

$$\sum_\alpha \xi_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p}) = m[\gamma^5\gamma_\mu g^\mu + \mathbb{I}], \tag{32a}$$

$$\sum_\alpha \zeta_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p}) = m[\gamma^5\gamma_\mu g^\mu - \mathbb{I}]. \tag{32b}$$

The γ^μ, in the Weyl realization, are taken to be

$$\gamma^0 \equiv \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}, \qquad \gamma^i \equiv \begin{pmatrix} \mathbb{O} & -\sigma^i \\ \sigma^i & \mathbb{O} \end{pmatrix},$$
$$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}. \tag{33}$$

III. ELKO FERMIONIC FIELDS OF MASS-DIMENSION ONE: LAGRANGIAN DENSITIES

Confining to the preferred frame, we now examine the physical and mathematical content of two quantum fields [25]:

$$\Lambda(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_\alpha [a_\alpha(\mathbf{p})\xi_\alpha(\mathbf{p})e^{-ip_\mu x^\mu} + b^\ddagger_\alpha(\mathbf{p})\zeta_\alpha(\mathbf{p})e^{+ip_\mu x^\mu}] \tag{34}$$

and

$$\lambda(x) \equiv \Lambda(x)|_{b^\ddagger(\mathbf{p})\to a^\ddagger(\mathbf{p})}. \tag{35}$$

We assume that the annihilation and creation operators satisfy the fermionic anticommutation relations [26]

$$\{a_\alpha(\mathbf{p}), a^\ddagger_{\alpha'}(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\alpha\alpha'}, \tag{36a}$$

$$\{a_\alpha(\mathbf{p}), a_{\alpha'}(\mathbf{p}')\} = 0, \qquad \{a^\ddagger_\alpha(\mathbf{p}), a^\ddagger_{\alpha'}(\mathbf{p}')\} = 0. \tag{36b}$$

Similar anticommutators are assumed for the $b_\alpha(\mathbf{p})$ and $b^\ddagger_\alpha(\mathbf{p})$. The adjoint field $\overset{\neg}{\Lambda}(x)$ is defined as

$$\overset{\neg}{\Lambda}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_\alpha [a^\ddagger_\alpha(\mathbf{p})\overset{\neg}{\xi}_\alpha(\mathbf{p})e^{+ip_\mu x^\mu} + b_\alpha(\mathbf{p})\overset{\neg}{\zeta}_\alpha(\mathbf{p})e^{-ip_\mu x^\mu}]. \tag{37}$$

The results contained in Eqs. (25a)–(25d) assure us that it is the Klein-Gordon, and not the Dirac, operator that annihilates the fields $\Lambda(x)$ and $\lambda(x)$. The associated Lagrangian densities are

$$\mathcal{L}^\Lambda(x) = \partial^\mu\overset{\neg}{\Lambda}(x)\partial_\mu\Lambda(x) - m^2\overset{\neg}{\Lambda}(x)\Lambda(x),$$
$$\mathcal{L}^\lambda(x) = \mathcal{L}^\Lambda(x)|_{\Lambda\to\lambda}. \tag{38}$$

The mass dimensionality of these Elko fields is thus one, and not 3/2.

The mass dimensionality of a field can also be deciphered from constructing the Feynman-Dyson propagator. This matter is discussed in App. A 2.

A. Identification of Elko with dark matter

These results open up an entirely new and unexpected possibility for the dark matter sector. The primary observations that suggest this are four-fold:

(1) Because of the mismatch in mass dimensionality of $\mathcal{D}_\Lambda = 1$ and $\mathcal{D}_\lambda = 1$ with the SM's matter fields $\mathcal{D}_\Psi = 3/2$, the new fermionic fields cannot enter the SM doublets.

(2) The Lagrangian densities associated with Elko fields do not carry the gauge symmetries of the SM. [See our remarks above Eq. (31).]

(3) The dimension four interactions of the $\Lambda(x)$ and $\lambda(x)$ with the standard model fields are restricted to those with the SM Higgs doublet $\phi(x)$. These are

$$\mathcal{L}^{\mathrm{int}}(x) = \phi^\dagger(x)\phi(x)[a_1\overset{\neg}{\Lambda}(x)\Lambda(x) + a_2\overset{\neg}{\lambda}(x)\lambda(x) + a_3(\overset{\neg}{\Lambda}(x)\lambda(x) + \overset{\neg}{\lambda}(x)\Lambda(x))],$$

where the a's are unknown coupling constants.

(4) By virtue of their mass dimensionality, the new dark matter fields are endowed with dimension four self-interactions,

$$\mathcal{L}^{\mathrm{self}}(x) = b_1(\overset{\neg}{\Lambda}(x)\Lambda(x))^2 + b_2(\overset{\neg}{\lambda}(x)\lambda(x))^2 + b_3[(\overset{\neg}{\Lambda}(x)\lambda(x))^2 + (\overset{\neg}{\lambda}(x)\Lambda(x))^2], \tag{39}$$

where the b's are unknown coupling constants. Observational evidence suggests that dark matter needs to be self-interacting [1–4,27].

Combined, the enumerated Elko properties not only render Elko dark with respect to the SM matter fields, but they also endow it with various observationally attractive properties. It is worth emphasizing that all of these properties are intrinsic to Elko, and arise in a natural way.

B. The locality structure of Elko

The canonically conjugate momenta to the Elko fields are

$$\Pi(x) = \frac{\partial \mathcal{L}^{\Lambda}}{\partial \dot{\Lambda}} = \frac{\partial}{\partial t}\vec{\Lambda}(x), \tag{40}$$

and similarly $\pi(x) = \frac{\partial}{\partial t}\vec{\lambda}(x)$. The calculational details for the two fields now differ significantly. We begin with the evaluation of the equal time anticommutator for the $\Lambda(x)$ and its conjugate momentum, and find

$$\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x}',t)\} = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \underbrace{\sum_{\alpha}[\xi_\alpha(\mathbf{p})\vec{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(-\mathbf{p})\vec{\zeta}_\alpha(-\mathbf{p})]}_{=2m[\mathbb{I}+\mathcal{G}(\mathbf{p})]} \tag{41}$$

or, equivalently,

$$\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x}',t)\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}). \tag{42}$$

The anticommutators for the particle/antiparticle annihilation and creation operators suffice to yield the remaining locality conditions,

$$\{\Lambda(\mathbf{x},t), \Lambda(\mathbf{x}',t)\} = \mathbb{O}, \qquad \{\Pi(\mathbf{x},t), \Pi(\mathbf{x}',t)\} = \mathbb{O}. \tag{43}$$

Since the integral on the right-hand side of Eq. (42) vanishes only along the $\pm\hat{z}_e$ axis, the preferred axis also becomes the *axis of locality.*

For the equal time anticommutator of the $\lambda(x)$ field with its conjugate momentum, we find

$$\{\lambda(\mathbf{x},t), \pi(\mathbf{x}',t)\} = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m}\sum_{\alpha}[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}(\xi_\alpha(\mathbf{p})\vec{\xi}_\alpha(\mathbf{p}) - \zeta_\alpha(-\mathbf{p})\vec{\zeta}_\alpha(-\mathbf{p}))], \tag{44}$$

which, using the same argument as before, yields

$$\{\lambda(\mathbf{x},t), \pi(\mathbf{x}',t)\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}). \tag{45}$$

The difference arises in the evaluation of the remaining anticommutators. The equal time λ-λ anticommutator reduces to

$$\{\lambda(\mathbf{x},t), \lambda(\mathbf{x}',t)\} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\mathbf{p})} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \underbrace{\sum_{\alpha}[\xi_\alpha(\mathbf{p})\zeta^T_\alpha(\mathbf{p}) + \zeta_\alpha(-\mathbf{p})\xi^T_\alpha(-\mathbf{p})]}_{\equiv\Omega(\mathbf{p})}. \tag{46}$$

Now using explicit expressions for $\xi_\alpha(\mathbf{p})$ and $\zeta_\alpha(\mathbf{p})$, we find that $\Omega(\mathbf{p})$ identically vanishes. Eq. (46) then implies

$$\{\lambda(\mathbf{x},t), \lambda(\mathbf{x}',t)\} = \mathbb{O}. \tag{47}$$

Finally, the equal time π-π anticommutator simplifies to

$$\{\pi(\mathbf{x},t), \pi(\mathbf{x}',t)\} = \int \frac{d^3p}{(2\pi)^3}\frac{E(\mathbf{p})}{2m} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \underbrace{\sum_{\alpha}[(\vec{\xi}_\alpha(\mathbf{p}))^T\vec{\zeta}_\alpha(\mathbf{p}) + (\vec{\zeta}_\alpha(-\mathbf{p}))^T\vec{\xi}_\alpha(-\mathbf{p})]}_{=\mathbb{O},\text{ by a direct evaluation}},$$

yielding

$$\{\pi(\mathbf{x},t), \pi(\mathbf{x}',t)\} = \mathbb{O}. \tag{48}$$

Eqs. (42), (43), and (45)–(48) establish that $\Lambda(x)$ and $\lambda(x)$ are *local* quantum fields along the preferred axis, $\hat{z}_e$. We propose to call $\hat{z}_e$ the *axis of locality* in the dark sector.

IV. CONCLUDING REMARKS

Modulo its specialization to the Majorana field, Weinberg's monographic work [8] establishes the uniqueness of the Dirac quantum field for spin 1/2 particles. Seen from that perspective the Ahluwalia-Grumiller work on Elko in 2005 was unexpected. Elko found significant interest among mathematical physicists and cosmologists [28–41]. In these papers one dealt with Elko as spinors and not as a quantum field. Hence, no contradiction with Weinberg's theoremlike work occurred. Gillard and Martin showed that if Elko were to be taken as "good" quantum fields, Poincaré symmetries would be violated in some form or the other [42]. The results presented in this communication explicitly confirm this and show that the violation occurs in a rather subtle way. Despite this, Elko stands as a natural dark matter candidate. Its darkness with respect to the SM matter and gauge fields follows immediately from its intrinsic mass dimensionality. It admits an unsuppressed quartic self coupling. Additionally, it points towards the existence of a preferred axis, along which the Elko quantum fields enjoy locality. Although Elko is nonlocal when the frame is not aligned to the preferred axis, Fabbri [41] has shown that the fields do not violate causality in the sense of Velo and Zwanziger [43].

Recent results seem to suggest that the Elko quantum fields satisfy the symmetry of very special relativity (VSR) proposed by Cohen and Glashow [44]. The HOM(2) and SIM(2) VSR groups naturally incorporate a preferred axis which may be identified with the axis of locality. This will be published in a forthcoming paper.

ACKNOWLEDGMENTS

We thank Adam Gillard, Ben Martin, and T. F. Watson for their constant questions and discussions, and also Karl-Henning Rehren for his helpful comments. The presentation of our draft was improved through a discussion with Matt Visser.

APPENDIX A

1. Dirac spin sums and a "misleading" derivation of the Dirac equation

With a minor departure from the historical path, the Dirac counterpart of Eqs. (32a) and (32b) may be constructed as follows. Instead of (6), we start with

$$\psi_D \equiv \begin{pmatrix} \phi_r \\ \phi_\ell \end{pmatrix}. \tag{A1}$$

The helicities of ϕ_r and ϕ_ℓ are identical and are determined by requiring that ψ_D be eigenspinors of the parity operator $S(P)$. Again, there are four independent rest spinors. (These differ from those mentioned in Sec. I A only in that we now work in the "helicity basis.")

$$u_{+1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} \phi_r^+(\boldsymbol{\epsilon}) \\ \phi_\ell^+(\boldsymbol{\epsilon}) \end{pmatrix}, \qquad u_{-1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} \phi_r^-(\boldsymbol{\epsilon}) \\ \phi_\ell^-(\boldsymbol{\epsilon}) \end{pmatrix}, \tag{A2}$$

$$v_{+1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} \phi_r^-(\boldsymbol{\epsilon}) \\ -\phi_\ell^-(\boldsymbol{\epsilon}) \end{pmatrix}, \qquad v_{-1/2}(\boldsymbol{\epsilon}) = \begin{pmatrix} -\phi_r^+(\boldsymbol{\epsilon}) \\ \phi_\ell^+(\boldsymbol{\epsilon}) \end{pmatrix}. \tag{A3}$$

The $u(\mathbf{p})$ and $v(\mathbf{p})$ for an arbitrary momentum are obtained via the action of the boost κ:

$$u(\mathbf{p}) = \kappa u(\boldsymbol{\epsilon}), \qquad v(\mathbf{p}) = \kappa v(\boldsymbol{\epsilon}). \tag{A4}$$

These lead to the spin sums

$$\sum_\beta u_\beta(\mathbf{p})\bar{u}_\beta(\mathbf{p}) = m\left[\frac{\gamma_\mu p^\mu}{m} + \mathbb{I}\right], \tag{A5a}$$

$$\sum_\beta v_\beta(\mathbf{p})\bar{v}_\beta(\mathbf{p}) = m\left[\frac{\gamma_\mu p^\mu}{m} - \mathbb{I}\right], \tag{A5b}$$

where β takes two values: $+1/2$ and $-1/2$. As before, the right-hand sides in the above expression simply express the result of a direct evaluation of the left-hand sides. These are covariant.

We thus see that in the Dirac construct (whether it be at the level of spinors or at the level of a quantum field), no preferred frame is introduced. For Majorana spinors, and Elko, the conclusion is both unexpected and inevitable. This difference—as pertaining to the existence of a preferred frame—between the Dirac and Majorana spinors, along with their cousins Elko, to our knowledge is completely unknown. This conclusion carries distinct echoes of the unpublished notes [45] which eventually, in collaboration with Grumiller, led to the discovery reported in Refs. [5,6].

If we multiply Eq. (A5a) by $u_{\beta'}(\mathbf{p})$ from the right, and use $\bar{u}_\beta(\mathbf{p})u_{\beta'}(\mathbf{p}) = 2m\delta_{\beta\beta'}$, and carry out a similar exercise with Eq. (A5b), then after a minor rearranging we obtain

$$(\gamma_\mu p^\mu - m\mathbb{I})u(\mathbf{p}) = 0, \tag{A6}$$

$$(\gamma_\mu p^\mu + m\mathbb{I})v(\mathbf{p}) = 0. \tag{A7}$$

These are indeed Dirac equations in momentum space. With $p^\mu \to i\partial^\mu$ and

$$\psi(x) \equiv \begin{cases} u(\mathbf{p}) & \exp(-ip_\mu x^\mu) \\ v(\mathbf{p}) & \exp(+ip_\mu x^\mu), \end{cases} \tag{A8}$$

these yield the well-known Dirac equation in the configuration space

$$(i\gamma_\mu \partial^\mu - m\mathbb{I})\psi(x) = 0. \tag{A9}$$

To associate these with the dynamics of spin 1/2 spinors, particularly in the context of quantum field theory [where $\psi(x)$ is elevated to a spinor field $\Psi(x)$] requires that, in addition, the vacuum expectation value, $\langle|\mathcal{T}[\Psi(x')\bar{\Psi}(x)]|\rangle$, be proportional to the relevant Green's function. That is to say, it is not sufficient to find an operator, such as $(i\gamma_\mu\partial^\mu - m\mathbb{I})$, or the Klein-Gordon operator, that annihilates $\Psi(x)$ for it to serve in the Lagrangian density of the field $\Psi(x)$. It must also satisfy the said requirement. This will become abundantly clear from what follows in the context of Elko.

While we do consider the above "derivation" of the Dirac equation misleading, it does serve to tell us that the Dirac spinors are eigenspinors of $\gamma_\mu p_\mu$ with eigenvalues $\pm m$:

$$\gamma_\mu p^\mu u(\mathbf{p}) = +mu(\mathbf{p}), \qquad \gamma_\mu p^\mu v(\mathbf{p}) = -mv(\mathbf{p}). \tag{A10}$$

The Elko counterpart is

$$\mathcal{G}(\mathbf{p})\xi(\mathbf{p}) = +\xi(\mathbf{p}), \qquad \mathcal{G}(\mathbf{p})\zeta(\mathbf{p}) = -\zeta(\mathbf{p}). \tag{A11}$$

It again emphasizes that identities such as these should not be mistaken for dynamical equations. In particular, $\mathcal{G}(\mathbf{p})$, unlike its Dirac counterpart $\gamma_\mu p^\mu$, contains no time derivative.

2. Elko time ordering and propagators

The mass dimensionality of a field can also be deciphered from constructing the Feynman-Dyson propagator. This involves defining a time-ordering operator. The existence of a preferred direction, however, raises questions with regard to the definition in the context of Elko. In what follows, we first adopt the standard definition of the fermionic time-ordering operator, and then we invoke a consistency argument to formulate a redefinition for Elko.

Let $\mathcal{T}$ be the standard fermionic time-ordering operator. Then, a straightforward calculation yields

$$\begin{aligned}&\langle\,|\mathcal{T}[\Lambda(x')\stackrel{\neg}{\Lambda}(x)]|\,\rangle\\ &\quad= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\mathbf{p})}\\ &\qquad\times\sum_\alpha[\theta(t'-t)\xi_\alpha(\mathbf{p})\stackrel{\neg}{\xi}_\alpha(\mathbf{p})e^{-ip_\mu(x'^\mu-x^\mu)}\\ &\qquad-\theta(t-t')\zeta_\alpha(\mathbf{p})\stackrel{\neg}{\zeta}_\alpha(\mathbf{p})e^{+ip_\mu(x'^\mu-x^\mu)}],\end{aligned}\tag{A12}$$

where the step function $\theta(t)$ equals unity for $t > 0$ and vanishes for $t < 0$.

Using the spin sums (27a) and (27b), setting $q^\mu = (q^0, \mathbf{q} = \mathbf{p})$, and using the standard integral representation for the $\theta(t)$, Eq. (A12) simplifies to

$$\begin{aligned}&\langle\,|\mathcal{T}[\Lambda(x')\stackrel{\neg}{\Lambda}(x)]|\,\rangle\\ &\quad= i\int \frac{d^4q}{(2\pi)^4}e^{-iq_\mu(x'^\mu-x^\mu)}\left[\frac{\mathbb{I}+\mathcal{G}(\mathbf{q})}{q_\mu q^\mu - m^2 + i\epsilon}\right],\end{aligned}\tag{A13}$$

where the limit $\epsilon \to 0^+$ is understood [46]. If there were no preferred axis, then the integral involving the $\mathcal{G}(\mathbf{q})$ term would have identically vanished. Consistency with result (38) suggests that, in Elko quantum field theory, one may need to modify the definition of the time-ordered product to $\mathcal{T}_\#$, such that

$$\begin{aligned}&\langle\,|\mathcal{T}_\#[\Lambda(x')\stackrel{\neg}{\Lambda}(x)]|\,\rangle\\ &\quad= i\int \frac{d^4q}{(2\pi)^4}e^{-iq_\mu(x'^\mu-x^\mu)}\left[\frac{\mathbb{I}}{q_\mu q^\mu - m^2 + i\epsilon}\right].\end{aligned}\tag{A14}$$

To decipher the mass dimensionality, let $\mathcal{D}_\Lambda$ be the mass dimensionality of $\Lambda(x)$. Then the left-hand side of the above equation has mass dimension $2\mathcal{D}_\Lambda$. As for the right-hand side, the mass dimensionality is 2. This gives $\mathcal{D}_\Lambda = 1$. Similarly, a simple computation shows that $\langle\,|\mathcal{T}_\#[\Lambda(x')\stackrel{\neg}{\Lambda}(x)]|\,\rangle = \langle\,|\mathcal{T}_\#[\lambda(x')\stackrel{\neg}{\lambda}(x)]|\,\rangle$. As such, $\mathcal{D}_\lambda = 1$.

Applying the operator $[\partial'^\mu\partial'_\mu + m^2]$ from the left on both sides of Eq. (A14) gives

$$[\partial'^\mu\partial'_\mu + m^2]\langle\,|\mathcal{T}_\#[\Lambda(x')\stackrel{\neg}{\Lambda}(x)]|\,\rangle = -i\delta^4(x'^\mu - x^\mu).\tag{A15}$$

In comparison, for the Dirac field,

$$\begin{aligned}&\langle\,|\mathcal{T}[\Psi(x')\bar{\Psi}(x)]|\,\rangle\\ &\quad= i\int \frac{d^4q}{(2\pi)^4}e^{-iq_\mu(x'^\mu-x^\mu)}\left[\frac{m\mathbb{I}+\gamma^\mu q_\mu}{q_\mu q^\mu - m^2 + i\epsilon}\right].\end{aligned}\tag{A16}$$

This well-known result gives $\mathcal{D}_\Psi = \frac{3}{2}$. The reader is reminded that the $\gamma^\mu q_\mu$ structure appears here through the spin sums which, in the logical framework of this communication, do not invoke any wave equation or a Lagrangian density. Applying the operator $[i\gamma^\mu\partial'_\mu - m]$ from the left on both sides of Eq. (A16) yields

$$[i\gamma^\mu\partial'_\mu - m]\langle\,|\mathcal{T}[\Psi(x')\bar{\Psi}(x)]|\,\rangle = i\delta^4(x'^\mu - x^\mu).\tag{A17}$$

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[26] Here we use ‡ for the creation operator instead of † to allow for the possibility that the state space for Elko may be different from the Dirac particle.

[27] We parenthetically remark that the interactions with the standard model gauge fields—with $F^{\rm SM}_{\mu\nu}(x)$ symbolically representing the associated field strength tensors—through Pauli terms

$$\mathcal{L}^{\rm Pauli}(x) = \vec{\neg}{\Lambda}(x)[\gamma^{\mu},\gamma^{\nu}]\lambda(x)F^{\rm SM}_{\mu\nu}(x), \quad \text{etc.}$$

may exist in principle. However, we consider them to have vanishing coupling strength, as $\mathcal{L}^{\Lambda}(x)$ and $\mathcal{L}^{\lambda}(x)$ do not carry invariance under SM gauge transformations.

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