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Effective gravitational wave stress-energy tensor in alternative theories of gravity

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The inspiral of binary systems in vacuum is controlled by the stress-energy of gravitational radiation and any other propagating degrees of freedom. For gravitational waves, the dominant contribution is characterized by an effective stress-energy tensor at future null infinity. We employ perturbation theory and the short-wavelength approximation to compute this stress-energy tensor in a wide class of alternative theories. We find that this tensor is generally a modification of that first computed by Isaacson, where the corrections can dominate over the general relativistic term. In a wide class of theories, however, these corrections identically vanish at asymptotically flat, future, null infinity, reducing the stress-energy tensor to Isaacson's. We exemplify this phenomenon by first considering dynamical Chern-Simons modified gravity, which corrects the action via a scalar field and the contraction of the Riemann tensor and its dual. We then consider a wide class of theories with dynamical scalar fields coupled to higher-order curvature invariants and show that the gravitational wave stress-energy tensor still reduces to Isaacson's. The calculations presented in this paper are crucial to perform systematic tests of such modified gravity theories through the orbital decay of binary pulsars or through gravitational wave observations.

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I. INTRODUCTION

Feynman has argued that no matter how beautiful or elegant a certain theory is, or how authoritative its proponents, if it does not agree with experiments, then it must be wrong. For the past 40 years, this philosophy has been applied to gravitational theories with great success. Many modified gravity theories that were prominent in the 1970s have now been essentially discarded, as they were found to disagree with Solar System experiments or binary pulsar observations [1]. Similarly, this decade is beginning to bring a wealth of astrophysical information that will be used to constrain new modified gravity theories. In fact, precision double binary pulsar observations [2–4] have already allowed us to constrain modified theories to exciting new levels [5,6]. Future gravitational wave (GW) observations on Earth, with the Advanced Laser Interferometer Gravitational Observatory [7–9], aVIRGO [10], and its collaborators, and in space, through the Laser Interferometer Space Antenna [11–14], will allow new precision tests of strong-field gravity [15].

Such tests of alternative theories of gravity will be very sensitive to the motion of compact bodies in a regime of spacetime where gravitational fields and velocities are large, i.e. the so-called strong-field. Gralla [16] has shown that motion in classical field theories that satisfy certain conditions (the existence of a Bianchi-like identity and field equations no higher than second order) is ''universally'' geodesic to leading order in the binary system's mass ratio, with possible deviations from geodesicity due to the bodies' internal structure. He also argues that one might be able to relax the second condition, as it does not seem necessary. In fact, motion in certain higher-order theories, such as Chern-Simons (CS) modified gravity [17], is already known to be purely geodesic to leading order in the mass ratio, without influence of internal structure due to additional symmetries in the theory.

Tests of modified gravity theories in the strong field, however, not only require a prescription for the conservative sector of motion, but also of the dissipative sector, that which describes how the objects inspiral. Geodesic motion must thus be naturally corrected by a radiation-reaction force that drives nongeodesic motion toward an ultimate plunge and merger [18]. Similarly, one can think of such motion as geodesic but with *varying* orbital elements [19–23] (energy, angular momentum, and the Carter constant). The rate of change of such orbital elements is governed by the rate at which all degrees of freedom (gravitational and nongravitational) radiate.

In the gravitational sector and to leading order in the metric perturbation, such a rate of change is controlled by an effective stress-energy tensor for GWs, first computed by Isaacson in general relativity (GR) [24,25]. In his approach, Isaacson expanded the Einstein equations to second order in the metric perturbation about an arbitrary background. The first-order equations describe the evolution of gravitational radiation. The second-order equation serves as a source to the zeroth-order field equations, just like a stress-energy tensor, and it depends on the square of the first-order perturbation. This tensor can then be averaged over several gravitational wavelengths, assuming the background length scale is much longer than the GW wavelength (the short-wavelength approximation). In this approximation, Isaacson found that the effective GW energy-momentum tensor is proportional to the square of first partial derivatives of the metric perturbation, i.e. proportional to the square of the gravitational frequency. Components of this stress-energy then provide the rate of change of orbital elements, leading to the well-known quadrupole formula.

Alternative theories of gravity generically lead to a modified effective GW stress-energy tensor. It is sometimes assumed that this stress-energy tensor will take the same form as in GR [26,27], but this need not be the case. In GR, the scaling of this tensor with the GW frequency squared can be traced to the Einstein-Hilbert action's dependence on second derivatives of the metric perturbation through the Ricci scalar. If the action is modified through the introduction of higher powers of the curvature tensor, then the stress-energy tensor will be proportional to higher powers of the frequency. Therefore, the consistent calculation of the modified Isaacson tensor needs to be carried out until terms similar to the GR contribution (proportional to frequency squared) are obtained. This in turn implies that calculations of effective energy-momentum tensors in modified gravity theories to *leading order in the GW frequency* can sometimes be insufficient for determining the rate of change of orbital elements.

In this paper, we present a formalism to compute the energy-momentum tensor consistently in generic classical field theories. We employ a scheme where the action itself is first expanded in the metric perturbation to second order, and the background metric and metric perturbation are treated as independent fields. Varying with respect to the background metric leads to an effective GW stress-energy tensor that can then be averaged over several wavelengths. This produces results equivalent to Isaacson's calculation.

We exemplify this formulation by first considering CS gravity [17]. This theory modifies the Einstein-Hilbert action through the addition of the product of a scalar field with the contraction of the Riemann tensor and its dual. This scalar field is also given dynamics through a kinetic term in the action. The leading-order contribution to the CS-modified GW stress-energy tensor should appear at order frequency to the fourth power, but Sopuerta and Yunes [28] have shown that this contribution vanishes at future null infinity.

We here continue this calculation through order frequency cubed and frequency squared and find that such CS modifications still vanish at future null infinity. This is because the background scalar field must decay at a certain rate for it to have a finite amount of energy in an asymptotically flat spacetime. If one insists on ignoring such a requirement, such as in the case of the nondynamical theory, then frequency-cubed CS modifications to the energy momentum do not vanish.

We explicitly calculate such modifications for a canonical embedding, where the scalar field is a linear function of time in inertial coordinates. This is similar to previous work [29] that calculated another effective stress-energy tensor for the nondynamical version of Chern-Simons. In this case, the dominant modification to the radiation-reaction force is in the rate of change of radiated momentum, which leads to so-called recoil velocities after binary coalescence. In GR, such recoil is proportional to the product of the (mass) quadrupole and octopole when multipolarly decomposing the radiation field. In nondynamical CS gravity with a canonical embedding, the recoil is proportional to the square of the mass quadrupole, which dominates over the GR term.

We then construct a wide class of alternative theories that differ from GR through higher-order curvature terms in the action coupled to a scalar field. We compute the GW stress-energy-momentum tensor in such theories and find that corrections to the Isaacson tensor vanish at future null infinity provided the following conditions are satisfied: (i) The curvature invariants in the modification are quadratic or higher order; (ii) the nonminimally coupled scalar field is dynamical; (iii) the modification may be modeled as a weak deformation away from GR; (iv) the spacetime is asymptotically flat at future null infinity. These results prove that the effective stress-energy tensor assumed in [26,27] is indeed correct.[1]

Even if the effective GW energy-momentum tensor is identical to that in GR, in terms of contractions of first derivatives of the metric perturbation, this does not imply that GWs will not be modified. First, background solutions could be modified. For example, in dynamical CS gravity, the Kerr metric is not a solution to the modified field equations for a rotating black hole (BH) [30], but it is instead modified in the shift sector [31]. Second, the solution to the GW evolution equation could also be modified. For example, in nondynamical CS gravity, GWs become amplitude birefringent as they propagate [32–34]. Third, additional degrees of freedom may also be present and radiate, thus changing the orbital evolution. All of these facts imply that even if the Isaacson tensor correctly describes the effective GW energy-momentum tensor, GWs themselves can and generically will be modified in such alternative theories.

In the remainder of this paper we use the following conventions. Background quantities are always denoted with an overhead bar, while perturbed quantities of first order with an overhead tilde. We employ decompositions of the type $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon \tilde{h}_{\mu\nu} + \mathcal{O}(\epsilon^2)$, where $g_{\mu\nu}$ is the full metric, $\bar{g}_{\mu\nu}$ is the background metric, and $\tilde{h}_{\mu\nu}$ is a small perturbation ($\epsilon \ll 1$ is a bookkeeping parameter). Covariant differentiation with respect to the background metric is denoted via $\bar{\nabla}_\mu B_\nu$, while covariant differentiation with respect to the full metric is denoted via $\nabla_\mu B_\nu$. Symmetrization and antisymmetrization are denoted with parentheses and square brackets around the indices, respectively, such as $A_{(\mu\nu)} \equiv [A_{\mu\nu} + A_{\nu\mu}]/2$ and

[1]The authors of [26,27] presented an energy loss formula which was not evaluated at $\mathcal{I}^+$. In the limit of $r \to \infty$, their energy loss formula reduces to the Isaacson formula.

$A_{[\mu\nu]} \equiv [A_{\mu\nu} - A_{\nu\mu}]/2$. We use the metric signature $(-, +, +, +)$ and geometric units, such that $G = c = 1$.

This paper is organized as follows: Section II describes the perturbed Lagrangian approach used in this paper to compute the effective GW stress-energy tensor. Section III applies this framework to GR. Section IV discusses dynamical CS gravity. Section V computes the full effective stress-energy tensor in this theory. Section VI generalizes the calculation to a wider class of alternative theories. Section VII concludes and points to future research.

II. PERTURBED LAGRANGIAN APPROACH

Isaacson [24,25] introduced what is now the standard technique to obtain an effective stress-energy tensor for gravitational radiation, via second-order perturbation theory on the equations of motion. This technique requires an averaging procedure to construct an *effective* stress-energy tensor. This is because of the inability to localize the energy of the gravitational field to less than several wavelengths of the radiation and because of the ambiguities of the metric perturbation on distances of order the wavelength due to gauge freedom.[2] Isaacson employed the Brill-Hartle [37] averaging scheme, although one can arrive at an identical quantity by using different schemes [38–40], e.g. Whitham or macroscopic gravity.

An alternative approach to derive field equations and an effective stress-energy tensor for GWs is to work at the level of the action. One possibility is to use the Palatini framework [41,42], where the connection is promoted to an independent field that is varied in the action, together with the metric tensor. Such a framework, however, is problematic in alternative theories of gravity, as it need not lead to the same field equations as variation of the action with respect to the metric tensor only.

A similar but more appropriate approach is that of *second variation* [43]. In this approach, the action is first expanded to second order in the metric perturbation, assuming the connection is the Christoffel one $\Gamma^{\sigma}_{\mu\nu}$. Then, the action is promoted to an effective one, by treating the background metric tensor and the metric perturbation as *independent fields*. Variation of this effective action with respect to the metric perturbation and the background metric yields the equations of motion. The former variation leads to the first-order field equations, when the background field equations are imposed. The latter variation leads to the background field equations to zeroth order in the metric perturbation and to an effective GW stress-energy tensor to second order.

Let us begin by expanding all quantities in a power series about a background solution

$$\varphi = \bar{\varphi} + \epsilon\tilde{\varphi} + \epsilon^2\tilde{\tilde{\varphi}} + \mathcal{O}(\epsilon^3), \qquad (1)$$

where $\epsilon \ll 1$ is an order counting parameter and φ represents all tensor fields of the theory with indices suppressed: $\bar{\varphi}$ is the background field, $\tilde{\varphi}$ is the first-order perturbation to φ, and $\tilde{\tilde{\varphi}}$ is the second-order perturbation. The action can then be expanded, as

$$S[\varphi] = S^{(0)}[\bar{\varphi}] + S^{(1)}[\bar{\varphi}, \tilde{\varphi}] + S^{(2)}[\bar{\varphi}, \tilde{\varphi}, \tilde{\tilde{\varphi}}] + \mathcal{O}(\epsilon^3), \qquad (2)$$

where $S^{(1)}$ is linear in $\tilde{\varphi}$ and $S^{(2)}$ is quadratic in $\tilde{\varphi}$ but linear in $\tilde{\tilde{\varphi}}$. We now define the effective action as Eq. (2) but promoting $\bar{\varphi}$ and $\tilde{\varphi}$ to independent fields.

One might wonder why the field $\tilde{\tilde{\varphi}}$ is not also treated as independent. First, variation of the action with respect to $\tilde{\tilde{\varphi}}$ would lead to second-order equations of motion, which we are not interested in here. Second, the variation of the action with respect to $\tilde{\varphi}$ cannot introduce terms that depend on $\tilde{\tilde{\varphi}}$, because the product of $\tilde{\varphi}$ and $\tilde{\tilde{\varphi}}$ never appears in Eq. (2), as this would be of $\mathcal{O}(\epsilon^3)$. Third, the variation of the action with respect to $\bar{\varphi}$ can only introduce terms linear in $\tilde{\tilde{\varphi}}$, which vanish upon averaging, as we describe in Sec. II A. This is because averages of any odd number of short-wavelength quantities generically vanish. Therefore, we can safely neglect all terms that depend on $\tilde{\tilde{\varphi}}$ in the effective action, which renders Eq. (2) a functional of only $\bar{\varphi}$ and $\tilde{\varphi}$.

As in the standard approach, the second-order variation method still requires that one performs a short-wavelength average of the effective stress-energy tensor. Upon averaging, the variation of the first-order piece of the action $S^{(1)}$ with respect to $\bar{\varphi}$ vanishes because it generates terms linear in $\tilde{\varphi}$. Since $S^{(1)}$ does not contribute to the effective stress-energy tensor, we can safely drop it from the effective action for now.

The effective action reduces to

$$S^{\rm eff}[\bar{\varphi}, \tilde{\varphi}] = S^{{\rm eff}(0)}[\bar{\varphi}] + S^{{\rm eff}(2)}[\bar{\varphi}, \tilde{\varphi}]. \qquad (3)$$

Naturally, the variation of $S^{{\rm eff}(0)}$ with respect to the background metric $\bar{g}^{\mu\nu}$ yields the background equations of motion. The effective stress-energy tensor comes from averaging the variation of $S^{{\rm eff}(2)}$ with respect to $\bar{g}^{\mu\nu}$:

$$\delta S^{{\rm eff}(2)} = \epsilon^2 \int d^4x\sqrt{-\bar{g}}\,\delta\bar{g}^{\mu\nu} t_{\mu\nu}, \qquad (4a)$$

$$T^{\rm eff}_{\mu\nu} \equiv -2\epsilon^2 \langle\langle t_{\mu\nu}\rangle\rangle, \qquad (4b)$$

where the factor of 2 is conventional for agreement with the canonical stress-energy tensor, and $\langle\langle\ \rangle\rangle$ is the averaging operator, which we discuss below. One of the immediate benefits of working from an action principle comes from the diffeomorphism invariance of the action. The diffeomorphism invariance immediately implies that the variation of the total action with respect to the metric is divergence-free [42]. When the matter stress-energy tensor is itself divergence-free, then the gravity sector—the sum of the stress-energy of nonminimally coupled degrees of

[2] Gauge freedom in perturbation theory stands for the freedom to identify points between the physical and "background" manifolds [35,36].

freedom and the effective stress-energy tensor of gravitational waves—will also be divergence-free.

A. Short-wavelength averaging

The goal of the averaging scheme is to distinguish radiative quantities, those which are rapidly varying functions of spacetime, from Coulomb-like quantities, those which are slowly varying functions. This is accomplished by defining the operator $\langle\!\langle\,\rangle\!\rangle$ as a linear integral operator. This operator may either be an average over the phase of the rapidly varying quantities or over spacetime. If the integral is over spacetime, there is an averaging kernel with characteristic length scale $L_{\rm ave}$ that separates the foreground short-wavelength $\lambda_{\rm GW}$ from the background length scale $L_{\rm bg}$, that is, $\lambda_{\rm GW} \ll L_{\rm ave} \ll L_{\rm bg}$.

The details of the averaging scheme are not as important as their properties [38–40], since one arrives at equivalent results using different schemes. The most useful properties of $\langle\!\langle\,\rangle\!\rangle$ are

(1) The average of a product of an odd number of short-wavelength quantities vanishes.

(2) The average of a derivative of a tensor vanishes, e.g. for some tensor expression $T^{\mu}{}_{\alpha\beta}$, $\langle\!\langle \bar{\nabla}_\mu T^{\mu}{}_{\alpha\beta}\rangle\!\rangle = 0$.

(3) As a corollary to the above, integration by parts can be performed, e.g. for tensor expressions $R^{\mu}{}_{\alpha}$, S_β, $\langle\!\langle R^{\mu}{}_{\alpha}\bar{\nabla}_\mu S_\beta\rangle\!\rangle = -\langle\!\langle S_\beta \bar{\nabla}_\mu R^{\mu}{}_{\alpha}\rangle\!\rangle$.

Let us briefly mention where some of these properties come from and some caveats. When considering monochromatic functions, an odd oscillatory integral has an average value about zero, while an even oscillatory integral has a nonzero average. This is enough to find that averages of expressions linear in a short-wavelength quantity will vanish. Expressions at third (and higher odd) order would vanish for monochromatic radiation but not in general. However, these are at sufficiently high order that we neglect them.

The vanishing of averages of derivatives is a subtle point. In the spacetime average approach, this is found by integrating by parts, leaving a term with a derivative on the averaging kernel. This term is smaller than nonvanishing averages by a factor of $\mathcal{O}(\lambda_{\rm GW}/L_{\rm ave})$ and depends on the averaging kernel. From physical grounds, the choice of averaging kernel should not affect any physical quantities, so the average should in fact vanish identically. From the action standpoint, the average of a derivative can be seen to arise from an action term which is a total divergence. Since total divergences in the action do not affect the equations of motion, the average of a derivative vanishes.

A similar argument holds for integration by parts. In the Brill-Hartle average scheme, integration by parts incurs an error of order $\mathcal{O}(\lambda_{\rm GW}/L_{\rm ave})$ from a derivative of the averaging kernel. From the action standpoint, though, integration by parts at the level of the action incurs no error, since there is no averaging kernel in the action.

The fact that the variation of $S^{(1)}$ with respect to $\bar{\varphi}$ does not contribute to the effective stress-energy tensor is a direct consequence of property (1) above. The vanishing of all terms linear in $\tilde{\varphi}$ upon averaging is also a consequence of (1). As one can see, these properties greatly simplify all further calculations.

B. Varying Christoffel and curvature tensors

Let us now consider what types of terms arise from the variation of the effective action with respect to $\bar{g}^{\mu\nu}$. In order to perform this variation properly, any implicit dependence of the action on $\bar{g}^{\mu\nu}$ must be explicitly revealed; for example, terms which contain the trace $\tilde{h}$ must be rewritten as $\bar{g}_{\mu\nu}\tilde{h}^{\mu\nu}$. Indices should appear in their "natural" positions (see Sec. II D), for indices raised and lowered with the metric have implicit dependence on it. Furthermore, since we are approaching gravity from the metric formulation, rather than the Palatini formulation, there will be contributions from the variation of $\bar{\nabla}_\mu$ and curvature tensors.

Consider a term in the effective action such as

$$S^{\rm ex.1} = \int d^4x\sqrt{-\bar{g}}\,T^{\gamma\cdots}{}_{\delta\ldots}\bar{\nabla}_\mu S^{\alpha\cdots}{}_{\beta\ldots}, \tag{5}$$

where g is the determinant of the metric and $T^{\gamma\cdots}{}_{\delta\ldots}$ and $S^{\alpha\cdots}{}_{\beta\ldots}$ are some tensor expressions, with all indices contracted in some fashion, as the action must be a scalar. When such a term is varied with $\bar{g}^{\mu\nu}\to\bar{g}^{\mu\nu}+\delta\bar{g}^{\mu\nu}$, besides the obvious contributions from $\delta\sqrt{-\bar{g}}$, and explicit dependence of $T^{\gamma\cdots}{}_{\delta\ldots}$ and $S^{\alpha\cdots}{}_{\beta\ldots}$ on $\bar{g}^{\mu\nu}$, there are also contributions from varying the Christoffel connection $\Gamma^{\lambda}_{\mu\beta}$ in $\bar{\nabla}_\mu$. The general expression is

$$\begin{aligned}\delta(\bar{\nabla}_\mu S^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}) = {}& \bar{\nabla}_\mu(\delta S^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}) \\ &+ \sum_{i=1}^{n}\delta\bar{\Gamma}^{\alpha_i}_{\mu\lambda}S^{\ldots\lambda\ldots}{}_{\beta_1\ldots\beta_m} \\ &- \sum_{j=1}^{m}\delta\bar{\Gamma}^{\lambda}_{\mu\beta_j}S^{\alpha_1\ldots\alpha_n}{}_{\ldots\lambda\ldots},\end{aligned} \tag{6}$$

where $\ldots\lambda\ldots$ in the ith term of a sum means replacing α_i or β_i in the index list with λ and where

$$\delta\bar{\Gamma}^{\sigma}_{\mu\nu} = -\tfrac{1}{2}[\bar{g}_{\lambda\mu}\bar{\nabla}_\nu\delta\bar{g}^{\lambda\sigma} + \bar{g}_{\lambda\nu}\bar{\nabla}_\mu\delta\bar{g}^{\lambda\sigma} - \bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\bar{\nabla}^\sigma\delta\bar{g}^{\alpha\beta}]. \tag{7}$$

Curvature tensors also depend on derivatives of the connection, so one naturally expects terms of the form $\bar{\nabla}_\rho\bar{\nabla}_\sigma\delta\bar{g}^{\mu\nu}$ from the variation of curvature quantities, i.e. the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}$, or Ricci scalar R. For example, one can show that

$$\delta\bar{R}^{\mu}{}_{\nu\alpha\beta} = 2\bar{\nabla}_{[\alpha}\delta\bar{\Gamma}^{\mu}_{\beta]\nu}, \tag{8}$$

where the contribution from $\Gamma \wedge \Gamma$ cancels [42]. Upon integration by parts any scalar in the action that contains curvature tensors, one can convert a term containing $\bar{\nabla}_\rho \bar{\nabla}_\sigma \delta \bar{g}^{\mu\nu}$ into

$$\delta S^{\text{ex.2}} = \int d^4x \sqrt{-\bar{g}} P^\sigma{}_{\mu\nu} \bar{\nabla}_\sigma \delta \bar{g}^{\mu\nu}. \qquad (9)$$

In fact, many terms in the variation of the action can be written in the form of Eq. (9).

The contribution of Eq. (9) to the effective stress-energy tensor is found by integrating by parts and then averaging, according to Eq. (4). Upon averaging, however, one finds that such terms do not contribute to the effective stress-energy tensor because

$$T^{\text{eff,ex.2}}_{\mu\nu} = 2\langle\langle \bar{\nabla}_\sigma P^\sigma{}_{\mu\nu} \rangle\rangle \qquad (10)$$

vanishes according to property (2) in Sec. II A.

The above arguments and results imply that the variations of curvature tensors and connection coefficients with respect to $\bar{g}_{\mu\nu}$ do not contribute to the effective stress-energy tensor. Only metric tensors which are raising, lowering, and contracting indices in the action contribute to this tensor. We can thus concentrate on these, when computing $T^{\text{eff}}_{\mu\nu}$.

C. Contributions at asymptotic infinity

When calculating the radiation-reaction force to leading order in the metric perturbation, it is crucial to account for all the energy-momentum loss in the system. The first contribution is straightforward: Energy momentum is radiated outward, toward future, null infinity $\mathcal{I}^+$. Since the stress-energy tensor is covariantly conserved, the energy momentum radiated to $\mathcal{I}^+$ can be calculated by performing a surface integral over a 2-sphere at future, null infinity.[3]

However, not all energy-momentum loss escapes to infinity, as energy can also be lost due to the presence of trapped surfaces in the interior of the spacetime. Trapped surfaces can effectively absorb GW energy momentum, which must also be accounted for, e.g. in the calculation of extreme mass-ratio inspiral (EMRI) orbits around supermassive BHs [44,45]. Calculations of such energy-momentum loss at the BH horizon are dramatically more complicated than those at $\mathcal{I}^+$ and we do not consider them here.

What is the relative importance of energy momentum lost to $\mathcal{I}^+$ and that lost into trapped surfaces? To answer this question, we can concentrate on the magnitude of the leading-order energy flux, as the argument trivially extends to momentum. The post-Newtonian approximation [46], which assumes weak-gravitational fields and slow velocities, predicts that the energy flux carried out to $\mathcal{I}^+$ is proportional to v^{10} to leading order in v, where v is the orbital velocity of a binary system in a quasicircular orbit (see e.g. [46]). On the other hand, a combination of the post-Newtonian approximation and BH perturbation theory predicts that, to leading order in v, the energy flux carried into trapped surfaces is proportional to v^{15} for spinning BHs and v^{18} for nonspinning BHs [47]. BH GW flux absorption is then clearly smaller than the GW flux carried out to $\mathcal{I}^+$ if $v < 1$, which is true for EMRIs for which the post-Newtonian approximation holds.

Intuitively, this hierarchy in the magnitude of energy-momentum flux lost by BH binaries can be understood by considering the BH as a geometric absorber in the radiation field. Radiation which is longer in wavelength than the size of the BH is very weakly absorbed. Only at the end of an inspiral will the orbital frequency be high enough that GWs will be significantly absorbed by the horizon. Notice that this argument is independent of the particular theory considered, only relying on the existence of trapped surfaces. This result does not imply that BH absorption should be neglected in EMRI modeling, but just that it is a smaller effect than the flux carried out to infinity [44,45].

In the remainder of the paper, we will only address energy momentum radiated to $\mathcal{I}^+$ and relegate any analysis of radiation lost into a trapped surface to future work. The only terms which can contribute to an energy-momentum flux integral on a 2-sphere at $\mathcal{I}^+$ are those which decay as r^{-2}, since the area element of the sphere grows as r^2. No terms may decay more slowly than r^{-2}, as the flux must be finite; i.e. the effective stress-energy cannot scale as r^{-1}, as a constant, or with positive powers of radius. Similarly, any terms decaying faster than r^{-2} do not contribute, as they would vanish at $\mathcal{I}^+$. Of course, to determine which terms contribute and which do not, one must know the leading asymptotic forms of all quantities in the effective stress-energy tensor.

In GR, as we shall see in Sec. III, the only fields appearing in the effective stress-energy tensor are the background metric $\bar{g}_{\mu\nu}$ and derivatives of the metric perturbation $\tilde{h}_{\mu\nu}$. As one approaches $\mathcal{I}^+$, $\bar{g}_{\mu\nu} \sim \eta_{\mu\nu}$ in Cartesian coordinates, while $|\tilde{h}_{\mu\nu}| \sim r^{-1} \sim |\bar{\nabla}_\rho \tilde{h}_{\mu\nu}|$. Curvature tensors scale as $|\bar{R}_{\mu\nu\delta\sigma}| \sim r^{-3}$, since they quantify tidal forces. For a theory that is a deformation away from GR, and far away from regions of strong curvature, these asymptotic forms cannot change.

Consider now terms in the general effective action at order $\mathcal{O}(\epsilon^2)$ that contain background curvature tensors. Because of their ordering, they would contribute to the effective stress-energy. One such term is

$$S^{\text{ex.3}} = \epsilon^2 \int d^4x \sqrt{-\bar{g}} \bar{\nabla}_\rho \tilde{\varphi}_1^\sigma \bar{\nabla}_\alpha \tilde{\varphi}_2^\beta \bar{R}^\rho{}_{\beta\sigma\kappa} \bar{g}^{\alpha\kappa}, \qquad (11)$$

where $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the first-order perturbations to two fields in the theory (e.g. the metric perturbation $\tilde{h}_{\mu\nu}$ and the CS scalar perturbation $\tilde{\vartheta}$ that we introduce in Sec. V).

[3] We will not consider spacetimes which are not asymptotically flat, e.g. de Sitter space; the calculations are more involved in such spacetimes.

Since there is no contribution to the effective stress-energy tensor from the variation of curvature quantities (see Sec. II B), the only contributions to the effective stress-energy come from

$$T^{\rm eff,ex.3}_{\mu\nu} = -2\epsilon^2 \left\langle\!\!\left\langle (-\tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\kappa} + \delta^{\alpha}{}_{(\mu}\delta^{\kappa}{}_{\nu)}) \cdot \underbrace{\bar{\nabla}_{\rho}\tilde{\varphi}^{\sigma}_{1}}_{r^{-1}} \underbrace{\bar{\nabla}_{\alpha}\tilde{\varphi}^{\beta}_{2}}_{r^{-1}} \underbrace{\bar{R}^{\rho}{}_{\beta\sigma\kappa}}_{r^{-3}} \right\rangle\!\!\right\rangle, \tag{12}$$

which has the same functional form as the integral. Note that the curvature tensor always remains when varying with respect to $\bar{g}^{\mu\nu}$.

Combining this result with the asymptotic arguments above, such terms can be ignored as one approaches $\mathcal{I}^+$. Each of the first-order fields possess a radiative part that scales as r^{-1}. The square of the first-order fields would then satisfy the r^{-2} scaling requirement for the flux integral. The curvature tensor, however, scales as r^{-3}, which implies that the term in Eq. (12) vanishes at $\mathcal{I}^+$.

We then conclude that terms in the action that contain background curvature quantities at $\mathcal{O}(\epsilon^2)$ may be ignored in calculating the effective stress-energy tensor at $\mathcal{I}^+$. As an immediate corollary to this simplification, we may also freely commute background covariant derivatives if we are interested in the stress-energy tensor at infinity only, since the commutator is proportional to background curvature tensors.

D. Imposing gauge in the effective action

We will choose as our dynamical field not $\tilde{h}_{\mu\nu}$ but rather $\underline{\tilde{h}}^{\mu\nu}$, where the underline stands for the trace-reverse operation, and we take the natural position of the indices to be contravariant. The resulting stress-energy tensor is equal to the one calculated using $\tilde{h}_{\mu\nu}$ *after* evaluating both of them on-shell, i.e. imposing the equations of motion.

We also impose a gauge condition to simplify future expressions: the Lorenz gauge condition

$$\bar{\nabla}_{\mu}\underline{\tilde{h}}^{\mu\nu} = 0. \tag{13}$$

Typically one may not impose a gauge condition at the level of the action. However, in our case, the gauge condition in Eq. (13) has the important property of having all of the indices in their natural positions: The contraction of the indices does not involve the metric.

Consider a term in the effective action that contains this divergence:

$$S^{\rm ex.4} = \epsilon^2 \int d^4x \sqrt{-\bar{g}}\, T_{\beta} \bar{\nabla}_{\alpha}\underline{\tilde{h}}^{\alpha\beta}, \tag{14}$$

with T_β some tensor expression at first order in ϵ. The α index that is contracted above does not require the metric for such contraction. Therefore, $\bar{\nabla}_{\alpha}\underline{\tilde{h}}^{\alpha\beta}$ always remains upon variation:

$$T^{\rm eff,ex.4}_{\mu\nu} = -2\epsilon^2 \left\langle\!\!\left\langle \left(-\frac{1}{2}\bar{g}_{\mu\nu}T_{\beta} + \frac{\delta T_{\beta}}{\delta \bar{g}^{\mu\nu}}\right)\bar{\nabla}_{\alpha}\underline{\tilde{h}}^{\alpha\beta} \right\rangle\!\!\right\rangle. \tag{15}$$

If we delayed imposing the Lorenz gauge condition until after the calculation of the effective stress-energy tensor, we would find the same effective tensor as if we had imposed the gauge condition at the level of the action. Having said that, one should not impose the gauge condition when varying with respect to $\underline{\tilde{h}}^{\mu\nu}$ as clearly $\bar{\nabla}_{\alpha}\underline{\tilde{h}}^{\alpha\beta}$ must also be varied.

III. EFFECTIVE STRESS-ENERGY IN GR

Let us now demonstrate the principles described in the previous section by deriving the standard Isaacson stress-energy tensor in GR. Consider the Einstein-Hilbert action

$$S_{\rm GR} = \kappa \int d^4x \sqrt{-g} R, \tag{16}$$

where $\kappa = (16\pi G)^{-1}$. Now perturb to second order to form the effective action

$$S^{\rm eff}_{\rm GR} = S^{\rm eff(0)}_{\rm GR} + S^{\rm eff(2)}_{\rm GR}, \tag{17a}$$

$$S^{\rm eff(0)}_{\rm GR} = \kappa \int d^4x \sqrt{-\bar{g}}\bar{R}, \tag{17b}$$

$$S^{\rm eff(2)}_{\rm GR} = \epsilon^2 \kappa \int d^4x \mathcal{L}^{\rm eff,1}_{\rm GR} + \mathcal{L}^{\rm eff,2}_{\rm GR}, \tag{17c}$$

where

$$\mathcal{L}^{\rm eff,1}_{\rm GR} = \tfrac{1}{8}\sqrt{-\bar{g}}\,[4\bar{R}_{\alpha\beta}(2\underline{\tilde{h}}^{\alpha}{}_{\mu}\underline{\tilde{h}}^{\beta\mu} - \underline{\tilde{h}}^{\alpha\beta}\underline{\tilde{h}}) + \bar{R}(\underline{\tilde{h}}^2 - 2\underline{\tilde{h}}_{\alpha\beta}\underline{\tilde{h}}^{\alpha\beta})] \tag{17d}$$

and

$$\begin{aligned}\mathcal{L}^{\rm eff,2}_{\rm GR} = \sqrt{-\bar{g}}[&-\underline{\tilde{h}}^{\alpha\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\mu}\underline{\tilde{h}}^{\mu}{}_{\beta} - \tfrac{1}{8}(\bar{\nabla}_{\mu}\underline{\tilde{h}})(\bar{\nabla}^{\mu}\underline{\tilde{h}}) \\ &- (\bar{\nabla}_{\mu}\underline{\tilde{h}}^{\mu}{}_{\alpha})(\bar{\nabla}_{\nu}\underline{\tilde{h}}^{\nu\alpha}) + \tfrac{1}{2}(\bar{\nabla}_{\nu}\underline{\tilde{h}})(\bar{\nabla}_{\mu}\underline{\tilde{h}}^{\mu\nu}) \\ &- \underline{\tilde{h}}^{\alpha\beta}\bar{\nabla}_{\mu}\bar{\nabla}_{\alpha}\underline{\tilde{h}}^{\mu}{}_{\beta} + \tfrac{1}{2}\underline{\tilde{h}}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\underline{\tilde{h}}^{\mu\nu} + \underline{\tilde{h}}^{\alpha\beta}\Box\underline{\tilde{h}}_{\alpha\beta} \\ &- \tfrac{1}{4}\underline{\tilde{h}}\Box\underline{\tilde{h}} - \tfrac{1}{2}(\bar{\nabla}_{\mu}\underline{\tilde{h}}_{\nu\alpha})(\bar{\nabla}^{\nu}\underline{\tilde{h}}^{\mu\alpha}) + \tfrac{3}{4}(\bar{\nabla}_{\mu}\underline{\tilde{h}}_{\alpha\beta})(\bar{\nabla}^{\mu}\underline{\tilde{h}}^{\alpha\beta})],\end{aligned} \tag{17e}$$

and $(\bar{R}_{\mu\nu}, \bar{R})$ refer to the background Ricci tensor and scalar, respectively. The integrands have been written in terms of the trace-reversed metric perturbation $\underline{\tilde{h}}^{\mu\nu}$. From Sec. II C, $\mathcal{L}^{\rm eff,1}_{\rm GR}$ does not contribute at $\mathcal{I}^+$ because it depends explicitly on curvature quantities, so we ignore it. The variation and averaging of $\mathcal{L}^{\rm eff,2}_{\rm GR}$ produces the Isaacson stress-energy tensor.

By integrating by parts, all terms in Eq. (17e) can be written as $(\bar{\nabla}_{\alpha}\underline{\tilde{h}}^{\rho\sigma})(\bar{\nabla}_{\beta}\underline{\tilde{h}}^{\kappa\lambda})$ (with indices contracted to form a scalar) rather than $\underline{\tilde{h}}^{\rho\sigma}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\underline{\tilde{h}}^{\kappa\lambda}$ (again, with indices contracted). $\mathcal{L}^{\rm eff,2}_{\rm GR}$ is thus rewritten in the more compact form

$$\mathcal{L}_{\rm MT} = \sqrt{-\bar{g}}[\tfrac{1}{2}(\bar{\nabla}_\mu \underline{\tilde{h}}^{\nu\alpha})(\bar{\nabla}_\nu \underline{\tilde{h}}^\mu{}_\alpha) - \tfrac{1}{4}(\bar{\nabla}_\mu \underline{\tilde{h}}_{\alpha\beta})(\bar{\nabla}^\mu \underline{\tilde{h}}^{\alpha\beta}) + \tfrac{1}{8}(\bar{\nabla}_\mu \underline{\tilde{h}})(\bar{\nabla}^\mu \underline{\tilde{h}})], \tag{18}$$

which is the expression that appears in MacCallum and Taub [43]. With this simplified expression at hand, we can promote $\underline{\tilde{h}}^{\mu\nu}$ to an independent dynamical field in Eq. (17) and vary it with respect to both $\bar{g}^{\mu\nu}$ and $\underline{\tilde{h}}^{\mu\nu}$ to obtain the effective stress-energy tensor and the first-order equations of motion, respectively.

Let us first derive the first-order equations of motion. Varying Eq. (17) with respect to $\underline{\tilde{h}}^{\mu\nu}$, we find

$$\bar{\Box}\underline{\tilde{h}}_{\mu\nu} - 2\bar{\nabla}^\alpha\bar{\nabla}_{(\mu}\underline{\tilde{h}}_{\nu)\alpha} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}\,\underline{\tilde{h}} = 0, \tag{19a}$$

whose trace is

$$2\bar{\nabla}_\alpha\bar{\nabla}_\beta\underline{\tilde{h}}^{\alpha\beta} + \bar{\Box}\,\underline{\tilde{h}} = 0. \tag{19b}$$

We can now impose the Lorenz gauge on Eq. (19b), which then leads to $\bar{\Box}\,\underline{\tilde{h}} = 0$. If $\underline{\tilde{h}} = 0$ is further imposed on an initial hypersurface while maintaining Lorenz gauge, then the evolution equation preserves the trace-free gauge [41]. The combination of these two gauge choices (Lorenz gauge plus trace-free) is the transverse-trace-free gauge, or TT gauge. After commuting derivatives in Eq. (19a) and imposing the TT gauge, the tensor equation of motion reads

$$\bar{\Box}\underline{\tilde{h}}_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta}\underline{\tilde{h}}^{\alpha\beta} = 0, \tag{19c}$$

where $\bar{R}_{\mu\alpha\nu\beta}$ is the background Riemann tensor. At $\mathcal{I}^+$, this equation reduces to $\bar{\Box}\underline{\tilde{h}}_{\mu\nu} = 0$, which leads to the standard dispersion relation for GWs, traveling at the speed of light.

Let us now calculate the effective stress-energy tensor. Note that the first term in $\mathcal{L}_{\rm MT}$ may be integrated by parts and covariant derivatives commuted to form the Lorenz gauge condition, so the first term may be ignored. Varying the action with respect to $\bar{g}^{\mu\nu}$, we find

$$\kappa G_{\mu\nu} = -\epsilon^2\kappa\left\langle\!\!\left\langle \frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\mathcal{L}_{\rm MT}\right\rangle\!\!\right\rangle \equiv \frac{1}{2}T^{\rm eff}_{{\rm MT}\mu\nu}, \tag{20a}$$

where

$$T^{\rm eff}_{{\rm MT}\mu\nu} = 2\epsilon^2\kappa\langle\!\langle \tfrac{1}{4}\bar{\nabla}_\mu\underline{\tilde{h}}^{\alpha\beta}\bar{\nabla}_\nu\underline{\tilde{h}}_{\alpha\beta} - \tfrac{1}{2}\bar{\nabla}_\alpha\underline{\tilde{h}}_{\beta\mu}\bar{\nabla}^\alpha\underline{\tilde{h}}_\nu{}^\beta - \tfrac{1}{8}\bar{\nabla}_\mu\underline{\tilde{h}}\bar{\nabla}_\nu\underline{\tilde{h}} + \tfrac{1}{4}\bar{\nabla}_\alpha\underline{\tilde{h}}_{\mu\nu}\bar{\nabla}^\alpha\underline{\tilde{h}} + \tfrac{1}{2}\bar{g}_{\mu\nu}(-\bar{g})^{-1/2}\mathcal{L}_{\rm MT}\rangle\!\rangle, \tag{20b}$$

which we refer to as the MacCallum-Taub tensor. Terms that depend on the trace $\underline{\tilde{h}}^\mu{}_\mu$ in this tensor can be eliminated in the TT gauge.

Let us now evaluate the MacCallum-Taub tensor on-shell, by imposing the equations of motion [Eq. (19)]. When short-wavelength averaging, derivatives that are contracted together can be converted into the d'Alembertian via integration by parts; such terms vanish at $\mathcal{I}^+$. What results is the usual Isaacson stress-energy tensor

$$T^{\rm eff}_{{\rm GR}\mu\nu} = \epsilon^2\frac{\kappa}{2}\langle\!\langle(\bar{\nabla}_\mu\underline{\tilde{h}}^{\alpha\beta})(\bar{\nabla}_\nu\underline{\tilde{h}}_{\alpha\beta})\rangle\!\rangle. \tag{20c}$$

Notice that this expression is only valid at $\mathcal{I}^+$ and in the TT gauge. As mentioned earlier, this tensor cannot be used to model energy-momentum loss through trapped surfaces, since then curvature quantities cannot be ignored.

IV. CHERN-SIMONS GRAVITY

CS gravity is a modified theory introduced first by Jackiw and Pi [32] (for a recent review see [17]). The dynamical version of this theory modifies the Einstein-Hilbert action through the addition of the following terms:

$$S = S_{\rm EH} + S_{\rm CS} + S_\vartheta + S_{\rm mat}, \tag{21}$$

where

$$\begin{aligned} S_{\rm EH} &= \kappa\int d^4x\sqrt{-g}R, \\ S_{\rm CS} &= \frac{\alpha}{4}\int d^4x\sqrt{-g}\vartheta\,{}^*RR, \\ S_\vartheta &= -\frac{\beta}{2}\int d^4x\sqrt{-g}g^{\mu\nu}(\nabla_\mu\vartheta)(\nabla_\nu\vartheta), \\ S_{\rm mat} &= \int d^4x\sqrt{-g}\mathcal{L}_{\rm mat}. \end{aligned} \tag{22}$$

The quantity $\kappa = (16\pi G)^{-1}$ is the gravitational constant, while α and β are coupling constants that control the strength of the CS coupling to the gravitational sector and its kinetic energy, respectively. In the nondynamical version of the theory, $\beta = 0$ and there are no dynamics for the scalar field, which is promoted to a prior-geometric quantity.

The quantity ϑ is the CS field, which couples to the gravitational sector via the parity-violating Pontryagin density ${}^*RR$, which is given by

$${}^*RR := R_{\alpha\beta\gamma\delta}{}^*R^{\alpha\beta\gamma\delta} = \tfrac{1}{2}\varepsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\gamma\delta}R^{\gamma\delta}{}_{\mu\nu}, \tag{23}$$

where the asterisk denotes the dual tensor, which we construct using the antisymmetric Levi-Civita tensor $\varepsilon^{\alpha\beta\mu\nu}$. This scalar is a topological invariant, as it can be written as the divergence of a current:

$${}^*RR = 4\nabla_\mu[\varepsilon^{\mu\alpha\beta\gamma}\Gamma^\sigma_{\alpha\tau}(\tfrac{1}{2}\partial_\beta\Gamma^\tau_{\gamma\sigma} + \tfrac{1}{3}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma})]. \tag{24}$$

Equation (21) contains several terms that we describe below: The first one is the Einstein-Hilbert action; the second one is the CS coupling to the gravitational sector; the third one is the CS kinetic term; and the fourth one stands for additional matter degrees of freedom. The CS kinetic term is precisely the one that distinguishes the nondynamical and the dynamical theory. In the former, the scalar field is *a priori* prescribed, while in the dynamical theory, the scalar field satisfies an evolution equation.

The field equations of this theory are obtained by varying the action with respect to all degrees of freedom:

$$G_{\mu\nu} + \frac{\alpha}{\kappa} C_{\mu\nu} = \frac{1}{2\kappa}(T^{\rm mat}_{\mu\nu} + T^{(\vartheta)}_{\mu\nu}), \tag{25a}$$

$$\beta \Box \vartheta = -\frac{\alpha}{4} {}^{*}RR, \tag{25b}$$

where $T^{\rm mat}_{\mu\nu}$ is the matter stress-energy tensor and $T^{(\vartheta)}_{\mu\nu}$ is the CS scalar stress-energy:

$$T^{(\vartheta)}_{\mu\nu} = \beta[(\nabla_\mu \vartheta)(\nabla_\nu \vartheta) - \tfrac{1}{2} g_{\mu\nu}(\nabla^\sigma \vartheta)(\nabla_\sigma \vartheta)]. \tag{26}$$

The C tensor $C^{\mu\nu}$ is given by

$$C^{\alpha\beta} = (\nabla_\sigma \vartheta)\varepsilon^{\sigma\delta\nu(\alpha}\nabla_\nu R^{\beta)}{}_\delta + (\nabla_\sigma \nabla_\delta \vartheta){}^{*}R^{\delta(\alpha\beta)\sigma}. \tag{27}$$

Many solutions to these field equations have been found. In their pioneering work, Jackiw and Pi showed that the Schwarzschild metric is also a solution in CS gravity [32]. Later on, a detailed analysis showed that all spherically symmetric spaces, such as the Friedman-Robertson-Walker metric, are also solutions [30]. Axially symmetric spaces, however, are not necessarily solutions, because the Pontryagin density does not vanish in this case, sourcing a nontrivial scalar field. Specifically, this implies the Kerr metric is not a solution.

A slowly rotating solution, however, does exist in dynamical CS gravity. Yunes and Pretorius [31] found that when the field equations are expanded in the Kerr parameter $a/M \ll 1$ and in the small-coupling parameter $\zeta \equiv \xi/M^4 = \alpha^2/(\beta\kappa M^4) \ll 1$, then the CS field equations have the solution

$$d\bar{s}^2 = ds^2_{\rm Kerr} + \frac{5}{8}\zeta \frac{Ma}{r^4}\left(1 + \frac{12}{7}\frac{M}{r} + \frac{27}{10}\frac{M^2}{r^2}\right)\sin^2\theta dt d\phi,$$

$$\bar{\vartheta} = \frac{5}{8}\frac{\alpha}{\beta}\frac{a}{M}\frac{\cos\theta}{r^2}\left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2}\right), \tag{28}$$

to second order in a/M and to first order in ζ, assuming no matter sources. These equations employ Boyer-Lindquist coordinates (t, r, θ, ϕ), and $ds^2_{\rm Kerr}$ is the Kerr line element. The solution for ϑ may also include an arbitrary additive constant, but this constant is unimportant, since only derivatives of ϑ enter the CS field equations. Recently, the same solution has been found to linear order in a/M in the Einstein-Cartan formulation of the nondynamical theory [48].

The divergence of the field equations reduce to $\nabla^\mu T^{\rm mat}_{\mu\nu} = 0$. This is because the divergence of the Einstein tensor vanishes by the Bianchi identities. Meanwhile, the divergence of the C tensor exactly cancels the divergence of the CS scalar field stress-energy tensor, upon imposition of the equations of motion [Eq. (25b)]. Therefore, test-particle motion in dynamical CS gravity is exactly geodesic.[4] This result automatically implies the weak-equivalence principle is satisfied.

The gravitational perturbation only possesses two independent, propagating degrees of freedom or polarizations. Jackiw and Pi showed that this was the case in the nondynamical theory [32], while Sopuerta and Yunes did the same in the dynamical version [28]. One can also show easily that a transverse and approximately traceless gauge exists in dynamical CS gravity. The trace of the field equations take the interesting form

$$-R = \frac{1}{2\kappa}(T^{\rm mat} + T^{(\vartheta)}), \tag{29}$$

where R is the Ricci scalar and T is the trace of the stress-energy tensor. Notice that the trace of the C tensor vanishes identically.

In vacuum ($T^{\rm mat}_{\mu\nu} = 0$) and when expanding to linear order about a Minkowski background, Eq. (29) reduces to

$$\bar{\Box}\tilde{h} = \frac{\beta}{2\kappa}(\bar{\nabla}^\sigma \bar{\vartheta})(\bar{\nabla}_\sigma \bar{\vartheta}), \tag{30}$$

where $\tilde{h} \equiv \eta^{\mu\nu}\tilde{h}_{\mu\nu}$ is the trace of the metric perturbation, $\bar{\Box}$ is the d'Alembertian operator with respect to the background metric, and $\bar{\vartheta}$ is the background scalar field. Since the latter must satisfy the evolution equation [Eq. (25b)], we immediately see that $\vartheta \propto \alpha/\beta$. This means that the right-hand side of Eq. (30) is proportional to ζ. To zeroth order in the small-coupling approximation, $\tilde{h}$ then satisfies a free wave equation and can thus be treated as vanishing. Deviations from the trace-free condition can only arise at $\mathcal{O}(\zeta)$ and they are suppressed by factors of the curvature tensor, as $\bar{\vartheta}$ must satisfy Eq. (25b). Approaching $\mathcal{I}^+$, the right-hand side of Eq. (30) vanishes. This allows one to impose the TT gauge at future null infinity.

V. EFFECTIVE STRESS-ENERGY IN CS GRAVITY

The perturbed Lagrangian for the Einstein-Hilbert action has already been calculated, so here we need only consider the contribution from $S_{\rm CS}$. At $\mathcal{O}(\epsilon^2)$, there are a large number of terms generated (we used the package XPERT [49–53] to calculate the perturbations). Many of these terms are irrelevant when considering their contribution at $\mathcal{I}^+$.

Let us classify the types of terms that arise in $S_{\rm CS}$. At $\mathcal{O}(\epsilon^2)$, these are of two types:

(1) the second-order part of one field, or

(2) the product of first-order parts of two fields.

As mentioned in Sec. II A, terms containing the second-order part of one field are linear in a short-wavelength quantity, which vanishes under averaging. Thus we only

[4] This statement is true only in the absence of spins, since otherwise the CS effective worldline action would contain new self-interaction terms.

need to consider the latter case. There are five fields in $S_{\rm CS}$ ($\sqrt{-g}$, ε, ϑ, R, and ${}^{*}R$), so one would at first think that there are $\binom{5}{2} = 10$ types of terms arising; however, from the definition of the Levi-Civita tensor, we have

$$\sqrt{-g}\varepsilon^{\alpha\beta\mu\nu} = {\rm sgn}(g)[\alpha\beta\mu\nu], \tag{31}$$

where $[\alpha\beta\mu\nu]$ is the Levi-Civita *symbol*, which is not a spacetime field. The combination $\sqrt{-g}\varepsilon^{\alpha\beta\mu\nu}$ therefore has no perturbation, and there are only three spacetime fields which contribute. We are left with only $\binom{3}{2} = 3$ possibilities for the types of terms that could appear, corresponding to two perturbed fields and one unperturbed one among the set $(\vartheta, R_{\alpha\beta\sigma\delta}, R_{\alpha\beta\sigma\delta})$. Two of these possibilities are actually the same by exchanging the two copies of $R_{\alpha\beta\sigma\delta}$.

Therefore, we are left with only the following two types of terms in the CS Lagrangian density:

$$\mathcal{L}_{\tilde{\vartheta}\tilde{R}} \sim \sqrt{-\bar{g}}\bar{\varepsilon}^{\alpha\beta\mu\nu}\tilde{\vartheta}\tilde{R}^{\gamma}{}_{\delta\alpha\beta}\bar{R}^{\delta}{}_{\gamma\mu\nu}, \tag{32a}$$

$$\mathcal{L}_{\tilde{R}\tilde{R}} \sim \sqrt{-\bar{g}}\bar{\varepsilon}^{\alpha\beta\mu\nu}\bar{\vartheta}\tilde{R}^{\gamma}{}_{\delta\alpha\beta}\tilde{R}^{\delta}{}_{\gamma\mu\nu}, \tag{32b}$$

where $\tilde{R}^{\gamma}{}_{\delta\alpha\beta}$ is the first-order perturbation to the Riemann tensor, in terms of $\underline{\tilde{h}}^{\mu\nu}$. Variation of these terms with respect to the background metric yields the CS contributions to the effective stress-energy tensor, while variation with respect to the metric perturbation yields CS corrections to the first-order equations of motion.

A. Variation with respect to the perturbation

Just as in GR, the final expression for the stress-energy tensor must be put on-shell by imposing the equations of motion. The first-order equations of motion of dynamical CS gravity, in vacuum and at $\mathcal{I}^{+}$, are

$$\bar{\Box}\underline{\tilde{h}}_{\mu\nu} = -\frac{1}{\kappa}\tilde{T}^{(\vartheta)}_{\mu\nu} + \frac{\alpha}{\kappa}[\bar{\nabla}_{\alpha}\bar{\vartheta}\bar{\nabla}_{\beta}\bar{\Box}\underline{\tilde{h}}_{\gamma(\mu}\bar{\varepsilon}^{\alpha\beta\gamma}{}_{\nu)} + \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\vartheta}\bar{\varepsilon}^{\alpha}{}_{\gamma\delta(\mu}\bar{\nabla}^{\delta}(\bar{\nabla}_{\nu)}\underline{\tilde{h}}^{\beta\gamma} - \bar{\nabla}^{\beta}\underline{\tilde{h}}_{\nu)}{}^{\gamma})]. \tag{33}$$

Imposing these equations of motion is easier when taking advantage of the weak-coupling limit $\zeta_{\rm GW} \ll 1$, where $\zeta_{\rm GW} \equiv \alpha\bar{\nabla}\vartheta/(\kappa\lambda_{\rm GW})$ quantifies the size of the deformation away from GR. Let us then expand the metric perturbation in a Taylor series

$$\underline{\tilde{h}}_{\mu\nu} = \sum_{n=0}^{\infty}(\zeta_{\rm GW})^{n}\underline{\tilde{h}}^{(n)}_{\mu\nu}. \tag{34}$$

To zeroth order, it is clear that Eq. (33) reduces to $\bar{\Box}\underline{\tilde{h}}^{(0)}_{\mu\nu} = 0$, which is the standard GR equation of motion. To next order, the leading-order piece of the right-hand side vanishes and one is then left with

$$\bar{\Box}\underline{\tilde{h}}^{(1)}_{\mu\nu} = \frac{\alpha}{\kappa}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\vartheta}\bar{\varepsilon}^{\alpha}{}_{\gamma\delta(\mu}\bar{\nabla}^{\delta}(\bar{\nabla}_{\nu)}\underline{\tilde{h}}^{\beta\gamma}_{(0)} - \bar{\nabla}^{\beta}\underline{\tilde{h}}_{\nu)}{}^{\gamma}{}_{(0)}) - \frac{1}{\kappa}\tilde{T}^{(\vartheta)}_{\mu\nu}. \tag{35}$$

In the remainder of this section, we drop the superscripts that indicate $\zeta_{\rm GW}$ ordering.

B. Variation with respect to the background

Let us first discuss terms of type $\mathcal{L}_{\tilde{\vartheta}\tilde{R}}$ under variation with respect to $\bar{g}^{\mu\nu}$. From Sec. II B, only total derivative terms arise from $\delta\bar{R}^{\gamma}{}_{\delta\alpha\beta}$, and these vanish upon averaging. The remaining terms contain $\bar{R}^{\gamma}{}_{\delta\alpha\beta}$, which must vanish at $\mathcal{I}^{+}$. Thus, as mentioned before, terms in the effective action which contain curvature tensors do not contribute to the effective stress-energy tensor at $\mathcal{I}^{+}$.

We are then only left with $\mathcal{L}_{\tilde{R}\tilde{R}}$. Writing these in terms of $\underline{\tilde{h}}^{\mu\nu}$, the effective action reads

$$S^{\rm eff(2)}_{\rm CS} = \epsilon^{2}\frac{\alpha}{4}\int d^{4}x\mathcal{L}^{\rm eff,1}_{\rm CS} + \mathcal{L}^{\rm eff,2}_{\rm CS}, \tag{36}$$

where

$$\mathcal{L}^{\rm eff,1}_{\rm CS} = +\sqrt{-\bar{g}}\bar{\varepsilon}^{\alpha\beta\gamma\delta}\bar{\vartheta}\bar{\nabla}^{\rho}\bar{\nabla}_{\beta}\underline{\tilde{h}}_{\alpha}{}^{\sigma}\bar{\nabla}_{\delta}\bar{\nabla}_{\rho}\underline{\tilde{h}}_{\sigma\gamma}, \tag{37a}$$

$$\mathcal{L}^{\rm eff,2}_{\rm CS} = -\sqrt{-\bar{g}}\bar{\varepsilon}^{\alpha\beta\gamma\delta}\bar{\vartheta}\bar{\nabla}_{\beta}\bar{\nabla}_{\rho}\underline{\tilde{h}}_{\alpha}{}^{\sigma}\bar{\nabla}_{\sigma}\bar{\nabla}_{\delta}\underline{\tilde{h}}^{\rho}{}_{\gamma}. \tag{37b}$$

Naively, one might think that these expressions lead to an effective stress-energy tensor at $\mathcal{O}(\lambda_{\rm GW}^{-4})$. This is premature, however, as there can be a cancellation of $\lambda_{\rm GW}^{-4}$ terms that lead to a less steep wavelength dependence. One should try to move as many derivatives away from the perturbed quantities as possible before proceeding. In fact, we know that this must be possible from [32]: The Pontryagin density can be written as the divergence of a 4-current, so at least one derivative can be moved off of $\underline{\tilde{h}}^{\mu\nu}$. This automatically implies that there cannot be $\lambda_{\rm GW}^{-4}$ terms in the effective stress-energy tensor, as shown explicitly by Sopuerta and Yunes [28].

Let us transform $\mathcal{L}^{\rm eff,1}_{\rm CS}$ in the following way. The Levi-Civita tensor is contracted onto two derivative operators ($\bar{\nabla}_{\beta}$ and $\bar{\nabla}_{\delta}$). One may integrate by parts to move one of these derivative operators onto the remaining terms in Eq. (37a). This generates two types of terms: one with three derivatives acting on the metric perturbation and one with one derivative on the CS scalar (the term acting on the Levi-Civita tensor or the determinant of the metric vanishes by metric compatibility). Let us focus on the former first. Because of the contraction onto the Levi-Civita tensor, only the antisymmetric part of the second derivative operator would contribute. Such a combination is nothing but the commutator of covariant derivatives, which can be written as the Riemann tensor, and thus vanishes at $\mathcal{I}^{+}$. The remaining term with a covariant derivative of the CS scalar does not generically vanish. Dropping terms proportional to the Riemann tensor, $\mathcal{L}^{\rm eff,1}_{\rm CS}$ becomes

$$\mathcal{L}^{\rm eff,1}_{\rm CS} = \sqrt{-\bar{g}}\bar{\varepsilon}^{\alpha\beta\gamma\delta}\bar{\nabla}_{\alpha}\bar{\vartheta}\bar{\nabla}^{\rho}\underline{\tilde{h}}_{\beta}{}^{\sigma}\bar{\nabla}_{\delta}\bar{\nabla}_{\rho}\underline{\tilde{h}}_{\sigma\gamma}. \tag{38a}$$

Equation (37b) can be analyzed with the property discussed in Sec. II D: The Lorenz gauge may be imposed at the level of the action for the purposes of calculating the effective stress-energy tensor. This means that if one integrates by parts, moving $\bar\nabla_\sigma$ and $\bar\nabla_\rho$ onto remaining terms, the only term that survives is proportional to $\bar\vartheta$, as the divergence of $\tilde{\underline{h}}^{\mu\nu}$ vanishes (after commuting derivatives, dropping Riemann terms, and imposing the Lorenz gauge). Thus $\mathcal{L}^{\rm eff,2}_{\rm CS}$ becomes

$$\mathcal{L}^{\rm eff,2}_{\rm CS} = \sqrt{-\bar{\mathfrak{g}}}\,\bar\varepsilon^{\alpha\beta\gamma\delta}\bar\nabla_\rho\bar\nabla_\sigma\bar\vartheta\bar\nabla_\alpha\tilde{\underline{h}}_\beta{}^\sigma\bar\nabla_\gamma\tilde{\underline{h}}_\delta{}^\rho. \tag{38b}$$

With these simplified Lagrangian densities at hand, we can now compute the total effective stress-energy tensor for GWs in CS gravity:

$$T^{\rm eff}_{{\rm CS}\mu\nu} = T^{\rm eff}_{{\rm MT}\mu\nu} + T^{\rm eff,1}_{{\rm CS}\mu\nu} + T^{\rm eff,2}_{{\rm CS}\mu\nu}, \tag{39}$$

where $T^{\rm eff,1}_{{\rm CS}\mu\nu}$ and $T^{\rm eff,2}_{{\rm CS}\mu\nu}$ are due to the variation of $\mathcal{L}^{\rm eff,1}_{\rm CS}$ and $\mathcal{L}^{\rm eff,2}_{\rm CS}$, respectively. These expressions are

$$\begin{aligned} T^{\rm eff,1}_{{\rm CS}\mu\nu} = -\epsilon^2\frac{\alpha}{2}\langle\!\langle \bar\nabla_\alpha\bar\vartheta[\bar\varepsilon^{\alpha\beta\gamma\delta}(\bar\nabla_{(\mu}\tilde{h}_{|\beta|}{}^\sigma\bar\nabla_{\nu)}\bar\nabla_\delta\tilde{h}_{\sigma\gamma} \\ -\bar\nabla^\rho\tilde{h}_{\beta(\mu}\bar\nabla_{|\delta|}\bar\nabla_{|\rho|}\tilde{h}_{\nu)\gamma}) \\ -2\bar\varepsilon^\alpha{}_{(\mu}{}^{\gamma\delta}\bar\nabla^\rho\tilde{h}_{\nu)\sigma}\bar\nabla_\delta\bar\nabla_\rho\tilde{h}^\sigma{}_\gamma]\rangle\!\rangle \end{aligned} \tag{40a}$$

and

$$T^{\rm eff,2}_{{\rm CS}\mu\nu} = -\epsilon^2\alpha\langle\!\langle\bar\nabla_\sigma\bar\nabla_\rho\bar\vartheta\bar\varepsilon^\alpha{}_{(\mu}{}^{\gamma\delta}\bar\nabla_{|\alpha|}\tilde{\underline{h}}_{\nu)}{}^\sigma\bar\nabla_\gamma\tilde{\underline{h}}^\rho{}_\delta\rangle\!\rangle. \tag{40b}$$

C. Imposing the on-shell condition

The equation of motion may be imposed anywhere $\bar\Box\tilde{\underline{h}}_{\alpha\beta}$ may be formed in $T^{\rm eff}_{{\rm CS}\mu\nu}$ via integration by parts. There is no contraction of derivative operators onto each other in $T^{\rm eff,2}_{{\rm CS}\mu\nu}$, so it remains unchanged. In the final two terms of $T^{\rm eff,1}_{{\rm CS}\mu\nu}$, the derivative operator $\bar\nabla_\rho$ may be moved onto $\bar\nabla_\alpha\bar\vartheta\bar\nabla^\rho\tilde{\underline{h}}_{\kappa\lambda}$. This would generally make two terms, but the term proportional to $\bar\nabla_\alpha\bar\vartheta\,\bar\Box\,\tilde{\underline{h}}_{\kappa\lambda}$ is $\mathcal{O}(\zeta^2_{\rm GW})$ relative to the Isaacson piece, so we only keep one term. This gives

$$\begin{aligned} T^{\rm eff,1}_{{\rm CS}\mu\nu} = -\epsilon^2\frac{\alpha}{2}\langle\!\langle\bar\varepsilon^{\alpha\beta\gamma\delta}(\bar\nabla_\alpha\bar\vartheta\bar\nabla_{(\mu}\tilde{\underline{h}}_{|\beta|}{}^\sigma\bar\nabla_{\nu)}\bar\nabla_\delta\tilde{\underline{h}}_{\sigma\gamma} \\ +\bar\nabla_\rho\bar\nabla_\alpha\bar\vartheta\bar\nabla^\rho\tilde{\underline{h}}_{\beta(\mu}\bar\nabla_{|\delta|}\tilde{\underline{h}}_{\nu)\gamma}) \\ +2\bar\nabla_\rho\bar\nabla_\alpha\bar\vartheta\bar\varepsilon^\alpha{}_{(\mu}{}^{\gamma\delta}\bar\nabla^\rho\tilde{\underline{h}}_{\nu)\sigma}\bar\nabla_\delta\tilde{\underline{h}}^\sigma{}_\gamma\rangle\!\rangle. \end{aligned} \tag{41}$$

Let us now evaluate $T^{\rm eff}_{{\rm MT}\mu\nu}$ on-shell. Since $T^{\rm eff}_{{\rm MT}\mu\nu}$ is $\mathcal{O}((\zeta_{\rm GW})^0)$, imposing the equation of motion Eq. (35) will introduce terms of $\mathcal{O}(\zeta_{\rm GW})$, which are kept since they are the same order as $T^{\rm eff,1}_{{\rm CS}\mu\nu}$ and $T^{\rm eff,2}_{{\rm CS}\mu\nu}$. We can also impose a gauge condition. We have already imposed the Lorenz gauge throughout at the level of the action. We may further specialize this to the TT gauge. While the TT gauge may not be imposed globally, it may be imposed at $\mathcal{I}^+$, where the effective stress-energy tensor is being evaluated. In the TT gauge,

$$\begin{aligned} T^{\rm eff}_{{\rm MT}\mu\nu} &= \epsilon^2\kappa\langle\!\langle\tfrac{1}{2}\bar\nabla_\mu\tilde{\underline{h}}_{\alpha\beta}\bar\nabla_\nu\tilde{\underline{h}}^{\alpha\beta} - \bar\nabla_\rho\tilde{\underline{h}}_{\alpha\mu}\bar\nabla^\rho\tilde{\underline{h}}_\nu{}^\alpha \\ &\quad -\tfrac{1}{4}\bar{\mathfrak{g}}_{\mu\nu}\bar\nabla_\rho\tilde{\underline{h}}_{\alpha\beta}\bar\nabla^\rho\tilde{\underline{h}}^{\alpha\beta}\rangle\!\rangle \\ &= T^{\rm eff}_{{\rm GR}\mu\nu} + T^{\rm eff,1}_{{\rm MT}\mu\nu} + T^{\rm eff,2}_{{\rm MT}\mu\nu}, \end{aligned} \tag{42a}$$

where

$$T^{\rm eff,1}_{{\rm MT}\mu\nu} = -\epsilon^2\kappa\langle\!\langle\bar\nabla_\rho\tilde{\underline{h}}_{\alpha\mu}\bar\nabla^\rho\tilde{\underline{h}}_\nu{}^\alpha\rangle\!\rangle, \tag{42b}$$

$$T^{\rm eff,2}_{{\rm MT}\mu\nu} = -\epsilon^2\frac{\kappa}{4}\langle\!\langle\bar{\mathfrak{g}}_{\mu\nu}\bar\nabla_\rho\tilde{\underline{h}}_{\alpha\beta}\bar\nabla^\rho\tilde{\underline{h}}^{\alpha\beta}\rangle\!\rangle. \tag{42c}$$

Integrating by parts, imposing the equations of motion Eq. (35), and integrating by parts again where appropriate, these contributions to the effective stress-energy tensor at $\mathcal{I}^+$ are

$$\begin{aligned} T^{\rm eff,1}_{{\rm MT}\mu\nu} = &-\epsilon^2\langle\!\langle\tilde{\underline{h}}^\alpha{}_{(\mu}\tilde{T}^{(\vartheta)}_{\nu)\alpha}\rangle\!\rangle \\ &-\epsilon^2\frac{\alpha}{2}\langle\!\langle\bar\nabla_\sigma\bar\nabla_\rho\bar\vartheta\bar\nabla^\delta\tilde{\underline{h}}^\alpha{}_\mu\bar\varepsilon^\sigma{}_{\gamma\delta(\alpha}(\bar\nabla_{\nu)}\tilde{\underline{h}}^{\rho\gamma} - \bar\nabla^\rho\tilde{\underline{h}}_{\nu)}{}^\gamma) \\ &+(\mu\leftrightarrow\nu)\rangle\!\rangle, \end{aligned} \tag{42d}$$

$$T^{\rm eff,2}_{{\rm MT}\mu\nu} = \tfrac{1}{4}\bar{\mathfrak{g}}_{\mu\nu}\bar{\mathfrak{g}}^{\alpha\beta}T^{\rm eff,1}_{{\rm MT}\alpha\beta}. \tag{42e}$$

Finally, we may write an expression for $T^{\rm eff}_{{\rm CS}\mu\nu}$ at $\mathcal{I}^+$ after imposing the equations of motion:

$$T^{\rm eff}_{{\rm CS}\mu\nu} = T^{\rm eff}_{{\rm GR}\mu\nu} + \delta T^{\rm eff}_{{\rm CS}\mu\nu}, \tag{43a}$$

$$\delta T^{\rm eff}_{{\rm CS}\mu\nu} = T^{\rm eff,1}_{{\rm MT}\mu\nu} + T^{\rm eff,2}_{{\rm MT}\mu\nu} + T^{\rm eff,1}_{{\rm CS}\mu\nu} + T^{\rm eff,2}_{{\rm CS}\mu\nu}, \tag{43b}$$

where $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$ contains the Chern-Simons correction at $\mathcal{O}(\zeta_{\rm GW})$. The summands are taken from Eqs. (40b), (41), (42d), and (42e). Putting them together for convenience, the final result is

$$\begin{aligned} \delta T^{\rm eff}_{{\rm CS}\mu\nu} = &-\epsilon^2\left\langle\!\!\left\langle\tilde{\underline{h}}^\alpha{}_{(\mu}\tilde{T}^{(\vartheta)}_{\nu)\alpha} + \frac{1}{4}\bar{\mathfrak{g}}_{\mu\nu}\tilde{\underline{h}}^{\alpha\beta}\tilde{T}^{(\vartheta)}_{\alpha\beta}\right\rangle\!\!\right\rangle \\ &-\epsilon^2\frac{\alpha}{2}\left\langle\!\!\left\langle\bar\nabla_\sigma\bar\nabla_\rho\bar\vartheta\Big[\bar\nabla^\delta\tilde{\underline{h}}^\alpha{}_\mu\bar\varepsilon^\sigma{}_{\gamma\delta(\alpha}(\bar\nabla_{\nu)}\tilde{\underline{h}}^{\rho\gamma} - \bar\nabla^\rho\tilde{\underline{h}}_{\nu)}{}^\gamma) + \bar\nabla^\delta\tilde{\underline{h}}^\alpha{}_\nu\bar\varepsilon^\sigma{}_{\gamma\delta(\alpha}(\bar\nabla_{\mu)}\tilde{\underline{h}}^{\rho\gamma} - \bar\nabla^\rho\tilde{\underline{h}}_{\mu)}{}^\gamma)\right. \\ &\quad + \frac{1}{2}\bar{\mathfrak{g}}_{\mu\nu}\bar\nabla^\delta\tilde{\underline{h}}^{\alpha\beta}\bar\varepsilon^\sigma{}_{\gamma\delta(\alpha}(\bar\nabla_{\beta)}\tilde{\underline{h}}^{\rho\gamma} - \bar\nabla^\rho\tilde{\underline{h}}_{\beta)}{}^\gamma) + 2\bar\varepsilon^\alpha{}_{(\mu}{}^{\gamma\delta}\bar\nabla_{|\alpha|}\tilde{\underline{h}}_{\nu)}{}^\sigma\bar\nabla_\gamma\tilde{\underline{h}}^\rho{}_\delta \\ &\quad + \bar\varepsilon^{\sigma\beta\gamma\delta}\bar\nabla^\rho\tilde{\underline{h}}_{\beta(\mu}\bar\nabla_{|\delta|}\tilde{\underline{h}}_{\nu)\gamma} + 2\bar\varepsilon^\sigma{}_{(\mu}{}^{\gamma\delta}\bar\nabla^\rho\tilde{\underline{h}}_{\nu)\sigma}\bar\nabla_\delta\tilde{\underline{h}}^\sigma{}_\gamma\Big] \\ &\quad \left. + \bar\varepsilon^{\alpha\beta\gamma\delta}\bar\nabla_\alpha\bar\vartheta\bar\nabla_{(\mu}\tilde{\underline{h}}_{|\beta|}{}^\sigma\bar\nabla_{\nu)}\bar\nabla_\delta\tilde{\underline{h}}_{\sigma\gamma}\right\rangle\!\!\right\rangle. \end{aligned} \tag{44}$$

In the above, we have organized the terms by their scaling with powers of wavelength. The first line contains terms which scale as $\lambda_{\rm GW}^0$ and $\lambda_{\rm GW}^{-1}$; the first of these corresponds to a "mass" term in the effective stress-energy tensor. Both of these scale more slowly with inverse wavelength than the GR contribution, so they are subdominant. The next three lines have the same scaling with inverse wavelength as GR: $\lambda_{\rm GW}^{-2}$. The final line scales more strongly with inverse wavelength $\lambda_{\rm GW}^{-3}$. This term in principle could dominate over the GR term in the high frequency limit.

Notice that the effective stress-energy tensor presented here is applicable to both the dynamical and the nondynamical versions of CS gravity. Also note that if ϑ were a constant, rather than a function, the effective stress-energy tensor would be identical to that of GR (which is expected, since, in that case, the modification to the action is purely a boundary or topological term).

D. In dynamical CS gravity

From asymptotic arguments, we can argue that $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$ does not contribute to dissipation laws at $\mathcal{I}^+$ in the dynamical version of CS gravity. As mentioned in Sec. II C, the dissipation of energy and linear and angular momentum of a system is computed by integrating components of the stress-energy tensor on a 2-sphere at $\mathcal{I}^+$. Since the area of the 2-sphere grows as r^2, for the dissipation integrals to be finite, the components of the stress-energy tensor must fall off at least as r^{-2}. In fact, only the r^{-2} part of the stress-energy contributes as one takes the $r \to \infty$ limit. Therefore, any part of the stress-energy tensor that decays faster than r^{-2} does not contribute to dissipation laws.

The CS correction to the effective stress-energy tensor, $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$, always falls off faster than r^{-2} in the dynamical theory. To see this, we must analyze the behavior of ϑ, which is restricted. This restriction comes from demanding that the field ϑ sourced by an isolated system and in an asymptotically flat space contains a finite amount of energy. The energy in ϑ is computed by integrating the time-time component of $T^{(\vartheta)}_{\mu\nu}$ on a hypersurface of constant time and over all space. For the energy to be finite, the integral $\int^\infty (\nabla\vartheta)^2 r^2 dr$ (in an asymptotically flat, Cartesian spatial slice, appropriate to $\mathcal{I}^+$) must be finite. This restricts $\nabla\vartheta$ to fall off at least faster than $r^{-3/2}$. We then conclude that the CS correction to the energy-momentum tensor must vanish at $\mathcal{I}^+$, as $T^{\rm eff,1}_{{\rm MT}\mu\nu}$, $T^{\rm eff,2}_{{\rm MT}\mu\nu}$, $T^{\rm eff,1}_{{\rm CS}\mu\nu}$, and $T^{\rm eff,2}_{{\rm CS}\mu\nu}$ decay at least as $r^{-7/2}$ or faster.

The only contribution at $\mathcal{I}^+$ to the effective stress-energy of GWs in dynamical CS gravity which decays as r^{-2} is the GR part:

$$T^{\rm eff}_{{\rm CS}\mu\nu} = T^{\rm eff}_{{\rm GR}\mu\nu}. \tag{45}$$

Again, we stress that this only accounts for the outgoing GW radiation. However, the same argument as in Sec. II C holds; the correction to the energy flux absorbed by trapped surfaces is only important at the end of an inspiral, in both GR and deformations away from GR. This is supported by the small velocity, small mass-ratio expansion of [47] (see also [44,45]).

E. In nondynamical CS gravity

In the dynamical theory, since the scalar field ϑ must carry a finite energy, we were able to argue for the vanishing of $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$ at $\mathcal{I}^+$. In the nondynamical theory, there is no such demand and no further simplification can be made beyond the vanishing of $T^{(\vartheta)}_{\mu\nu}$. However, for a particular choice of $\bar\vartheta$ field, the effective stress-energy tensor may be evaluated. We demonstrate this below.

In the canonical embedding

The canonical embedding of nondynamical CS gravity is given by [32]

$$v_\mu \equiv \bar\nabla_\mu \bar\vartheta \doteq (1/\mu, 0, 0, 0), \tag{46}$$

in Cartesian coordinates in the asymptotically flat part of the spacetime. Approaching infinity, this yields

$$\bar\nabla_\alpha \bar\nabla_\beta \bar\vartheta = 0, \tag{47}$$

so by extension $T^{\rm eff,2}_{{\rm CS}\mu\nu} = 0$, the first-order equation of motion becomes $\bar\Box \underline{\tilde h}_{\mu\nu} = 0 + \mathcal{O}(\zeta_{\rm GW}^2)$, the final two terms of $T^{\rm eff,1}_{{\rm CS}\mu\nu}$ vanish, and $T^{\rm eff,1}_{{\rm MT}\mu\nu} = 0 = T^{\rm eff,2}_{{\rm MT}\mu\nu}$. Notice that here there is no amplitude birefringence in flat spacetime as $\ddot\vartheta = 0$ [33,34,54,55].

The first term of $T^{\rm eff,1}_{{\rm CS}\mu\nu}$ is the only $\mathcal{O}(\zeta_{\rm GW})$ correction which survives. The total stress-energy tensor in the canonical embedding of nondynamical Chern-Simons gravity at $\mathcal{I}^+$, with this correction, is

$$\begin{aligned} T^{\rm eff}_{{\rm CS}\mu\nu} &= T^{\rm eff}_{{\rm GR}\mu\nu} + \delta T^{\rm eff}_{{\rm CS}\mu\nu}, \\ \delta T^{\rm eff}_{{\rm CS}\mu\nu} &= -\epsilon^2 \frac{\alpha}{2} \langle\langle \bar\nabla_\alpha \bar\vartheta \bar\varepsilon^{\alpha\beta\gamma\delta} \bar\nabla_{(\mu} \underline{\tilde h}_{|\beta|}{}^\sigma \bar\nabla_{\nu)} \bar\nabla_\delta \underline{\tilde h}_{\sigma\gamma} \rangle\rangle \\ &= +\epsilon^2 \frac{\alpha}{2\mu} \langle\langle \bar\varepsilon^{ijk} \bar\nabla_{(\mu} \underline{\tilde h}_{|i|}{}^\sigma \bar\nabla_{\nu)} \bar\nabla_k \underline{\tilde h}_{\sigma j} \rangle\rangle, \end{aligned} \tag{48}$$

where $\bar\varepsilon^{ijk}$ is the Levi-Civita tensor on the 3-space orthogonal to $(\partial/\partial t)^\mu$, and the sign change arises from the factor of sgn(g) in Eq. (31).

From the form of the correction $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$, we can briefly mention the leading modification to radiation reaction in a binary inspiral at Newtonian order. At this order, there is no modification to the trajectories of the two bodies from the $\bar\vartheta$ field. Since the first-order equation of motion is identical to that of GR at order $\mathcal{O}(\zeta_{\rm GW})$, the leading solution to $\underline{\tilde h}_{\mu\nu}$ is the same as in GR: $\underline{\tilde h}_{\mu\nu} = \underline{\tilde h}^{\rm GR}_{\mu\nu}$.

Inserting this solution in the TT gauge into $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$, the energy, linear momentum, and angular momentum radiated by the system can be computed. Adopting a Cartesian coordinate system at asymptotic infinity, the correction to the radiated quantities is given by

$$\delta \dot{E}^{\rm CS} = -\int d\Omega r^2 \delta T^{\rm eff}_{{\rm CS}0j} n_j = +\int d\Omega r^2 \delta T^{\rm eff}_{\rm CS00}, \tag{49a}$$

$$\delta \dot{P}^{\rm CS}_i = +\int d\Omega r^2 \delta T^{\rm eff}_{{\rm CS}ij} n_j = -\int d\Omega r^2 \delta T^{\rm eff}_{{\rm CS}i0}, \tag{49b}$$

$$\delta \dot{J}^{\rm CS}_i = -\int d\Omega r^2 \varepsilon_{ijk} x_j \delta T^{(-3)}_{{\rm CS}kl} n_l, \tag{49c}$$

where $\delta T^{(-3)}_{{\rm CS}\mu\nu}$ is the part of $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$ which decays as r^{-3} [56]. In evaluating these integrals, the only angular dependence is in factors of n_i or x_i. An angular integral of an odd number of such factors vanishes, while an integral of an even number of them reduces to a symmetrized product of Kronecker delta tensors. These factors arise explicitly in the definitions of Eqs. (49) and from spatial derivatives acting on $\underline{\tilde{h}}_{\mu\nu}$ in $T^{\rm eff}_{\mu\nu}$. The most important difference between $T^{\rm eff}_{{\rm GR}\mu\nu}$ and $\delta T^{\rm eff}_{{\rm CS}\mu\nu}$ is the parity of the number of derivatives, which leads to the following behavior.

In GR, the leading contribution to $\dot{E}^{\rm GR}$ is from the (mass quadrupole)^2 combination. Compare this with the same integral for $\delta T^{\rm eff}_{\rm CS00}$, where the (mass quadrupole)^2 term has an odd number of factors of n_i and thus vanishes. The leading contribution is then from the product of the mass quadrupole and mass octupole.

The same situation takes place in calculating $\dot{J}_i$. In GR, the leading contribution is from the product of mass quadrupole with itself. In the correction from CS gravity, the mass quadrupole squared term has an odd number of factors of n_i; the dominant contribution is again from the mass quadrupole times the mass octupole.

Finally, the situation is different in the calculation of $\dot{P}_i$. In GR, the quadrupole squared contribution to $\dot{P}_i$ has an odd number of n_i factors. The dominant contribution is from the mass quadrupole times the mass octupole. However, for the CS correction, the quadrupole squared term has an even number of factors of n_i. Using

$$\underline{\tilde{h}}^{\rm TT}_{ij} = \frac{1}{8\pi\kappa r} \ddot{I}^{\rm TT}_{ij}(t-r), \tag{50}$$

this evaluates to

$$\delta \dot{P}^{\rm CS}_i = -\frac{\alpha}{120\pi\kappa^2\mu} \varepsilon_{ijk} I^{(3)}_{lj} I^{(4)}_{lk}, \tag{51}$$

where I_{ij} is the reduced quadrupole moment of the matter, and $I^{(n)}_{ij} \equiv (d/dt)^n I_{ij}$.

For a binary in a circular orbit about the $\hat{z}$ axis with masses m_1 and m_2, total mass $m = m_1 + m_2$, symmetric mass ratio $\eta = m_1 m_2/m^2$, separation d, and orbital frequency ω, we find the momentum flux correction to be

$$\delta \dot{P}^{\rm CS}_z = -\frac{8\alpha}{15\pi\kappa^2\mu} (\eta m d^2)^2 \omega^7, \tag{52a}$$

or, in terms of the velocity $v = \omega d$, with Kepler's third law $v^2 = m/d$,

$$\delta \dot{P}^{\rm CS}_z = -\frac{128}{15}\left(\frac{\alpha}{\kappa\mu m}\right)\eta^2 v^{13}, \tag{52b}$$

where we notice that the quantity in parentheses is dimensionless. This is to be compared with the leading momentum luminosity in GR, which is proportional to $\dot{P}^{\rm GR}_z \propto \eta^2 v^{11} \delta m/m$, where $\delta m = m_1 - m_2$ [57]. Although the GR effect is two powers of v stronger, it depends on the difference in masses, whereas the nondynamical CS correction only depends on the total mass. This implies that in the limit of comparable masses $m_1 \approx m_2$, the recoil velocity would not asymptote to zero in CS gravity, as it does in GR for nonspinning binaries.

A physical interpretation of this effect is related to the parity-violating nature of the theory. When one chooses a canonical embedding, the action becomes parity-violating as the Pontryagin density is parity odd. The embedding coordinate chooses a (temporal) direction to which the modification to the Einstein equations can couple, inducing a new term in the stress-energy that is proportional to the curl of the metric perturbation. Because kicks are predominantly generated during merger, the CS modification is indeed dominant over the GR result, leading to the first, nonlinear, strong-field modification computed in CS gravity.

VI. EFFECTIVE STRESS-ENERGY TENSOR OF MODIFIED GRAVITY THEORIES

Let us now consider a broader class of modified gravity theories. There is an infinite variety of GR modifications one could construct. However, there are several properties that are desirable and that we require here:

(1) Metric theories.—The action depends on a symmetric metric tensor that controls the spacetime dynamics.
(2) Deformations of GR.—Analytically controllable and small corrections to the Einstein-Hilbert action with a continuous GR limit.
(3) High-rank curvature.—Corrections depend on quadratic or higher products of the Riemann tensor, Ricci tensor, or Ricci scalar.
(4) Minkowski stable.—The theory must admit Minkowski spacetime as a stable vacuum solution, and future null infinity should be asymptotically flat for isolated matter spacetimes.

Besides the metric, there may be new fields introduced which are considered part of the "gravity sector." This distinction means that said fields are not minimally coupled; i.e. they may be coupled to connection and curvature quantities. These additional fields may be of any spin: scalars, spinors, vectors, etc. For simplicity, we will only consider scalar fields here, but the results may also be extended to higher spin fields. Scalar fields are well-motivated from quantum completions of GR; e.g. moduli fields are common appearances in string theoretical models [58].

A. Action

In defining a modified gravity theory, let us consider what terms may arise in the action. These terms must include the Einstein-Hilbert and matter terms, along with modifications built from additional scalar fields and curvature invariants. Additionally, it ought to contain a dynamical term for the scalars that couple to the curvature invariants, as we will motivate in Sec. VI B.

In principle, there are an infinite number of curvature invariants to consider. The first few of these are simple to construct: $\Lambda, R, R^2, \nabla_\mu R \nabla^\mu R, R_{\mu\nu}R^{\mu\nu}, R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \ldots,$ where Λ is any scalar constant, e.g. the cosmological constant. These may be specified by their rank r, which is the number of curvature tensors which are contracted together, and by further specifying a list of r non-negative integers $\{\lambda_1, \ldots, \lambda_r\}$, where λ_i specifies the number of derivatives acting on the ith curvature tensor. For a rank r and case $\{\lambda_i\}_{i=1}^r$, there are a finite number of independent curvature invariants corresponding to the number of ways to contract indices. Thus all curvature invariants may be countably enumerated, assigning some number n to each independent invariant.

We here consider only combinations of *algebraic* curvature invariants, i.e. $\lambda_i = 0$ for all cases. This means we do not allow modifications that depend on derivatives of curvature tensors. Such a simplification is a good one, from the standpoint that it automatically guarantees the field equations to be no higher than fourth order.

Consider then a modified gravity theory defined by the action

$$S = S_{\rm EH} + S_{\rm mat} + S_{\rm int} + S_\vartheta, \tag{53a}$$

where S_ϑ is the canonical kinetic term for ϑ,

$$S_\vartheta = -\frac{\beta}{2}\int d^4x\sqrt{-g}[g^{\mu\nu}(\nabla_\mu\vartheta)(\nabla_\nu\vartheta) + 2V(\vartheta)], \tag{53b}$$

with V an arbitrary potential function, and where $S_{\rm int}$ is the interaction term between the scalar ϑ and some algebraic combination of curvature tensors, for example,

$$S_{\rm int,0} = \alpha_0 \int d^4x\sqrt{-g} f_0(\vartheta)\Lambda, \tag{53c}$$

$$S_{\rm int,1} = \alpha_1 \int d^4x\sqrt{-g} f_1(\vartheta)R, \tag{53d}$$

$$S_{\rm int,2} = \alpha_2 \int d^4x\sqrt{-g} f_2(\vartheta)R^2, \tag{53e}$$

or generally

$$S_{\rm int} = \alpha \int d^4x\sqrt{-g} f(\vartheta)\mathcal{R}, \tag{53f}$$

with f an arbitrary "coupling function" and $\mathcal{R}$ an algebraic combination of curvature invariants. Alternatively, notice that we could have assigned each term proportional to α_i a separate ϑ_i coupling with its associated kinetic and potential terms. The arguments presented below would also hold for such constructions.

B. Dynamical scalar fields

The requirement for the scalar ϑ to be dynamical arises from demanding diffeomorphism invariance in the theory. Consider the infinitesimal transformation of the action under a diffeomorphism generated by the vector field υ^μ. Specifically, look at the terms containing ϑ, i.e. the sum $S_{\rm mod} = S_{\rm int} + S_\vartheta$. The infinitesimal transformation under the diffeomorphism is

$$\delta S_{\rm mod} = \int d^4x\left(\frac{\delta}{\delta g_{\mu\nu}}\mathcal{L}_{\rm int}\right)\mathcal{L}_\upsilon g_{\mu\nu} + \left(\frac{\delta}{\delta\vartheta}\mathcal{L}_{\rm int}\right)\mathcal{L}_\upsilon\vartheta + \int d^4x\left(\frac{\delta}{\delta g_{\mu\nu}}\mathcal{L}_\vartheta\right)\mathcal{L}_\upsilon g_{\mu\nu} + \left(\frac{\delta}{\delta\vartheta}\mathcal{L}_\vartheta\right)\mathcal{L}_\upsilon\vartheta, \tag{54}$$

where $\mathcal{L}_{\rm int}$ is the interaction Lagrangian density, $\mathcal{L}_\vartheta$ is the kinetic Lagrangian density, and $\mathcal{L}_\upsilon$ stands for the Lie derivative along υ^μ.

For a theory to be diffeomorphism invariant, the infinitesimal transformation in the total action must vanish, $\delta S = 0$. Since $\mathcal{L}_\upsilon\vartheta$ may be arbitrary for some ϑ and some υ^μ, the functional multiplying $\mathcal{L}_\upsilon\vartheta$ must vanish for δS to vanish. This means

$$\frac{\delta}{\delta\vartheta}(\mathcal{L}_{\rm int} + \mathcal{L}_\vartheta) = 0. \tag{55}$$

When the scalar field ϑ has dynamics, i.e. $\beta \neq 0$, then Eq. (55) is identical to the equations of motion of the field ϑ and is therefore automatically satisfied. However, if the field is *not* dynamical, $\beta = 0$, then Eq. (55) gives

$$f'(\vartheta)\mathcal{R} = 0. \tag{56}$$

Except in the case where $f'(\vartheta) = 0$, this is an additional constraint on the geometry of spacetime, namely, that $\mathcal{R} = 0$. Given that the equations of motion already saturate the number of equations for the degrees of freedom present, this would be an overconstrained system. This is in fact the case in the nondynamical version of CS gravity, as discussed in [30,59]. We therefore only admit dynamical scalar fields or terms with no scalar field dependence [$f(\vartheta) = $ const].

C. Special cases: Zeroth and first rank

Before doing a calculation for a general curvature invariant $\mathcal{R}$, let us briefly discuss some special cases. As we will see, curvature invariants of zeroth and first rank will not be considered.

1. Zeroth rank

At zeroth rank, there is only one algebraic curvature invariant: a constant. The nonconstant part of $f_0(\vartheta)$ may simply be reabsorbed into the potential $V(\vartheta)$. This gives a minimally coupled scalar field, which may be absorbed into $S_{\rm mat}$. The constant part of f_0 leads to a "cosmological constant." Since we are only considering theories which are Minkowski stable and asymptotically flat, this cosmological part must vanish.

2. First rank

There is only one algebraic curvature invariant of rank 1, the Ricci scalar R. If we allow f to be a nonconstant function, we would have a classical scalar-tensor theory, akin to the Brans-Dicke theory (see e.g. [1]). The effective GW stress-energy tensor for the scalar-tensor theory has been computed, for example, in the Brans-Dicke theory (see [60]), so we do not consider it here. Since we already include the Einstein-Hilbert term in Eq. (53a), there can be no additional term linear in R without affecting Newton's constant. Thus we only consider quadratic and higher rank curvature invariants.

D. Cubic and higher ranks

At cubic rank, one can easily show that there are five algebraic invariants that may not be factored as products of lower rank invariants ($R^{\mu}{}_{\nu}R^{\nu}{}_{\rho}R^{\rho}{}_{\mu}$, $R^{\mu\nu}R^{\alpha\beta}R_{\mu\alpha\nu\beta}$, $R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$, ${}^*R^{\rho\sigma\mu\nu}R^{\kappa\lambda}{}_{\mu\nu}R_{\rho\sigma\kappa\lambda}$, and ${}^*R_{\rho\sigma}{}^{\alpha\beta}R^{\rho}{}_{\alpha}R^{\sigma}{}_{\beta}$) and four that may be factorized (R^3, $RR_{\mu\nu}R^{\mu\nu}$, $RR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, and $R^*R^{\rho}{}_{\sigma}{}^{\mu\nu}R^{\sigma}{}_{\rho\mu\nu}$). The arguments that follow work for all of them, so for concreteness we choose just one: $R^{\mu}{}_{\nu}R^{\nu}{}_{\rho}R^{\rho}{}_{\mu}$. The modification to the action arising from this term is

$$S_{\rm ex.5} = \alpha\int d^4x\sqrt{-g}f(\vartheta)R^{\mu}{}_{\nu}R^{\nu}{}_{\rho}R^{\rho}{}_{\mu}. \tag{57}$$

The contribution from this term to the effective action at second order is

$$\begin{aligned}S^{\rm eff(2)}_{\rm ex.5} = \epsilon^2\alpha\int d^4x\sqrt{-\bar{g}}\bigg[&\frac{\tilde{h}}{2}(f'(\bar{\vartheta})\tilde{\vartheta}\bar{R}^{\mu}{}_{\nu}\bar{R}^{\nu}{}_{\rho}\bar{R}^{\rho}{}_{\mu}\\ &+3f(\bar{\vartheta})\tilde{R}^{\mu}{}_{\nu}\bar{R}^{\nu}{}_{\rho}\bar{R}^{\rho}{}_{\mu}) + 3f'(\bar{\vartheta})\tilde{\vartheta}\tilde{R}^{\mu}{}_{\nu}\bar{R}^{\nu}{}_{\rho}\bar{R}^{\rho}{}_{\mu}\\ &+3f(\bar{\vartheta})\tilde{R}^{\mu}{}_{\nu}\tilde{R}^{\nu}{}_{\rho}\bar{R}^{\rho}{}_{\mu}\bigg].\end{aligned} \tag{58}$$

Immediately we see that all terms have at least one power of background curvature tensors. This means that each term can be written similarly to an earlier example in Sec. II C, in Eq. (11). When evaluating this effective stress-energy tensor at $\mathcal{I}^+$, all of the background curvature tensors vanish. This automatically implies that cubic and higher rank terms in the action do not contribute to the effective stress-energy tensor at asymptotically flat, future null infinity.

The stress-energy tensor is then given by the MacCallum-Taub tensor, Eq. (20b), which need not be identical to the GR one yet, as one must first impose the first-order equations of motion at $\mathcal{I}^+$. These equations could be modified by the introduction of higher-order operators in the action. Let us analyze such equations again through the example of Eq. (57). As the calculation depends only on the rank of the curvature invariant appearing in the action and not on its specific form, the results shown below extend to all cubic and higher rank algebraic curvature invariants as well.

The equation of motion arising from Eq. (57) is

$$\begin{aligned}&\kappa G_{\mu\nu} + 3\alpha f(\vartheta)R_{\mu\beta}R^{\beta}{}_{\gamma}R^{\gamma}{}_{\nu} - \frac{\alpha}{2}g_{\mu\nu}f(\vartheta)R^{\alpha}{}_{\beta}R^{\beta}{}_{\gamma}R^{\gamma}{}_{\alpha}\\ &\quad+\frac{3\alpha}{2}[g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}(f(\vartheta)R^{\alpha}{}_{\gamma}R^{\gamma\beta}) + \Box(f(\vartheta)R_{\mu\gamma}R^{\gamma}{}_{\nu})\\ &\quad-2\nabla_{\beta}\nabla_{(\mu}(f(\vartheta)R^{\gamma}{}_{\nu)}R^{\beta}{}_{\gamma})] = T^{\rm mat}_{\mu\nu} + T^{(\vartheta)}_{\mu\nu}.\end{aligned} \tag{59}$$

The important feature to note is that all terms containing α, that is, all terms deforming away from GR, are cubic or quadratic in curvature tensors. This is a general feature: From a term in the action of rank r, terms in the equations of motion will be of rank r and rank $r-1$.

Now consider evaluating the first-order equations of motion at asymptotically flat, future null infinity, which we need in order to put the MacCallum-Taub stress-energy tensor on-shell. We will not write out the full first-order equations of motion; it suffices to say that the modification terms (those terms containing α) are of rank r, $r-1$, and $r-2$ in the first-order equations of motion. When going to $\mathcal{I}^+$, only the terms of rank 0 survive, e.g. $\bar{\Box}\underline{\tilde{h}}_{\mu\nu}$.

Immediately we see that the only modifications to the action that affect the first-order equations of motion at $\mathcal{I}^+$ are those of rank 2 and lower. Thus for modifications that are cubic and higher, the first-order equations of motion at $\mathcal{I}^+$ are simply those of GR: $\bar{\Box}\underline{\tilde{h}}_{\mu\nu} = 0$.

Inserting this asymptotically flat, on-shell condition into the MacCallum-Taub stress-energy tensor yields the Isaacson stress-energy tensor. Cubic and higher rank modifications to the Lagrangian do not modify the effective stress-energy tensor due to GWs. We again emphasize that radiation reaction will still be different in a higher-order theory because of different motion in the strong field, additional energy carried in the scalar field ϑ, and energy carried down horizons being different. But the energy of a GW at $\mathcal{I}^+$ is the same as in GR.

E. Quadratic terms

Let us now consider quadratic deformations to the action, as these are the only ones left to study, and let us classify the types of modifications possible. There are two important characteristics that we use for such a classification. The first depends on the nature of the curvature quantity $\mathcal{R}$. This quantity may either be topological or not. A curvature quantity that is topological may be expressed as the divergence of a current: $\mathcal{R} = \nabla_\mu \mathcal{K}^\mu$. As we mentioned in Sec. IV, Eq. (24), the Pontryagin density ${}^*RR$ is a topological curvature invariant. In metric gravity, the only other nonvanishing, algebraic, second rank curvature invariant that is topological is the Gauss-Bonnet term

$$\mathcal{G} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{60}$$

as the Nieh-Yan invariant vanishes in torsion-free theories.

The second characteristic we can use to classify theories is the behavior of the scalar field ϑ, which depends on the potential $V(\vartheta)$. The two possibilities are a potential that is flat, $V(\vartheta) = 0$, or one that is nonflat, V varying with ϑ. A flat potential does not choose out any preferred values of the scalar field, whereas a nonflat potential must be bounded from below for stability and thus has a global minimum (or several minima). The presence or absence of a preferred field value is important in the limit going to $\mathcal{I}^+$.

For nonflat potentials, without loss of generality, the global minimum can be shifted to $\vartheta = 0$ by simultaneously shifting the potential function and the coupling function f. Such a shift does not affect the derivative term in the kinetic term for ϑ, since it simply adds a global constant to the field. The only problematic situation is if the global minimum is in the limit $\vartheta \to \pm\infty$, which we do not allow here.

We begin by discussing the asymptotic behavior of ϑ, which satisfies the sourced wave equation

$$\beta(\Box\vartheta - V'(\vartheta)) = -\alpha f'(\vartheta)\mathcal{R}. \tag{61a}$$

At $\mathcal{I}^+$, the right-hand side vanishes. Furthermore, if we are interested in static or quasistatic background solutions for $\bar{\vartheta}$ around which we can expand, time derivatives in the d'Alembertian will vanish leaving only the Laplacian:

$$\bar{\nabla}^2\bar{\vartheta} - V'(\bar{\vartheta}) = 0. \tag{61b}$$

For a nonflat potential, $V'(\vartheta) \neq 0$, the zeroth-order asymptotic solution will be $\bar{\vartheta}$ going to the minimum of the potential, which we have shifted to $\bar{\vartheta} = 0$.

For a flat potential, the background equation of motion for ϑ, Eq. (61b), at $\mathcal{I}^+$ becomes

$$\bar{\nabla}^2\bar{\vartheta} = 0. \tag{61c}$$

There are two asymptotic solutions: ϑ asymptotes to a constant or ϑ asymptotes to a function linear in Cartesian coordinates. The latter case would contribute a constant stress-energy tensor $T^{(\vartheta)}_{\mu\nu}$ at $\mathcal{I}^+$. This would lead to an asymptotically de Sitter spacetime, not an asymptotically flat spacetime. Therefore, we only consider the case where ϑ asymptotes to a constant.

The equation of motion Eq. (61c) does not determine to what value $\bar{\vartheta}$ asymptotes. A boundary condition is required in this case. Again, without loss of generality, for some given asymptotic value determined by some boundary condition, the field and coupling function $f(\vartheta)$ may be shifted so as to redefine the asymptotic value to be $\vartheta \to 0$ without changing the physics.

A boundary condition is *not* required if the theory is "shift-symmetric." In a shift-symmetric theory, the translation operation $\vartheta \to \vartheta + c$, where c is a constant, leaves the equations of motion invariant. Such a theory, therefore, must have equations of motion that depend only on the derivative $\nabla_\mu\vartheta$. Such is the case, for example, if the action depends on a topological term multiplied by some scalar field, $f(\vartheta)\nabla_\mu\mathcal{K}^\mu$, as then the action can be rewritten as $(\nabla_\mu f(\vartheta))\mathcal{K}^\mu$ via integration by parts. Of course, in this case, the potential must also be flat and f must be linear for the theory to be shift-symmetric. Such types of corrections arise naturally in the low-energy limit of string theory [17,61–63].

Let us rewrite the action and split the interaction term into a dynamical and a nondynamical part. Since we can always shift the field, potential, and coupling function so that the asymptotic value is $\vartheta \to 0$, let us define

$$\alpha' \equiv \alpha f(0), \tag{62a}$$

$$F(\vartheta) \equiv f(\vartheta) - f(0). \tag{62b}$$

Then, the interaction term in Eq. (53f) may be rewritten as

$$S_{\rm int} = S_{\rm n\text{-}d} + S_{\rm dyn} = \alpha' \int d^4x\sqrt{-g}\mathcal{R} + \alpha \int d^4x\sqrt{-g}F(\vartheta)\mathcal{R}. \tag{62c}$$

The first term is the nondynamical part, i.e. the part that does not couple to the scalar field, while the second part is the dynamical part. If $\mathcal{R}$ is a topological curvature invariant, then the first term in Eq. (62c) does not contribute to the equations of motion, as it is the integral of a total derivative.

1. Dynamical contribution

Let us perturb $S_{\rm dyn}$ to second order to calculate the contribution to the effective action, keeping in mind that $\tilde{\tilde{\vartheta}}$ and $\tilde{\tilde{h}}$ do not contribute. This part of the effective action is

$$S^{\rm eff(2)}_{\rm dyn} = \epsilon^2\alpha \int d^4x\sqrt{-\bar{g}}\Big[F(\bar{\vartheta})\tilde{\tilde{\mathcal{R}}} + \Big(\frac{\tilde{h}}{2}F(\bar{\vartheta}) + F'(\bar{\vartheta})\tilde{\vartheta}\Big)\tilde{\mathcal{R}} + \frac{1}{8}(\tilde{h}^2 - 2\tilde{h}^{\mu\nu}\tilde{h}_{\mu\nu})F(\bar{\vartheta})\bar{\mathcal{R}} + \frac{1}{2}(\tilde{h}F'(\bar{\vartheta})\tilde{\vartheta} + F''(\bar{\vartheta})\tilde{\vartheta}^2)\bar{\mathcal{R}}\Big]. \tag{63}$$

To determine the contribution to the effective stress-energy tensor at $\mathcal{I}^+$, again analyze the asymptotic form of all of

TABLE I. The asymptotic forms of fields appearing in the effective action for a rank 2 modification to the action. All tensor indices have been suppressed.

Field	Asymptotic form	Field	Asymptotic form
$\bar{g}$	$\mathcal{O}(1 + r^{-1})$	$\bar\nabla\bar\vartheta$	At least $r^{-3/2}$
$\bar\Gamma$	r^{-2}	$\bar\vartheta$	At least $r^{-1/2}$
$\bar R$	r^{-3}	$F(\bar\vartheta)$	At least $r^{-1/2}$
$\bar{\mathcal{R}} \sim \bar R^2$	r^{-6}	$F^{(n)}(\bar\vartheta)$	$\mathcal{O}(1)$
$\tilde h$, $\bar\nabla^{(n)}\tilde h$	r^{-1}	$\tilde\vartheta$, $\bar\nabla^{(n)}\tilde\vartheta$	r^{-1}
$\tilde{\mathcal{R}} \sim \bar R(\bar\nabla^2\tilde h)$	r^{-4}	$\tilde{\tilde{\mathcal{R}}} \sim (\bar\nabla^2\tilde h)^2$	r^{-2}

the fields appearing in Eq. (63). The asymptotic forms are summarized in Table I.

The simplest way to see that the dynamical part of the effective action does not contribute at $\mathcal{I}^+$ is to examine the asymptotics of $\bar{\mathcal{R}}$, $\tilde{\mathcal{R}}$, and $\tilde{\tilde{\mathcal{R}}}$. Since curvature tensors $\bar R_{\alpha\beta\mu\nu}$ are tidal tensors, they goes as r^{-3}; since $\bar{\mathcal{R}}$ contains two curvature tensors, it scales as r^{-6}. The slowest decaying (i.e. leading) part of $\tilde{\mathcal{R}}$ roughly comes from $\bar R\tilde R$ (with indices suppressed); the leading part of $\tilde R$ is $\bar\nabla^2\tilde h$, which, being radiative, goes as r^{-1}. This means that $\tilde{\mathcal{R}} \sim r^{-4}$. Similarly, the leading part of $\tilde{\tilde{\mathcal{R}}}$ goes as $(\bar\nabla^2\tilde h)(\bar\nabla^2\tilde h)$, so $\tilde{\tilde{\mathcal{R}}} \sim r^{-2}$.

Examining the effective action, Eq. (63), we see that there are no terms that decay as r^{-2}, which are the only ones that can contribute to the GW effective stress-energy tensor at $\mathcal{I}^+$. Any term with $\bar{\mathcal{R}}$ or $\tilde{\mathcal{R}}$ already decay too quickly; only terms with $\tilde{\tilde{\mathcal{R}}}$ could remain, and only if they were multiplied by terms that asymptote as $\mathcal{O}(1)$.

Having performed the splitting into the dynamical and nondynamical parts, $\tilde{\tilde{\mathcal{R}}}$ is multiplied by $\sqrt{-\bar g}F(\bar\vartheta)$ in the effective action. This splitting was specifically constructed so that $F(0) = 0$. Since F must be differentiable at $\vartheta = 0$, $F(\bar\vartheta)$ must go to zero at least as fast as $\bar\vartheta$ goes to zero, which is at least $r^{-1/2}$.

We have thus shown that the dynamical part of the interaction term does not contribute to the effective stress-energy tensor at $\mathcal{I}^+$ directly. However, it could still contribute indirectly through the imposition of the first-order field equations. We examine this in a later section.

2. Nondynamical contribution

Let us now consider $S_{\text{n-d}}$ in Eq. (62c). This term generically contributes to the effective stress-energy tensor of GWs at $\mathcal{I}^+$. To show this contribution, consider the general rank 2 modification as the linear combination of the four independent rank 2 curvature invariants

$$\alpha\mathcal{R} \equiv \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \alpha_4 {}^*RR \tag{64a}$$

and absorb $f(0)$ into the coefficients α_i' in the nondynamical part:

$$\alpha'\mathcal{R} \equiv \alpha_1' R^2 + \alpha_2' R_{\mu\nu}R^{\mu\nu} + \alpha_3' R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \alpha_4' {}^*RR. \tag{64b}$$

Note that this form also includes the Weyl squared invariant, which is a dependent linear combination of the above terms: $C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu} = R^2/3 - 2R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$, which is considered in [26,27]. The Pontryagin density ${}^*RR$, being a topological invariant, does not contribute to the action in $S_{\text{n-d}}$, so we may drop the final term. Similarly, if the linear combination is proportional to the Gauss-Bonnet (or Euler) invariant, which has $\alpha_1 = 1 = \alpha_3$, $\alpha_2 = -4$, then $\mathcal{R}$ would be topological and there would be no contribution to $S_{\text{n-d}}$ and hence no contribution to the effective stress-energy tensor of GWs.

The calculation of the effective action for the nondynamical term is straightforward but long, so we do not show the steps here. An outline of the calculation is to perturb $\sqrt{-g}\mathcal{R}$ to second order; the only parts that may contribute to an effective stress-energy tensor at $\mathcal{I}^+$ are of the form $\sqrt{-\bar g}\tilde R\tilde R$, where again we have suppressed indices on the perturbed curvature tensor $\tilde R$. This is calculated in terms of the trace-reversed metric perturbation $\tilde{\underline{h}}^{\mu\nu}$. As before, the Lorenz gauge may be imposed at the level of the action. All terms that remain will be of the form $\bar\nabla_\alpha\bar\nabla_\beta\tilde{\underline{h}}^{\mu\nu}\bar\nabla_\kappa\bar\nabla_\lambda\tilde{\underline{h}}^{\rho\sigma}$ with all indices contracted to form a scalar. If any derivative is contracted onto $\tilde{\underline{h}}^{\mu\nu}$, by integrating by parts and commuting covariant derivatives, one may form the Lorenz gauge condition $\bar\nabla_\mu\tilde{\underline{h}}^{\mu\nu} = 0$ and ignore the term in the effective action. Thus the only surviving terms have derivatives contracted together, which can be put into one of two forms: $\bar\Box\tilde{\underline{h}}^{\mu\nu}\bar\Box\tilde{\underline{h}}_{\mu\nu}$ and $\bar\Box\,\tilde{\underline{h}}\,\bar\Box\,\tilde{\underline{h}}$. After the explicit calculation, the prefactors are found and

$$S^{\text{eff}(2)}_{\text{n-d}} = \frac{\epsilon^2}{4}\int d^4x\sqrt{-\bar g}\,[(\alpha_1' - \alpha_3')\bar\Box\,\tilde{\underline{h}}\,\bar\Box\,\tilde{\underline{h}} + (\alpha_2' + 4\alpha_3')\bar\Box\tilde{\underline{h}}^{\mu\nu}\bar\Box\tilde{\underline{h}}_{\mu\nu}]. \tag{65}$$

Note again that α_4 does not appear, and if α_1, α_2, and α_3 are in the Gauss-Bonnet ratio, then the effective action of Eq. (65) vanishes.

Putting all indices in their natural positions, so as to expose implicit metric dependence, and varying the effective action of Eq. (65) with respect to $\bar g^{\mu\nu}$, the contribution to the effective stress-energy tensor is

$$T^{\text{eff}}_{\text{n-d}\mu\nu} = \epsilon^2\langle\langle(\alpha_1' - \alpha_3')\bar\Box\tilde{\underline{h}}(\bar\Box\tilde{\underline{h}}_{\mu\nu} - \bar\nabla_\mu\bar\nabla_\nu\tilde{\underline{h}}) + (\alpha_2' + 4\alpha_3')(\bar\Box\tilde{\underline{h}}_{\alpha\mu}\bar\Box\tilde{\underline{h}}_\nu{}^\alpha - \bar\Box\tilde{\underline{h}}_{\alpha\beta}\bar\nabla_{(\mu}\bar\nabla_{\nu)}\tilde{\underline{h}}^{\alpha\beta})\rangle\rangle. \tag{66}$$

We then find that the effective stress-energy tensor of GWs at $\mathcal{I}^+$ is given by the MacCallum-Taub stress-energy tensor (coming from the Einstein-Hilbert action) plus the direct contribution from the nondynamical part of the rank 2 interaction term, $T^{\rm eff}_{\text{n-d}\mu\nu}$:

$$T^{\rm eff}_{\mu\nu} = T^{\rm eff}_{{\rm MT}\mu\nu} + T^{\rm eff}_{\text{n-d}\mu\nu}. \tag{67}$$

The only remaining part of the calculation is to put the stress-energy tensor on-shell, that is, to impose the first-order equations of motion at $\mathcal{I}^+$.

3. First-order equations of motion

We need the first-order equations of motion at $\mathcal{I}^+$ of the full theory, including both the dynamical and nondynamical terms of the action. At $\mathcal{I}^+$, however, $\bar{\vartheta}$ and $\bar{\nabla}_\mu \bar{\vartheta}$ decay at least as $r^{-1/2}$ and $r^{-3/2}$. The only remaining dependence on $\bar{\vartheta}$ is through $f(\bar{\vartheta}) \to f(0)$.

The general zeroth- and first-order equations of motion are quite long, so we do not reproduce them here, but they do simplify as $r \to \infty$. These equations are linear in $\underline{\tilde{h}}_{\mu\nu}$ and $\tilde{\vartheta}$, which are both radiative and decay as r^{-1}. This r^{-1} scaling is the asymptotic scaling of the first-order equations of motion, as there are terms of the form $\bar{\Box}\underline{\tilde{h}}_{\mu\nu}$ that appear with no curvature tensors or background scalar field $\bar{\vartheta}$ multiplying them. However, all terms containing $\tilde{\vartheta}$ have curvature tensors multiplying them, so they decay faster than the leading behavior of r^{-1}.

Keeping only the terms that go as r^{-1}, the first-order equation of motion in the Lorenz gauge at asymptotically flat, future null infinity is

$$\begin{aligned}\kappa\bar{\Box}\underline{\tilde{h}}_{\mu\nu} = &-(2\alpha_1 + \alpha_2 + 2\alpha_3)f(\bar{\vartheta})(\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\Box} - \bar{g}_{\mu\nu}\bar{\Box}\bar{\Box})\underline{\tilde{h}} \\ &- (\alpha_2 + 4\alpha_3)f(\bar{\vartheta})\bar{\Box}\bar{\Box}\underline{\tilde{h}}_{\mu\nu},\end{aligned} \tag{68a}$$

and the trace of this equation is

$$\kappa\bar{\Box}\underline{\tilde{h}} = 2(3\alpha_1 + \alpha_2 + \alpha_3)f(\bar{\vartheta})\bar{\Box}\bar{\Box}\underline{\tilde{h}}. \tag{68b}$$

Again we see that if the α coefficients are in the Gauss-Bonnet ratio, the GR equation of motion is recovered at $\mathcal{I}^+$.

This wave equation can be seen to be a massive wave equation for the auxiliary variable $\tilde{r}_{\mu\nu} \equiv \bar{\Box}\underline{\tilde{h}}_{\mu\nu}$, with mass $m \sim 1/\bar{\lambda}$, where $\bar{\lambda}^2 \sim |\alpha_i| f(0)/\kappa$. In the weak-coupling limit, $\bar{\lambda}/\lambda_{\rm GW} \ll 1$, the equations simplify considerably. This simplification comes from treating the solution to the full theory as a *deformation* away from GR; this means expanding the fields as power series in a small parameter, namely, $\bar{\zeta} = (\bar{\lambda}/\lambda_{\rm GW})^2$. As in Eq. (34), we impose

$$\underline{\tilde{h}}_{\mu\nu} = \sum_{n=0}^{\infty} \bar{\zeta}^n \underline{\tilde{h}}^{(n)}_{\mu\nu}, \tag{69}$$

and similarly for other fields, where the zeroth field $\underline{\tilde{h}}^{(0)}_{\mu\nu}$ is the GR solution. Inserting this expansion in the first-order equation of motion Eq. (68) and matching order by order gives

$$\begin{aligned}\kappa\bar{\Box}\underline{\tilde{h}}^{(n+1)}_{\mu\nu} = &-(2\alpha_1 + \alpha_2 + 2\alpha_3)f(\bar{\vartheta})(\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\Box} - \bar{g}_{\mu\nu}\bar{\Box}\bar{\Box})\underline{\tilde{h}}^{(n)} \\ &- (\alpha_2 + 4\alpha_3)f(\bar{\vartheta})\bar{\Box}\bar{\Box}\underline{\tilde{h}}^{(n)}_{\mu\nu},\end{aligned} \tag{70a}$$

for all orders $n \geq 0$, and

$$\bar{\Box}\underline{\tilde{h}}^{(0)}_{\mu\nu} = 0, \tag{70b}$$

for the GR solution. Substituting Eq. (70b) into Eq. (70a) and iteratively solving the field equations one order at a time, we find at all orders that

$$\bar{\Box}\underline{\tilde{h}}^{(n)}_{\mu\nu} = 0. \tag{70c}$$

This is the GR first-order equation of motion, and just as in GR, we may specialize the Lorenz gauge to the TT gauge at $\mathcal{I}^+$.[5] This expansion has discontinuously turned the massive wave equation into a massless one by killing the massive modes in the limit of $m \to \infty$. Such an order-reduction procedure, where certain solutions are eliminated through perturbative constraints, has been shown to select the physically correct ones in all studied cases [31,65,66].

We can now evaluate the complete effective stress-energy tensor of GWs at $\mathcal{I}^+$. As shown in Sec. VI E 1, there is no direct contribution from the dynamical part of the interaction term. Section VI E 2 showed that the nondynamical part does contribute directly, but imposing Eq. (70c) forces this contribution to also vanish. Since the MacCallum-Taub tensor on-shell is equal to the Isaacson tensor, we then have

$$T^{\rm eff}_{\mu\nu} = (T^{\rm eff}_{{\rm MT}\mu\nu} + T^{\rm eff}_{\text{n-d}\mu\nu})|_{(\bar{\Box}\underline{\tilde{h}}_{\alpha\beta}=0)} = T^{\rm eff}_{{\rm GR}\mu\nu}. \tag{71}$$

That is, the effective GW stress-energy tensor is identical to the Isaacson one at $\mathcal{I}^+$ for this wide class of modified gravity theories.

VII. CONCLUSIONS

We have here addressed the energy content of GWs in a wide class of modified gravity theories. We focused on theories that are weak deformations away from GR and calculated the effective stress-energy tensor where GWs are extracted: in the asymptotically flat region of spacetime.

The main calculation tool we employed was the perturbed Lagrangian approach. We demonstrated the calculation explicitly for GR, recovering the Isaacson effective stress-energy tensor. We also explicitly calculated this effective tensor in dynamical modified CS gravity, where again the result at $\mathcal{I}^+$ reduces to the Isaacson tensor. The features of CS gravity that lead to the effective stress-energy tensor being identical to the one in GR are the

[5]To prove that the TT gauge exists at $\mathcal{I}^+$ for this theory, the proof in Appendix A of Flanagan and Hughes [64] must be extended. Their Eq. (A.12) must be replaced by our (68a) and a small-coupling expansion performed. The result will again be (70c), which is identical to Flanagan and Hughes's Eq. (A.12).

dynamical nature of the scalar field and the topological nature of the curvature correction to the action.

We then generalized this finding to all action modifications of a similar nature: a dynamical scalar field coupled to a scalar curvature invariant of rank 2 or higher in a spacetime that is asymptotically flat. For scalar curvature invariants of rank 3 or higher, we showed that there is no modification to the stress-energy tensor or the equations of motion at $\mathcal{I}^+$. For rank 2, we calculated the contribution to the effective stress-energy tensor and to the first-order equation of motion. In the weak-coupling limit, the only solutions to the first-order equations of motion satisfy the GR first-order equations of motion at $\mathcal{I}^+$, namely, $\bar{\Box}\underline{\tilde{h}}_{\mu\nu} = 0$. Evaluating the effective stress-energy tensor on-shell with these solutions leads, again, to the Isaacson stress-energy tensor.

A few caveats are in order. As we have stressed before, this result is evaluated at asymptotically flat, future null infinity, so it does not apply to cosmological spacetimes, e.g. de Sitter spacetime. Not all of the energy that is lost by a system is carried away by GWs to $\mathcal{I}^+$: There is also radiation in the scalar field (which is calculated straightforwardly from $T^{(\vartheta)}_{\mu\nu}$), and both GWs and the scalar field radiation are lost to trapped surfaces. All of these effects must be accounted for in calculating the radiation reaction of a system. Finally, we did not address modifications to the action of the form $f(\vartheta)R$, which reduce to a classical scalar-tensor theory.

There are several avenues open for future work. Considering classical scalar-tensor modifications is one possible extension. The work should also be extended to the next simplest spacetimes, those that are asymptotically de Sitter. This is appropriate for calculating GWs from inflation, for example. Extending this approach to calculating energy lost to trapped surfaces is another possibility.

The most natural application of this work is in tests of GR with pulsar binaries and with GWs emitted by EMRIs. The former problem requires performing a post-Keplerian expansion of the motion of bodies orbiting each other. The latter requires knowing the BH spacetime (background) solution in the class of modified gravity theories and the geodesic or nongeodesic motion on that spacetime. Both of these programs require knowledge of radiation reaction in GWs at $\mathcal{I}^+$, which we have here computed for a large class of modified gravity theories.

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