Information content of nonautonomous free fields in curved space-time

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We show that it is possible to quantify the information content of a nonautonomous free field state in curved space-time. A covariance matrix is defined and it is shown that, for symmetric Gaussian field states, the matrix is connected to the entropy of the state. This connection is maintained throughout a quadratic nonautonomous (including possible phase transitions) evolution. Although particle-antiparticle correlations are dynamically generated, the evolution is isoentropic. If the current standard cosmological model for the inflationary period is correct, in absence of decoherence such correlations will be preserved, and could potentially lead to observable effects, allowing for a test of the model.

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I. INTRODUCTION

Quantum information theory [1], which promises important technological advances, is usually formulated in the framework of nonrelativistic quantum mechanics. It has achieved remarkable success both from the formal as well as from the experimental point of view. The generalization of typical quantum information concepts, such as entanglement, in a relativistic context has been attempted in many papers [2–5], although a definitive answer is not yet clear. It is only natural to say that, no matter how complicated this problem may seem, quantum field theory is the appropriate framework to formalize it. Gaussian states, completely defined by their second moments, allow for the generalization of important theoretical tools from information theory to field theory. Moreover, they are of special interest in cosmology, where particle creation is usually treated in the inflationary period as parametric oscillators of time-dependent frequencies. Thus, the general theoretical scenario is that of a nonautonomous field in curved space-times [6]. The purpose of the present work is twofold: to show that entropy, an important quantity in the cosmological context, is related to a quantum field theoretical covariance matrix, whose elements are two-point functions; and to exemplify the result in a model used to describe the inflationary period. We also find a generalization of the Robertson-Schrödinger uncertainty principle and show that for Gaussian field states entropy is preserved, even through phase transitions. Thereby, entropy production is not necessarily related to particle production, at least for nonautonomous free fields in curved space-times. Besides, in the absence of decoherence mechanisms, some particle-particle quantum correlations will be present in the final stages of the inflationary evolution and could, in principle, have observable consequences and may help us to test the validity of the standard cosmological model.

This paper is organized as follows. The connection between the entropy and the uncertainty of Gaussian states in nonrelativistic quantum mechanics, reviewed in Sec. II, is used in III to describe the entropy dynamics of two-mode Gaussian states in the presence of decoherence. We apply the generalization of the covariance matrix used in these two sections to quantum fields IV to show that, in a cosmologically inspired model, Gaussian fields in curved spaces preserve entropy through their evolution V. Final considerations and conclusions can be found in Sec. VI.

II. GAUSSIAN STATES: ENTROPY AND SCHRÖDINGER DETERMINANT

We define the covariance matrix Σ, for an arbitrary quantum state $\hat{\rho}$ of a one-dimensional particle with moment operator $\hat{p}$ and position operator $\hat{q}$, as

$$\Sigma = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{qp} & \sigma_{pp} \end{pmatrix} = \begin{pmatrix} \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 & \frac{1}{2}\langle\{\hat{p}, \hat{q}\}\rangle - \langle \hat{p} \rangle\langle \hat{q} \rangle \\ \frac{1}{2}\langle\{\hat{p}, \hat{q}\}\rangle - \langle \hat{p} \rangle\langle \hat{q} \rangle & \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \end{pmatrix}. \quad (1)$$

The average values $\langle \hat{O} \rangle$ are defined as $\mathrm{Tr}(\hat{\rho}\,\hat{O})$. As we show in this section, the determinant of this matrix is related to the entropy, for Gaussian states. The general Gaussian density operator corresponds to a displaced squeezed thermal density operator

$$\hat{\rho}_{\mathrm{G}} = \mathcal{D}(\alpha)\mathcal{S}(r, \phi)\hat{\rho}_\nu \mathcal{S}^\dagger(r, \phi)\mathcal{D}^\dagger(\alpha), \quad (2)$$

where $\mathcal{D}(\alpha)$ is the displacement operator, $\mathcal{S}(r, \phi)$ is the squeezing operator, and $\hat{\rho}_\nu$ the thermal density operator with average number of excitations ν. This decomposition plays an important role to establish the connection between entropy and the determinant of the covariance matrix. More explicitly we have

$$\mathcal{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}), \quad (3)$$

$$\mathcal{S}(r, \phi) = \exp\left(\frac{1}{2} r e^{i\phi} \hat{a}^\dagger - \frac{1}{2} r e^{i\phi} \hat{a}\right), \quad (4)$$

$$\hat{\rho}_{\nu} = \frac{1}{1+\nu} \exp\left(\ln\left(\frac{\nu}{\nu+1}\right)\hat{a}^\dagger \hat{a}\right). \quad (5)$$

The dimensionless position $\hat{q}$ and momentum $\hat{p}$ operators are linear combinations of creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators, and as usual $\hat{p} = -i(\hat{a} - \hat{a}^\dagger)/\sqrt{2}$, $\hat{q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$. In order to compute the elements of the covariance matrix for a Gaussian state, it is sufficient to know that the following averages for thermal states:

$$\langle \hat{a}\hat{a}^\dagger_{\hat{\rho}_\nu}\rangle = \langle \hat{a}^\dagger \hat{a}_{\hat{\rho}_\nu}\rangle + 1 = \nu + 1, \quad (6)$$

and the following transformation properties. The average value of a function of creation and annihilation operators is

$$\begin{aligned}\langle f(\hat{a}, \hat{a}^\dagger)\rangle_{\rho_{\mathrm{G}}} &= \mathrm{Tr}(f(\hat{a}, \hat{a}^\dagger)\mathcal{D}(\alpha)\mathcal{S}(r,\phi)\hat{\rho}_\nu \mathcal{S}^\dagger(r,\phi)\mathcal{D}^\dagger(\alpha)) \\ &= \mathrm{Tr}(\mathcal{S}^\dagger(r,\phi)\mathcal{D}^\dagger(\alpha) f(\hat{a}, \hat{a}^\dagger)\mathcal{D}(\alpha)\mathcal{S}(r,\phi)\hat{\rho}_\nu) \\ &= \mathrm{Tr}(f(\hat{A}, \hat{A}^\dagger)\hat{\rho}_\nu),\end{aligned}$$

where $\hat{A} = \mathcal{S}^\dagger(r,\phi)\mathcal{D}^\dagger(\alpha)\hat{a}\mathcal{D}(\alpha)\mathcal{S}(r,\phi) = \cosh(r)\hat{a} + e^{i\phi}\sinh(r)\hat{a}^\dagger + \alpha$. If $f(x,y)$ is a quadratic polynomial we need only the nonzero averages (6). Now, the elements of the covariance matrix

$$\begin{aligned}\sigma_{pp} &= \left(\nu + \frac{1}{2}\right)(\cosh(2r) - \cos(\phi)\sinh(2r)), \\ \sigma_{qq} &= \left(\nu + \frac{1}{2}\right)(\cosh(2r) + \cos(\phi)\sinh(2r)), \\ \sigma_{qp} &= \left(\nu + \frac{1}{2}\right)\sin(\phi)\sinh(2r),\end{aligned}$$

enable us to compute the determinant,

$$D = \sigma_{pp}\sigma_{qq} - \sigma_{qp}^2 = \left(\nu + \frac{1}{2}\right)^2, \quad (7)$$

which turns out to be a function of only ν, the mean number of excitations of the initial thermal state, ρ_ν. The Schrödinger determinant (7) contains the famous Heisenberg's uncertainty principle as a particular case.

The entropy of the Gaussian state is equal to the entropy of the corresponding thermal state,

$$\begin{aligned}S[\hat{\rho}_{\mathrm{G}}] &= -\mathrm{Tr}(\hat{\rho}_{\mathrm{G}} \ln\hat{\rho}_{\mathrm{G}}) \\ &= -\mathrm{Tr}(\mathcal{D}\mathcal{S}\hat{\rho}_\nu\mathcal{S}^\dagger\mathcal{D}^\dagger \ln(\mathcal{D}\mathcal{S}\hat{\rho}_\nu\mathcal{S}^\dagger\mathcal{D}^\dagger)) \\ &= -\mathrm{Tr}(\mathcal{D}\mathcal{S}\hat{\rho}_\nu\mathcal{S}^\dagger\mathcal{D}^\dagger\mathcal{D}\mathcal{S}\ln(\hat{\rho}_\nu)\mathcal{S}^\dagger\mathcal{D}^\dagger) \\ &= -\mathrm{Tr}(\mathcal{S}^\dagger\mathcal{D}^\dagger\mathcal{D}\mathcal{S}\hat{\rho}_\nu\ln(\hat{\rho}_\nu)) = -\mathrm{Tr}(\hat{\rho}_\nu\ln(\hat{\rho}_\nu)) \\ &= S[\hat{\rho}_\nu].\end{aligned}$$

Here the arguments of the displacement $\mathcal{D}$ and squeezing $\mathcal{S}$ operators were omitted. The thermal state $\hat{\rho}_\nu$ is diagonal in the basis of number of excitations, $\hat{\rho}_\nu = \sum_k \xi^k |k\rangle \times \langle k|/(1+\nu)$, where $\xi = \nu/(\nu+1)$. The entropy of the thermal state, on the other hand, is given by

$$\begin{aligned}S[\hat{\rho}_\nu] &= -\sum_{k=0}^{\infty}\frac{\xi^k}{\nu+1}\ln\left(\frac{\xi^k}{\nu+1}\right) \\ &= \frac{\ln(\nu+1)}{\nu+1}\sum_{k=0}^{\infty}\xi^k - \frac{\ln\xi}{\nu+1}\sum_{k=0}^{\infty}k\xi^k.\end{aligned}$$

Taking into account the well-known results for infinite sums we obtain the entropy of the thermal state,

$$S[\hat{\rho}_\nu] = (\nu+1)\ln(\nu+1) - \nu\ln\nu = S[\hat{\rho}_{\mathrm{G}}], \quad (8)$$

which also is a function of ν, the mean number of thermal excitations. We have shown that the entropy of Gaussian states ρ_{G} is a function of D, the determinant of its covariance matrix Σ,

$$\begin{aligned}S[\hat{\rho}_{\mathrm{G}}] = &\left(\sqrt{D} + \frac{1}{2}\right)\ln\left(\sqrt{D} + \frac{1}{2}\right) \\ &- \left(\sqrt{D} - \frac{1}{2}\right)\ln\left(\sqrt{D} - \frac{1}{2}\right).\end{aligned} \quad (9)$$

Now, a technical digression is in order. It is important to realize that major simplifications were possible due to the specific decomposition of the Gaussian state employed here [see (2)]. Although (9) is a kinematical relationship, it is more useful in the study of dynamical evolution, whether unitary or nonunitary, driven by quadratic Hamiltonians in the former case, or quadratic Liouvillians in the latter.

III. DYNAMICAL EVOLUTION OF ENTROPY

In this section we show how our approach can be used in dynamical situations. We know, from the previous section, that in order to obtain the entropy dynamics all we need to calculate is the time evolution of the covariance matrix elements, which turns out to be much easier than the dynamics of the whole state, at least for Gaussian states. The full Gaussian state can be reconstructed at any time, as a function of the covariance matrix elements, even for nonunitary quadratic dynamics, making it possible to study how quantum correlations are washed out by dissipation.

We consider two initially entangled modes subject to a Lindblad-type dynamics. This example is simulating the physical situation when two correlated particles are created, e.g. in the interior of two colliding heavy nuclei, and then cross a dissipative medium before they are released to the detector. The Gaussian state $\rho_G(t)$ that we consider,

$$\hat{\rho}_G(t) = \mathcal{S}_{12}(z(t))\hat{\rho}_{\nu1(t)}\hat{\rho}_{\nu2(t)}\mathcal{S}_{12}^\dagger(z(t)), \quad (10)$$

is not the most general two-mode Gaussian state. However it is adequate for the purpose of this section. Here $\hat{\rho}_{\nu1(t)}$ ($\hat{\rho}_{\nu2(t)}$) is a thermal state (5) for the first (second) mode with $\nu_1(t)$ ($\nu_2(t)$) mean excitations, and

$$\mathcal{S}_{12}(z) = \exp(z\hat{a}_1^\dagger\hat{a}_2^\dagger - z^*\hat{a}_1\hat{a}_2) \quad (11)$$

is the two-mode squeezing operator.

Our nonunitary dynamical evolution

$$\frac{d\hat{\rho}}{dt} = (\mathcal{L}_1 + \mathcal{L}_2)\hat{\rho}(t)$$

preserves the chosen form of the initial state (10). The Liouvillian superoperator $\mathcal{L}_i$, $i = 1, 2$

$$\mathcal{L}_i = -i\omega_i[\hat{a}_i^\dagger \hat{a}_i \cdot] + \gamma_i(\bar{n}_i + 1)(2\hat{a}_i \cdot \hat{a}_i^\dagger - \cdot \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_i \cdot) + \gamma_i \bar{n}_i (2\hat{a}_i^\dagger \cdot \hat{a}_i - \cdot \hat{a}_i \hat{a}_i^\dagger - \hat{a}_i \hat{a}_i^\dagger \cdot),$$

corresponds to a harmonic mode with frequency ω_i coupled to a heat bath, with mean number of excitations $\bar{n}_i$. The relaxation time of the first mode τ_1 (second mode τ_2) is the inverse of the relaxation rate γ_1 (γ_2).

The elements of the covariance matrix Σ_2

$$\Sigma_2 = \begin{pmatrix} \sigma_{q_1q_1} & \sigma_{q_1q_2} & \sigma_{q_1p_1} & \sigma_{q_1p_2} \\ \sigma_{q_1q_2} & \sigma_{q_2q_2} & \sigma_{q_2p_1} & \sigma_{q_2p_2} \\ \sigma_{q_1p_1} & \sigma_{q_2p_1} & \sigma_{p_1p_1} & \sigma_{p_1p_2} \\ \sigma_{q_1p_2} & \sigma_{q_2p_2} & \sigma_{p_1p_2} & \sigma_{p_2p_2} \end{pmatrix} \tag{12}$$

are defined by

$$\sigma_{r_ir_j} = \frac{1}{2}\mathrm{Tr}(\{\hat{r}_i, \hat{r}_j\}\hat{\rho}) - \mathrm{Tr}(\hat{r}_i\hat{\rho})\,\mathrm{Tr}(\hat{r}_j\hat{\rho}),$$

where $\hat{r}$ can be $\hat{q}$ or $\hat{p}$, and $i = 1, 2$. Here, besides the averages of $\hat{a}_i\hat{a}_i^\dagger$ (and $\hat{a}_i^\dagger\hat{a}_i$) in the thermal state, we need the following transformation relations:

$$\mathcal{S}_{12}^\dagger(z)\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}\mathcal{S}_{12}(z) = \cosh(|z|)\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} + e^{i\phi}\sinh(|z|)\begin{pmatrix}\hat{a}_2^\dagger\\ \hat{a}_1^\dagger\end{pmatrix},$$

where $z = |z|e^{i\phi}$.

Under the assumed conditions, the elements of the symmetric covariance matrix (12) acquires a particularly simple form,

$$\sigma_{q_1q_1} = \sigma_{p_1p_1} = \frac{1}{2}(\nu_1 - \nu_2) + \frac{1}{2}(\nu_1 + \nu_2 + 1)\cosh(2|z|),$$

$$\sigma_{q_2q_2} = \sigma_{p_2p_2} = \frac{1}{2}(\nu_2 - \nu_1) + \frac{1}{2}(\nu_1 + \nu_2 + 1)\cosh(2|z|),$$

$$\sigma_{q_1q_2} = -\sigma_{p_1p_2} = \frac{1}{2}(\nu_1 + \nu_2 + 1)\sinh(2|z|)\cos(\phi), \quad \sigma_{q_1p_1} = \sigma_{q_2p_2} = 0,$$

$$\sigma_{q_1p_2} = \sigma_{q_2p_1} = \frac{1}{2}(\nu_1 + \nu_2 + 1)\sinh(2|z|)\sin(\phi).$$

Initially we assume a two-mode squeezed state, that is $\nu_1(t=0) = 0 = \nu_2(t=0)$, with $z(t=0) = |z|_0 \geq 0$. The covariance matrix of this state is

$$\frac{1}{2}\begin{pmatrix} \cosh(2|z|_0) & \sinh(2|z|_0) & 0 & 0 \\ \sinh(2|z|_0) & \cosh(2|z|_0) & 0 & 0 \\ 0 & 0 & \cosh(2|z|_0) & -\sinh(2|z|_0) \\ 0 & 0 & -\sinh(2|z|_0) & \cosh(2|z|_0) \end{pmatrix}.$$

Now, we need the equations of motion for the elements of the covariance matrix. In this case, it is easier to solve the equations of motion of the expected values of $\hat{a}_i^{(\dagger)}\hat{a}_j^{(\dagger)}$. Thus, it is worth knowing that, for the considered form of the Gaussian state (10), we have

$$\langle \hat{a}_1^\dagger \hat{a}_1\rangle + \langle \hat{a}_2^\dagger \hat{a}_2\rangle + 1 = (1 + \nu_1 + \nu_2)\cosh(2|z|),$$

$$\langle \hat{a}_1^\dagger \hat{a}_1\rangle - \langle \hat{a}_2^\dagger \hat{a}_2\rangle = \nu_1 - \nu_2,$$

$$2\langle \hat{a}_1 \hat{a}_2\rangle = (1 + \nu_1 + \nu_2)\sinh(2|z|)e^{i\phi}.$$

We set $n_i(t) = \langle \hat{a}_i^\dagger \hat{a}_i\rangle(t)$ and $\mu(t) = \langle \hat{a}_1 \hat{a}_2\rangle(t)$. The equations of motion of these expectation values,

$$\frac{dn_i}{dt} = -2\gamma_i n_i + \bar{n}_i, \qquad i = 1, 2,$$

$$\frac{d\mu}{dt} = -(\gamma_1 + \gamma_2 + i\omega_1 + i\omega_2)\mu,$$

are easily solvable. The initial conditions are given by

$$n_1(0) + n_2(0) + 1 = \cosh(2|z|_0), \qquad n_1(0) - n_2(0) = 0,$$

$$2\mu(0) = \sinh(2|z|_0).$$

From the first and second equations above we find

$$n_1(0) = n_2(0) = \sinh^2(|z|_0).$$

The solutions of the dynamical equations for the elements of the covariance matrix are

$$n_i(t) = \sinh^2(|z|_0)e^{-2\gamma_i t} + \bar{n}_i(1 - e^{-2\gamma_i t}),$$

$$\mu(t) = \frac{\sinh(2|z|_0)}{2}e^{-(\gamma_1+\gamma_2)t}e^{-i(\omega_1+\omega_2)t}.$$

The time dependence parameters of the Gaussian state $\hat{\rho}_G(t)$ are functions of these elements

$$z(t) = \frac{e^{-i(\omega_1+\omega_2)t}}{4}\ln\left(\frac{1 + n_1(t) + n_2(t) + 2|\mu(t)|}{1 + n_1(t) + n_2(t) - 2|\mu(t)|}\right)$$

$$\nu_{1,2}(t) + \frac{1}{2} = \pm\frac{n_1(t) - n_2(t)}{2} + \frac{1}{2}\sqrt{(1 + n_1(t) + n_2(t))^2 - (2|\mu(t)|)^2}.$$

The relation between the entropy and the covariance matrix can be introduced via a generalization of the Robertson-Schrödinger uncertainty relation for two or more degrees of freedom [7],

$$\mathbb{C}\,\mathbb{J}\mathbb{C} = \frac{\hbar^2}{4}\mathbb{J}. \tag{13}$$

Here the simplectic matrix $\mathbb{J}$ is defined as

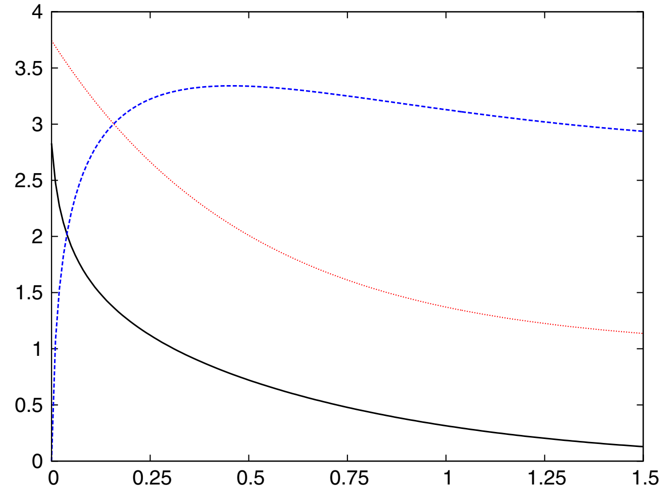

FIG. 1 (color online). Dynamics of a symmetric two-mode Gaussian state coupled to local identical baths with mean number of excitations $\bar{n} = 1$ and relaxation rates $\gamma = 1$. The parameters of the initial state (10) are $z = \sqrt{2}$ and $\nu_1 = 0 = \nu_2$. The solid (black) line depicts the evolution of twice the norm of the two-mode squeezing parameter ($2|z|$), the dashed (blue) line corresponds to the entropy of the state S, and the dotted (red) line corresponds to the mean number of excitations of one of the modes, n.

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The uncertainty relation (13) works if the state is symmetric under the exchange of the 2 degrees of freedom.

For the sake of simplicity we consider identical baths. In this case the entropy of the Gaussian state, $S = 2(\nu(t) + 1)\log(\nu(t) + 1) - 2\nu(t)\log\nu(t)$, can also be written in terms of the determinant of the covariance matrix $D = (\frac{1}{2} + \nu(t))^4$, or in terms of the uncertainty relation (13), which gives $\hbar^2(\nu + 1/2)^2\mathbb{J}$. We have assumed $\nu_1 = \nu_2 = \nu$. In the asymmetric case it is still possible to quantify the Gaussian state information using the determinants of the 2×2 submatrices of the covariance matrix. A full dynamics of this sort is given in [8].

As can be seen in Fig. 1, even in this simple case one observes two time scales. One of them, related to the particle production rate, is larger than the time scale for decoherence.

IV. QUANTUM FIELDS

As in II, in this section we study only kinematics, that is, we generalize the concept of a covariance matrix to quantum fields, and derive a relation between its determinant and the entropy, for symmetric Gaussian states of complex scalar fields. We find a generalization of the Robertson-Schrödinger uncertainty principle. We show that it is possible to quantify the information content of a Gaussian symmetric field state in terms of its second moments only.

We will consider a complex scalar field subjected to a flat-space Robertson-Walker metric. The associated metric is given by

$$d^2s = -d\tau^2 + a^2(z)(dx^2 + dy^2 + dz^2). \tag{14}$$

It is very convenient to define the conformal time as

$$\eta(t) = \int_{t_0}^{t} \frac{d\tau}{a(z)} \tag{15}$$

and to explicitly write down the original Lagrangian

$$S = \int a^3(t)\left(\dot{\hat{\phi}}\dot{\hat{\phi}}^* - \frac{1}{a^2(t)}\nabla\hat{\phi}\nabla\hat{\phi}^* - m^2\hat{\phi}\hat{\phi}^*\right) \tag{16}$$

in terms of the auxiliary field $\hat{\chi} = a(\eta)\hat{\phi}$. We get

$$S = \int d^3x d\eta\left(\hat{\chi}'\hat{\chi}'^* - \nabla\hat{\chi}\nabla\hat{\chi}^* - \left(m^2a^2 - \frac{a''}{a}\right)\hat{\chi}\hat{\chi}^*\right), \tag{17}$$

where the prime denotes derivation with respect to η. We can define the Hamiltonian in the usual way and expand the fields and their conjugated momenta in their Fourier components

$$\hat{\chi}(\boldsymbol{x}, \eta) = \int \frac{d^3k}{(2\hat{\pi})^3}\hat{\chi}_{\boldsymbol{k}}(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{18a}$$

$$\hat{\pi}(\boldsymbol{x}, \eta) = \int \frac{d^3k}{(2\hat{\pi})^3}\hat{\pi}_{\boldsymbol{k}}(\eta)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{18b}$$

$$\hat{\chi}^\dagger(\boldsymbol{x}, \eta) = \int \frac{d^3k}{(2\hat{\pi})^3}\hat{\chi}^\dagger_{\boldsymbol{k}}(\eta)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{18c}$$

$$\hat{\pi}^\dagger(\boldsymbol{x}, \eta) = \int \frac{d^3k}{(2\hat{\pi})^3}\hat{\pi}^\dagger_{\boldsymbol{k}}(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{18d}$$

It is usual to define the time-dependent mass

$$m^2(\eta) = m^2a^2(\eta) - \frac{a''(\eta)}{a(\eta)}, \tag{19}$$

and the following annihilation and creation operators:

$$\hat{a}_{\boldsymbol{k}}(\eta) = i(\varphi^*_{\boldsymbol{k}}(\eta)\hat{\pi}^\dagger_{\boldsymbol{k}} - \varphi'^*_{\boldsymbol{k}}(\eta)\hat{\chi}_{\boldsymbol{k}}) = (\hat{a}^\dagger_{\boldsymbol{k}}(\eta))^\dagger, \tag{20a}$$

$$\hat{b}_{\boldsymbol{k}}(\eta) = i(\varphi^*_{\boldsymbol{k}}(\eta)\hat{\pi}_{\boldsymbol{k}} - \varphi'^*_{\boldsymbol{k}}(\eta)\hat{\chi}^\dagger_{\boldsymbol{k}}) = (\hat{b}^\dagger_{\boldsymbol{k}}(\eta))^\dagger. \tag{20b}$$

The time-dependent functions $\varphi_{\boldsymbol{k}}(\eta)$ satisfy the equation of motion

$$\varphi''_{\boldsymbol{k}}(\eta) + (\boldsymbol{k}^2 + m^2(\eta))\varphi_{\boldsymbol{k}}(\eta) = 0, \tag{21}$$

and the condition

$$\hbar(\varphi'^*_{\boldsymbol{k}}(\eta)\varphi_{\boldsymbol{k}}(\eta) - \varphi'_{\boldsymbol{k}}(\eta)\varphi^*_{\boldsymbol{k}}(\eta)) = i. \tag{22}$$

Adopting the classical solution as expansion coefficients for the field in this context is of extreme technical importance, as will become clear later. They are specially important when one includes a relativistic background. This is one of the vital technical points of this paper.

Since the equation of motion does not depend on $\boldsymbol{k}$ but only on its magnitude, the coefficients $\varphi_{\boldsymbol{k}}$ can be simply written as φ_k. For example, the condition (22) reduces to

$$\hbar\,(\varphi_k'^*(\eta)\varphi_k(\eta) - \varphi_k'(\eta)\varphi_k^*(\eta)) = i. \tag{23}$$

For future reference we also give the expressions of the creation and annihilation operator in terms of the fields

$$\hat{\chi}_{\boldsymbol{k}} = \hbar(\varphi_k(\eta)\hat{a}_{\boldsymbol{k}}(\eta) + \varphi_k^*(\eta)\hat{b}_{\boldsymbol{k}}^\dagger(\eta)), \tag{24a}$$
$$\hat{\chi}_{\boldsymbol{k}}^\dagger = \hbar(\varphi_k^*(\eta)\hat{a}_{\boldsymbol{k}}^\dagger(\eta) + \varphi_k(\eta)\hat{b}_{\boldsymbol{k}}(\eta)), \tag{24b}$$
$$\hat{\pi}_{\boldsymbol{k}} = \hbar(\varphi_k'^*(\eta)\hat{a}_{\boldsymbol{k}}^\dagger(\eta) + \varphi_k'(\eta)\hat{b}_{\boldsymbol{k}}(\eta)), \tag{24c}$$
$$\hat{\pi}_{\boldsymbol{k}}^\dagger = \hbar(\varphi_k'(\eta)\hat{a}_{\boldsymbol{k}}(\eta) + \varphi_k'^*(\eta)\hat{b}_{\boldsymbol{k}}^\dagger(\eta)). \tag{24d}$$

In order to evaluate the two-point correlation functions we assume the most general Gaussian state, which besides being very general, implies a considerable technical simplification. This state is not necessarily pure and this impurity can mask quantum effects present in the initial Gaussian states, such as squeezing or, more generally, sub-Poissonian statistics, as will become clear in the examples.

We now define a covariance matrix $\mathbb{C}(\boldsymbol{x}, \boldsymbol{y}, \eta)$, which is a direct generalization of the one proposed in [7], as

$$\mathbb{C}(\boldsymbol{x}, \boldsymbol{y}, \eta) = \begin{pmatrix} \langle\hat{\chi}\,\hat{\chi}\rangle & \langle\hat{\chi}\hat{\chi}^\dagger\rangle & \frac{1}{2}\langle\{\hat{\chi}, \hat{\pi}\}\rangle & \frac{1}{2}\langle\{\hat{\chi}, \hat{\pi}^\dagger\}\rangle \\ \langle\hat{\chi}^\dagger\hat{\chi}\rangle & \langle\hat{\chi}^\dagger\hat{\chi}^\dagger\rangle & \frac{1}{2}\langle\{\hat{\chi}^\dagger, \hat{\pi}\}\rangle & \frac{1}{2}\langle\{\hat{\chi}^\dagger, \hat{\pi}^\dagger\}\rangle \\ \frac{1}{2}\langle\{\hat{\chi}, \hat{\pi}\}\rangle & \frac{1}{2}\langle\{\hat{\pi}, \hat{\chi}^\dagger\}\rangle & \langle\hat{\pi}\,\hat{\pi}\rangle & \langle\hat{\pi}\hat{\pi}^\dagger\rangle \\ \frac{1}{2}\langle\{\hat{\pi}^\dagger, \hat{\chi}\}\rangle & \frac{1}{2}\langle\{\hat{\pi}^\dagger, \hat{\chi}^\dagger\}\rangle & \langle\hat{\pi}^\dagger\hat{\pi}\rangle & \langle\hat{\pi}^\dagger\hat{\pi}^\dagger\rangle \end{pmatrix}, \tag{25}$$

where the dependence of the entries of the matrix on $\boldsymbol{x}$, $\boldsymbol{y}$, and η has been ignored, for the sake of clarity. The example below shows how the two-point functions are calculated. We have

$$\begin{aligned}\langle\{\hat{\chi}, \hat{\pi}\}\rangle &= \langle\{\hat{\chi}, \hat{\pi}\}\rangle(\boldsymbol{x}, \boldsymbol{y}, \eta) \\ &= \langle\hat{\chi}\,\hat{\pi}\rangle(\boldsymbol{x}, \boldsymbol{y}, \eta) + \langle\hat{\pi}\,\hat{\chi}\rangle(\boldsymbol{x}, \boldsymbol{y}, \eta) \\ &= \mathrm{Tr}(\hat{\chi}(\boldsymbol{x}, \eta)\hat{\pi}(\boldsymbol{y}, \eta)\hat{\rho}) + \mathrm{Tr}(\hat{\pi}(\boldsymbol{x}, \eta)\hat{\chi}(\boldsymbol{y}, \eta)\hat{\rho}),\end{aligned}$$

with $\hat{\rho} = \hat{\rho}(\eta)$ being the density operator for the field, at time η. The average values of the fields were assumed to vanish $\langle\hat{\chi}\rangle = 0 = \langle\hat{\pi}\rangle$ without losing generality. For fields of nonvanishing average it suffices to subtract the mean values. For example, instead of $\langle\{\hat{\chi}, \hat{\pi}\}\rangle$, we would consider $\langle\{\hat{\chi} - \langle\hat{\chi}\rangle, \hat{\pi} - \langle\hat{\pi}\rangle\}\rangle$.

In terms of the covariance matrix, we will show that the condition for saturation of Robertson-Schrödinger uncertainty relation in quantum field theory reads

$$\int d^3\boldsymbol{u}\,d^3\boldsymbol{v}\,\mathbb{C}(\boldsymbol{x}, \boldsymbol{u}, \eta)\mathbb{J}(\boldsymbol{u}, \boldsymbol{v})\mathbb{C}(\boldsymbol{v}, \boldsymbol{y}, \eta) = \frac{\hbar^2}{4}\mathbb{J}(\boldsymbol{x}, \boldsymbol{y}), \tag{26}$$

where $\mathbb{J}(\boldsymbol{x}, \boldsymbol{y}) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\tilde{\mathbb{J}}$, and

$$\tilde{\mathbb{J}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

is a simplectic matrix. For the sake of economy, the left-hand side of Eq. (26) will be referred to as $\mathbb{D}(\boldsymbol{x}, \boldsymbol{y}, \eta)$.

In what follows we consider the following Gaussian state, defined as

$$\hat{\rho}(\eta_0) = \prod_{\boldsymbol{k}} \mathcal{S}_1(r_k\phi_k)\mathcal{S}_2(r_k\phi_k)\hat{\rho}_{\nu k}^{(1)}\hat{\rho}_{\nu k}^{(2)}\mathcal{S}_2^\dagger(r_k\phi_k)\mathcal{S}_1^\dagger(r_k\phi_k), \tag{27}$$

where $\mathcal{S}_1(r_k\phi_k)$ and $\mathcal{S}_2(r_k\phi_k)$ are squeezing operators [defined in (4)], acting on the modes a and b, respectively. The mean number of excitations of the thermal state $\hat{\rho}_{\nu k}^{(1)}$, corresponding to mode a, is ν_k. The Gaussian state defined here, a squeezed thermal state, includes as particular cases the vacuum, squeezed states, and thermal states. It is worth pointing out the symmetry between particles and antiparticles, given by the symmetry of the state (27) under exchange of the modes a and b. An additional rotation symmetry can be seen: although the product is over $\boldsymbol{k}$, the squeezing parameters and the number of excitations depend only on its norm, k. For example,

$$\hat{\rho}_{\nu k}^{(1)} = \frac{1}{1 + \nu_k}\exp\left(\ln\left(\frac{\nu_k}{\nu_k + 1}\right)\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}\right).$$

The calculation of the covariance matrix is a straightforward procedure. As an illustration we presently calculate its element $\mathbb{C}_{1,2}(\boldsymbol{x}, \boldsymbol{t}, \eta)$,

$$\begin{aligned}\langle\hat{\chi}\hat{\chi}^\dagger\rangle_{\boldsymbol{xy}} &= \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{Tr}(\hat{\chi}_{\boldsymbol{q}}(\eta)\hat{\chi}_{\boldsymbol{k}}^\dagger(\eta)\hat{\rho}) \\ &= \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\langle\hat{\chi}_{\boldsymbol{q}}\hat{\chi}_{\boldsymbol{k}}^\dagger\rangle,\end{aligned}$$

where we used the Fourier decomposition of the field, given by (18). If we use the expressions (24a) and (24b) we find

$$\langle\hat{\chi}_{\boldsymbol{q}}\hat{\chi}_{\boldsymbol{k}}^\dagger\rangle = \hbar^2\langle(\varphi_q(\eta)\hat{a}_{\boldsymbol{q}}(\eta) + \varphi_q^*(\eta)\hat{b}_{\boldsymbol{q}}^\dagger(\eta))(\varphi_k^*(\eta)\hat{a}_{\boldsymbol{k}}^\dagger(\eta) + \varphi_k(\eta)\hat{b}_{\boldsymbol{k}}(\eta))\rangle. \tag{28}$$

This expression is of the form $f = f\langle(\hat{a}_{\boldsymbol{q}}, \hat{b}_{\boldsymbol{q}}^\dagger, \hat{a}_{\boldsymbol{k}}^\dagger, \hat{b}_{\boldsymbol{k}})\rangle$

$$f = \mathrm{Tr}\left(f(\hat{a}_{\boldsymbol{q}}, \hat{b}_{\boldsymbol{q}}^\dagger, \hat{a}_{\boldsymbol{k}}^\dagger, \hat{b}_{\boldsymbol{k}})\prod_{\boldsymbol{l}}\mathcal{S}_{1l}\mathcal{S}_{2l}\hat{\rho}_{\nu l}^{(1)}\hat{\rho}_{\nu l}^{(2)}\mathcal{S}_{2l}^\dagger\mathcal{S}_{1l}^\dagger\right),$$

which can be recast as

$$f = \mathrm{Tr}\Big(\prod_{l} \mathcal{S}_{2l}^{\dagger}\mathcal{S}_{1l}^{\dagger} f(\hat{a}_{\boldsymbol{q}}, \hat{b}_{\boldsymbol{q}}^{\dagger}, \hat{a}_{\boldsymbol{k}}^{\dagger}, \hat{b}_{\boldsymbol{k}})\mathcal{S}_{1l}\mathcal{S}_{2l}\hat{\rho}_{\nu l}^{(1)}\hat{\rho}_{\nu l}^{(2)}\Big)$$
$$= \mathrm{Tr}\Big(\prod_{l} f(\hat{A}_{\boldsymbol{q}}, \hat{B}_{\boldsymbol{q}}^{\dagger}, \hat{A}_{\boldsymbol{k}}^{\dagger}, \hat{B}_{\boldsymbol{k}})\hat{\rho}_{\nu l}^{(1)}\hat{\rho}_{\nu l}^{(2)}\Big),$$

where the transformed operator $\hat{a}_{\boldsymbol{q}}$ is

$$\hat{A}_{\boldsymbol{q}} = \mathcal{S}_{1q}^{\dagger}\hat{a}_{\boldsymbol{q}}\mathcal{S}_{1q} = \cosh(r_q)\hat{a}_{\boldsymbol{q}} + e^{i\phi_q}\sinh(r_q)\hat{a}_{\boldsymbol{q}}^{\dagger}$$
$$= x_q\hat{a}_{\boldsymbol{q}} - y_q^{*}\hat{a}_{\boldsymbol{q}}^{\dagger}.$$

The short notation $x_q = \cosh(r_q)$, $y_q = -e^{i\phi_q}\sinh(r_q)$ is useful to perform this calculation. Taking into account the equality above, and similar relationships which hold for the other operators $(\hat{B}_{\boldsymbol{q}}^{\dagger}, \hat{A}_{\boldsymbol{k}}^{\dagger}, \hat{B}_{\boldsymbol{k}})$, we have

$$\begin{aligned}\langle \hat{\chi}_{\boldsymbol{q}}\hat{\chi}_{\boldsymbol{k}}^{\dagger}\rangle_{\rho_G} &= \hbar^2\langle(\varphi_q(x_q\hat{a}_{\boldsymbol{q}} - y_q^{*}\hat{a}_{\boldsymbol{q}}^{\dagger}) + \varphi_q^{*}(x_q^{*}\hat{b}_{\boldsymbol{q}}^{\dagger} - y_q\hat{b}_{\boldsymbol{q}}))\\ &\quad\times(\varphi_k^{*}(x_k^{*}\hat{a}_{\boldsymbol{k}}^{\dagger} - y_k\hat{a}_{\boldsymbol{k}}) + \varphi_k(x_k\hat{b}_{\boldsymbol{k}} - y_k^{*}\hat{b}_{\boldsymbol{k}}^{\dagger}))\rangle_{\mathrm{Th}}\\ &= \hbar^2\langle\varphi_q x_q\hat{a}_{\boldsymbol{q}}\varphi_k^{*}x_k^{*}\hat{a}_{\boldsymbol{k}}^{\dagger} + \varphi_q y_q^{*}\hat{a}_{\boldsymbol{q}}^{\dagger}\varphi_k^{*}y_k\hat{a}_{\boldsymbol{k}}\rangle_{\mathrm{Th}}\\ &\quad+ \hbar^2\langle\varphi_q^{*}x_q^{*}\hat{b}_{\boldsymbol{q}}^{\dagger}\varphi_k x_k\hat{b}_{\boldsymbol{k}} + \varphi_q^{*}y_q\hat{b}_{\boldsymbol{q}}\varphi_k y_k^{*}\hat{b}_{\boldsymbol{k}}^{\dagger}\rangle_{\mathrm{Th}},\end{aligned} \tag{29}$$

with the subscript reminding us that the averages on the right-hand side are taken on the thermal density operator. Taking into account that

$$\langle\hat{a}_{\boldsymbol{q}}^{\dagger}\hat{a}_{\boldsymbol{k}}\rangle_{\mathrm{Th}} = (2\pi)^3\nu_k\delta(\boldsymbol{q}-\boldsymbol{k}) = \langle\hat{b}_{\boldsymbol{q}}^{\dagger}\hat{b}_{\boldsymbol{k}}\rangle_{\mathrm{Th}}$$

we can write

$$\begin{aligned}\langle\hat{\chi}_{\boldsymbol{q}}\hat{\chi}_{\boldsymbol{k}}^{\dagger}\rangle &= \hbar^2\delta(\boldsymbol{q}-\boldsymbol{k})\varphi_k\varphi_k^{*}(x_k x_k^{*}(n_k+1) + y_k^{*}y_k n_k\\ &\quad+ x_k^{*}x_k n_k + y_k y_k^{*}(n_k+1))\\ &= \hbar^2\delta(\boldsymbol{q}-\boldsymbol{k})|\varphi_k|^2(|x_k|^2 + |y_k|^2)(2n_k+1).\end{aligned}$$

Finally we can write the first term as

$$\begin{aligned}\langle\hat{\chi}\hat{\chi}^{\dagger}\rangle_{\boldsymbol{xy}} &= \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{d^3\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\hbar^2\\ &\quad\times\delta(\boldsymbol{q}-\boldsymbol{k})|\varphi_k|^2(|x_k|^2+|y_k|^2)(2n_k+1)\\ &= \hbar^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}|\varphi_k|^2(|x_k|^2+|y_k|^2)(2n_k+1).\end{aligned}$$

After evaluation of all of the elements of the covariance matrix we get

$$\mathbb{C}(\boldsymbol{x},\boldsymbol{y},\eta) = \hbar^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}(2n_k+1)\mathbb{C}_1(\boldsymbol{k},\boldsymbol{x},\boldsymbol{y},\eta),$$

where the new matrix $\mathbb{C}_1$ is

$$\begin{pmatrix} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_1 & e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}c_2 & e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}c_3 & e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_4\\ e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}c_2 & e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_1 & e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_4 & e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}c_3\\ e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}c_3 & e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_4 & e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_5 & e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}c_6\\ e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_4 & e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}c_3 & e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}c_6 & e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}c_5\end{pmatrix}.$$

The explicit expressions for the coefficients c_i are as follows:

$$\begin{aligned}c_1 &= -\varphi_k^2 x_k y_k^{*} - (\varphi_k^{*})^2 x_k^{*} y_k c_2 = |\varphi_k|^2(|x_k|^2 + |y_k|^2)\\ c_3 &= \frac{1}{2}(\varphi_k'\varphi_k^{*} + \varphi_k\varphi_k'^{*})(|x_k|^2 + |y_k|^2)\\ c_4 &= x_k^{*}y_k\varphi_k^{*}\varphi_k'^{*} + x_k y_k^{*}\varphi_k\varphi_k'\\ c_5 &= -\varphi_k'\varphi_k' x_k y_k^{*} - \varphi_k'^{*}\varphi_k'^{*}x_k^{*}y_k\\ c_6 &= |\varphi_k'|^2(|x_k|^2 + |y_k|^2).\end{aligned}$$

Now we are able to compute $\mathbb{D}(\boldsymbol{x},\boldsymbol{y},\eta)$, the left-hand side of (26),

$$\begin{aligned}\mathbb{D}(\boldsymbol{x},\boldsymbol{y},\eta) &= \int d^3\boldsymbol{u}d^3\boldsymbol{v}\mathbb{C}(\boldsymbol{x},\boldsymbol{u},\eta)\mathbb{J}(\boldsymbol{u},\boldsymbol{v})\mathbb{C}(\boldsymbol{v},\boldsymbol{y},\eta)\\ &= \hbar^4\int d^3\boldsymbol{u}\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{d^3\boldsymbol{q}}{(2\pi)^3}(2n_k+1)\\ &\quad\times(2n_q+1)\mathbb{C}_1(\boldsymbol{k},\boldsymbol{x},\boldsymbol{u},\eta)\tilde{\mathbb{J}}\mathbb{C}_1(\boldsymbol{q},\boldsymbol{u},\boldsymbol{y},\eta)\\ &= \hbar^4\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}(2n_k+1)^2\mathbb{D}_1(\boldsymbol{k},\boldsymbol{x},\boldsymbol{y},\eta).\end{aligned}$$

After integration over $\boldsymbol{u}$ and $\boldsymbol{q}$ matrix $\mathbb{D}_1(\boldsymbol{x},\boldsymbol{y},\eta)$ reads

$$\begin{pmatrix} 0 & 0 & -d_1e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} & d_2e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}\\ 0 & 0 & d_2e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})} & -d_1e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\ d_1e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} & -d_2e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})} & 0 & 0\\ -d_2e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})} & d_1e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} & 0 & 0\end{pmatrix},$$

with

$$\begin{aligned}d_1 &= c_3^2 + c_4^2 - c_1c_5 - c_2c_6,\\ d_2 &= c_1c_6 - 2c_3c_4 + c_2c_5.\end{aligned}$$

After the use of the explicit expressions of the coefficients c_i we obtain

$$\begin{aligned}d_1 &= \frac{1}{4}(\varphi_k'^{*}(\eta)\varphi_k(\eta) - \varphi_k'(\eta)\varphi_k^{*}(\eta))^2(|x_k|^2 - |y_k|^2)^2\\ &= \frac{1}{4}\Big(\frac{-1}{\hbar^2}\Big)\\ d_2 &= 0,\end{aligned}$$

where the Wronskian (23) and the explicit form of the coefficients x_k and y_k was used.

Making the appropriate simplifications, we find that the left-hand side of (26), evaluated at time η_0, is given by

$$\mathbb{D}(\boldsymbol{x},\boldsymbol{y},\eta) = \frac{\hbar^2}{4}\tilde{\mathbb{J}}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{z})}(2\nu_k(\eta_0)+1)^2. \tag{30}$$

The integral above is a smoothed version of a Dirac delta function. For the vacuum state (and for squeezed vacuum states) $\nu_k = 0$, we recover the delta function behavior. We have written η_0 instead of η to stress the fact that, although

η is an arbitrary instant of time, it is fixed. Dynamical considerations are left to the following section.

It is easy to find a relationship between entropy and the covariance matrix in quantum field theory. Let us remember that entropy is defined, for separable field states, as

$$\mathcal{S}\Big(\prod_{\boldsymbol{k}}\hat{\rho}_{\boldsymbol{k}}\Big) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \mathrm{Tr}(\hat{\rho}_{\boldsymbol{k}} \ln\hat{\rho}_{\boldsymbol{k}}).$$

If the field happens to be in a Gaussian state like (27) then we can write

$$\mathcal{S}\Big(\prod_{\boldsymbol{k}}\hat{\rho}_{\boldsymbol{k}}\Big) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} 2((\nu_k + 1)\ln(\nu_k + 1) - \nu_k \ln\nu_k),$$

where the factor of 2 comes from the existence of two modes, a and b for the same value of $\boldsymbol{k}$. Now, given the result (30)

$$\begin{aligned}\int d^3\boldsymbol{\xi} e^{-i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathbb{D}(z+\boldsymbol{\xi}, z, \eta) &= \int d^3\boldsymbol{\xi} e^{-i\boldsymbol{k}\cdot\boldsymbol{\xi}} \frac{\hbar^2}{4}\tilde{\mathbb{J}} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{\xi}}(2\nu_k + 1)^2 \\ &= \frac{\hbar^2}{4}\tilde{\mathbb{J}}(2\nu_k + 1)^2.\end{aligned}$$

Since $-\tilde{\mathbb{J}}$ is the inverse of $\tilde{\mathbb{J}}$, we can write

$$\nu_k = \frac{1}{2\hbar}\sqrt{-\mathrm{Tr}\Big(\int d^3\boldsymbol{\xi} e^{-i\boldsymbol{k}\cdot\boldsymbol{\xi}}\tilde{\mathbb{J}}\mathbb{D}(z+\boldsymbol{\xi}, z, \eta_0)\Big)} - \frac{1}{2}.$$

Now, entropy depends only on the numbers ν_k and hence only on the covariant matrix.

Thermal state

A particularly important example is a thermal state, with inverse temperature β. The average value of the excitations is

$$\nu_k(\eta_0) = \frac{e^{-\beta\hbar\omega_k(\eta_0)}}{1 - e^{-\beta\hbar\omega_k(\eta_0)}}, \qquad \omega_k^2(\eta_0) = \boldsymbol{k}^2 + m^2(\eta_0),$$

so that the function $\Delta(\boldsymbol{x}, z, \eta_0)$ can be written as

$$\Delta(\boldsymbol{x}, z, \eta_0) = \frac{\hbar^2}{4}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-z)}\coth^2\Big(\frac{\beta\omega_k(\eta_0)}{2}\Big).$$

For any temperature the argument of the hyperbolic cotangent function can be small, but not smaller than $\beta\hbar m/2$, or large, depending on the value of k. Hence, the asymptotic behavior of this function

$$\coth^2\Big(\frac{\beta\hbar\omega_k}{2}\Big) \to \frac{4}{\beta^2\hbar^2\omega_k^2} \quad \text{if } \beta\hbar\omega_k \ll 1, \tag{31}$$

$$\coth^2\Big(\frac{\beta\hbar\omega_k}{2}\Big) \to 1 + 4e^{-((\beta\hbar\omega_k)/2)} \quad \text{if } \beta\hbar\omega_k \gg 1, \tag{32}$$

can be used to approximate it as follows:

$$\coth^2\Big(\frac{\beta\hbar\omega_k}{2}\Big) \approx \frac{4e^{-((\beta^2\hbar^2\omega_k^2)/4)}}{\beta^2\hbar^2\omega_k^2} + 1 + 4e^{-((\beta\hbar\omega_k)/2)},$$

where the maximum fractional error is of the order of 20%. Note that the asymptotic result (31) is small for large arguments and the asymptotic result (32) is small for small arguments, so the simpler approximation [9],

$$\coth^2\Big(\frac{\beta\hbar\omega_k}{2}\Big) \approx \frac{4}{\beta^2\hbar^2\omega_k^2} + 1,$$

can be used, giving a maximum fractional error of the same magnitude (although the integrated fractional error is larger). Employing the second approximation we obtain

$$\begin{aligned}\Delta(\boldsymbol{x}, z, \eta_0) &\approx \frac{\hbar^2}{4}\delta^3(\boldsymbol{x}-z) \\ &\quad + \frac{\hbar^2}{4}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{4e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-z)}}{\beta^2\hbar^2(k^2 + m^2(\eta_0))} \\ &\approx \frac{\hbar^2}{4}\delta^3(\boldsymbol{x}-z) + \frac{e^{-m(\eta_0)|\boldsymbol{x}-z|}}{4\pi\beta^2|\boldsymbol{x}-z|},\end{aligned}$$

where the second term can be interpreted as a classical correction, because it does not depend on Planck's constant.

V. QUANTUM FIELD EVOLUTION

The calculation of the left-hand side of (26) can be extended to other times as follows. After the promotion of the fields to field operators we obtain the Hamiltonian

$$\hat{H}(\eta) = \int \frac{d^3k}{(2\pi)^3}(\hat{\pi}_{\boldsymbol{k}}^\dagger\hat{\pi}_{\boldsymbol{k}} + (\boldsymbol{k}^2 + m^2(\eta))\hat{\chi}_{\boldsymbol{k}}^\dagger\hat{\chi}_{\boldsymbol{k}}),$$

where we have defined the time-dependent mass

$$m^2(\eta) = m^2a^2(\eta) - \frac{a''(\eta)}{a(\eta)}.$$

In terms of the annihilation and creation operators

$$\begin{aligned}&\hat{\pi}_{\boldsymbol{k}}^\dagger\hat{\pi}_{\boldsymbol{k}} + (\boldsymbol{k}^2 + m^2(\eta))\hat{\chi}_{\boldsymbol{k}}^\dagger\hat{\chi}_{\boldsymbol{k}} \\ &= \hbar^2(\dot{\chi}_k^*\dot{\chi}_k + \omega_k^2\chi_k^*\chi_k)(\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}} + \hat{b}_{\boldsymbol{k}}^\dagger\hat{b}_{\boldsymbol{k}} + 1) \\ &\quad + \hbar^2(\dot{\chi}_k^*\dot{\chi}_k^* + \omega_k^2\chi_k^*\chi_k^*)\hat{a}_{\boldsymbol{k}}^\dagger\hat{b}_{\boldsymbol{k}}^\dagger \\ &\quad + \hbar^2(\dot{\chi}_k\dot{\chi}_k + \omega_k^2\chi_k\chi_k)\hat{a}_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}} \\ &= A(\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}} + \hat{b}_{\boldsymbol{k}}^\dagger\hat{b}_{\boldsymbol{k}} + 1) + B\hat{a}_{\boldsymbol{k}}^\dagger\hat{b}_{\boldsymbol{k}}^\dagger + B^*\hat{a}_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}}.\end{aligned}$$

The equations of motion for the annihilation and creation operators are

$$i\hbar\dot{\hat{a}}_{\boldsymbol{k}} = [\hat{H}(\eta), \hat{a}_{\boldsymbol{k}}] \propto -A\hat{a}_{\boldsymbol{k}} - B\hat{b}^{\dagger}_{\boldsymbol{k}}$$
$$i\hbar\dot{\hat{b}}^{\dagger}_{\boldsymbol{k}} = [\hat{H}(\eta), \hat{b}^{\dagger}_{\boldsymbol{k}}] \propto B^{*}\hat{a}_{\boldsymbol{k}} + A\hat{b}^{\dagger}_{\boldsymbol{k}}$$
$$i\hbar\dot{\hat{b}}_{\boldsymbol{k}} = [\hat{H}(\eta), \hat{b}_{\boldsymbol{k}}] \propto -A\hat{b}_{\boldsymbol{k}} - B\hat{a}^{\dagger}_{\boldsymbol{k}}$$
$$i\hbar\dot{\hat{a}}^{\dagger}_{\boldsymbol{k}} = [\hat{H}(\eta), \hat{a}^{\dagger}_{\boldsymbol{k}}] \propto B^{*}\hat{b}_{\boldsymbol{k}} + A\hat{a}^{\dagger}_{\boldsymbol{k}}.$$

Taking into account the symmetry of the equations the solutions can be written as

$$\begin{pmatrix} \hat{a}_{\boldsymbol{k}}(\eta) \\ \hat{a}^{\dagger}_{\boldsymbol{k}}(\eta) \\ \hat{b}_{\boldsymbol{k}}(\eta) \\ \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta) \end{pmatrix} = \begin{pmatrix} u_k & 0 & 0 & v_k \\ 0 & u_k^* & v_k^* & 0 \\ 0 & v_k & u_k & 0 \\ v_k^* & 0 & 0 & u_k^* \end{pmatrix} \begin{pmatrix} \hat{a}_{\boldsymbol{k}}(\eta_0) \\ \hat{a}^{\dagger}_{\boldsymbol{k}}(\eta_0) \\ \hat{b}_{\boldsymbol{k}}(\eta_0) \\ \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta_0) \end{pmatrix},$$

with time-dependent coefficients $u_k = u_k(\eta, \eta_0)$, $v_k = v_k(\eta, \eta_0)$. In order to preserve the canonical commutation relations we must have

$$|u_k|^2 - |v_k|^2 = 1.$$

Notice that this evolution corresponds to a time-dependent two-mode squeezing, defined in (11),

$$\hat{a}_{\boldsymbol{k}}(\eta) = \mathcal{S}^{\dagger}_{12,k}(z(\eta, \eta_0))\hat{a}_{\boldsymbol{k}}(\eta_0)\mathcal{S}_{12,k}(z(\eta, \eta_0)).$$

Physically, we have a mechanism of production of correlated particles and antiparticles.

The calculation of the covariance matrix (25) proceeds as in the previous section. However, the two-point functions are now understood as

$$\langle \hat{\chi}\,\hat{\chi} \rangle = \langle \hat{\chi}\,\hat{\chi} \rangle(\boldsymbol{x}, \boldsymbol{y}, \eta) = \mathrm{Tr}(\hat{\chi}(\boldsymbol{x}, \eta)\hat{\chi}(\boldsymbol{y}, \eta)\hat{\rho}(\eta_0)).$$

The fields $\hat{\chi}(\boldsymbol{x}, \eta)$, $\hat{\chi}^{\dagger}(\boldsymbol{x}, \eta)$, etc., are functions of the creation and annihilation operators $\hat{a}_{\boldsymbol{k}}(\eta)$, $\hat{b}^{\dagger}_{\boldsymbol{k}}(\eta)$, etc. To highlight this fact we write $\hat{\chi}(\hat{a}_{\boldsymbol{k}}(\eta), \ldots, \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta))$. In the two-point function used above as an example, we would write

$$\hat{\chi}\,\hat{\chi} = \mathrm{Tr}(\hat{\chi}(\hat{a}_{\boldsymbol{k}}(\eta), \ldots, \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta))\hat{\chi}(\hat{a}_{\boldsymbol{k}}(\eta), \ldots, \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta))\hat{\rho}(\eta_0))$$
$$= \mathrm{Tr}(f(\hat{a}_{\boldsymbol{k}}(\eta), \ldots, \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta))\hat{\rho}(\eta_0))$$

$$= \mathrm{Tr}(\mathcal{S}^{\dagger}_{12} f(\hat{a}_{\boldsymbol{k}}(\eta_0), \ldots, \hat{b}^{\dagger}_{\boldsymbol{k}}(\eta_0))\mathcal{S}_{12}\hat{\rho}(\eta_0)).$$

We see that the effect of evolution is an additional transformation on the operators. For example, the calculation of the element $\mathbb{C}_{12}$ of the previous section remains essentially unaltered. For example, (29), now reads

$$\langle \hat{\chi}_{\boldsymbol{q}}\hat{\chi}^{\dagger}_{\boldsymbol{k}} \rangle_{\rho_G} = \hbar^2 \langle \mathcal{S}^{\dagger}_{12}(\varphi_q(x_q\hat{a}_{\boldsymbol{q}} - y_q^*\hat{a}^{\dagger}_{\boldsymbol{q}}) + \varphi_q^*(x_q^*\hat{b}^{\dagger}_{\boldsymbol{q}} - y_q\hat{b}_{\boldsymbol{q}}))(\varphi_k^*(x_k^*\hat{a}^{\dagger}_{\boldsymbol{k}} - y_k\hat{a}_{\boldsymbol{k}}) + \varphi_k(x_k\hat{b}_{\boldsymbol{k}} - y_k^*\hat{b}^{\dagger}_{\boldsymbol{k}}))\mathcal{S}_{12} \rangle_{\mathrm{Th}}.$$

Proceeding with the evaluation of $\mathbb{C}_{12}$ we finally obtain

$$\langle \hat{\chi}\hat{\chi}^{\dagger} \rangle_{\boldsymbol{xy}} = \hbar^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} c_2(k),$$

where the coefficient $c_2(k)$ is slightly more complicated,

$$(2n_k + 1)(|x_k|^2 + |y_k|^2)(\varphi^* u_k^* + \varphi v_k)(\varphi u_k + \varphi^* v_k^*).$$

After obtaining all of the elements of the covariance matrix we follow the procedure sketched in the previous section. We finally obtain that the left-hand side of (26), evaluated at time η, is given by $\hbar^2\Delta(\boldsymbol{x}, \boldsymbol{z}, \eta_0)\tilde{\mathbb{J}}/4$, that is, it is conserved in the course of the evolution.

VI. CONCLUSIONS

We established a quantum field theoretical counterpart of well-known quantum mechanical measures of information. We consider 1 and 2 degrees of freedom and establish the connection between entropy and the covariance matrix. We generalize this important concept (covariance matrix) to a quantum field theoretical scenario of nonautonomous fields in curved space-time. We work with symmetric complex fields, showing that in this case it is also possible to implement a relationship between the covariance matrix and the state entropy. We show that unitary (even nonautonomous) evolutions are isoentropic, a result also obtained in [10]. This does not mean, however, that the dynamics does not produce quantum correlations. In the case studied we show that particle-antiparticle correlations are dynamically generated and, provided environmental effects are weak enough, such correlations remain as a hallmark of the inflationary period.

There are some scenarios in which it would be natural to use the tools introduced here, such as variational Gaussian approximations (however, the evolution will be always isoentropic), proposals of a decoherence mechanism, by interaction with other fields or by lack of information of correlation functions higher than the second order. Systematic approximations beyond Gaussian states have been derived [11,12], but their technical complexities prevent current use. Thus it would be highly desirable to find suitable approximation schemes in which the decoherence mechanisms, and their consequences, become more transparent. We hope our results to be useful as a first step in that direction, because they will be of use for interacting fields.

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