

**Exotic singularities and spatially curved loop quantum cosmology**Parampreet Singh<sup>1,2,\*</sup> and Francesca Vidotto<sup>3,4,†</sup><sup>1</sup>*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, USA*<sup>2</sup>*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*<sup>3</sup>*Centre de Physique Théorique de Luminy, Case 907, F-13288 Marseille, France*<sup>4</sup>*Dipartimento di Fisica Nucleare e Teorica, Università degli Studi di Pavia, and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via A. Bassi 6, 27100 Pavia, Italy*

(Received 20 December 2010; published 17 March 2011)

We investigate the occurrence of various exotic spacelike singularities in the past and the future evolution of  $k = \pm 1$  Friedmann-Robertson-Walker model and loop quantum cosmology using a sufficiently general phenomenological model for the equation of state. We highlight the nontrivial role played by the intrinsic curvature for these singularities and the new physics which emerges at the Planck scale. We show that quantum gravity effects generically resolve all strong curvature singularities including big rip and big freeze singularities. The weak singularities, which include sudden and big brake singularities, are ignored by quantum gravity when spatial curvature is negative, as was previously found for the spatially flat model. Interestingly, for the spatially closed model there exist cases where weak singularities may be resolved when they occur in the past evolution. The spatially closed model exhibits another novel feature. For a particular class of equation of state, this model also exhibits an additional physical branch in loop quantum cosmology, a baby universe separated from the parent branch. Our analysis generalizes previous results obtained on the resolution of strong curvature singularities in flat models to isotropic spacetimes with nonzero spatial curvature.

DOI: [10.1103/PhysRevD.83.064027](https://doi.org/10.1103/PhysRevD.83.064027)

PACS numbers: 04.60.Pp, 04.20.Dw, 04.60.Kz

**I. INTRODUCTION**

A fundamental limitation of the cosmological models based on general relativity (GR) is the occurrence of singularities. Perhaps one of the simplest examples is the case of an expanding homogeneous and isotropic universe filled with a matter satisfying strong energy condition such as dust or radiation. Independent of the intrinsic geometry of the universe, be it closed, flat or open, the past evolution of such a universe from arbitrary initial conditions leads to an initial singularity: the big bang, where the classical dynamical equations break down and the physics stops. Another example is the case of an inflationary universe in which even though the evolution is almost de Sitter, the classical spacetime is past incomplete [1].

In recent years, various new singularities have been found in classical cosmology [2–9]. Unlike the big bang and big crunch singularities, these singularities do not occur when the scale factor vanishes. These occur either at finite values of the scale factor or when it diverges. Recall that for the matter satisfying weak energy condition, the latter is not possible as the spacetime curvature goes to zero when the scale factor goes to infinity. However, if matter violates the weak energy condition, for example, in the case of phantom fields, then spacetime curvature will diverge as the scale factor becomes infinite. For homogeneous and isotropic models with matter equation of state in the form of a perfect fluid these exotic singularities come in

four types. Following Refs. [6,7] these are classified as: Big rip (type I) where the energy density and pressure diverge along with a divergence in the scale factor, sudden singularity (type II) occurring at a finite value of the scale factor and energy density with a divergence in pressure, big freeze (type III) where energy density and pressure diverge at a finite value of the scale factor, and big brake (type IV) singularity where scale factor, energy density and pressure are finite but there is a divergence in the time derivative of the pressure or rate of change of energy density.<sup>1</sup>

Lack of successful resolution of these singularities renders classical cosmological models incomplete. It is generally believed that existence of these singularities is a result of assuming the validity of GR even in the regime of large spacetime curvature where the effects due to quantum gravity may become important and lead to significant departures from the classical theory. It is thus hoped that incorporation of quantum gravitational effects may result in a possible resolution of these singularities.

Loop quantum gravity (LQG) is one of the candidate theories of quantum gravity which attempts to address this issue. It is a nonperturbative and background independent quantization, with a key prediction that the continuum differential geometry of the classical theory is replaced by a

<sup>1</sup>In certain modified gravity scenarios, it is also possible to obtain solutions with a type V singularity where equation of state matter diverges even though pressure and energy density remain finite (see for e.g. [8]). To our knowledge such a solution is yet to be found in loop quantum cosmology and will not be considered in the present analysis.

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discrete quantum geometry in the quantum theory. Perhaps one of the best illustrations of the novel effects of quantum geometry is captured in loop quantum cosmology (LQC) which is a quantization of homogeneous spacetimes based on LQG [10–12]. A key prediction of LQC is that the big bang singularity is replaced by a big bounce, which is a direct consequence of the underlying quantum geometry [13–16]. These results which were first obtained for homogeneous and isotropic models (for all values of spatial curvature) with a massless scalar field have been extended to inflationary potential [17], anisotropic spacetimes [18,19], and also certain inhomogeneous situations [20]. Further, using an exactly solvable model it has been shown that the expectation values of energy density have a universal upper bound for a dense subspace in the physical Hilbert space [21]. There are strong constraints on the change in relative fluctuations of quantum observables across the bounce [22]. Recently, much stronger constraints on the change in dispersions have been obtained by Kaminski and Pawłowski [23]. These results show that a universe like ours, i.e. macroscopic at late times, bounces from a similar universe at very early times (in the contracting branch,) and the universe recalls almost of all its state through the bounce.

Interestingly, the loop quantum dynamics admits an effective description on a continuum spacetime which can be obtained using coherent state techniques [24,25]. An important feature of this analysis is that one can obtain an effective Hamiltonian from which one can obtain modified Friedmann and Raychaudhuri equations as the Hamilton's equations. The modified set of dynamical equations inherit quantum geometric effects via higher order nonperturbative corrections which vanish at small spacetime curvatures. It is important to note that various numerical simulations have shown that effective equations capture the underlying quantum evolution very accurately for universes which become macroscopic at late times. These thus prove to be useful tools to understand the physics in LQC, such as whether the underlying theory has well-defined ultraviolet and infrared limits. It turns out that even though there exist various quantization ambiguities, there is a unique quantization leading to a consistent unambiguous physical description [26,27] (the improved dynamics [15,16]: which is being considered here).

Using effective equations, we can ask various questions regarding the generality of singularity resolution in LQC. For example, one can ask whether spacetime curvature is always bounded in LQC? Here we should note that a universal bound on energy density (as in LQC), does not imply that the spacetime curvature is also bounded. This is easy to understand for the classical cosmology, where the Ricci scalar, which provides us with complete information about the spacetime curvature in the homogeneous and isotropic spacetime, depends both on the energy density and pressure. Though for most matter-energy configurations, the behavior of equation of state is such that an upper

bound in energy density is sufficient to control the divergence in pressure and hence the spacetime curvature, it is not difficult to come up with counter examples with a more general equation of state [28,29]. Hence, an upper bound in energy density is not sufficient to prevent a divergence in the spacetime curvature.

A pertinent question is whether this divergence signals the end of spacetime in LQC. In order to answer this question, we recall that even in GR we encounter events where spacetime curvature blows up but there is no associated physical singularity. This can happen if the tidal forces are not strong enough to cause a complete destruction of infalling objects into the singularity and geodesics can be extended beyond such events. It turns out that the events where spacetime curvature diverges in flat isotropic LQC are weak singularities and geodesics can be extended beyond them. In flat isotropic LQC, the divergence of spacetime curvature occurs only for sudden singularities which are caused by a divergence in pressure at a finite scale factor and energy density. It is straightforward to show that the expansion parameter in this case is bounded and the spacetime is geodesically complete in the flat isotropic LQC [29].

In this paper we take the first step to generalize the above result by including intrinsic curvature in the spacetime. This is done by considering the effective dynamics of loop quantized spatially closed and open models in the Robertson-Walker geometry. In the classical Friedmann dynamics, the intrinsic curvature term enters in the form of a  $1/a^2$  term in the dynamical equations. Thus it is quite straightforward to understand the expected modifications from the results in the spatially flat model. In LQC, the quantization of intrinsic curvature brings nontrivial modifications to the effective description and makes the resulting form of effective dynamical equations less straightforward to analyze. Though one expects that at small intrinsic curvatures one recovers the results of flat isotropic LQC, physics may bring up surprises when intrinsic curvature is large. As we will show, this is indeed what happens in the case of  $k = \pm 1$  models in LQC.

Our analysis will be based on considering a sufficiently general phenomenological model for the equation of state which was proposed in Ref. [7]. This model allows a study of all of the exotic singularities by the choice of different parameters and was earlier used for investigation of resolution of strong curvature singularities in the flat isotropic LQC [29]. As we will see, the effective dynamical equations of spatially curved models approximate those of the flat model in LQC at large volumes because the contribution from intrinsic curvature becomes negligible in this limit. Thus for future singularities, at large volumes, the resulting physics is similar for models with or without spatial curvature in LQC. However for certain values of parameters the spatially closed model in LQC permits two separate physical branches, a short-lived baby universe at small volume and a parent universe which evolves to a

macroscopic size. This branch is absent in the classical theory and has a pure quantum geometric origin.

To completely capture the new physics from inclusion of intrinsic curvature, it is important to study exotic singularities in the past evolution when they occur at small volumes. Our analysis of past and future exotic singularities shows that all strong curvature singularities are resolved in  $k = \pm 1$  isotropic LQC. The scale factors at which big rip and big freeze singularities occur are excluded from the allowed range by loop quantum effects. As in the flat model, the spacetime curvature can diverge in spatially curved LQC, however whenever this happens one has a weak singularity which is known to be harmless. In almost all cases these singularities are ignored by LQC. The only exception to this occurs in spatially closed model where weak singularities occurring in the past evolution may be resolved. This occurs purely because of the non-trivial role of intrinsic curvature effects in LQC.

We organize our paper as follows: In Sec. II we revisit the classical equations for a spatially curved model in classical cosmology and introduce the phenomenological ansatz of the equation of state. To facilitate the reader in following the derivation of effective equations in LQC, we derive the Friedmann and Raychaudhuri equations in the Hamiltonian framework. In Sec. III, we derive the effective equations for spatially curved LQC starting from the effective Hamiltonian [16,30,31]. Using these modified equations, we numerically obtain solutions and discuss the new physics in Sec. IV. Here we show that all exotic strong curvature singularities, irrespective of whether they occur in the past or the future, are resolved in spatially curved LQC. We summarize our results in Sec. V.

## II. CLASSICAL THEORY

### A. Hamiltonian cosmology

The fundamental equations of cosmology describe the evolution of the scale factor  $a$  in proper time  $t$ . These equations can be derived in a simple Hamiltonian framework. To this purpose, consider the conjugate variables  $(c, p)$  where  $c$  is the Ashtekar-Barbero connection and  $p$  is the triad (which without any loss of generality will be chosen with positive orientation). These are related to the metric variables as

$$c = \gamma \dot{a} + k \quad p = a^2, \quad (1)$$

with the relation between  $c$  and  $\dot{a}$  valid only in the classical theory and it modifies when quantum gravity effects are considered. Here  $\dot{a} = da/dt$ ,  $k = 0, \pm 1$  is the normalized intrinsic curvature and  $\gamma \in \mathbb{R}$  is the Barbero-Immirzi parameter.<sup>2</sup> The conjugate variables satisfy the following Poisson bracket

<sup>2</sup>The value of  $\gamma$  can be fixed by computing the black hole entropy in LQG. In our numerical analysis we set  $\gamma \approx 0.2375$  [32].

$$\{c, p\} = \frac{8\pi G}{3} \gamma. \quad (2)$$

In these variables, the Hamiltonian for gravity in a homogeneous and isotropic spacetime becomes

$$\mathcal{H}_g = -\frac{3}{8\pi G} \gamma^{-2} \sqrt{p} [(c-k)^2 + k\gamma^2]. \quad (3)$$

We introduce a generic matter field with Hamiltonian  $\mathcal{H}_m = p^{3/2} \rho$ , where  $\rho$  is the matter-energy density. The total Hamiltonian of the system  $\mathcal{H} = \mathcal{H}_g + \mathcal{H}_m$  is constrained to vanish.

The dynamics is given by the Hamilton's equation

$$\dot{p} = \{p, \mathcal{H}\} = -\{c, p\} \frac{\partial \mathcal{H}}{\partial c} = 2\gamma^{-1} \sqrt{p} (c-k). \quad (4)$$

Since the Hamiltonian is a constraint, namely  $\mathcal{H} \approx 0$ , we find that

$$(c-k)^2 = \frac{8\pi G}{3} \gamma^2 p \rho - k\gamma^2. \quad (5)$$

We can now recover the usual Friedmann equation as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{p}}{2p}\right)^2 = \left(\frac{c-k}{\gamma\sqrt{p}}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{p}. \quad (6)$$

In order to obtain the Raychaudhuri equation for the acceleration, we compute the equation of motion for  $c$

$$\begin{aligned} \dot{c} &= \{c, \mathcal{H}\} = \{c, p\} \frac{\partial \mathcal{H}}{\partial p} \\ &= -\frac{(c-k)^2 + k\gamma^2}{2\gamma\sqrt{p}} + \frac{8\pi G\gamma}{3} \frac{\partial \mathcal{H}_m}{\partial p} \\ &= -\frac{4\pi G\gamma}{3} \sqrt{p} (\rho + 3P) \end{aligned} \quad (7)$$

where we have introduced the thermodynamic pressure  $P$  as the derivative of  $\mathcal{H}_m$  with respect to the volume. This gives  $\partial \mathcal{H}_m / \partial p = -\frac{3}{2} \sqrt{p} P$ . From (4) and (6) we compute

$$\ddot{p} = \frac{16\pi G}{3} p \rho - 2k + \frac{2}{\gamma} \sqrt{p} \dot{c} \quad (8)$$

and we obtain<sup>3</sup>

$$\dot{H} = \frac{\ddot{p}}{2p} = \frac{4\pi G}{3} (\rho - 3P) - \frac{k}{p} \quad (9)$$

which yields the Raychaudhuri equation

<sup>3</sup>Equivalently, one can obtain  $\dot{H}$  using  $2H\dot{H} = \frac{8\pi G}{3} \dot{\rho} + 2H\frac{\dot{k}}{p}$  and  $\dot{\rho} = -3H(\rho + P)$ . The conservation equation will hold also in the case with quantum gravity modifications, since it results from the Hamilton's equation for the matter part, without involving the gravitational part.

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4}{3}\pi G(\rho + 3P). \quad (10)$$

Note that a divergence in  $\rho$  and/or  $P$  can lead to a divergence in the Ricci scalar

$$R = 6\left(\dot{H} + 2H^2 + \frac{k}{p}\right). \quad (11)$$

Analogously, a divergence in  $\dot{P}$ , the derivative of the pressure, can lead to a divergence in the derivative of the curvature

$$\dot{R} = 6\left(\ddot{H} + 4H\dot{H} - 2H\frac{k}{p}\right). \quad (12)$$

Divergences can similarly be computed for the higher derivatives.

### B. Phenomenological model

Different types of singularities are categorized depending upon whether the divergences appear in the scale factor  $a$ , in the energy density  $\rho$ , in the pressure  $P$  or in its derivative  $\dot{P}$ . For our investigation we choose a specific expression for the pressure

$$P = -\rho - \frac{AB\rho^{2\alpha-1}}{A\rho^{\alpha-1} + B} \quad (13)$$

that allows to obtain various singularities by varying the parameters of the model  $A$ ,  $B$  and  $\alpha$  [7]. The derivative of the pressure is given by the expression

$$\dot{P} = 3H(\rho + P)\left[1 + \frac{(2\alpha - 1)AB\rho^{2\alpha-2}}{A\rho^{\alpha-1} + B} + \frac{(1 - \alpha)A^2B\rho^{3\alpha-3}}{(A\rho^{\alpha-1} + B)^2}\right]. \quad (14)$$

The equations of motion of the model can be integrated and the scale factor can be expressed as a function of the energy density  $\rho$  as

$$a = a_o \exp\left(\frac{(2A + B\rho^2)\rho^{1-\alpha}}{6AB(1 - \alpha)}\right). \quad (15)$$

Here  $a_o$  is an integration constant, and for the singularities at finite scale factor, it corresponds to the value where the singularity appears. Inverting this expression we obtain

$$\rho = \left(-\frac{A}{B} \pm \sqrt{\frac{A^2}{B^2} - 6(\alpha - 1)A \ln\left(\frac{a}{a_o}\right)}\right)^{1/(1-\alpha)}. \quad (16)$$

The following table summarizes the relation between the parameters of the model and the quantities that diverge, while the other quantities remain finite.

Singularity	Divergences	Parameters
Type I	$a \rightarrow \infty, \rho \rightarrow \infty, P \rightarrow \infty$	$3/4 < \alpha < 1 \forall A, B$
Type II	$P \rightarrow \infty$	$\alpha < 0$ or $A/B > 1$
Type III	$\rho \rightarrow \infty, P \rightarrow \infty$	$\alpha > 1 \forall A, B$
Type IV	$\dot{P} \rightarrow \infty$	$0 < \alpha < 1/2 \forall A, B$

In the classical theory, singularities can appear in the past or in the future, depending on the choice of the parameters  $A$  and  $B$ . If  $A$  and  $B$  have different sign, they always give rise to a sudden singularity. The other singularities depend on the value of  $\alpha$ , irrespectively on the value of  $A$  and  $B$ . No exotic singularity appears for  $1/2 < \alpha < 3/4$ .

In Sec. IV we study these singularities, in particular, by showing the behavior of the Hubble rate, the Ricci curvature, and its derivative, since these quantities are closely related to the energy density, the pressure, and its derivative.

### III. EFFECTIVE DYNAMICS IN LQC

The underlying dynamics in LQC captures the quantum discreteness of LQG and is dictated by a quantum difference equation with a uniform step size in the volume representation. However, using suitably chosen coherent states, one can obtain an effective *continuum* spacetime description which leads to a modified Friedmann (differential) dynamics capturing the underlying quantum geometric effects. Such an analysis has been carried out for different types of matter and an effective Hamiltonian has been obtained using the so-called embedding approach [24,25]. In this approach, the Hilbert space is treated as a quantum phase space on which the quantum dynamics generates a Hamiltonian flow. The quantum phase space has a fiber bundle structure with the base space as the classical phase space. One can then obtain an embedding of the classical phase space in to the quantum phase space by finding an approximately horizontal section preserved under the time evolution. This procedure can be performed under controlled approximations which results in an effective Hamiltonian, whose solutions can then be compared with the expectation values of operators in the quantum theory.

Extensive numerical simulations show that the effective dynamics is an excellent approximation to full quantum dynamics for states which correspond to a macroscopic universe at late times [15–17] (where macroscopic here refers to large compared to the Planck size). As an example of the efficiency of the effective dynamics, let us consider a closed universe sourced with a massless scalar field having  $p_\phi \approx 5000\hbar$ , resulting in a maximum value of the scale factor as approximately 25 times the Planck length. Even for such a universe, effective Friedmann equations reproduce the underlying quantum dynamics and the bounce at

approximately  $15\ell_{\text{Pl}}$  to an extraordinary accuracy [16]. Further, the universe undergoes a very large number of cycles of expansion and contraction without losing coherence and the quantum state remaining peaked on the effective dynamical trajectory with relative fluctuations remaining small. For more general matter, it turns out that until one is probing scales larger than the Planck length, even though at Planck curvature, effective Friedmann equations as derived in the embedding approach are very reliable to study singularity resolution [24,25]. The effective Hamiltonian we will be working with will be under this approximation and it be valid up to terms of the order of square of relative fluctuations (which are very small).

The nonlocal quantum gravitational effects originating from the underlying quantum geometry primarily modify the gravitational part of the Hamiltonian. The matter part of the Hamiltonian constraint remains unaffected. (This is consistent with our underlying approximation as discussed later in a remark at the end of this section). The effective Hamiltonian can be written as [15,16,24,25,30,31]:

$$\mathcal{H}_{\text{eff}} := \frac{A(v)}{16\pi G} [\sin^2(\bar{\mu}(c-k)) - k\chi] + \mathcal{H}_M \quad (17)$$

where

$$\bar{\mu}^2 p = 4\sqrt{3}\pi\gamma\ell_{\text{Pl}}^2 =: \Delta \quad (18)$$

and  $\chi$  has different expressions for  $k=1$  and  $k=-1$  geometries. For  $k=1$  it is given by [16]

$$\chi := \sin^2\bar{\mu} - (1 + \gamma^2)\bar{\mu}^2 \quad (19)$$

and for  $k=-1$  it becomes [30,31]

$$\chi := -\gamma^2\bar{\mu}^2. \quad (20)$$

In above equations,  $\Delta$  denotes the minimum eigenvalue of the area operator in LQG and  $v$  denotes the eigenvalues of the volume operator  $\hat{V} = \hat{p}^{3/2}$  in LQC.<sup>4</sup> Without any loss of generalization we restrict ourselves to the positive eigenvalues of the volume operator.

$$V = p^{3/2} = a^3 = \left(\frac{8\pi\gamma}{6}\right)^{3/2} \frac{v}{K} \quad (21)$$

with  $K = 2/3\sqrt{3}\sqrt{3}$ .

For  $v > 1$ , the expression for  $A(v)$  yields [15]

$$A(v) = -\frac{27K\ell_{\text{Pl}}}{2\gamma^{3/2}} \sqrt{\frac{8\pi}{6}} |v| = -\frac{6p^{1/2}}{\bar{\mu}^2\gamma^2} \quad (22)$$

where in the last step we have used Eq. (21). Thus, the effective Hamiltonian becomes

<sup>4</sup>Note that the value of  $K$  is different from various previous works. This is due to change in the expression for the area gap in LQC [19]. In comparison to Ref. [15], we have  $\bar{\mu}^2 p^2 = 4\sqrt{3}\pi\gamma\ell_{\text{Pl}}^2$ .

$$\mathcal{H}_{\text{eff}} = -\frac{3}{8\pi G\gamma^2} \frac{\sqrt{p}}{\bar{\mu}^2} (\sin^2(\bar{\mu}(c-k)) - k\chi) + \mathcal{H}_M. \quad (23)$$

The vanishing of the Hamiltonian constraint,  $\mathcal{H}_{\text{eff}} \approx 0$ , leads to

$$\sin^2(\bar{\mu}(c-k)) = \frac{8\pi G}{3} \frac{\gamma^2 \bar{\mu}^2}{\sqrt{p}} \mathcal{H}_M + k\chi = \frac{\rho}{\rho_{\text{crit}}} + k\chi \quad (24)$$

where we have defined the critical energy density  $\rho_{\text{crit}}$  as

$$\rho_{\text{crit}} = \frac{3}{8\pi G\gamma^2\Delta}. \quad (25)$$

The modified Friedmann equation can be obtained from the effective Hamiltonian (23), by computing the Hamilton's equation for  $p$ :

$$\dot{p} = \{p, \mathcal{H}_{\text{eff}}\} = \frac{2}{\gamma\bar{\mu}} \sqrt{p} \sin(\bar{\mu}(c-k)) \cos(\bar{\mu}(c-k)). \quad (26)$$

The Hubble rate  $H^2 = \dot{a}^2/a^2$ , then becomes

$$\begin{aligned} H^2 &= \left(\frac{\dot{p}}{2p}\right)^2 = \frac{1}{\gamma^2\Delta} \sin^2(\bar{\mu}(c-k))(1 - \sin^2(\bar{\mu}(c-k))) \\ &= \left(\frac{8}{3}\pi G\rho + \frac{k\chi}{\gamma^2\Delta}\right) \left(1 - \frac{\rho}{\rho_{\text{crit}}} - k\chi\right). \end{aligned} \quad (27)$$

This equation can be rewritten as

$$H^2 = \frac{8\pi G}{3} (\rho - \rho_1) \left(\frac{1}{\rho_{\text{crit}}} (\rho_2 - \rho)\right) \quad (28)$$

where  $\rho_1$  and  $\rho_2$  are defined as

$$\rho_1 := \frac{-3k\chi}{8\pi G\gamma^2\Delta} = -k\chi\rho_{\text{crit}} \quad (29)$$

and

$$\rho_2 := \rho_{\text{crit}}(1 - k\chi). \quad (30)$$

In the classical limit,  $\Delta \rightarrow 0$ , we obtain

$$\begin{aligned} \chi &\rightarrow -\gamma^2\bar{\mu}^2, \\ \rho_1 &\rightarrow \frac{3k}{8\pi Gp} \quad \text{and} \quad \frac{1}{\rho_{\text{crit}}} (\rho_2 - \rho) \rightarrow 1. \end{aligned} \quad (31)$$

Thus we recover back the classical Friedmann equation in the limit  $\Delta \rightarrow 0$ .

Similarly, using the Hamilton's equation for  $c$ , we can obtain the modified Raychaudhuri equation:

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} (\rho + 3P) + \frac{16\pi G}{3} \left(\frac{\rho}{\rho_{\text{crit}}} + k\chi\right) \left(\rho + \frac{3}{2}P\right) \\ &\quad + \frac{k\chi}{\gamma^2\Delta} \left(\frac{\rho}{\rho_{\text{crit}}} + k\chi\right) - \frac{2\zeta k}{\gamma^2\Delta} \left(\frac{\rho}{\rho_{\text{crit}}} + k\chi - \frac{1}{2}\right) \end{aligned} \quad (32)$$

where

$$\zeta := \sin^2(\bar{\mu}) - \bar{\mu} \sin(\bar{\mu}) \cos(\bar{\mu}). \quad (33)$$

Using Eqs. (32) and (27) we obtain the equation for the rate of change of the Hubble rate

$$\dot{H} = \left( -4\pi G(\rho + P) + \frac{k(\zeta - \chi)}{\gamma^2 \Delta} \right) \left( 1 - 2 \left( \frac{\rho}{\rho_{\text{crit}}} + k\chi \right) \right). \quad (34)$$

It is then straightforward to verify that the conservation law

$$\dot{\rho} + 3H(\rho + P) = 0 \quad (35)$$

remains unchanged in the effective dynamics of LQC.

We can now write down the expression for the Ricci scalar which captures the complete behavior of the space-time curvature for the homogeneous and isotropic model:

$$\begin{aligned} R &= 6 \left( H^2 + \frac{\ddot{a}}{a} + \frac{k}{a^2} \right) \\ &= 6 \left[ \frac{4\pi G}{3} (\rho - 3P) + \frac{8\pi G}{3} \left( \frac{\rho}{\rho_{\text{crit}}} + k\chi \right) (\rho + 3P) \right. \\ &\quad \left. + \frac{k\chi}{\gamma^2 \Delta} - \frac{2\zeta k}{\gamma^2 \Delta} \left( \frac{\rho}{\rho_{\text{crit}}} + k\chi - \frac{1}{2} \right) + \frac{k}{a^2} \right]. \end{aligned} \quad (36)$$

Further, as for the case of modified Friedmann equation, the modified Raychaudhuri (32),  $\dot{H}$  (34) equations and Ricci scalar go to the classical GR versions in the limit  $\Delta \rightarrow 0$ .

It is important to point out a notable difference between the Ricci scalar in the classical theory and LQC. In the classical theory, the expression for  $R$  is *independent* of the curvature index  $k$

$$R = 8\pi G(\rho - 3P). \quad (37)$$

Whereas in LQC, the Ricci scalar is *dependent* on the curvature index. The expression for the Ricci scalar is *different* for flat, open, and closed models in LQC. We will see in the next section that it leads to interesting distinctions for singularity resolution for different spatial geometries in LQC.

We conclude this section with following remarks:

*Remark 1:* In the derivation of effective equations we have worked under the approximation that contributions from the inverse scale factor effects are negligible. The approximation is well motivated due to two reasons. First, from insights gained from various numerical simulations which show that in LQC, modifications originating from the nonlocal field strength of Ashtekar-Barbero connection, which results in the trigonometric function in Eq. (17), significantly overwhelm the modifications originating from inverse scale factor [15,16,30]. Second, for  $\nu > 1$  inverse scale factor effects are in any case negligible (in the fundamental representation of the theory).<sup>5</sup> This

<sup>5</sup>The situation changes when one is considering higher representations where inverse scale factor effects can lead to novel phenomenological effects (see for e.g. Refs. [33–35]). See, however Ref. [36] for subtleties in dealing with higher representations.

corresponds to  $a > 1.5\ell_{\text{pl}}$  from (21). In our analysis we will only consider scale factors large than this value. Thus in our analysis it will be safe to make this approximation. We expect the results to hold true in general as inverse scale factor effects tend to weaken the strength of gravity (or make it effectively more “repulsive”), thus aiding singularity resolution.

*Remark 2:* It is to be emphasized that the effective Hamiltonian (17) is *not* obtained from the classical Hamiltonian (3) by the so-called “polymerization” which amounts to  $c \rightarrow \sin(\lambda c)/\lambda$ . It can be easily seen that such a “polymerization,” which is sometimes discussed in the literature as a shortcut to write an effective Hamiltonian in loop quantization, does not lead to (17) starting from (3). The effective Hamiltonian for the closed model, in particular, shows that the effective Hamiltonian obtained from the quantum constraint can be much richer than as expected from a naive polymerization of the connection. Further, it turns out that an effective Hamiltonian obtained from polymerization of connection for the closed model has serious problems with singularity resolution [37].

#### IV. PHENOMENOLOGICAL MODEL: NUMERICAL RESULTS

The phenomenological model introduced in Sec. II B allows us to study different kinds of singularities by choosing appropriate values for various parameters in Eq. (13). If  $A$  and  $B$  have a different sign, then there is always a sudden singularity independently of the value of  $\alpha$ . If  $A$  and  $B$  have the same sign, then we obtain the following classification:

(i) Type I singularity (Big Rip):  $A > 0$  and  $3/4 < \alpha < 1$ .

Type II singularity (Sudden):  $\alpha < 0$

(ii) Type III singularity (Big Freeze):  $\alpha > 1$

Type IV singularity (Big Brake):  $0 < \alpha < 1/2$

If  $\alpha = 0$  or  $3/4 < \alpha < 1$ , there are no type I-IV singularities. Further, apart from the type I singularity which occurs with an associated divergence in the scale factor, type II-IV singularities can occur both in the past or the future of an expanding branch in the classical FRW model.

Using the effective loop quantum dynamics as elaborated in the previous section, we carried out extensive numerical simulations for various parameters with the phenomenological equation of state (13). Below we discuss various exotic singularities and show the representative results for different cases. (We use Planck units to show results in various plots.)

##### A. Type I singularity: The big rip

A type I singularity is also called “Big Rip” because there exists a finite time in which the scale factor, energy density, and pressure diverge, tearing apart the universe. The dominant energy condition is broken and the equation of state converges to  $w = -1$  when approaching the singularity.

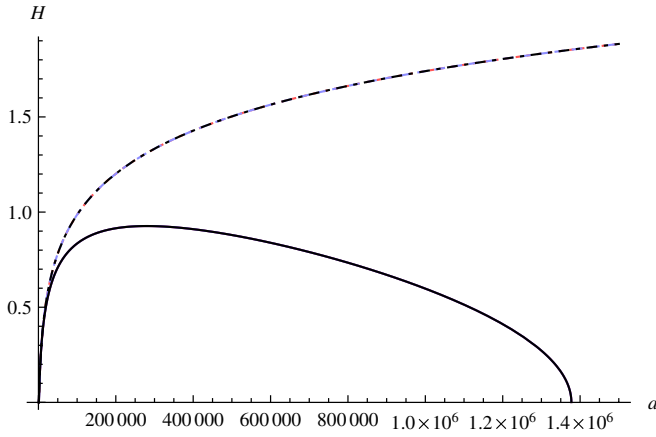


FIG. 1 (color online). *Type I future singularity*: Hubble rates for  $k = 0, \pm 1$  models are shown. Dashed curves correspond to the classical theory and the solid ones to LQC. (Curves for different values of the curvature index coincide with each other for the resolution in the plot). In the classical theory, the Hubble rate diverges in a finite time with  $a \rightarrow \infty$ . We see that the effective LQC curves behave similarly to the classical ones for small values of the scale factor  $a$  when the energy density is very small compared to the Planck scale. As the energy density increases, departures from classical trajectories become significant and we see a quantum recollapse. In this figure the parameters are  $a_o = 1000$ ,  $A = 0.1$ ,  $B = 1$  and  $\alpha = 0.8$ .

In LQC, the energy density is bounded by a maximal value, therefore all type I singularities are resolved by the quantum theory. The energy density grows as in the classical theory as the singularity is approached, but when it reaches close to  $\rho_{\text{crit}}$ , quantum effects lead to significant modifications to the dynamical trajectory. The acceleration  $\ddot{a}$  becomes negative and the Hubble rate goes to zero (Fig. 1). Instead of ripping apart in finite time, the loop quantum universe recollapses and the evolution continues. The presence of a maximal density affects also the curvature invariants. In particular the Ricci scalar and its derivatives remain bounded during the whole evolution.

This behavior which was first noted in the  $k = 0$  model [29] (see also Refs. [38,39] for earlier results) remains essentially the same in the  $k = 1$  and  $k = -1$ , as is shown by Fig. 1. This is not surprising because type I singularities occur in the future in the expanding branch where the scale factor is very large and the effects due to intrinsic curvature are expected to be negligible. (We will see below that effects due to intrinsic curvature can indeed produce surprising results for other exotic singularities).

The only notable difference between the  $k = 0$  and  $k = \pm 1$  cases is that the exact value of the density at the bounce changes, but the correction is small (of the order of  $\ell_{\text{Pl}}^2/a_{\text{min}}^2$ ) and has no effect on the qualitative behavior of the trajectory. The three cases differ only around the value  $a_o$  of the scale factor as shown in Fig. 2. As one would expect for this equation of state, at large scale factors the

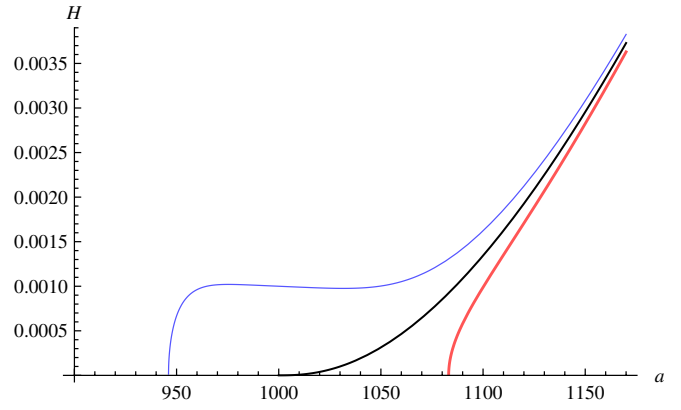


FIG. 2 (color online). Zoom of the Hubble rate around  $a_o = 1000$ . The curves are, respectively, thin (blue) for  $k = -1$  on the left, black for  $k = 0$ , and thick (red) for  $k = +1$  on the right. The parameters are the same as in Fig. 1. This zoom reveals the differences for evolution in Hubble rates in the region around  $a_o$ , which corresponds in this case to the “initial point” of cosmic evolution.

evolution in the  $k = \pm 1$  model mimics that in the flat model and the type I singularity is resolved.

### B. Type II singularity: The sudden singularity

In the type II singularity, the energy density does not diverge. The singularity is caused by a divergence in the pressure which results in a divergence in spacetime curvature at a finite value of the scale factor. To understand this singularity in more detail, let us consider Eq. (13): the pressure diverges when the energy density is  $\rho = \rho_s \equiv (-A/B)^{-1/(\alpha-1)}$ . Inserting this expression into (15) we find that the singularity appears for  $a \rightarrow a_o$ . It is important to note that in LQC, quantum geometric effects do not regulate any divergence in pressure and the spacetime curvature can diverge [29]. It turns out that such a divergence for the type II singularity does not signal the end of spacetime. Geodesics can be extended beyond this singularity. Sudden singularity also turns out to be a weak singularity because the tidal forces are not strong enough to cause a complete destruction of arbitrary detectors [6,29]. Thus weak singularities signal neither a breakdown of the underlying theory nor the end of spacetime.

In Fig. 3, we show the evolution of the Hubble rate for different values of spatial curvature index for the type II singularity occurring in the future of an expanding branch. We see that both in the classical theory (dashed curves) and LQC (solid curves), the Hubble rate vanishes in a finite time for all values of the spatial curvature at scale factor  $a = a_o$ . In contrast, the plot of the Ricci scalar shows that it diverges at this value of the scale factor for all values of the spatial curvature in the classical theory as well as in LQC.

As is clear from Fig. 3, the results about future sudden singularity are on expected lines of the  $k = 0$  model when the scale factor is very large, where the effects due to

intrinsic curvature become negligible. However, a surprising result emerges for the  $k = 1$  model. Our numerical analysis shows that, for certain values of the parameters, there exists an additional baby evolutionary branch separated from the main evolutionary branch. For  $k = +1$ ,  $H^2$  is positive in two disjoint intervals. Consequently there exist two disjoint branches corresponding to real solutions of the Friedmann equation. The extra branch occurs at small scale factors and signifies the nontrivial new physics which emerges from the quantization of intrinsic curvature in LQC. The new branch, that we can see in the Fig. 3, is bounded both in the Hubble rate and in the spacetime curvature.

Another interesting feature which can be seen from Fig. 3 is that for  $k = 1$  model, the classical curve (shown

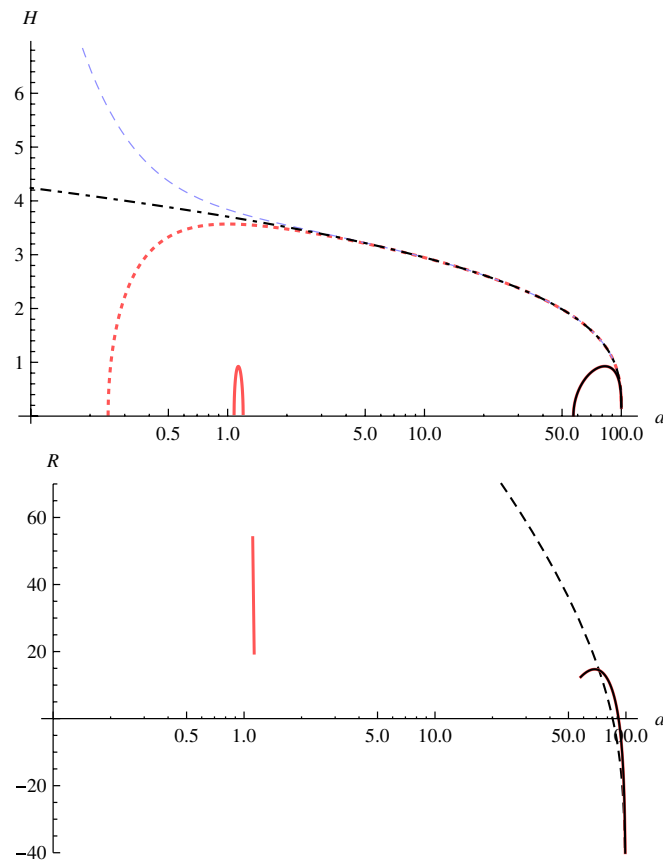


FIG. 3 (color online). *Type II future singularity*: Hubble rate and Ricci scalar for  $k = 0$  in black,  $k = +1$  in thick (red), and  $k = -1$  in thin (blue) curves are shown. Classical curves are dashed and LQC curves are solid. The Hubble rate goes to zero both in the classical theory and LQC for all values of the curvature index at  $a = a_o = 100$ . The Ricci scalar diverges at  $a_o$  in the classical theory and LQC. (Since Ricci scalar is independent of the value of curvature index in the classical model, there is only curve for the classical theory). For  $k = +1$  there exist two disjoint solutions of the Friedmann equation, and the further branch appearing there is bounded both in the Hubble rate and in the spacetime curvature. The parameters are  $A = -0.1$ ,  $B = 10$  and  $\alpha = 1/4$ .

with a dotted curve) depicts that there is no initial singularity. The Hubble rate vanishes both in the past and the future of the classical evolution. The classical universe faces a sudden singularity in future, but since it is a weak singularity, the classical  $k = 1$  universe is geodesically complete for the considered equation of state. This does not hold true for the  $k = 0$  and  $k = -1$  case, and these classical spacetimes are past incomplete.

Figure 4 shows the plot of Hubble rate and Ricci scalar for different values of curvature index when the sudden singularity occurs in past. In such a case the effects due to intrinsic curvature lead to novel features in the physics of singularity resolution. We find that classically the Hubble rate vanishes for all values of  $k$  at  $a = a_o$  and the Ricci scalar diverges. However in LQC, though the Hubble rate vanishes for values of  $k$ , the divergence in Ricci scalar does not occur for a closed universe. It turns out that as the scale factor approaches  $a_o$ , the Ricci scalar increases to large values but it does not diverge for the allowed values of the

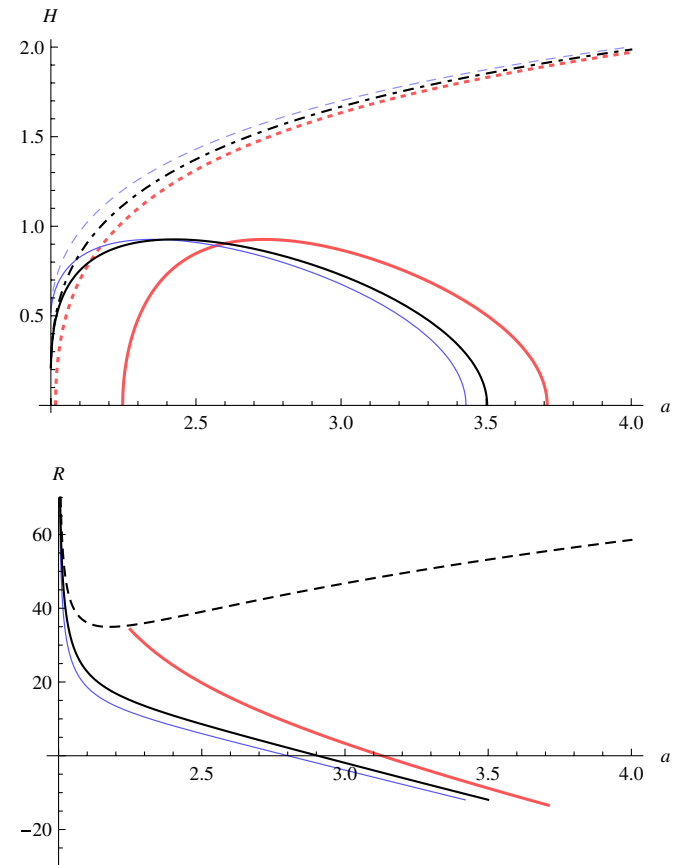


FIG. 4 (color online). *Type II past singularity*: Hubble rate and Ricci scalar for  $k = 0$  and  $k = \pm 1$  are shown. Classical curves are dashed and LQC curves are solid.  $k = 0$  is in black,  $k = +1$  is the thick (red) curve, and  $k = -1$  in thin (blue) curve. The singularity is at  $a_o = 2$ , where the Ricci scalar diverges. Note that for  $k = +1$  the universe begins at  $a > a_o$ , and the Ricci scalar is always finite. In this figure the parameters are  $A = 0.1$ ,  $B = -10$  and  $\alpha = 1/4$ .



scale factor. (The divergence occurs in the forbidden region of effective LQC dynamics where the Hubble rate is imaginary). This result implies that in the closed model for the parameters considered here, quantization of intrinsic curvature leads to a resolution of the past sudden singularity. This is a direct consequence of the way intrinsic curvature terms enter in to the expression of the Ricci scalar in LQC. We emphasize that type II singularities whether in future or past, even though they are harmless, are not resolved in LQC for  $k = 0$  and  $k = -1$  models. In contrast we find that these singularities are resolved, for a certain choice of parameters, when occurring in the past for  $k = 1$  model.

**C. Type III singularity: The big freeze**

A type III singularity occurs at a finite value of scale factor where both the Hubble rate and the Ricci scalar diverge. In LQC, the presence of a maximal energy density

$\rho_{\text{crit}}$  bounds the Hubble rate, giving rise to a recollapse when the singularity is approached. Also  $\ddot{a}/a$  and the Ricci scalar are bounded and finite. Thus type III singularities are generically resolved in LQC for all values of the curvature index.

This can be seen from the plots in Fig. 5, where we have shown the plots of the Hubble rate and the Ricci scalar when the big freeze singularity occurs in the past at  $a = a_o$ . We see that in the classical theory, Hubble rate and Ricci scalar diverge as  $a \rightarrow a_o$  in the classical theory. On the other hand, in LQC the Hubble rate vanishes in the past showing that there is a bounce. Further, the Ricci scalar remains finite for all the values of the curvature index.

We now make an interesting observation for the  $k = -1$  case. For a certain choice of parameters which lead to a type III singularity in the past, we find that there exists an additional classical branch at the small scale factors. The additional branch faces a big bang singularity in the past of classical evolution and is free from singularity in future

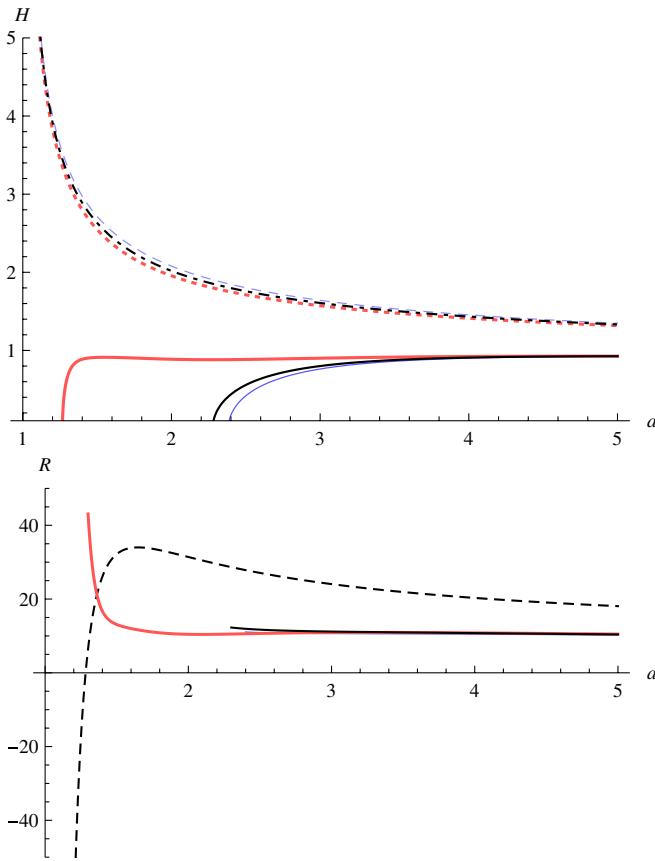


FIG. 5 (color online). *Type III past singularity*: Classical (dashed) and effective LQC (solid) Hubble rate and Ricci scalar are shown. The curves are, respectively, thick (red) for  $k = +1$  starting on the left, black for  $k = 0$  in the middle, and thin (blue) for  $k = -1$  starting a bit more on the right. In the classical case there is a divergence of both  $H$  and  $R$  at  $a_o = 1$  while the curves for LQC remain bounded (in the figure the Ricci scalar for  $k = -1$  is covered by the black line of the  $k = 0$  case). Here the parameters used are  $A = -100$ ,  $B = -1$  and  $\alpha = 2$ .

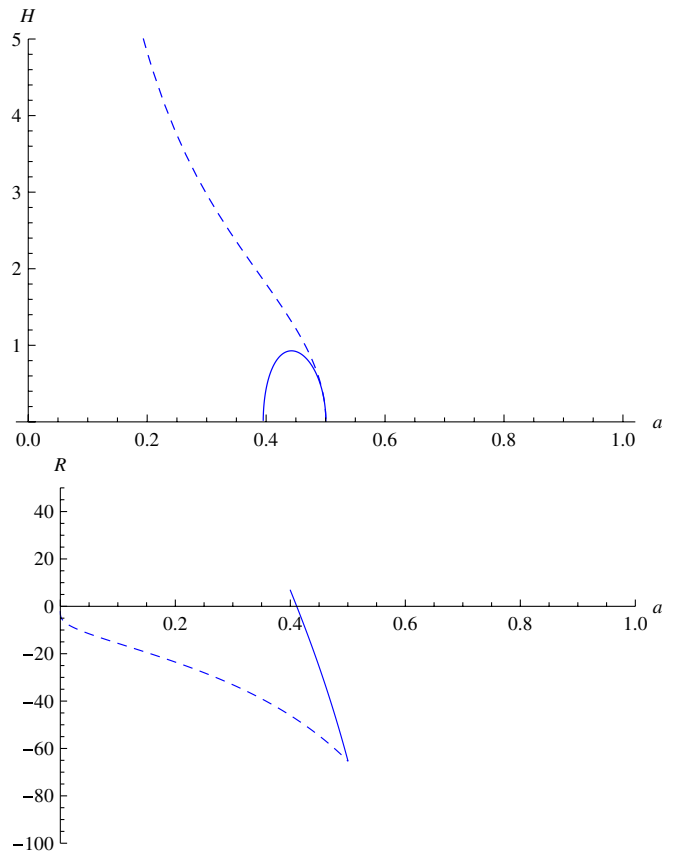


FIG. 6 (color online). This figure shows the additional branch occurring at small scale factors in the  $k = -1$  model for equation of state leading to a type III past singularity in Fig. 5. The additional branch occurs for certain values of parameters both in the classical theory (dashed curve) and LQC (solid curve) when the scale factor is below the value  $a_o$ , where the big freeze occurs classically. Note that, also in this case, only LQC is immune from primordial singularity. The parameters used are  $A = -100$ ,  $B = -1$  and  $\alpha = 2$ .

evolution (as is shown by the dashed curves for the Hubble rate and the Ricci scalar in Fig. 6). However the additional branch occurs in LQC when the scale factor is less than the Planck length. Since the length scale involved is below where we expect the effective dynamics in LQC to be valid, a more detailed analysis is needed, by including modifications pertaining to the inverse scale factor, in order to understand the physics emerging from LQC in this special case. Nevertheless, if we assume the validity of the effective Hamiltonian in this regime, then LQC resolves the past singularity in the additional branch (as depicted by the solid curves for the Hubble rate and the Ricci scalar in Fig. 6).

The type III singularity is also resolved for all values of the spatial curvature when it occurs in the future. This is on the expected lines, as the effects due to intrinsic curvature become small at large scale factors. In Fig. 7 we depict results from such an evolution. In the classical theory, there is a big freeze singularity in the future where the Hubble

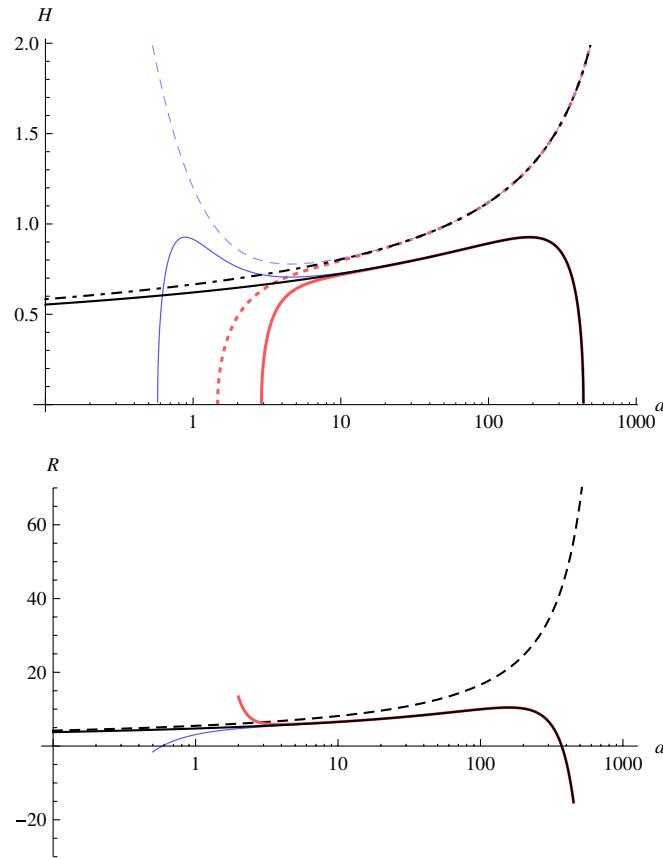


FIG. 7 (color online). *Type III future singularity*: Classical (dashed) and effective LQC (solid) Hubble rate and Ricci scalar with  $k = 0, \pm 1$  are compared. In the classical case there is a divergence of both  $H$  and  $R$  at  $a_0$  while the curves for LQC remain bounded. Notice that LQC cures also the initial singularity in the curve case, and, in particular, the thin (blue) solid curves ( $k = -1$ ) shows a characteristic “tilt” for small  $a$ . The parameters are  $a_0 = 1000$ ,  $A = 100$ ,  $B = 1$  and  $\alpha = 2$ .

rate and the Ricci scalar diverge. For the  $k = -1$  universe, there also exists a singularity (big bang) in the past evolution. We can see that in LQC there is no type III singularity as the Hubble rate vanishes at  $a = a_0$  and Ricci scalar reaches a finite value. Further, the past big bang singularity in the  $k = -1$  case, as pointed above, is also resolved. The LQC universe is bounded for all values of the curvature index for the equation of state which leads to a type III singularity in the classical theory.

After performing various numerical simulations we reach the conclusion that type III singularities, irrespective of them occurring in the past or the future, are always resolved in LQC for all values of the spatial curvature index.

#### D. Type IV singularity: The big brake

Type IV singularity is a derivative-curvature-singularity where none of the curvature invariants diverge. (In this sense, it does not qualify as a curvature singularity). Though the energy density and pressure remain finite, a

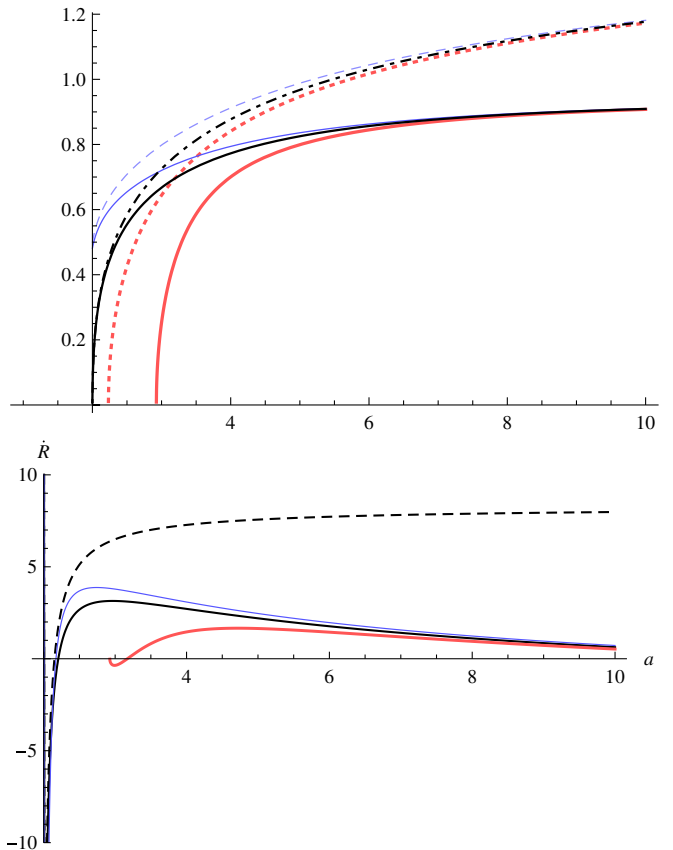


FIG. 8 (color online). *Type IV past singularity*: Hubble rate and Ricci scalar for  $k = 0, \pm 1$ . The singularity is at  $a_0 = 2$ , where the Ricci scalar diverges for  $k = 0$  and  $k = -1$ . Note that for  $k = +1$  (dotted line for the classical solutions and thick red line for LQC effective solutions) the universe begins at  $a > a_0$ , thus the Ricci scalar is always finite. The parameters are  $A = 0.01$ ,  $B = 1$  and  $\alpha = 1/4$ .

higher derivative of the curvature diverges as  $a \rightarrow a_o$ . The value of  $\alpha$  determines the order of derivative which blows up [7].

As in the type II case, the geodesic equations are well-behaved since the Hubble rate is finite at  $a = a_o$ . The singularity is weak because it occurs at a finite volume and the theory is geodetically complete since the Hubble rate is bounded. Quantum geometric effects have little influence on this harmless extremal event beyond which geodesics can be extended even in the classical theory [6,29].

The behavior of the curves is very similar to the type II case. In Fig. 8 we have shown the results from the past singularity. (We have chosen  $\alpha = 1/4$  so that the divergence appears in  $\dot{R}$ ). We find that the Hubble rate vanishes at  $a = a_o$  both in the classical theory and LQC. However,  $\dot{R}$  diverges except for the spatially closed model in LQC. This result is similar to what we obtain for type II

singularity. It turns out that LQC resolves the type IV weak singularity when occurring in the past for the closed model for certain choices of parameters. As in the type II case, this result highlights the surprises which quantization of intrinsic curvature may bring in comparison to earlier studies [29]. For the  $k = 0$  and  $k = -1$  model, this weak singularity is ignored by the quantum geometry for all values of the parameters.

We show the results for type IV singularity occurring in future evolution in Fig. 9. Here we find that for all the values of spatial curvature, the Hubble rate vanishes at  $a = a_o$  with a divergence in  $\dot{R}$  for both the classical theory and LQC. We see that not only the LQC curves have no big bang, but this happens also classically for the closed universe (Fig. 9, dotted line). Furthermore, for  $k = +1$  it is possible to obtain a baby universe for an appropriate choice of the parameters, as for type II singularity (see Fig. 3). In this case  $\dot{R}$  is strongly negative but bounded, thus there is no singularity in the past and in the future.

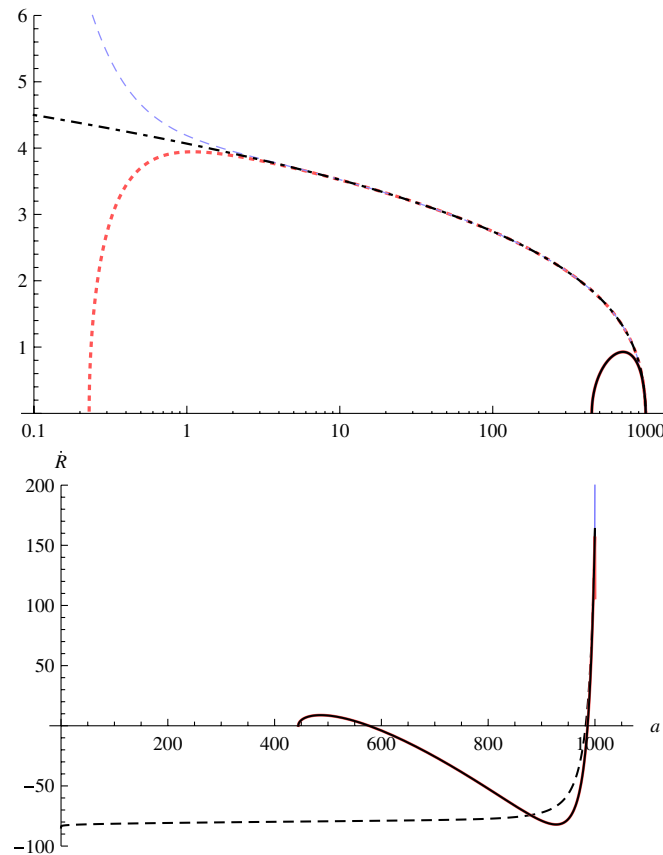


FIG. 9 (color online). *Type IV future singularity*: Comparison of Hubble rates and  $\dot{R}$  for the classical theory and LQC (in the figure LQC curves overlap). The singularity appears in the future at  $a_o = 1000$ . The LQC universes (solid lines) do not have primordial singularity. Interestingly, for  $k = 1$  this happens also classically (dotted line). In this picture there are no baby universes, but they can be obtained for  $k = 1$  for a different choice of the parameters. Here the parameters are  $A = -0.1$ ,  $B = -1$  and  $\alpha = 1/4$ .

## V. SUMMARY

A fundamental question in quantum gravity is whether spacelike singularities of the classical theory are resolved. Since not all such singularities signal end of the spacetime, it is important to understand the role of quantum gravitational effects in the resolution of strong singularities (those beyond geodesics can not be extended) and weak singularities (those beyond which geodesics can be extended). These issues were addressed in the loop quantization of cosmological models which are spatially flat and it was found that the nonperturbative loop quantum effects resolve all strong singularities and ignore weak singularities [29].

The aim of the present analysis was to investigate these issues for spatially curved models using the phenomenological model of equation of state permitting exotic singularities such as big rip, sudden singularities, big freeze singularity, and the big brake singularity. As we pointed out earlier, in certain models of modified gravity one also has the possibility to obtain w-singularities where equation of state diverges but pressure and energy density are regular. Such a solution is so far unknown in LQC and was not considered here. In any case, these singularities are weak curvature singularities and are harmless [9].

In order to capture the role of intrinsic curvature, we considered exotic singularities both in the future and the past evolution. To our knowledge, even in the classical theory exotic singularities had not been studied earlier for the spatially curved model. For the singularities occurring in the future, the contribution of the intrinsic curvature is expected to become very small and we expect results to agree with the spatially flat case. This turns out to be true. However, more interesting are cases where the exotic singularities occur in past. Here one would expect effects due to quantization of intrinsic curvature to play a

nontrivial role. In fact we encounter some surprising results. Though strong singularities are always resolved in LQC, it turns out that for the closed model weak singularities occurring in the past evolution may also be resolved. Thus LQC does not always ignore weak singularities. This is an intriguing result which deserves further investigation.

Another peculiar feature of the curved models is the appearance of a small branch for type II and type IV singularities for certain values of the parameters. This “baby universe” is bounded, and is devoid of any singularities. It will be interesting to analyze these additional branches which we find for both spatially open and closed models in more detail, in particular, by taking into account inverse scale factor effects which may play some role when scale factor is below the Planck length.

Our results extend to earlier results on generic resolution of strong curvature singularities in a spatially flat model in LQC to the spatially curved models. They also bring some important lessons, the primary one being that quantization of intrinsic curvature may throw some novel unexpected results and we need to gain more insights on when quantum

gravity may ignore or resolve a weak curvature singularity. Another lesson is that as for the spatially flat model, space-time curvature invariants may diverge for the spatially curved models and yet there may be no physical singularity.

These results strengthen the case for a generic resolution of strong singularities in LQG. To achieve this goal, the next step will be to include anisotropies and then inhomogeneities. The latter will require us to go beyond the minisuperspace approximation considered here. This brings up additional challenges such as the complete classification of the strong and weak singularities in inhomogeneous situations. Two promising avenues where such an analysis can be undertaken would be the Gowdy models [20] and in the spinfoam paradigm [40]. We hope that these studies will also provide insights on the deeper relation of these frameworks with LQC.

### ACKNOWLEDGMENTS

F. V. thanks Elena Magliaro, Frank Hellmann and Carlo Rovelli for useful discussions.

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- [1] A. Borde, A. H. Guth, and A. Vilenkin, *Phys. Rev. Lett.* **90**, 151301 (2003); arXiv:gr-qc/0110012.
  - [2] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, *Phys. Rev. Lett.* **91**, 071301 (2003).
  - [3] J. D. Barrow, *Classical Quantum Gravity* **21**, L79 (2004).
  - [4] J. D. Barrow, *Classical Quantum Gravity* **21**, 5619 (2004); J. D. Barrow and C. G. Tsagas, *Classical Quantum Gravity* **22**, 1563 (2005).
  - [5] See, for example, P. Singh, M. Sami, and N. Dadhich, *Phys. Rev. D* **68**, 023522 (2003); S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **562**, 147 (2003); K. Lake, *Classical Quantum Gravity* **21**, L129 (2004); S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **595**, 1 (2004); *Phys. Rev. D* **70**, 103522 (2004); H. Wei and R-G. Kai, *Phys. Rev. D* **72**, 123507 (2005); S. Cotsakis and I. Klaudatou, *J. Geom. Phys.* **57**, 1303 (2007); M. P. Dabrowski, C. Kiefer, and B. Sandhoefer, *Phys. Rev. D* **74**, 044022 (2006); S. Cotsakis and J. D. Barrow, *J. Phys. Conf. Ser.* **68**, 012004 (2007); M. Dabrowski, T. Denkiewicz, and M. A. Hendry, *Phys. Rev. D* **75**, 123524 (2007); A. Kamenshchik, C. Kiefer, and B. Sandhoefer, *Phys. Rev. D* **76**, 064032 (2007); M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz, and P. Martin-Moruno, *Int. J. Mod. Phys. D* **17**, 2269 (2008); F. Cannata, A. Yu. Kamenshchik, and D. Regoli, *Phys. Lett. B* **670**, 241 (2009); J. D. Barrow, A. B. Batista, J. C. Fabris, and S. Houndjo, arXiv:0808.1382; J. D. Barrow and S. Z. W. Lip, *Phys. Rev. D* **80**, 043518 (2009); J. D. Barrow, S. Cotsakis, and A. Tsokaros, arXiv:1003.1027; *Classical Quantum Gravity* **27**, 165017 (2010).
  - [6] L. Fernandez-Jambrina and R. Lazkoz, *Phys. Rev. D* **70**, 121503 (2004); *Phys. Rev. D* **74**, 064030 (2006); *Phys. Lett. B* **656**, 9 (2007); *J. Phys. Conf. Ser.* **229**, 012037 (2010).
  - [7] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, *Phys. Rev. D* **71**, 063004 (2005).
  - [8] M. P. Dabrowski and T. Dekiewicz, *Phys. Rev. D* **79**, 063521 (2009).
  - [9] L. Fernandez-Jambrina, *Phys. Rev. D* **82**, 124004 (2010).
  - [10] A. Ashtekar, *Gen. Relativ. Gravit.* **41**, 707 (2009).
  - [11] M. Bojowald, *Living Rev. Relativity* **8**, 11 (2005).
  - [12] P. Singh, *J. Phys. Conf. Ser.* **140**, 012005 (2008).
  - [13] A. Ashtekar, T. Pawlowski, and P. Singh, *Phys. Rev. Lett.* **96**, 141301 (2006).
  - [14] A. Ashtekar, T. Pawlowski, and P. Singh, *Phys. Rev. D* **73**, 124038 (2006).
  - [15] A. Ashtekar, T. Pawlowski, and P. Singh, *Phys. Rev. D* **74**, 084003 (2006).
  - [16] A. Ashtekar, T. Pawlowski, P. Singh, and K. Vandersloot, *Phys. Rev. D* **75**, 024035 (2007).
  - [17] A. Ashtekar, T. Pawlowski, and P. Singh (to be published).
  - [18] D.-W. Chiou and K. Vandersloot, *Phys. Rev. D* **76**, 084015 (2007).
  - [19] A. Ashtekar and E. Wilson-Ewing, *Phys. Rev. D* **79**, 083535 (2009).
  - [20] D. Brizuela, G. A. D. Mena Marugan, and T. Pawlowski, *Classical Quantum Gravity* **27**, 052001 (2010).
  - [21] A. Ashtekar, A. Corichi, and P. Singh, *Phys. Rev. D* **77**, 024046 (2008).
  - [22] A. Corichi and P. Singh, *Phys. Rev. Lett.* **100**, 161302 (2008).

- [23] W. Kaminski and T. Pawłowski, *Phys. Rev. D* **81**, 084027 (2010).
- [24] V. Taveras, *Phys. Rev. D* **78**, 064072 (2008).
- [25] P. Singh and V. Taveras (to be published).
- [26] A. Corichi and P. Singh, *Phys. Rev. D* **78**, 024034 (2008).
- [27] A. Corichi and P. Singh, *Phys. Rev. D* **80**, 044024 (2009).
- [28] T. Cailleteau, A. Cardoso, K. Vandersloot, and D. Wands, *Phys. Rev. Lett.* **101**, 251302 (2008).
- [29] P. Singh, *Classical Quantum Gravity* **26**, 125005 (2009).
- [30] K. Vandersloot, *Phys. Rev. D* **75**, 023523 (2007).
- [31] L. Szulc, *Classical Quantum Gravity* **24**, 6191 (2007).
- [32] Krzysztof A. Meissner, *Classical Quantum Gravity* **21**, 5245 (2004).
- [33] S. Tsujikawa, P. Singh, and R. Maartens, *Classical Quantum Gravity* **21**, 5767 (2004).
- [34] P. Singh, *Classical Quantum Gravity* **22**, 4203 (2005).
- [35] J. Magueijo and P. Singh, *Phys. Rev. D* **76**, 023510 (2007).
- [36] A. Perez, *Phys. Rev. D* **73**, 044007 (2006).
- [37] B. Gupt, P. Singh (to be published).
- [38] M. Sami, P. Singh, and S. Tsujikawa, *Phys. Rev. D* **74**, 043514 (2006).
- [39] D. Samart and B. Gumjudpai, *Phys. Rev. D* **76**, 043514 (2007); T. Naskar and J. Ward, *Phys. Rev. D* **76**, 063514 (2007); P. Wu and S.N. Zhang, *J. Cosmol. Astropart. Phys.* **06** (2008) 007; X. Fu and H. Yu. P. Wu, *Phys. Rev. D* **78**, 063001 (2008); R. Lamon and A. Woehr, *Phys. Rev. D* **81**, 024026 (2010).
- [40] C. Rovelli and F. Vidotto, *Classical Quantum Gravity* **25**, 225024 (2008); M.V. Battisti, A. Marciano, and C. Rovelli, *Phys. Rev. D* **81**, 064019 (2010); E. Bianchi, C. Rovelli, and F. Vidotto, *Phys. Rev. D* **82**, 084035 (2010); F. Vidotto, *J. Phys.: Conf. Ser.*, arXiv:1011.4705.