Quantum statistical relation for black holes in nonlinear electrodynamics coupled to Einstein-Gauss-Bonnet AdS gravity

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We consider curvature-squared corrections to Einstein-Hilbert gravity action in the form of a Gauss-Bonnet term in $D > 4$ dimensions. In this theory, we study the thermodynamics of charged static black holes with anti–de Sitter (AdS) asymptotics, and whose electric field is described by nonlinear electrodynamics. These objects have received considerable attention in recent literature on gravity/gauge dualities. It is well-known that, within the framework of anti–de Sitter/conformal field theory (AdS/CFT) correspondence, there exists a nonvanishing Casimir contribution to the internal energy of the system, manifested as the vacuum energy for global AdS spacetime in odd dimensions. Because of this reason, we derive a quantum statistical relation directly from the Euclidean action and not from the integration of the first law of thermodynamics. To this end, we employ a background-independent regularization scheme which consists, in addition to the bulk action, of counterterms that depend on both extrinsic and intrinsic curvatures of the boundary (Kounterterm series). This procedure results in a consistent inclusion of the vacuum energy and chemical potential in the thermodynamic description for Einstein-Gauss-Bonnet AdS gravity regardless of the explicit form of the nonlinear electrodynamics Lagrangian.

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I. INTRODUCTION

Anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1] provides a computational tool to obtain physical properties of a field theory at strong coupling by studying its gravity dual.

In hydrodynamic models, for example, gauge/gravity duality is used to calculate the ratio of shear viscosity η to entropy density s in a strongly interacting quantum theory. In particular, it predicts a universal bound $1/4\pi$ of this ratio for a large class of theories, dual to Einstein AdS gravity [2].

However, it has been shown that the addition of higher-derivative terms to the gravitational action in AdS space modifies the dynamics of the boundary theory and it violates the Kovtun-Son-Starinets bound [3]. On the other hand, higher-curvature terms in Abelian fields described by nonlinear electrodynamics (NED) do not change the value η/s [4], unless gravity action is augmented by a Gauss-Bonnet (GB) term, which also causes instabilities in the large-momentum regime [5].

In a similar fashion, when studying gravity duals to high critical temperature superconductivity models, the inclusion of GB and NED terms alters the ratio between T_c and the energy gap [6–8].

In the examples of AdS/CFT duality mentioned above, the mapping between the bulk and boundary theories makes extensive use of thermal properties of black holes in the gravity side of the correspondence. This provides the motivation to investigate the thermodynamics of black holes in Einstein-Gauss-Bonnet (EGB) gravity with a negative cosmological constant coupled to an arbitrary NED theory. In particular, we prove here that any black hole which is a solution to the theory will satisfy the quantum statistical relation (QSR) [9]

$$TI^E = U - TS + \Phi Q, \tag{1}$$

where T is the Hawking temperature, U is the black hole energy, and S its entropy that includes a correction due to the GB term. The addition of a NED Lagrangian to pure gravity action brings in electric charge Q to the solution, whose conjugate variable Φ is the difference in electric potential between the horizon and infinity. In other words, the Euclidean action is the Legendre transformation of the entropy with respect to chemical (electric) potential. As it is standard within the framework of AdS/CFT, the Euclidean action has to be regularized by means of a background-independent procedure.

In gravity with AdS asymptotics, there is a subtle difference between the relation (1) and the first law of thermodynamics

$$dU = TdS - \Phi dQ, \tag{2}$$

because U contains in general a Casimir contribution in $D = 2n + 1$ dimensions [11]. Clearly, the latter formula remains the same if the total energy is shifted by an additive constant $E_{\rm vac}$. This does not mean that the presence of a vacuum energy is irrelevant. On the contrary, it is expected that $E_{\rm vac}$ will appear in a consistent thermodynamic description of asymptotically AdS (AAdS) black holes, when we work out their thermal properties from the partition function in semiclassical approximation

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$$Z = e^{-I^E_{\rm clas}}. \tag{3}$$

Here, $I^E_{\rm clas}$ stands for the classical Euclidean action, which is divergent in AdS gravity. If one intends to identify any thermodynamic potential with the Euclidean action, one must resort to a regularization mechanism that eliminates the divergences in the infrared sector of the theory, but not the vacuum energy. It is only in this way that we can extract I^E as the finite part of $I^E_{\rm clas}$, and relate it to the asymptotic charges that appear in Eq. (1).

In the context of AdS/CFT correspondence, the regularization of the Euclidean action requires the addition of counterterms which are intrinsic functionals of the boundary metric and curvature [12,13], and constructed using holographic techniques for AAdS spacetimes [14,15]. When a GB term is included in the gravity action—despite the fact the field equations are still second-order in the metric—one faces almost unsurmountable obstacles to apply the holographic renormalization procedure. Thus, it is necessary to postulate the explicit form of the local counterterm series, and then the coefficients in the series are adjusted by the convergence of charges and Euclidean action for particular solutions [16,17]. This seems to work well for low enough dimensions, but it is far from giving a general prescription for all cases.

In this paper, we apply an *extrinsic*, background-independent regularization method for EGB-AdS gravity to render the Euclidean action finite in all dimensions [18]. This scheme involves the addition to the action of counterterms depending on both the intrinsic and extrinsic curvatures (which, for that reason, have been suggestively called *Kounterterms*). This procedure results in the obtention of a quantum statistical relation for black hole solutions to Einstein-Gauss-Bonnet AdS gravity coupled to an arbitrary NED. As we shall show below, the picture in odd dimensions is consistent only if the total energy is shifted as $U = M + E_{\rm vac}$ with respect to the *Hamiltonian* mass M, in a similar fashion as for Einstein-Born-Infeld case studied in Ref. [19].

II. FIELD EQUATIONS

Nonlinear actions for electromagnetic field have been a subject of research for many years, since it was noted by Heisenberg and Euler that quantum corrections to electrodynamics lead to nonlinear equations for the field strength [20]. The physics of objects described by nonlinear effective Lagrangians, as the one of Born and Infeld [21], possesses remarkable properties, such as a regular electric field at the origin and a finite self-energy. Furthermore, Born-Infeld electrodynamics can be obtained in exact one-loop computations from open Bose strings [22].

Gravity coupled to NED leads to a large class of charged black hole solutions. In Einstein-Hilbert gravity coupled to Born-Infeld electrodynamics, static, spherically symmetric black holes were derived in, e.g., Refs. [23,24]. Among other gravitating NED models that exhibit electrically charged solutions, we can mention the logarithmic theory investigated in [25], and the conformally invariant electrodynamics found in [26].

We will study gravitating NED in any spacetime dimension $D > 4$ which is described by the action

$$I_0 = \int_{\mathcal{M}} d^D x \sqrt{-g}\,\mathcal{L}_0 = I_{\rm grav} + I_{\rm NED}. \tag{4}$$

The first part of the bulk action corresponds to the one of Einstein-Hilbert (EH) with negative cosmological constant $\Lambda = -(D-1)(D-2)/2\ell^2$ and a higher-curvature correction given by the Gauss-Bonnet term

$$I_{\rm grav} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{-g}[R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma})]. \tag{5}$$

Here, ℓ is the AdS radius, G is the gravitational constant, and α is the GB coupling, which has to be positive if one regards this action as obtained in the low-energy limit of string theory.

The Abelian gauge field $A_\mu(x)$ defines a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which couples minimally to gravity through the quadratic invariant $F^2 = g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}$. We consider an arbitrary Lagrangian density $\mathcal{L}(F^2)$, such that the NED action has the form

$$I_{\rm NED} = \int_{\mathcal{M}} d^D x\sqrt{-g}\,\mathcal{L}(F^2). \tag{6}$$

The equations of motion follow from the variation of the bulk action with respect to the dynamic fields $g_{\mu\nu}$ and A_μ, that is, $\delta I_0/\delta g_{\mu\nu} = 0$ and $\delta I_0/\delta A_\mu = 0$,

$$\mathcal{E}^\mu_\nu \equiv G^\mu_\nu + H^\mu_\nu - 8\pi G T^\mu_\nu = 0, \tag{7}$$

$$\mathcal{E}^\mu \equiv \nabla_\nu\left(F^{\mu\nu}\frac{d\mathcal{L}}{dF^2}\right) = 0. \tag{8}$$

In Eq. (7), G^μ_ν is the Einstein tensor which includes the contribution from the cosmological constant

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R + \Lambda\delta^\mu_\nu, \tag{9}$$

whereas H^μ_ν denotes the Lanczos tensor

$$H^\mu_\nu = -\frac{\alpha}{2}\delta^\mu_\nu(R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\lambda\sigma}R_{\alpha\beta\lambda\sigma}) + 2\alpha(RR^\mu_\nu - 2R^{\mu\lambda}R_{\lambda\nu} - 2R^\mu_{\lambda\nu\sigma}R^{\lambda\sigma} + R^{\mu\alpha\lambda\sigma}R_{\nu\alpha\lambda\sigma}), \tag{10}$$

which comes from the GB part of the gravity action. The right-hand side of Eq. (7) corresponds to the matter stress tensor $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta I_{\rm NED}}{\delta g_{\mu\nu}}$ which has the form

$$T_{\nu}^{\mu} = \delta_{\nu}^{\mu} \mathcal{L} - 4 \frac{d\mathcal{L}}{dF^2} F^{\mu\lambda} F_{\nu\lambda}. \tag{11}$$

For a given solution ansatz, the generalized Maxwell equation (8) determines the dynamics of the electromagnetic field.

The presence of the higher-curvature term H_{ν}^{μ} in the equation of motion (7) changes the vacua structure of AdS gravity. Indeed, a nonvanishing GB coupling α modifies the AdS radius ℓ of maximally symmetric spaces to an effective value $\ell_{\rm eff}$ given by

$$\ell_{\rm eff}^{(\pm)2} = \frac{2\alpha(D-3)(D-4)}{1 \pm \sqrt{1 - \frac{4\alpha}{\ell^2}(D-3)(D-4)}},$$
$$\alpha \leq \frac{\ell^2}{4(D-3)(D-4)}. \tag{12}$$

A multiplicity of solutions may exist in either branch of the theory, some of them yet to be discovered. For the present paper we will consider spacetimes which are AAdS. In order to impose this condition in a way independent of any coordinate frame, we will assume the AAdS behavior in the curvature rather than in the metric, that is,

$$R_{\mu\nu}^{\alpha\beta} \rightarrow -\frac{1}{\ell_{\rm eff}^2} \delta_{[\mu\nu]}^{[\alpha\beta]}, \tag{13}$$

for the corresponding branch.

For static black holes with constant-curvature transversal section Γ_{D-2}, described by the metric γ_{nm}, the line element in the local coordinates $x^{\mu} = (t, r, \varphi^m)$ reads

$$ds^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}$$
$$= -f^2(r) dt^2 + \frac{dr^2}{f^2(r)} + r^2 \gamma_{mn}(\varphi) d\varphi^m d\varphi^n. \tag{14}$$

The curvature of Γ_{D-2} is labeled by the topological parameter $k = 0$, $+1$, or -1 for the flat, spherical or hyperbolic case, respectively. The spacetime boundary $\partial\mathcal{M}$ has the topology of a cylinder and is placed at $r \rightarrow \infty$. A necessary condition for a solution to be a black hole is the existence of a horizon r_+ such that $f^2(r_+) = 0$. In case the latter relation possesses more than one root, r_+ denotes the outermost one.

A charged static solution is given in terms of a gauge field which depends only on the radial coordinate

$$A_{\mu} = \phi(r) \delta_{\mu}^{t}, \tag{15}$$

which generates the electric field

$$E(r) = -\phi'(r), \tag{16}$$

such that the field strength is

$$F_{\mu\nu} = E(r)(\delta_{\mu}^{t} \delta_{\nu}^{r} - \delta_{\nu}^{t} \delta_{\mu}^{r}). \tag{17}$$

Here, the prime stands for radial derivative.

In the static ansatz (14)–(17), the gravitational equation (7) gives rise to a single independent differential equation for f^2,

$$\mathcal{E}_t^t = \mathcal{E}_r^r$$
$$= \frac{D-2}{2r^2} \left[r(f^2)' + (D-3)(f^2 - k) - (D-1)\frac{r^2}{\ell^2} \right]$$
$$+ \alpha(D-2)(D-3)(D-4)\frac{k - f^2}{r^3}$$
$$\times \left[(f^2)' - (D-5)\frac{k - f^2}{2r} \right] - 8\pi G T_r^r, \tag{18}$$

where

$$T_t^t = T_r^r = \left(\mathcal{L} + 4E^2 \frac{d\mathcal{L}}{dF^2} \right) \Bigg|_{F^2 = -2E^2}. \tag{19}$$

On the other hand, the form of the electric field (17) leads to the differential version of a generalized Gauss law

$$\mathcal{E}^{\,t} = -\frac{d}{dr}\left(r^{D-2} E \frac{d\mathcal{L}}{dF^2} \Bigg|_{F^2=-2E^2} \right) = 0, \tag{20}$$

whose solution introduces q as an integration constant related to the electric charge,

$$E \frac{d\mathcal{L}}{dF^2} \Bigg|_{F^2=-2E^2} = -\frac{q}{r^{D-2}}. \tag{21}$$

The electric field decouples from the metric and one can determine E just from this algebraic equation. Therefore, the electric potential at the distance r measured with respect to radial infinity is calculated from

$$\phi(r) = -\int_{\infty}^{r} dv E(v). \tag{22}$$

For the analysis of thermodynamic properties of charged black holes, the conjugate variable to the electric charge Q is the difference in electric potential between infinity and the event horizon r_+, that is, $\Phi = \phi(\infty) - \phi(r_+)$.

The first integral of the equation of motion for the metric (18) is

$$(f^2 - k)\left(1 - \alpha(D-3)(D-4)\frac{f^2 - k}{r^2} \right)$$
$$= \frac{r^2}{\ell^2} - \frac{\mu}{r^{D-3}} + \frac{16\pi G \mathcal{T}(q, r)}{(D-2) r^{D-3}}, \tag{23}$$

where μ is related to the black hole mass, whereas the electromagnetic flux through the surface $r = $ const is given by the function

$$\mathcal{T}(q, r) = \int_{\infty}^{r} dv v^{D-2} T_r^r = \int_{\infty}^{r} dv (v^{D-2} \mathcal{L} - 4qE), \tag{24}$$

for an arbitrary NED Lagrangian. In the second line, the charge q appears due to the Gauss law (21).

As mentioned above, the theory possesses two branches, which are manifested as two solutions in the metric function f^2 of Eq. (23) for EGB gravity coupled to NED,

$$f_{\pm}^2(r) = k + \frac{r^2}{2\alpha(D-3)(D-4)}\left[1 \pm \sqrt{1 - 4\alpha(D-3)(D-4)\left(\frac{1}{\ell^2} - \frac{\mu}{r^{D-1}} + \frac{16\pi G \mathcal{T}(q,r)}{(D-2)r^{D-1}}\right)}\right]. \quad (25)$$

The branch with positive sign $f_{+}^2(r)$ has an ill-defined limit $\alpha \to 0$, and it does not recover the solutions of EH AdS gravity. Furthermore, perturbations around the corresponding vacuum state are proved to have negative mass [27]. Henceforth, we will focus on the stable and ghost-free branch $f^2(r) \equiv f_{-}^2(r)$. Asymptotically, this branch and its radial derivative behave as

$$f^2 = k + \frac{r^2}{\ell_{\text{eff}}^2} - \frac{\mu}{1 - \frac{2\alpha}{\ell_{\text{eff}}^2}(D-3)(D-4)}\frac{1}{r^{D-3}} + \mathcal{O}\left(\frac{1}{r^{2D-6}}\right), \quad (26)$$

$$(f^2)' = \frac{2r}{\ell_{\text{eff}}^2} + \frac{(D-3)\mu}{1 - \frac{2\alpha}{\ell_{\text{eff}}^2}(D-3)(D-4)}\frac{1}{r^{D-2}} + \mathcal{O}\left(\frac{1}{r^{2D-5}}\right). \quad (27)$$

The fact that there are no additional contributions to the energy of charged black holes in any physically sensible NED theory is a consequence of the metric falloff, which is the same as in the Reissner-Nordstrom case.

The procedure outlined above can be regarded as an algorithm to construct explicit solutions to various NED theories: conformally invariant electrodynamics, Born-Infeld, logarithmic electrodynamics, etc., as was pointed out in Ref. [28].

III. IYER-WALD CHARGES AND BLACK HOLE ENTROPY

At this point, we proceed to evaluate the Euclidean continuation of the action for EGB gravity coupled to NED in Eqs. (5) and (6).

As a first step, without loss of generality, we take Gauss-normal coordinates to foliate the spacetime along a radial direction

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(r)dr^2 + h_{ij}(r,x)dx^i dx^j. \quad (28)$$

This frame is obtained as a particular gauge-fixing ($N^i = 0$) of a general ADM form of the metric. We assume the manifold to have a boundary $\partial\mathcal{M}$, located at radial infinity, endowed with an induced metric h_{ij}.

The identities for the totally antisymmetric product of Kronecker deltas listed in Appendix A enable us to write down the pure gravity part of the bulk action as

$$I_{\text{grav}} = \frac{1}{16\pi G(D-2)!}\int_{\mathcal{M}} d^D x\sqrt{-g}\,\delta^{[\mu_1\cdots\mu_D]}_{[\nu_1\cdots\nu_D]} \times\left(\frac{1}{2}R^{\nu_1\nu_2}_{\mu_1\mu_2}\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \frac{D-2}{D\ell^2}\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \frac{\alpha(D-2)(D-3)}{4}R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}\right)\delta^{\nu_5}_{\mu_5}\cdots\delta^{\nu_D}_{\mu_D}. \quad (29)$$

In the foliation (28), the indices split as $\mu = (r, i)$, where Latin letters denote boundary components. Plugged in the total action Eq. (4), this evaluation produces

$$I_0 = \frac{1}{16\pi G(D-3)(D-4)}\int_{\mathcal{M}} d^{D-1}x dr\sqrt{-h}N\delta^{[j_1\cdots j_4]}_{[i_1\cdots i_4]} \times\left[\left(\frac{1}{2}R^{i_1i_2}_{j_1j_2} + \frac{1}{\ell^2}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \frac{2}{D-2}R^{ri_1}_{ri_1}\delta^{i_2}_{j_2}\right)\delta^{i_3}_{j_3}\delta^{i_4}_{j_4} + \alpha(D-3)\left(\frac{D-4}{4}R^{i_1i_2}_{j_1j_2} + 2R^{ri_1}_{rj_1}\delta^{i_2}_{j_2}\right)R^{i_3i_4}_{j_3j_4}\right] + \int_{\mathcal{M}} d^{D-1}x dr\sqrt{-h}N\mathcal{L}(F^2), \quad (30)$$

where we have used the fact that $\delta^{[rj_1\cdots j_4]}_{[ri_1\cdots i_4]} = \delta^{[j_1\cdots j_4]}_{[i_1\cdots i_4]}$. One can recognize the component $\mathcal{E}^r_r$ of the equations of motion in the form (A5) (see Appendix A) from the first two terms in the second line and the first term in the third one. In doing so, the action in radial normal frame reads

$$I_0 = \frac{1}{8\pi G(D-2)(D-3)} \times\int_{\mathcal{M}} d^{D-1}x dr\sqrt{-h}N\delta^{[j_1\cdots j_3]}_{[i_1\cdots i_3]}R^{ri_1}_{ri_1}(\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \alpha(D-2)(D-3)R^{i_2i_3}_{j_2j_3}) - \int_{\mathcal{M}} d^{D-1}x dr\sqrt{-h}N\left(\frac{1}{8\pi G}\mathcal{E}^r_r + T^r_r - \mathcal{L}(F^2)\right). \quad (31)$$

We consider a spacetime given by the Euclidean continuation of the static black hole metric (14), where the Euclidean time $\tau = it$ appears as identified in a period β, which is the inverse of the Hawking temperature

$$T \equiv \beta^{-1} = \frac{1}{4\pi}\frac{df^2(r)}{dr}\bigg|_{r=r_+}. \quad (32)$$

As the horizon is reduced to a single point, the Euclidean period in Eq. (32) smooths out the origin of the radial coordinate $r = r_+$, avoiding the presence of a conelike singularity. We stress the fact that no boundary is introduced at the horizon, such that no extra surface terms are needed there in order to reproduce the correct black hole thermodynamics.

More explicitly, the Hawking temperature is given by

$$T=\frac{1}{4\pi r_+}\frac{(D-3)k+\frac{(D-1)r_+^2}{\ell^2}+\alpha(D-3)(D-4)(D-5)\frac{k^2}{r_+^2}+\frac{16\pi G r_+^2}{D-2}T_r^r(q,r_+)}{1+2\alpha(D-3)(D-4)\frac{k}{r_+^2}}, \tag{33}$$

where the electromagnetic field contributes to the black hole temperature trough the tensor $T_r^r(q,r_+)=\mathcal{L}+4E^2\frac{d\mathcal{L}}{dF^2}$ evaluated at $F^2=-2E^2$ and for $r=r_+$.

The Wick rotation implies $I^E=-iI$ for the Euclidean action, which can be expressed in terms of the black hole metric function f^2 and its derivatives as

$$\begin{aligned}I_0^E=&\frac{1}{16\pi G}\int_0^\beta d\tau\int_{\Gamma_{D-2}}\sqrt{\gamma}d^{D-2}\varphi\int_{r_+}^\infty dr r^{D-2}\\&\times\bigg[(f^2)''+(D-2)\frac{(f^2)'}{r}+\frac{2\alpha(D-2)(D-3)}{r^2}\\&\times\bigg((f^2)''(k-f^2)-(f^2)'^2+(D-4)\frac{(f^2)'(k-f^2)}{r}\bigg)\bigg]\\&+4\int_0^\beta d\tau\int_{\Gamma_{D-2}}\sqrt{\gamma}d^{D-2}\varphi\\&\times\int_{r_+}^\infty dr r^{D-2}E^2\frac{d\mathcal{L}}{dF^2}\bigg|_{F^2=-2E^2},\end{aligned} \tag{34}$$

where we have substituted the expressions for the Riemann tensor in Appendix C and the electromagnetic stress tensor (19). Upon a trivial integration on the boundary coordinates, all the integrand can be written as a total derivative, and the bulk Euclidean action becomes

$$\begin{aligned}I_0^E=&\frac{\beta\mathrm{Vol}(\Gamma_{D-2})}{16\pi G}[(f^2)'(r^{D-2}+2\alpha(D-2)\\&\times(D-3)r^{D-4}(k-f^2))]|_{r_+}^\infty-\beta\mathrm{Vol}(\Gamma_{D-2})\\&\times\bigg(4r^{D-2}\phi E\frac{d\mathcal{L}}{dF^2}\bigg)\bigg|_{r_+}^\infty,\end{aligned} \tag{35}$$

where, in the last line, the equation of motion for the NED field (20) was used.

By definition of the Euclidean period β, the first line evaluated at the horizon r_+ produces $-S$, where S is the standard value of black hole entropy in EGB gravity [29,30]

$$S=\frac{\mathrm{Vol}(\Gamma_{D-2})r_+^{D-2}}{4G}\bigg(1+\frac{2k\alpha}{r_+^2}(D-2)(D-3)\bigg), \tag{36}$$

whereas the second line is $\beta Q\Phi$, where

$$Q=4\mathrm{Vol}(\Gamma_{D-2})q. \tag{37}$$

Notice that, what is physically relevant for the thermodynamic description of NED action, is the potential difference between the horizon and infinity. Therefore, we assume $\phi(\infty)=0$ without loss of generality.

Thus, the Euclidean continuation of the bulk action for EGB-AdS gravity coupled to NED is written as

$$\begin{aligned}I_0^E=&\beta Q\Phi-S+\frac{\beta\mathrm{Vol}(\Gamma_{D-2})}{16\pi G}\lim_{r\to\infty}[(f^2)'\\&\times(r^{D-2}+2\alpha(D-2)(D-3)r^{D-4}(k-f^2))],\end{aligned} \tag{38}$$

which, in terms of the black hole mass [18,31–33]

$$M=(D-2)\frac{\mathrm{Vol}(\Gamma_{D-2})\mu}{16\pi G}, \tag{39}$$

can be recast as

$$\begin{aligned}I_0^E=&\beta Q\Phi-S+\beta M\frac{D-2}{D-3}\frac{\ell_{\mathrm{eff}}^2-2\alpha(D-2)(D-5)}{\ell_{\mathrm{eff}}^2-2\alpha(D-3)(D-4)}\\&+\frac{\beta\mathrm{Vol}(\Gamma_{D-2})}{16\pi G}\lim_{r\to\infty}\bigg[\frac{r^{D-3}}{\ell_{\mathrm{eff}}^2}(1+2\alpha(D-2)\\&\times(D-3)r^{D-4}(k-f^2))\bigg].\end{aligned} \tag{40}$$

The unusual factor multiplying βM indicates that a background-subtraction method would not necessarily give rise to a correct QSR. Indeed, subtracting the value of I_0^E evaluated for the AdS vacuum (with the corresponding AdS radius ℓ_{eff}) gets rid of the divergences at $r=\infty$, but does not reproduce the mass in the asymptotic region. The quantity that appears at radial infinity is the analog of the Komar formula for EGB gravity [34].

It is possible to understand the above result in the light of the Iyer-Wald definition of conserved quantities in gravity theories [35]. This time-honored procedure interprets the black hole entropy as the Noether charge $\tilde{Q}[\xi]$ for a Killing vector $\xi=\xi^\mu\partial_\mu$, evaluated at the horizon. This quantity, however, is derived exclusively from the bulk Lagrangian of the gravity theory (which in our case is denoted by $\mathcal{L}_{\mathrm{grav}}$), without additional boundary terms. More concretely, the charge $\tilde{Q}[\xi]$ is written in terms of the derivative of the bulk Lagrangian with respect to the Lorentz curvature two-form. In tensorial notation, we can work out an alternative form if the Lagrangian density is rewritten as $\mathcal{L}_{\mathrm{grav}}=\delta^{[\mu_1\cdots\mu_D]}_{[\nu_1\cdots\nu_D]}L^{\nu_1\cdots\nu_D}_{\mu_1\cdots\mu_D}$. In this way, the charge is given by the fully covariant formula in terms of derivatives of the Riemann tensor

$$\begin{aligned}\tilde{Q}[\xi]=&\int_{\partial\mathcal{M}_r\cap\Xi_t}d^{D-2}x\sqrt{-g}\hat{n}_\mu\hat{u}_\nu\xi^\lambda\delta^{[\mu_1\cdots\mu_D]}_{[\nu_1\cdots\nu_D]}\\&\times g^{\alpha\sigma}\Gamma^\beta_{\lambda\sigma}\frac{\delta L^{\nu_1\cdots\nu_D}_{\mu_1\cdots\mu_D}}{\delta R^{\alpha\beta}_{\mu\nu}},\end{aligned} \tag{41}$$

which must be evaluated at the intersection of a $r=$ const boundary $\partial\mathcal{M}_r$ with a constant-time slice Ξ_t [35–37].

The normal vectors $\hat{n}_\mu$ and $\hat{u}_\mu$ generate the corresponding foliations which describe $\partial\mathcal{M}_r$ and Ξ_t, respectively [38].

The form of the action (29) for EGB gravity is particularly useful to compute the above expression, which produces

$$\tilde{Q}[\xi] = -\frac{1}{16\pi G(D-2)(D-3)} \times \int_{\partial\mathcal{M}_r\cap\Xi_t} d^{D-2}x\sqrt{-g}\hat{n}_{\mu_1}\hat{u}_{\mu_2}\xi^\lambda \delta^{[\mu_1\mu_2\mu_3\mu_4]}_{[\nu_1\nu_2\nu_3\nu_4]} \times g^{\nu_1\alpha}\Gamma^{\nu_2}_{\lambda\alpha}(\delta^{[\nu_3\nu_4]}_{[\mu_3\mu_4]} + 2(D-2)(D-3)\alpha R^{\nu_3\nu_4}_{\mu_3\mu_4}). \tag{42}$$

In the Gauss-normal frame (28), the normal derivative of the induced metric h_{ij} describes the extrinsic properties of the boundary,

$$K_{ij} = -\frac{1}{2N}h'_{ij}, \tag{43}$$

which defines the extrinsic curvature. Then the Christoffel symbol can be expressed in terms of K_{ij} (see Appendix B), such that the Iyer-Wald charge is

$$\tilde{Q}[\xi] = \frac{1}{16\pi G(D-2)(D-3)} \times \int_{\partial\mathcal{M}_r\cap\Xi_t} d^{D-2}x\sqrt{-h}\hat{u}_j\xi^i \delta^{[jj_2\cdots j_{D-1}]}_{[i_1\cdots i_{D-1}]}K^{i_1}_i \times (\delta^{[i_2i_3]}_{[j_2j_3]} + 2\alpha(D-2)(D-3)R^{i_2i_3}_{j_2j_3}), \tag{44}$$

where $\sqrt{-g} = N\sqrt{-h}$. It is not difficult to show that, for the black hole metric (14), the Noether charge (44) leads to

$$\tilde{Q}[\partial_t] = \frac{\mathrm{Vol}(\Gamma_{D-2})}{16\pi G}(f^2)'(r^{D-2} + 2\alpha(D-2) \times (D-3)r^{D-4}(k-f^2)), \tag{45}$$

for an arbitrary surface Γ_{D-2} of radius r. This quantity clearly reproduces the value of the entropy (36) when evaluated at the horizon $r = r_+$, and a finite contribution (which cannot be identified with the mass) plus a divergent term at infinity, which are the same terms present in Eq. (40). The fact that I_0^E can be written down as

$$I_0^E = \beta Q\Phi + \beta\tilde{Q}[\partial_t]|^\infty_{r_+} \tag{46}$$

means that the problem of finiteness of the Euclidean action is connected to the regularization of the Noether charges in the asymptotic region.

It is possible to correct the Iyer-Wald formula, such that it provides the correct mass and angular momentum for black holes, through the construction of a Hamiltonian $H[\xi]$ which describes the dynamics generated by the vector field ξ^μ. Generally speaking, when one varies the action with respect to the fields ϕ, one identifies the equations of motion plus a surface term

$$\delta I = \int_{\mathcal{M}}(\mathrm{EOM})\delta\phi + \int_{\partial\mathcal{M}}\Theta(\phi, \delta\phi). \tag{47}$$

Then, the Hamiltonian is related to the Iyer-Wald charge at radial infinity $\tilde{Q}_\infty[\xi] = \int_{\Sigma_\infty}\tilde{\mathcal{Q}}(\xi)$ and the surface term in Eq. (47) by

$$\delta H[\xi] = \delta\int_{\Sigma_\infty}\tilde{\mathcal{Q}}(\xi) - \int_{\Sigma_\infty}\xi\cdot\Theta(\phi, \delta\phi), \tag{48}$$

where $\Sigma_\infty = \partial\mathcal{M}\cap\Xi_t$. The Hamiltonian exists if there is a $(D-1)$-form $\mathcal{B}$ such that the second term is a total variation,

$$\int_{\Sigma_\infty}\xi\cdot\Theta(\phi, \delta\phi) = \delta\int_{\Sigma_\infty}\xi\cdot\mathcal{B}(\phi). \tag{49}$$

Whenever this is possible, one can write down the Wald Hamiltonian as [35]

$$H[\xi] = \int_{\Sigma_\infty}(\tilde{\mathcal{Q}}(\xi) - \xi\cdot\mathcal{B}). \tag{50}$$

However, the procedure outlined above in general breaks covariance at the boundary. As a result, the correction $\mathcal{B}$ to the charge has a noncovariant form, and has to be built on a case-by-case basis for different solutions. Moreover, if it exists, $\mathcal{B}$ cannot be used as an appropriate boundary term to render the Euclidean action finite.

In what follows, we employ covariant counterterms given as polynomials in the extrinsic and intrinsic curvatures to regularize the Euclidean action and to obtain the QSR given by Eq. (1).

IV. KOUNTERTERM REGULARIZATION AND QUANTUM STATISTICAL RELATION

The regularization of gravitational action prescribed by AdS/CFT correspondence leads to the addition of covariant functionals of the boundary metric and curvature, known as counterterm series. In a dual quantum field theory, this counterterm series corresponds to a standard UV divergence removal by adding finite polynomials in the fields.

However, in EGB-AdS gravity, holographic renormalization becomes too involved to provide a general answer to this problem.

Here, we use an alternative regularization procedure, where the counterterms depend explicitly on the extrinsic curvature. The choice of such terms in AdS gravity is justified by the asymptotic expansions of the fields. Indeed, the extrinsic curvature and the boundary metric are proportional at the leading order. In that way, the boundary term B_{D-1} is given as a unique geometrical structure depending only on the dimension. This approach circumvents the technicalities of holographic methods for higher-curvature theories of the Lovelock class [39].

We will work with gravity-NED action in D dimensions supplemented by a boundary term

$$I = I_0 + c_{D-1}\int_{\partial\mathcal{M}} d^{D-1}xB_{D-1}, \tag{51}$$

where the coupling c_{D-1} is fixed demanding a well-defined action principle [18,28]. The explicit form of the Kounterterm series as a polynomial of the extrinsic and intrinsic curvatures for EH AdS gravity was introduced in Refs. [40,41].

For the current discussion, we consider the grand canonical ensemble, where the temperature T and the electric potential Φ are held fixed at the horizon. The Gibbs free energy

$$G(T,\Phi) = U - TS + \Phi Q, \tag{52}$$

which satisfies the differential equation

$$dG = -SdT + Qd\Phi, \tag{53}$$

is given in terms of the Euclidean action as

$$G = TI^E \tag{54}$$

such that the partition function in semiclassical approximation is written in terms of Gibbs energy as

$$Z = e^{-G/T}. \tag{55}$$

In what follows, we employ Kounterterms to regulate the value of the Euclidean action for EGB-AdS coupled to NED and to obtain the QSR (1) for charged black hole solutions.

A. Even dimensions

In a manifold $\mathcal{M}$ without boundary in even dimensions $D = 2n$, the integration of the Euler topological invariant produces the Euler characteristic, $\chi(\mathcal{M})$. If a boundary $\partial\mathcal{M}$ is introduced, there appears a correction to $\chi(\mathcal{M})$ given by the nth Chern form

$$B_{2n-1} = 2n\sqrt{-h}\int_0^1 dt\,\delta^{[j_1\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]}K^{i_1}_{j_1}\left(\frac{1}{2}\mathcal{R}^{i_2i_3}_{j_2j_3} - t^2K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\cdots\left(\frac{1}{2}\mathcal{R}^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}} - t^2K^{i_{2n-2}}_{j_{2n-2}}K^{i_{2n-1}}_{j_{2n-1}}\right). \tag{56}$$

Here, $\mathcal{R}^{ij}_{kl}(h)$ is the intrinsic curvature of the boundary, related to the spacetime Riemann tensor by $R^{ij}_{kl} = \mathcal{R}^{ij}_{kl} - K^i_kK^j_l + K^i_lK^j_k$ (see Appendix B). In asymptotically AdS spacetimes, the addition of the above boundary term to the gravity action defines an extrinsic regularization scheme in even dimensions. In EGB-AdS gravity, it has been shown in Ref. [18] that the constant c_{2n-1} in front of the boundary term B_{2n-1} has to be fixed in terms of the effective AdS radius as

$$c_{2n-1} = -\frac{1}{16\pi G}\frac{(-\ell^2_{\text{eff}})^{n-1}}{n(2n-2)!}\left(1 - \frac{2\alpha}{\ell^2_{\text{eff}}}(2n-2)(2n-3)\right), \tag{57}$$

in order to cancel the divergences in the Euclidean action. Notice that the integration on the continuous parameter t generates the coefficients if one wants to express the boundary term as a polynomial. This compact way of writing B_{2n-1} is not a mere formality but it reflects the relation to topological invariants and provides a useful tool for explicit evaluations, as well.

The boundary term (56) evaluated on the Euclidean continuation of the black hole metric (14) becomes

$$\begin{aligned}\int_{\partial\mathcal{M}} d^{2n-1}xB^E_{2n-1} &= -2n\lim_{r\to\infty}\int_0^\beta d\tau\int_{\Gamma_{D-2}}\sqrt{\gamma}d^{D-2}\varphi r^{2n-2}f \\ &\times\int_0^1 dt\,\delta^{[m_1\cdots m_{2n-2}]}_{[n_1\cdots n_{2n-2}]}K^\tau_\tau\left(\frac{1}{2}\mathcal{R}^{n_1n_2}_{m_1m_2} - (2n-1)t^2K^{n_1}_{m_1}K^{n_2}_{m_2}\right) \\ &\times\left(\frac{1}{2}\mathcal{R}^{n_3n_4}_{m_3m_4} - t^2K^{n_3}_{m_3}K^{n_4}_{m_4}\right)\cdots \\ &\times\left(\frac{1}{2}\mathcal{R}^{n_{2n-3}n_{2n-2}}_{m_{2n-3}m_{2n-2}} - t^2K^{n_{2n-3}}_{m_{2n-3}}K^{n_{2n-2}}_{m_{2n-2}}\right).\end{aligned} \tag{58}$$

Substituting the components of the extrinsic curvature (C1) and intrinsic curvature (C2) (see Appendix C) and performing the integral

$$\int_0^1 dt[k - (2n-1)t^2f^2](k - t^2f^2)^{n-2} = (k - f^2)^{n-1}, \tag{59}$$

the boundary term takes the form

$$\begin{aligned}c_{2n-1}\int_{\partial\mathcal{M}} d^{2n-1}xB^E_{2n-1} &= \frac{\beta\text{Vol}(\Gamma_{D-2})\ell^{2n-2}_{\text{eff}}}{16\pi G}\left(1 - \frac{2\alpha}{\ell^2_{\text{eff}}}(2n-2)(2n-3)\right) \\ &\times (f^2)'(f^2-k)^{n-1}|^{r=\infty}.\end{aligned} \tag{60}$$

In consequence, the total Euclidean action

$$I^E_{2n} = I^E_0 + c_{2n-1}\int_{\partial\mathcal{M}} d^{2n-1}xB^E_{2n-1} \tag{61}$$

can be written as

$$\begin{aligned}I^E_{2n} = \frac{\beta\text{Vol}(\Gamma_{D-2})}{16\pi G}\Big\{&(f^2)'(r^{D-2} + 2\alpha(2n-2) \\ &\times(2n-3)r^{D-4}(k-f^2))|^\infty_{r_+} - \ell^{2n-2}_{\text{eff}} \\ &\times\left(1 - \frac{2\alpha}{\ell^2_{\text{eff}}}(2n-2)(2n-3)\right) \\ &\times[(f^2)'(f^2-k)^{n-1}]|^{r=\infty}\Big\} + \beta Q\Phi.\end{aligned} \tag{62}$$

The contribution at infinity from the bulk action combines with the one from the boundary to produce β times the Noether mass M

$$M = \frac{\mathrm{Vol}(\Gamma_{2n-2})}{16\pi G}\lim_{r\to\infty} r^{2n-2}(f^2)'\Big[1-2\alpha(2n-2)(2n-3)\times\frac{f^2-k}{r^2}-\Big(1-\frac{2\alpha(2n-2)(2n-3)}{\ell_{\mathrm{eff}}^2}\Big)\ell_{\mathrm{eff}}^{2n-2}\times\Big(\frac{f^2-k}{r^2}\Big)^{n-1}\Big], \quad (63)$$

which agrees with the formula (39) when expressed in terms of μ [28]. In other words, because the Kounterterm series leads to the cancellation of divergences in the asymptotic charges, the finiteness of the Euclidean action is ensured for any static black hole and satisfies the QSR

$$G(T,\Phi) = TI_{2n}^E = U - TS + Q\Phi, \quad (64)$$

where the EGB black hole entropy is given by Eq. (36).

B. Odd dimensions

In $D = 2n+1$ dimensions, Kounterterm regularization provides the explicit form of the boundary terms which makes the Euclidean action finite in EH AdS gravity [40]. The universality of the Kounterterm series ensures that the action will be also finite in any Lovelock gravity with AdS branches. That means that the information on a particular theory is incorporated in the regularization scheme through effective AdS radius ℓ_{eff} and the coupling constant c_{2n}. Thus, in general, the Kounterterms series is given in terms of the parametric integrations

$$B_{2n} = \frac{n}{2^{n-2}}\sqrt{-h}\int_0^1 dt\int_0^t ds\,\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}K^{i_1}_{j_1}\delta^{i_2}_{j_2}\mathcal{F}^{i_3i_4}_{j_3j_4}(t,s)\cdots\mathcal{F}^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}}(t,s), \quad (65)$$

where we introduce an auxiliary quantity with the symmetries of the Riemann tensor

$$\mathcal{F}^{ij}_{kl}(t,s) = \mathcal{R}^{ij}_{kl} - t^2(K^i_kK^j_l - K^i_lK^j_k) + \frac{s^2}{\ell_{\mathrm{eff}}^2}\delta^{[ij]}_{[kl]}, \quad (66)$$

such that in EGB-AdS theory the coupling reads [18]

$$c_{2n} = -\frac{1}{16\pi G}\frac{(-\ell_{\mathrm{eff}}^2)^{n-1}}{n(2n-1)!}\Big(1-\frac{2\alpha(2n-1)(2n-2)}{\ell_{\mathrm{eff}}^2}\Big)\times\Big[\int_0^1 dt(1-t^2)^{n-1}\Big]^{-1} = -\frac{1}{16\pi G}\frac{2(-\ell_{\mathrm{eff}}^2)^{n-1}}{n(2n-1)!\beta(n,\frac{1}{2})}\Big(1-\frac{2\alpha(2n-1)(2n-2)}{\ell_{\mathrm{eff}}^2}\Big), \quad (67)$$

where $\beta(n,\frac{1}{2}) = \frac{2^{2n-1}(n-1)!^2}{(2n-1)!}$ is the Beta function for those arguments.

In the Euclidean sector, the boundary term in the black hole ansatz (14) produces

$$\int_{\partial\mathcal{M}} d^{2n}xB_{2n}^E = -\frac{n}{2^{n-3}}\lim_{r\to\infty}\int_0^\beta d\tau\int_{\Gamma_{D-2}}\sqrt{\gamma}d^{D-2}\varphi r^{2n-1}\times f\int_0^1 dt\int_0^t ds\,\delta^{[m_1\cdots m_{2n-1}]}_{[n_1\cdots n_{2n-1}]}\mathcal{F}^{n_4n_5}_{m_4m_5}\cdots\mathcal{F}^{n_{2n-2}n_{2n-1}}_{m_{2n-2}m_{2n-1}}\times\Big[(K^\tau_\tau\delta^{n_1}_{m_1}+K^{n_1}_{m_1})\mathcal{F}^{n_2n_3}_{m_2m_3} + 2(n-1)K^{n_1}_{m_1}\delta^{n_2}_{m_2}\Big(-t^2K^\tau_\tau K^{n_3}_{m_3}+\frac{s^2}{\ell_{\mathrm{eff}}^2}\delta^{n_3}_{m_3}\Big)\Big], \quad (68)$$

or, more explicitly, after using the expressions (C1) and (C2),

$$\int_{\partial\mathcal{M}} d^{2n}xB_{2n}^E = 2n(2n-1)!\beta\mathrm{Vol}(\Gamma_{2n-1})\lim_{r\to\infty}\int_0^1 dt\int_0^t ds\times\Big(k-t^2f^2+s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)^{n-2}\times\Big[\frac{r(f^2)'}{2}\Big(k-(2n-1)t^2f^2+s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big) + f^2\Big(k-t^2f^2+(2n-1)s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)\Big]. \quad (69)$$

In order to recognize the asymptotic charges from the Euclidean action, it is especially convenient to convert the above double integrals into a single-parameter integration, i.e.,

$$\int_0^1 dt\int_0^t ds\Big(k-(2n-1)t^2f^2+s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)\times\Big(k-t^2f^2+s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)^{n-2} = \int_0^1 dt\Big(k-f^2+t^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)^{n-1} - \int_0^1 dtt\Big(k-t^2f^2+t^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)^{n-1}$$

and

$$\int_0^1 dt\int_0^t ds\Big(k-t^2f^2+(2n-1)s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)\times\Big(k-t^2f^2+s^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)^{n-2} = \int_0^1 dtt\Big(k-t^2f^2+t^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\Big)^{n-1}.$$

Thus, the surface term is expressed as

$$\int_{\partial\mathcal{M}} d^{2n}x B_{2n}^{E} = n(2n-1)!\beta\mathrm{Vol}(\Gamma_{2n-1})\lim_{r\to\infty}\times\left[r^{2n-1}(f^2)'\int_0^1 dt\left(\frac{k-f^2}{r^2}+\frac{t^2}{\ell_{\mathrm{eff}}^2}\right)^{n-1} + 2\left(f^2-\frac{r(f^2)'}{2}\right)\int_0^1 dtt\left(k-t^2f^2+t^2\frac{r^2}{\ell_{\mathrm{eff}}^2}\right)^{n-1}\right]. \quad (70)$$

When the above boundary term is added to the bulk Euclidean action (35) with a suitable coupling constant,

$$I_{2n+1}^{E} = I_0^E + c_{2n}\int_{\partial\mathcal{M}} d^{2n}x B_{2n}^{E}, \quad (71)$$

one gets

$$I_{2n+1}^{E} = \frac{\beta\mathrm{Vol}(\Gamma_{2n-1})}{16\pi G}\times\left[r^{2n-1}(f^2)'\left(1-2\alpha(D-2)(D-3)\frac{f^2-k}{r^2}\right)\right]\Bigg|_{r_+}^{\infty} + \beta\mathrm{Vol}(\Gamma_{2n-1})nc_{2n}(2n-1)!\Bigg[r^{2n-1}(f^2)'\times\int_0^1 dt\left(\frac{k-f^2}{r^2}+\frac{t^2}{\ell_{\mathrm{eff}}^2}\right)^{n-1} + 2\int_0^1 dtt\left(f^2-\frac{r(f^2)'}{2}\right)\times\left(k+t^2\left(\frac{r^2}{\ell_{\mathrm{eff}}^2}-f^2\right)\right)^{n-1}\Bigg]\Bigg|^{\infty} + \beta Q\Phi. \quad (72)$$

Thus, the contribution coming from radial infinity in the first two lines can be identified with βM, where the mass is [28]

$$M = \frac{\mathrm{Vol}(\Gamma_{2n-1})}{16\pi G}\lim_{r\to\infty} r^{2n-1}(f^2)'\Bigg[1+2\alpha(2n-1)(2n-2)\times\frac{k-f^2}{r^2} + 16\pi G(2n-1)!nc_{2n}\times\int_0^1 dt\left(\frac{k-f^2}{r^2}+\frac{t^2}{\ell_{\mathrm{eff}}^2}\right)^{n-1}\Bigg], \quad (73)$$

which agrees with the mass formula (39) when the metric function is expanded. The term with parametric integration in the third line of Eq. (73) is β times the vacuum energy E_{vac}, which is written in the form

$$E_{\mathrm{vac}} = 2n(2n-1)!c_{2n}\mathrm{Vol}(\Gamma_{2n-1})\lim_{r\to\infty}\times\int_0^1 dtt\left(f^2-\frac{r(f^2)'}{2}\right)\left[k+\left(\frac{r^2}{\ell_{\mathrm{eff}}^2}-f^2\right)t^2\right]^{n-1}. \quad (74)$$

The consistency of the black hole thermodynamics is therefore verified through the QSR, which involves the black hole entropy (36)

$$G(T,\Phi) = TI_{2n+1}^{E} = U - TS + Q\Phi, \quad (75)$$

for a total energy which includes the zero-point energy E_{vac},

$$U = M + E_{\mathrm{vac}}. \quad (76)$$

C. Thermodynamic charges

Until now, we have obtained the QSR (1) for electrically charged black holes in gravitating NED, which involves the total energy U and the electric charge Q as asymptotic Noether charges, computed in Ref. [28], and the entropy S as the Wald charge at the horizon. However, strictly speaking, one should be able to reproduce these quantities from thermodynamic relations in an independent way.

From that point of view, the gravitational entropy should be calculated from the Gibbs free energy as

$$S_{\mathrm{TD}} = -\left(\frac{\partial G}{\partial T}\right)_{\Phi}, \quad (77)$$

and the electric charge

$$Q_{\mathrm{TD}} = \left(\frac{\partial G}{\partial \Phi}\right)_{T}, \quad (78)$$

whereas the internal energy can be derived as a thermodynamic quantity from

$$U_{\mathrm{TD}} = G - T\left(\frac{\partial G}{\partial T}\right)_{\Phi} - \Phi\left(\frac{\partial G}{\partial \Phi}\right)_{T}. \quad (79)$$

In general, arbitrary variations of G in Eq. (52) produce in terms of the Noether charges

$$dG = dU - TdS + \Phi dQ - SdT + Qd\Phi. \quad (80)$$

It is clear that if Noether charges satisfy the first law (2), then the relations (77)–(79) identify thermodynamic with Noether charges.

In order to prove the first law, it is convenient to introduce the variable η,

$$\eta \equiv \frac{q}{r_+^{D-2}}, \quad (81)$$

and write the electric charge as

$$Q(r_+,\eta) = 4\mathrm{Vol}(\Gamma_{D-2})\eta r_+^{D-2}. \quad (82)$$

A more explicit expression for the Hawking temperature is

$$T(r_+,\eta)=\frac{(D-3)k+\frac{(D-1)r_+^2}{\ell^2}+\alpha(D-3)(D-4)(D-5)\frac{k^2}{r_+^2}+\frac{16\pi G r_+^2}{D-2}(\mathcal{L}_+-4\eta E_+)}{4\pi r_+(1+2\alpha(D-3)(D-4)\frac{k}{r_+^2})}, \tag{83}$$

where the index "+" denotes quantities evaluated at $r=r_+$.

The generalized Gauss law,

$$E_+\frac{\partial\mathcal{L}}{\partial F^2}\bigg|_{r_+}=-\eta, \tag{84}$$

determines the electric field $E_+=E(r_+,q)$ such that it is a function of η only, and so are $\mathcal{L}_+$ and its first derivative. A useful relation is

$$\frac{\partial\mathcal{L}_+}{\partial\eta}=\frac{\partial\mathcal{L}}{\partial F^2}\bigg|_{r_+}\frac{\partial F^2}{\partial\eta}=4\eta\frac{\partial E_+}{\partial\eta}. \tag{85}$$

From the definition of horizon, $f(r_+)=0$, we also obtain

$$U(r_+,\eta)=E_{\mathrm{vac}}+(D-2)\frac{\mathrm{Vol}(\Gamma_{D-2})}{16\pi G}\bigg[kr_+^{D-3}+k^2\alpha(D-3)\times(D-4)r_+^{D-5}+\frac{r_+^{D-1}}{\ell^2}+\frac{16\pi G\mathcal{T}_+}{D-2}\bigg], \tag{86}$$

where E_{vac} is vanishing for even dimensions and $\mathcal{T}_+$ is the function (24) evaluated at the horizon,

$$\begin{aligned}\mathcal{T}_+&=\frac{1}{D-1}(r^{D-1}\mathcal{L}-4qrE+(D-2)4q\phi)|_\infty^{r_+}\\&=\frac{1}{D-1}(r_+^{D-1}\mathcal{L}_+-4r_+^{D-1}\eta E_+-(D-2)4r_+^{D-2}\eta\Phi).\end{aligned} \tag{87}$$

Finally, the electric potential between infinity and the event horizon has the form

$$\Phi(r_+,\eta)=-\frac{r_+\eta^{1/(D-2)}}{D-2}\int_0^\eta du\,u^{-((D-1)/(D-2))}E(r_+,u), \tag{88}$$

where $u=q/r^{D-2}=\eta(r_+/r)^{D-2}$.

When varied, the entropy, charge, and total energy change as

$$dS=\frac{\mathrm{Vol}(\Gamma_{D-2})}{4G}(D-2)r_+^{D-3}\times\bigg(1+2\alpha(D-3)(D-4)\frac{k}{r_+^2}\bigg)dr_+, \tag{89}$$

$$dQ=4\mathrm{Vol}(\Gamma_{D-2})r_+^{D-3}((D-2)\eta dr_++r_+d\eta), \tag{90}$$

$$dU=(D-2)\frac{\mathrm{Vol}(\Gamma_{D-2})}{16\pi G}r_+^{D-4}\bigg[(D-3)k+\alpha(D-3)\times(D-4)(D-5)\frac{k^2}{r_+^2}+(D-1)\frac{r_+^2}{\ell^2}\bigg]dr_++\mathrm{Vol}(\Gamma_{D-2})d\mathcal{T}_+. \tag{91}$$

The last line involves the variation of $\mathcal{T}_+$ which, using Eq. (87) and the variation of Φ,

$$d\Phi=\frac{\Phi}{r_+}dr_++\frac{\Phi-r_+E_+}{(D-2)\eta}d\eta, \tag{92}$$

can be left in the more suitable form

$$d\mathcal{T}_+=[r_+^{D-2}(\mathcal{L}_+-4\eta E_+)-4(D-2)r_+^{D-3}\eta\Phi]dr_+-4r_+^{D-2}\Phi d\eta. \tag{93}$$

With the formula for T given by Eq. (83), we finally find that

$$dU=T(D-2)\frac{\mathrm{Vol}(\Gamma_{D-2})}{4G}r_+^{D-3}\times\bigg(1+2\alpha(D-3)(D-4)\frac{k}{r_+^2}\bigg)dr_+-4\mathrm{Vol}(\Gamma_{D-2})r_+^{D-3}((D-2)\eta dr_++r_+d\eta)\Phi, \tag{94}$$

from where we recognize the variations dS and dQ as in Eqs. (89) and (90), such that the first law holds.

In this case, the Gibbs free energy satisfies Eq. (53), and as a consequence, the thermodynamic relations (77)–(79) are also valid for the corresponding Noether charges, which implies

$$S_{\mathrm{TD}}=S,\qquad Q_{\mathrm{TD}}=Q,\qquad U_{\mathrm{TD}}=U=M+E_{\mathrm{vac}}. \tag{95}$$

It is worthwhile stressing that, while the proof of the first law is insensitive to the presence of the vacuum energy E_{vac}, the last equation defines the internal energy as the total energy of the black hole, as expected in the context of AdS/CFT correspondence.

V. CONCLUSIONS

In this paper, we have studied the thermodynamics of topological static AdS black holes in Einstein-Gauss-Bonnet theory in the presence of an electric field described by an arbitrary nonlinear electrodynamics. This is done using a background-independent regularization scheme which considers the addition of counterterms depending on both the intrinsic and extrinsic curvature of the boundary metric, to the action.

The fact that the Kounterterm series is given as a compact expression, i.e., in a parametric representation of the polynomials, provides us with a practical tool to evaluate the regularized action in all dimensions. Likewise, this also makes it easier to recognize the quantities appearing at radial infinity as the conserved quantities of the theory. This issue is particularly relevant, because it connects the finiteness of the Euclidean action with the problem of regularization of the asymptotic charges treated in Ref. [28] and, ultimately, with a well-posed variational principle.

In the standard counterterms method, it is not possible to separate the contributions to the quasilocal stress tensor that produce the black hole mass M from those which give rise to the vacuum energy. This means that in the evaluation of the Euclidean action I^E one cannot clearly identified the term $\beta E_{\rm vac}$ from the terms at $r = \infty$ in all odd dimensions. Nonetheless, one could work out a relation between the Kounterterms and the intrinsic prescription given by holographic renormalization, by taking the asymptotic expansion of the extrinsic curvature (for a similar procedure in Einstein-Hilbert gravity, see Ref. [42]).

We also performed an independent checking, which is obtaining the total energy U and electric charge Q as thermodynamic quantities from Eqs. (79) and (78), respectively. As expected, the internal energy contains a Casimir contribution in odd dimensions, consistent with the Noether approach.

The generality of the procedure is a consequence of the fact that, in the context of Kounterterm regularization, the quantum statistical relation appears as a thermodynamic identity of the gravity-NED action plus boundary terms. This suggests an extension of the above results to other Lovelock theories and, probably, to other matter couplings, as long as one can define the asymptotic behavior in terms of the curvature as in Eq. (13) (for a thermodynamic study of higher-derivative corrections in the Abelian gauge field of the form F^4, see Ref. [43]).

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APPENDIX A: USEFUL IDENTITIES

The totally antisymmetric Kronecker delta of rank p is defined as the determinant

$$\delta^{[\nu_1\cdots\nu_p]}_{[\mu_1\cdots\mu_p]} := \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots & \delta^{\nu_p}_{\mu_1} \\ \delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & & \delta^{\nu_p}_{\mu_2} \\ \vdots & & \ddots & \\ \delta^{\nu_1}_{\mu_p} & \delta^{\nu_2}_{\mu_p} & \cdots & \delta^{\nu_p}_{\mu_p} \end{vmatrix}. \tag{A1}$$

A contraction of $k \leq p$ indices in the Kronecker delta of rank p produces a delta of rank $p - k$,

$$\delta^{[\nu_1\cdots\nu_k\cdots\nu_p]}_{[\mu_1\cdots\mu_k\cdots\mu_p]}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_k}_{\nu_k} = \frac{(N-p+k)!}{(N-p)!}\delta^{[\nu_{k+1}\cdots\nu_p]}_{[\mu_{k+1}\cdots\mu_p]}, \tag{A2}$$

where N is the range of indices.

Using this compact notation, the Einstein tensor (9) can be rewritten in terms of the AdS radius as

$$G^\mu_\nu = -\frac{1}{2}\delta^{[\mu\mu_1\mu_2]}_{[\nu\nu_1\nu_2]}\left(\frac{1}{2}R^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell^2}\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\right), \tag{A3}$$

and in a similar fashion, the Laczos tensor (10) adopts the form

$$H^\mu_\nu = -\frac{\alpha}{8}\delta^{[\mu\mu_1\cdots\mu_4]}_{[\nu\nu_1\cdots\nu_4]}R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}. \tag{A4}$$

In order to identify the equation of motion for the metric in the evaluation of the Euclidean action, it is convenient to employ the above relations to convert (7) into

$$\begin{aligned}\mathcal{E}^\mu_\nu = &-\frac{1}{2(D-3)(D-4)}\delta^{[\mu\mu_1\cdots\mu_4]}_{[\nu\nu_1\cdots\nu_4]} \\ &\times\bigg[\frac{1}{2}R^{\nu_1\nu_2}_{\mu_1\mu_2}\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \frac{1}{\ell^2}\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} \\ &+\frac{\alpha}{4}(D-3)(D-4)R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}\bigg] - 8\pi G T^\mu_\nu.\end{aligned} \tag{A5}$$

APPENDIX B: GAUSS-NORMAL COORDINATE FRAME

In the Gauss-normal coordinate system (28), the only relevant components of the connection $\Gamma^\alpha_{\mu\nu}$ are expressed in terms of the extrinsic curvature $K_{ij} = -\frac{1}{2N}h'_{ij}$ as

$$\Gamma^r_{ij} = \frac{1}{N}K_{ij}, \qquad \Gamma^i_{rj} = -NK^i_j, \qquad \Gamma^r_{rr} = \frac{N'}{N}. \tag{B1}$$

The radial foliation (28) implies the Gauss-Codazzi relations for the spacetime curvature, as well:

$$R^{ir}_{kl} = \frac{1}{N}(\nabla_l K^i_k - \nabla_k K^i_l), \tag{B2}$$

$$R^{ir}_{kr} = \frac{1}{N}(K^i_k)' - K^i_l K^l_k, \tag{B3}$$

$$R^{ij}_{kl} = \mathcal{R}^{ij}_{kl}(h) - K^i_k K^j_l + K^i_l K^j_k, \tag{B4}$$

where $\nabla_i = \nabla_i(h)$ is the covariant derivative defined in the Christoffel symbol of the boundary $\Gamma^k_{ij}(g) = \Gamma^k_{ij}(h)$ and $\mathcal{R}^{ij}_{kl}(h)$ is the intrinsic curvature of the boundary.

APPENDIX C: TOPOLOGICAL BLACK HOLE METRIC

In the static topological black hole ansatz (14), the extrinsic curvature takes the form

$$K^i_j = -\frac{1}{2N} h^{ik} h'_{kj} = \begin{pmatrix} -f' & 0 \\ 0 & -\frac{f}{r}\delta^m_n \end{pmatrix}, \qquad \text{(C1)}$$

where prime denotes radial derivative, and the nonvanishing components of the boundary curvature are

$$\mathcal{R}^{\,n_1 n_2}_{\,m_1 m_2}(h) = \frac{k}{r^2}\delta^{[n_1 n_2]}_{[m_1 m_2]}. \qquad \text{(C2)}$$

The spacetime Riemann tensor $R^{\mu\nu}_{\lambda\rho}$ is then given by

$$R^{tr}_{tr} = -\frac{1}{2}(f^2)'', \qquad R^{tn}_{tm} = R^{rn}_{rm} = -\frac{1}{2r}(f^2)'\delta^n_m, \qquad R^{mn}_{kl} = \frac{1}{r^2}(k - f^2)\delta^{[mn]}_{[kl]}. \qquad \text{(C3)}$$

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