

Hydrostatic equilibrium and stellar structure in $f(R)$ gravityS. Capozziello,^{1,*} M. De Laurentis,^{1,†} S. D. Odintsov,^{2,‡} and A. Stabile^{3,§}¹*Dipartimento di Scienze Fisiche, Università di Napoli “Federico II”, INFN Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy*²*Institució Catalana de Recerca i Estudis Avançats (ICREA) and Institut de Ciències de l’Espai (IEEC-CSIC), Campus UAB, Facultat de Ciències, Torre C5-Par-2a pl, E-08193 Bellaterra (Barcelona), Spain*³*Dipartimento di Ingegneria, Università del Sannio, Palazzo Dell’Aquila Bosco Lucarelli, Corso Garibaldi, 107-82100, Benevento, Italy*

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We investigate the hydrostatic equilibrium of stellar structure by taking into account the modified Lané-Emden equation coming out from $f(R)$ gravity. Such an equation is obtained in a metric approach by considering the Newtonian limit of $f(R)$ gravity, which gives rise to a modified Poisson equation, and then introducing a relation between pressure and density with polytropic index n . The modified equation results an integro-differential equation, which, in the limit $f(R) \rightarrow R$, becomes the standard Lané-Emden equation. We find the radial profiles of the gravitational potential by solving for some values of n . The comparison of solutions with those coming from general relativity shows that they are compatible and physically relevant.

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I. INTRODUCTION

Extended Theories of Gravity (ETG) [1] are a new paradigm of modern physics aimed to address several shortcomings coming out in the study of gravitational interaction at ultraviolet and infrared scales. In particular, instead of introducing unknown fluids, the approach consists in extending general relativity (GR) by taking into account generic functions of curvature invariants. These functions can be physically motivated and capable of addressing phenomenology at galactic, extragalactic, and cosmological scales [2].

This viewpoint does not require to find candidates for dark energy and dark matter at a fundamental level (not detected up to now), but takes into account only the observed ingredients (i.e., gravity, radiation and baryonic matter), changing the left-hand side of the field equations. Despite this modification, it is in agreement with the spirit of GR since the only request is that the Hilbert-Einstein action should be generalized when asking for a gravitational interaction, acting, in principle, in different ways at different scales but preserving the robust results of GR at local and solar system scales (see [1] for a detailed discussion). This is the case of $f(R)$ gravity, which reduces to GR as soon as $f(R) \rightarrow R$.

Other issues such as, for example, the observed Pioneer anomaly problem [3] can be framed into the same approach [4] and then, apart from the cosmological dynamics,

a systematic analysis of such theories urges at short scales and in the low energy limit.

On the other hand, the strong gravity regime [5] is another way to check the viability of these theories. In general the formation and the evolution of stars can be considered suitable test beds for Alternative Theories of Gravity. Considering the case of $f(R)$ gravity, divergences stemming from the functional form of $f(R)$ may prevent the existence of relativistic stars in these theories [6], but thanks to the chameleon mechanism, introduced by Khoury and Weltman [7], possible problems jeopardizing the existence of these objects may be avoided [8]. Furthermore, there are also numerical solutions corresponding to static star configurations with strong gravitational fields [9] where the choice of the equation of state is crucial for the existence of solutions.

Furthermore, some observed stellar systems are incompatible with the standard models of stellar structure. We refer to anomalous neutron stars, the so-called “magnetars” [10] with masses larger than their expected Volkoff mass. It seems that, on particular length scales, the gravitational force is larger or smaller than the corresponding GR value. For example, a modification of the Hilbert-Einstein Lagrangian, consisting of R^2 terms, enables a major attraction while a $R_{\alpha\beta}R^{\alpha\beta}$ term gives a repulsive contribution [11]. Understanding which scales the modifications to GR are working on or what is the weight of corrections to gravitational potential is a crucial point that could confirm or rule out these extended approaches to gravitational interaction.

The plan of this paper is the following: In Sec. II, we review briefly the classical hydrostatic problem for stellar structures. In Sec. III we derive the Newtonian limit of $f(R)$ gravity obtaining the modified Poisson equation.

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The modified Lané-Emden equation is obtained in Sec. IV and its structure is compared with respect to the standard one. In Sec. V, we show the analytical solutions of standard Lané-Emden equation and compare them with those obtained perturbatively from $f(R)$ gravity. With the help of a plot we can compare all results between them. Discussion and conclusions are drawn in Sec. VI.

II. HYDROSTATIC EQUILIBRIUM OF STELLAR STRUCTURES

The condition of hydrostatic equilibrium for stellar structures in Newtonian dynamics is achieved by considering the equation

$$\frac{dp}{dr} = \frac{d\Phi}{dr} \rho, \quad (1)$$

where p is the pressure, $-\Phi$ is the gravitational potential, and ρ is the density [12]. Together with the above equation, the Poisson equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = -4\pi G \rho, \quad (2)$$

gives the gravitational potential as a solution for given matter density ρ . Since we are taking into account only static and stationary situations, here we consider only time-independent solutions.¹ In general, the temperature τ appears in Eqs. (1) and (2) and the density satisfies an equation of state of the form $\rho = \rho(p, \tau)$. In any case, we assume that there exists a polytropic relation between p and ρ of the form

$$p = K\rho^\gamma, \quad (3)$$

where K and γ are constant. Note that $\Phi > 0$ is in the interior of the model, since we define the gravitational potential as $-\Phi$. The polytropic constant K is fixed and can be obtained as a combination of fundamental constants. However, there are several realistic cases where K is not fixed and another equation for its evolution is needed. The constant γ is the *polytropic exponent*. Inserting the polytropic equation of state into Eq. (1), we obtain

$$\frac{d\Phi}{dr} = \gamma K \rho^{\gamma-2} \frac{d\rho}{dr}. \quad (4)$$

For $\gamma \neq 1$, the above equation can be integrated giving

$$\begin{aligned} \frac{\gamma K}{\gamma-1} \rho^{\gamma-1} &= \Phi \rightarrow \rho = \left[\frac{\gamma-1}{\gamma K} \right]^{1/(\gamma-1)} \Phi^{1/(\gamma-1)} \\ &\doteq A_n \Phi^n, \end{aligned} \quad (5)$$

where we have chosen the integration constant to give $\Phi = 0$ at surface ($\rho = 0$). The constant n is called the *polytropic index* and is defined as $n = \frac{1}{\gamma-1}$. Inserting the

relation (5) into the Poisson equation, we obtain a differential equation for the gravitational potential:

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r} \frac{d\Phi}{dr} = -4\pi G A_n \Phi^n. \quad (6)$$

Let us define now the dimensionless variables:

$$z = |x| \sqrt{\frac{\chi A_n \Phi_c^{n-1}}{2}} \quad w(z) = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c} \right)^{1/n}, \quad (7)$$

where the subscript c refers to the center of the star and the relation between ρ and Φ is given by Eq. (5). At the center ($r = 0$), we have $z = 0$, $\Phi = \Phi_c$, $\rho = \rho_c$ and therefore $w = 1$. Then Eq. (6) can be written as

$$\frac{d^2w}{dz^2} + \frac{2}{z} \frac{dw}{dz} + w^n = 0. \quad (8)$$

This is the standard *Lane-Emden equation* describing the hydrostatic equilibrium of stellar structures in the Newtonian theory [12].

III. THE NEWTONIAN LIMIT OF $f(R)$ GRAVITY

Let us start with a general class of Extended Theories of Gravity given by the action

$$\mathcal{A} = \int d^4x \sqrt{-g} [f(R) + \chi \mathcal{L}_m], \quad (9)$$

where $f(R)$ is an analytic function of the curvature invariant R . \mathcal{L}_m is the minimally coupled ordinary matter Lagrangian density. In the metric approach, the field equations are obtained by varying the action (9) with respect to $g_{\mu\nu}$. We get

$$f'R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - f_{;\mu\nu} + g_{\mu\nu} \square f' = \chi T_{\mu\nu} \quad (10)$$

$$3\square f' + f'R - 2f = \chi T,$$

where the second equation is the trace of the field equations. Here, $T_{\mu\nu} = \frac{-1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of matter; $T = T^\sigma{}_\sigma$ is the trace; $f' = \frac{df(R)}{dR}$, $\square = \nabla_{;\sigma}{}^{;\sigma}$ the d'Alembert operator and $\chi = 8\pi G$. We assume $c = 1$ is adopted. The conventions for Ricci's tensor are $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$; the Riemann tensor is $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \dots$. The affine connections are the Christoffel's symbols of the metric $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} (g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$. The signature is $(+ - - -)$.

In order to achieve the Newtonian limit of the theory the metric tensor $g_{\mu\nu}$ have to be approximated as follows:

$$g_{\mu\nu} \sim \begin{pmatrix} 1 - 2\Phi(t, \mathbf{x}) + \mathcal{O}(4) & \mathcal{O}(3) \\ \mathcal{O}(3) & -\delta_{ij} + \mathcal{O}(2) \end{pmatrix}, \quad (11)$$

¹The radius r is assumed as the spatial coordinate. It varies from $r = 0$ at the center to $r = \xi$ at the surface of the star

where $\mathcal{O}(n)$ (with $n = \text{integer}$) denotes the order of the expansion (see [13] for details). The set of coordinates² adopted is $x^\mu = (t, x^1, x^2, x^3)$. The Ricci scalar formally becomes

$$R \sim R^{(2)}(t, \mathbf{x}) + \mathcal{O}(4). \quad (12)$$

The n -th derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + \mathcal{O}(4)) \sim f^n(0) + f^{n+1}(0)R^{(2)} + \mathcal{O}(4); \quad (13)$$

here $R^{(n)}$ denotes a quantity of order $\mathcal{O}(n)$. From the lowest order of field in Eqs. (10), we have $f(0) = 0$ which trivially follows from the above assumption (11) for the metric. This means that the space-time is asymptotically Minkowskian and we are discarding a cosmological constant term in this analysis.³ Equation (10) at $\mathcal{O}(2)$ -order, that is at the Newtonian level, are

$$\begin{aligned} R_{tt}^{(2)} - \frac{R^{(2)}}{2} - f''(0) \Delta R^{(2)} &= \mathcal{X}T_{tt}^{(0)} \\ -3f''(0) \Delta R^{(2)} - R^{(2)} &= \mathcal{X}T^{(0)}, \end{aligned} \quad (14)$$

where Δ is the Laplacian in the flat space, $R_{tt}^{(2)} = -\Delta \Phi(t, \mathbf{x})$ and, for the sake of simplicity, we set $f'(0) = 1$. We recall that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (15)$$

where p is the pressure and ϵ is the energy density. Being the pressure contribution negligible in the field equations in the Newtonian approximation, we have

$$\begin{aligned} \Delta \Phi + \frac{R^{(2)}}{2} + f''(0) \Delta R^{(2)} &= -\mathcal{X}\rho \\ 3f''(0) \Delta R^{(2)} + R^{(2)} &= -\mathcal{X}\rho, \end{aligned} \quad (16)$$

where ρ is now the mass density.⁴ We note that for $f''(0) = 0$ we have the standard Poisson equation $\Delta \Phi = -4\pi G\rho$. This means that as soon as the second derivative of $f(R)$ is different from zero, deviations from the Newtonian limit of GR emerge.

The gravitational potential $-\Phi$, solution of Eqs. (16), has in general a Yukawa-like behavior depending on the characteristic length on which it evolves [13]. Then, as is evident, the Gauss theorem is not valid⁵ since the force law is not $\propto |\mathbf{x}|^{-2}$. The equivalence between a spherically symmetric distribution and pointlike distribution is not valid and how the matter is distributed in the space is very important [13–15].

²The Greek index runs between 0 and 3; the Latin index between 1 and 3.

³This assumption is quite natural since the contribution of a cosmological constant term is irrelevant at stellar level.

⁴Generally it is $\epsilon = \rho c^2$.

⁵It is worth noticing that if the Gauss theorem also does not hold, the Bianchi identities are always valid so the conservation laws are guaranteed.

Besides the Birkhoff theorem results modified at the Newtonian level, the solution can only be factorized by a space-depending function and an arbitrary time-depending function [13]. Furthermore, the correction to the gravitational potential is dependent upon only the first two derivatives of $f(R)$ in $R = 0$. This means that different analytical theories, from the third derivative perturbation terms on, admit the same Newtonian limit [13,14].

Equation (16) can be considered the *modified Poisson equation* for $f(R)$ gravity. They do not depend on gauge condition choice [15].

IV. STELLAR HYDROSTATIC EQUILIBRIUM IN $f(R)$ GRAVITY

From the Bianchi identity, satisfied by the field Eq. (10)), we have

$$T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2}(p + \epsilon) \frac{\partial \ln g_{tt}}{\partial x^k}. \quad (17)$$

If the dependence on the temperature τ is negligible, i.e. $\rho = \rho(p)$, this relation can be introduced into Eq. (16), which becomes a system of three equations for p , Φ and $R^{(2)}$, and can be solved without the other structure equations.

Let us suppose that matter still satisfies a polytropic equation $p = K\rho^\gamma$. If we introduce Eq. (5) into Eq. (16) we obtain an integro-differential equation for the gravitational potential $-\Phi$, that is

$$\begin{aligned} \Delta \Phi(\mathbf{x}) + \frac{2\mathcal{X}A_n}{3}\Phi(\mathbf{x})^n \\ = -\frac{m^2\mathcal{X}A_n}{6} \int d^3\mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}')\Phi(\mathbf{x}')^n, \end{aligned} \quad (18)$$

where $\mathcal{G}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ is the Green function of the field operator $\Delta_{\mathbf{x}} - m^2$ for systems with spherical symmetry and $m^2 = -\frac{1}{3f''(0)}$ (for details see [14,15]). The integro-differential nature of Eq. (18) is proof of the nonviability of the Gauss theorem for $f(R)$ gravity. Adopting again the dimensionless variables

$$z = \frac{|\mathbf{x}|}{\xi_0} \quad w(z) = \frac{\Phi}{\Phi_c} \quad (19)$$

where

$$\xi_0 \doteq \sqrt{\frac{3}{2\mathcal{X}A_n\Phi_c^{n-1}}} \quad (20)$$

is a characteristic length linked to stellar radius ξ . Eq. (18) becomes

$$\begin{aligned} \frac{d^2 w(z)}{dz^2} + \frac{2}{z} \frac{dw(z)}{dz} + w(z)^n \\ = \frac{m\xi_0}{8} \frac{1}{z} \int_0^{\xi/\xi_0} dz' z' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} w(z')^n, \end{aligned} \quad (21)$$

which is the *modified Lané-Emden equation* deduced from $f(R)$ gravity. Clearly the particular $f(R)$ model is specified by the parameters m and ξ_0 . If $m \rightarrow \infty$ (i.e. $f(R) \rightarrow R$), Eq. (21) becomes Eq. (8). We are only interested in the solutions of Eq. (21) that are finite at the center, that is for $z = 0$. Since the center must be an equilibrium point, the gravitational acceleration $|\mathbf{g}| = -d\Phi/dr \propto dw/dz$ must vanish for $w'(0) = 0$. Let us assume we have solutions $w(z)$ of Eq. (21) that fulfill the boundary conditions $w(0) = 1$ and $w(\xi/\xi_0) = 0$; then according to the choice (19), the radial distribution of density is given by

$$\rho(|\mathbf{x}|) = \rho_c w^n, \quad \rho_c = A_n \Phi_c^n \quad (22)$$

and the pressure by

$$p(|\mathbf{x}|) = p_c w^{n+1}, \quad p_c = K \rho_c^\gamma. \quad (23)$$

For $\gamma = 1$ (or $n = \infty$) the integro-differential Eq. (21) is not correct. This means that the theory does not contain the case of an isothermal sphere of ideal gas. In this case, the polytropic relation is $p = K\rho$. Putting this relation into Eq. (17) we have

$$\frac{\Phi}{K} = \ln \rho - \ln \rho_c \rightarrow \rho = \rho_c e^{\Phi/K}, \quad (24)$$

where the constant of integration is chosen in such a way that the gravitational potential is zero at the center. If we introduce Eq. (24) into Eq. (16), we have

$$\begin{aligned} \Delta \Phi(\mathbf{x}) + \frac{2\mathcal{X}\rho_c}{3} e^{\Phi(\mathbf{x})/K} \\ = -\frac{m^2 \mathcal{X}\rho_c}{6} \int d^3 \mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}') e^{\Phi(\mathbf{x}')/K}. \end{aligned} \quad (25)$$

Assuming the dimensionless variables $z = \frac{|\mathbf{x}|}{\xi_1}$ and $w(z) = \frac{\Phi}{K}$ where $\xi_1 \doteq \sqrt{\frac{3K}{2\mathcal{X}\rho_c}}$, Eq. (25) becomes

$$\begin{aligned} \frac{d^2 w(z)}{dz^2} + \frac{2}{z} \frac{dw(z)}{dz} + e^{w(z)} \\ = \frac{m\xi_1}{8} \frac{1}{z} \int_0^{\xi/\xi_1} dz' z' \left\{ e^{-m\xi_1|z-z'|} - e^{-m\xi_1|z+z'|} \right\} e^{w(z')}, \end{aligned} \quad (26)$$

which is the *modified ‘‘isothermal’’ Lané-Emden equation* derived from $f(R)$ gravity.

V. SOLUTIONS OF THE STANDARD AND MODIFIED LANÉ-EMDEN EQUATIONS

The task is now to solve the modified Lané-Emden equation and compare its solutions to those of the standard Newtonian theory. Only for three values of n , the solutions of Eq. (8) have the analytical expressions [12]

$$\begin{aligned} n = 0 &\rightarrow w_{\text{GR}}^{(0)}(z) = 1 - \frac{z^2}{6} \\ n = 1 &\rightarrow w_{\text{GR}}^{(1)}(z) = \frac{\sin z}{z} \\ n = 5 &\rightarrow w_{\text{GR}}^{(5)}(z) = \frac{1}{\sqrt{1 + \frac{z^2}{3}}}. \end{aligned} \quad (27)$$

We label these solution with GR since they agree with the Newtonian limit of GR. The surface of the polytrope of index n is defined by the value $z = z^{(n)}$, where $\rho = 0$ and thus $w = 0$. For $n = 0$ and $n = 1$ the surface is reached for a finite value of $z^{(n)}$. The case $n = 5$ yields a model of infinite radius. It can be shown that for $n < 5$ the radius of polytropic models is finite; for $n > 5$ they have infinite radius. From Eq. (27) one finds $z_{\text{GR}}^{(0)} = \sqrt{6}$, $z_{\text{GR}}^{(1)} = \pi$ and $z_{\text{GR}}^{(5)} = \infty$. A general property of the solutions is that $z^{(n)}$ grows monotonically with the polytropic index n . In Fig. 1 we show the behavior of solutions $w_{\text{GR}}^{(n)}$ for $n = 0, 1, 5$. Apart from the three cases where analytic solutions are known, the classical Lané-Emden Eq. (8) has to be solved numerically, considered with the expression

$$w_{\text{GR}}^{(n)}(z) = \sum_{i=0}^{\infty} a_i^{(n)} z^i \quad (28)$$

for the neighborhood of the center. Inserting Eq. (28) into Eq. (8) and by comparing coefficients, one finds, at lowest orders, a classification of solutions by the index n , that is

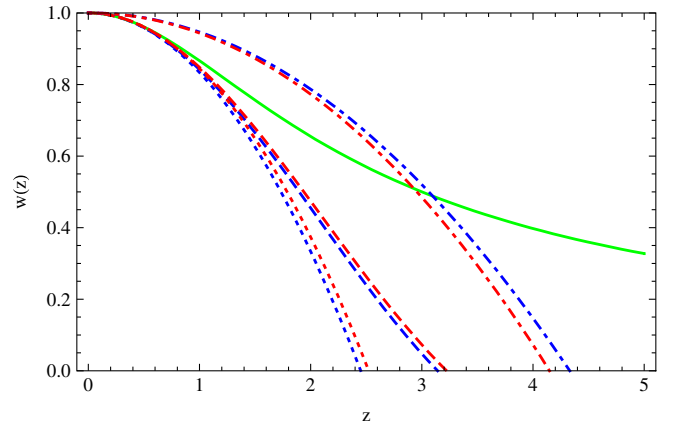


FIG. 1 (color online). Plot of solutions (blue lines) of standard Lané-Emden Eq. (8): $w_{\text{GR}}^{(0)}(z)$ (dotted line) and $w_{\text{GR}}^{(1)}(z)$ (dashed line). The green line corresponds to $w_{\text{GR}}^{(5)}(z)$. The red lines are the solutions of modified Lané-Emden Eq. (21): $w_{f(R)}^{(0)}(z)$ (dotted line) and $w_{f(R)}^{(1)}(z)$ (dashed line). The blue dashed-dotted line is the potential derived from GR ($w_{\text{GR}}(z)$) and the red dashed-dotted line is the potential derived from $f(R)$ gravity ($w_{f(R)}(z)$) for a uniform spherically symmetric mass distribution. The assumed values are $m\xi = 1$ and $m\xi_0 = .4$. From a rapid inspection of these plots, the differences between GR and $f(R)$ gravitational potentials are clear and the tendency is that at larger radius z they become more evident.

$$w_{\text{GR}}^{(n)}(z) = 1 - \frac{z^2}{6} + \frac{n}{120}z^4 + \dots \quad (29)$$

The case $\gamma = 5/3$ and $n = 3/2$ is the nonrelativistic limit while the case $\gamma = 4/3$ and $n = 3$ is the relativistic limit of a completely degenerate gas.

Also for the modified Lané-Emden Eq. (21), we have an exact solution for $n = 0$. In fact, it is straightforward to find

$$w_{f(R)}^{(0)}(z) = 1 - \frac{z^2}{8} + \frac{(1 + m\xi)e^{-m\xi}}{4m^2\xi^2} \left[1 - \frac{\sinh m\xi_0 z}{m\xi_0 z} \right], \quad (30)$$

where the boundary conditions $w(0) = 1$ and $w'(0) = 0$ are satisfied. A comment on the GR limit (that is $f(R) \rightarrow R$) of solution (30) is necessary. In fact, when we perform the limit $m \rightarrow \infty$, we do not recover exactly $w_{\text{GR}}^{(0)}(z)$. The difference is in the definition of quantity ξ_0 . In $f(R)$ gravity we have the definition (20) while in GR it is $\xi_0 = \sqrt{\frac{2}{\chi_{A_n} \Phi_c^{n-1}}}$, since in the first equation of (16), when we perform $f(R) \rightarrow R$, we have to eliminate the trace equation condition. In general, this means that the Newtonian limit and the Eddington parameterization of different relativistic theories of gravity cannot coincide with those of GR (see [16] for further details on this point).

The point $z_{f(R)}^{(0)}$ is calculated by imposing $w_{f(R)}^{(0)}(z_{f(R)}^{(0)}) = 0$, and by considering the Taylor expansion

$$\frac{\sinh m\xi_0 z}{m\xi_0 z} \sim 1 + \frac{1}{6}(m\xi_0 z)^2 + \mathcal{O}(m\xi_0 z)^4 \quad (31)$$

we obtain $z_{f(R)}^{(0)} = \frac{2\sqrt{6}}{\sqrt{3+(1+m\xi)e^{-m\xi}}}$. Since the stellar radius ξ is given by definition $\xi = \xi_0 z_{f(R)}^{(0)}$, we obtain the constraint

$$\xi = \sqrt{\frac{3\Phi_c}{2\pi G}} \frac{1}{\sqrt{1 + \frac{1+m\xi}{3}e^{-m\xi}}}. \quad (32)$$

By solving numerically the constraint,⁶ Eq. (32), we find the modified expression of the radius ξ . If $m \rightarrow \infty$ we have the standard expression $\xi = \sqrt{\frac{3\Phi_c}{2\pi G}}$ valid for the Newtonian limit of GR. Besides, it is worth noticing that in the $f(R)$ gravity case, for $n = 0$, the radius is smaller than in GR. On the other hand, the gravitational potential $-\Phi$ gives rise to a deeper potential well than the corresponding Newtonian potential derived from GR [14].

In the case $n = 1$, Eq. (21) can be recast as follows:

$$\frac{d^2 \tilde{w}(z)}{dz^2} + \tilde{w}(z) = \frac{m\xi_0}{8} \int_0^{\xi/\xi_0} dz' \times \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} \tilde{w}(z'), \quad (33)$$

where $\tilde{w} = zw$. If we consider the solution of (33) as a small perturbation to the one of GR, we have

$$\tilde{w}_{f(R)}^{(1)}(z) \sim \tilde{w}_{\text{GR}}^{(1)}(z) + e^{-m\xi} \Delta \tilde{w}_{f(R)}^{(1)}(z). \quad (34)$$

The coefficient $e^{-m\xi} < 1$ is the parameter with respect to which we perturb Eq. (33). Besides, these position ensure us that when $m \rightarrow \infty$ the solution converge to something like $\tilde{w}_{\text{GR}}^{(1)}(z)$. Substituting Eq. (34) in Eq. (33), we have

$$\begin{aligned} \frac{d^2 \Delta \tilde{w}_{f(R)}^{(1)}(z)}{dz^2} + \Delta \tilde{w}_{f(R)}^{(1)}(z) \\ = \frac{m\xi_0 e^{m\xi}}{8} \int_0^{\xi/\xi_0} dz' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} \tilde{w}_{\text{GR}}^{(1)}(z') \end{aligned} \quad (35)$$

and the solution is easily found to be

$$\begin{aligned} w_{f(R)}^{(1)}(z) \sim \frac{\sin z}{z} \left\{ 1 + \frac{m^2 \xi_0^2}{8(1 + m^2 \xi_0^2)} \left[1 + \frac{2e^{-m\xi}}{1 + m^2 \xi_0^2} \right. \right. \\ \left. \left. \times (\cos \xi / \xi_0 + m\xi_0 \sin \xi / \xi_0) \right] \right\} \\ - \frac{m^2 \xi_0^2}{8(1 + m^2 \xi_0^2)} \left[\frac{2e^{-m\xi}}{1 + m^2 \xi_0^2} \right. \\ \left. \times (\cos \xi / \xi_0 + m\xi_0 \sin \xi / \xi_0) \frac{\sinh m\xi_0 z}{m\xi_0 z} + \cos z \right]. \end{aligned} \quad (36)$$

Also in this case, for $m \rightarrow \infty$, we do not recover exactly $w_{\text{GR}}^{(1)}(z)$. The reason is the same of the previous $n = 0$ case [16]. Analytical solutions for other values of n are not available.

To conclude this section, we report the gravitational potential profile generated by a spherically symmetric source of uniform mass with radius ξ . We can impose a mass density of the form

$$\rho = \frac{3M}{4\pi\xi^3} \Theta(\xi - |\mathbf{x}|), \quad (37)$$

where Θ is the Heaviside function and M is the mass [14,15]. By solving field Eqs. (16) inside the star and considering the boundary conditions $w(0) = 1$ and $w'(0) = 0$, we get

$$\begin{aligned} w_{f(R)}(z) = \left[\frac{3}{2\xi} + \frac{1}{m^2 \xi^3} - \frac{e^{-m\xi}(1 + m\xi)}{m^2 \xi^3} \right]^{-1} \left[\frac{3}{2\xi} + \frac{1}{m^2 \xi^3} \right. \\ \left. - \frac{\xi_0^2 z^2}{2\xi^3} - \frac{e^{-m\xi}(1 + m\xi)}{m^2 \xi^3} \frac{\sinh m\xi_0 z}{m\xi_0 z} \right]. \end{aligned} \quad (38)$$

In the limit $m \rightarrow \infty$, we recover the GR case $w_{\text{GR}}(z) = 1 - \frac{\xi_0^2 z^2}{3\xi^2}$. In Fig. 1 we show the behaviors of $w_{f(R)}^{(0)}(z)$ and $w_{f(R)}^{(1)}(z)$ with respect to the corresponding GR cases. Furthermore, we plot the potential generated by a uniform spherically symmetric mass distribution in GR and $f(R)$ gravity and the case $w_{\text{GR}}^{(5)}(z)$.

⁶In principle, there is a solution for any value of m .

VI. DISCUSSION AND CONCLUSIONS

In this paper the hydrostatic equilibrium of a stellar structure in the framework of $f(R)$ gravity has been considered. The study has been performed starting from the Newtonian limit of $f(R)$ field equations. Since the field equations satisfy in any case the Bianchi identity, we can use the conservation law of energy-momentum tensor. In particular, adopting a polytropic equation of state relating the mass density to the pressure, we derive the *modified Lané-Emden equation* and its solutions for $n = 0, 1$ which can be compared to the analogous solutions coming from the Newtonian limit of GR. When we consider the limit $f(R) \rightarrow R$, we obtain the standard hydrostatic equilibrium theory coming from GR. A peculiarity of $f(R)$ gravity is the nonviability of the Gauss theorem, and then the *modified Lané-Emden equation* is an integro-differential equation where the mass distribution plays a crucial role. Furthermore, the correlation between two points in the star is given by a Yukawa-like term of the corresponding Green function.

These solutions have been matched with those coming from GR and the corresponding density radial profiles have been derived. In the case $n = 0$, we find an exact solution, while, for $n = 1$, we used a perturbative analysis with respect to the solution coming from GR. It is possible to demonstrate that density radial profiles coming from $f(R)$ gravity analytic models and close to those coming from GR are compatible. This result rules out some wrong claims in the literature stating that $f(R)$ gravity is not compatible with self-gravitating systems. Obviously the choice of the free parameter of the theory has to be consistent with boundary conditions, and then the solutions are

parameterized by a suitable “wavelength” $m = \sqrt{-\frac{1}{3f''(0)}}$ that should be experimentally fixed.

The next step is to derive self-consistent numerical solutions of the *modified Lané-Emden equation* and build up realistic star models where further values of the polytropic index n and other physical parameters, e.g. temperature, opacity, transport of energy, are considered. Interesting cases are the nonrelativistic limit ($n = 3/2$) and relativistic limit ($n = 3$) of completely degenerate gas. These models are a challenging task, since, up to now, there is no self-consistent, final explanation for compact objects (e.g. neutron stars) with masses larger than Volkoff mass, while observational evidence widely indicates these objects [10]. In fact, it is plausible that the gravity manifests itself on different characteristic lengths and also other contributions in the gravitational potential should be considered for these exotic objects. As we have seen above, the gravitational potential well results modified by higher-order corrections in the curvature. In particular, it is possible to show that if we put in the action (9) other curvature invariants that are also repulsive contributions can emerge [11,15]. These situations have to be seriously taken into account in order to address several issues of relativistic astrophysics that seem to be out of the explanation range of the standard theory.

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