PHYSICAL REVIEW D **83,** 063004 (2011)

Stability of the anisotropically inflating Bianchi type VI expanding solutions

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(Received 10 February 2011; published 21 March 2011)

A special class of the Bianchi type VI expanding solutions was speculated to break the cosmic no-hair theorem that will not approach the late-time de Sitter solution. We will show that an unstable mode always exists when the perturbation of the field equations is applied to the system. In addition to a model-independent perturbation formula, a simplification is also achieved by the introduction of a $\delta R = 0$ solution good for quadratic models in all Bianchi spaces. The result shows that this special class of anisotropically expanding solutions is unstable.

DOI: 10.1103/PhysRevD.83.063004 PACS numbers: 95.30.Sf, 04.50.Kd, 98.80.Bp, 98.80.Jk

I. INTRODUCTION

The inflationary scenario is known to be a successful model working properly with the cosmological standard model. There should have been a brief moment of accelerated expansion during the epoch of the early universe [1]. For example, a simple physically motivated inflationary scenario can be induced by the acceleration driven by a scalar field with a constant potential serving as the cosmological constant. It can also be induced by higher derivative pure gravity models with natural graceful exit. On the other hand, late-time de Sitter space appears to be a natural consequence of the evolutionary universe. Therefore, it is important to find out whether universal acceleration and an asymptotic approach to the de Sitter metric always occurs in these models.

In short, the field equations of any gravitational system with a cosmological constant Λ can always be written as

$$G_{ab} = T_{ab} - \Lambda g_{ab}. \tag{1}$$

The Einstein tensor G_{ab} on the left-hand side of the above equation signifies the geometric impact of the gravitational effect due to the contribution of the energy-momentum tensor T_{ab} shown on the right-hand side of the above equation.

Gibbons and Hawking [2] and Hawking and Moss [3] conjecture that all models with a positive cosmological constant will tend to a late-time de Sitter space. This is later known as the cosmic no-hair theorem for the Einstein gravity. Partial proof was given by Robert Wald [4] which shows clearly that any model with a positive cosmological constant will drive the late-time evolution towards the de Sitter spacetime, at least locally, for all non-type-IX Bianchi spaces provided that the matter sources obey (i) the dominant energy condition (DEC)

$$T_{ab} t^a t^b \geq 0 \tag{2}$$

and (ii) the strong-energy condition (SEC)

$$(T_{ab} - \tfrac{1}{2} g_{ab} T) t^a t^b \geq 0 \tag{3}$$

for any timelike vector t^a [4]. Here T_{ab} and T denote the energy momentum and its trace for any fields coupled to the gravitational system. The behavior of the type IX Bianchi space is similar if Λ is sufficiently large [4].

In addition, a series of cosmic no-hair theorems of varying strengths and degrees of applicability have been proved in support of certain constraints on the field parameters for its occurrence [4–13]. It is also known, however, that counterexamples exist where these energy conditions do not hold exactly [14,15]. Many of these solutions had later been shown to be unstable [9,16–19]. These examples appear to be in favor of the Hawking's no-hair conjecture. By all means, it is important to verify or confirm any existing claim that anisotropically expanding solutions could be stable. These may further our understanding of the limit and constraints on the fate of the evolution of our universe.

In particular, a new type of cosmological solutions was shown to arise naturally when $\Lambda > 0$ which has no counterpart in general relativity both in the Bianchi type II and type VI_h spaces [20] once quadratic terms are added to the Lagrangian of general relativity. Theses solutions inflate anisotropically and do not approach the late-time de Sitter spacetime. Hence, they could be counterexamples to the hope that the cosmic no-hair theorem will continue to hold in higher-order extensions of general relativity. Additional studies of higher-order theories can also be found in [21–25]. We have been, however, able to show that the inflationary solutions found [20] in the Bianchi II space are in fact unstable in the presence of anisotropic perturbations [26]. Note that the Bianchi type II solutions (and some Bianchi type I inflating solutions) were also found to be unstable by Barrow and Hervik in Ref. [27].

In this paper, we will try to show that the inflationary solutions found in the Bianchi VI space are also unstable in the presence of anisotropic perturbations. In particular, we will provide a comparably simple method in deriving the perturbation solutions for all quadratic models considered here that can be generalized to all Bianchi spaces.

A pure gravity theory which is quadratic in the scalar curvature and the Ricci tensor is considered in Ref. [20] for

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the model consists of the four-dimensional gravitational action

$$S_{\rm BH} = \frac{1}{2}\int d^4x\sqrt{g}L = \frac{1}{2}\int d^4x\sqrt{g}(R + \alpha R^2 + \beta R_{ab}R^{ab} - 2\Lambda). \quad (4)$$

The Einstein equations can be shown to be [20]

$$H_{ab} \equiv G_{ab} + \Phi_{ab} + \Lambda g_{ab} = 0, \quad (5)$$

where $G_{ab} \equiv R_{ab} - Rg_{ab}/2$ and

$$\Phi_{ab} \equiv 2\alpha R(R_{ab} - \tfrac{1}{4}Rg_{ab}) + (2\alpha + \beta)(g_{ab}D^2 - D_aD_b)R + \beta D^2(R_{ab} - \tfrac{1}{2}Rg_{ab}) + 2\beta(R_{acbd} - \tfrac{1}{4}g_{ab}R_{cd})R^{cd}. \quad (6)$$

Here the tensor Φ_{ab} incorporates the deviation from regular Einstein gravity related to the coupling constants α and β.

A new class of exact solutions is found in a spatially homogeneous universes of the Bianchi types VI (BVI) space given by the metric

$$ds^2_{VI} = -dt^2 + dx^2 + \exp[2(r-s)t + 2a(1-h)x]dx_2^2 + \exp[2(r+s)t + 2a(1+h)x]dx_3^2, \quad (7)$$

where

$$r^2 = \frac{8\beta s^2 + (3+h^2)(1+8\Lambda\alpha) + 8\Lambda\beta(1+h^2)}{8\beta h^2}, \quad a^2 = \frac{8\beta s^2 + 8\Lambda(3\alpha+\beta)+3}{8\beta h^2}. \quad (8)$$

Here r, s, a, and h are all constants. These solutions represent homogeneous universes with a four-dimensional group acting transitively on the spacetime [20,28–30]. Note that the solution inflates anisotropically if $\beta \neq 0$. These solutions exist when α or Λ vanishes but not in the limit $\beta \to 0$. Interesting discussions related to these solutions can be found in Ref. [20].

Note also that there is a symmetry [$s \to -s$ and $h \to -h$] which is equivalent to the transformation [$x_2 \to x_3$ and $x_3 \to x_2$] in the Barrow-Hervik (BH) metric solution (7).

The Bianchi type VI solutions given above inflate in the presence of a positive cosmological constant Λ. They are also known to violate the energy conditions that secure the cosmic no-hair theorem [4]. Hence, it is important to find out whether these expanding solutions are stable or not [9,16–19].

Higher derivative terms may come from the low energy limit of the string effective theories. From the point of view of an effective theory of gravity, quadratic curvature terms can also be treated as perturbative corrections to Einstein gravity valid in the higher energy scale. In particular, quadratic theories also give field equations with higher order in time derivatives. Many of them are known to have run away solutions and, hence, are supposed to be unphysical [31,32]. For example, the BH expanding solution in both Bianchi type II and type VI models is not defined for $\beta \to 0$. The BH solution in type II Bianchi space has been shown to be unstable.

Because of the complication of the expanding solutions studied here for the Bianchi type VI models, it is not easy to obtain or analyze the stability equation of the system. We will, however, introduce a useful method to derive the anisotropic perturbation equations of this BH solution for the quadratic models. In addition, we will also show that an unstable mode always exists without writing down the complicated expression of the perturbation solutions. As a result, we will be able to show that these new classes of anisotropically expanding solutions are not stable.

This paper will be organized as follows: (i) We will first derive the universal formula of the Friedmann equation, and the trace equation as the base of our stability analysis on a BVI metric space. This new set of equations can be shown to agree with the $H_{tt} = 0$ and $H_{11} = 0$ components of Eq. (5) for the quadratic curvature model (4). It can also be verified directly that BH solutions are also solutions to these new equations. (ii) Anisotropically perturbations can be obtained by perturbing these two field equations against any BVI background metric. (iii) A complete set of perturbation equations against the BH background metric solutions (7) can therefore be obtained directly. (iv) As a result, we end up with a polynomial equation of degree 3 for the perturbation equations. We will also show that a unstable mode can be shown to exist from a simple observation of this stability equation. Therefore, we do not need to solve the perturbation equations for a solution with a complicated expression that is difficult to analyze. (v) Finally, we conclude that the BH expanding solutions are always unstable against these anisotropic perturbations. Conclusions and discussions will also be drawn at the end of this paper. We also listed some useful derivations in the Appendix.

II. THE BVI METRIC AND THE FIELD EQUATIONS

The dynamical field of the BH solution can be described by the metric component $g_{22}(t,x)$ and $g_{33}(t,x)$. We are about to show that small perturbations in these two metric components lead to the existence of an unstable mode. This will be enough to prove that the BH solution is unstable. Therefore, we will only need to consider the dynamics of these two metric components in this paper. The field equations (5) given above are not easy to handle for the purpose of perturbation. Instead of directly applying perturbation to the field equations, we will derive the Friedmann equation of the higher derivative model from an effective action approach. Since there are two dynamical fields, we will need two field equations for a complete analysis. In addition to the Friedmann equation we will derive in a moment, we will also take the trace equation of (5) as another differential equation for our complete analysis. Indeed, the trace equation gives

$$R = (3\alpha + \beta)D^2R + 4\Lambda. \tag{9}$$

We will show in a moment that this is a very useful equation for the purpose of our analysis. Note also that the other components of the field equations (5) are redundant and can be derived from the Friedmann equation and the trace equation (9) following directly from the Bianchi identity $D_aG^a_b = 0$ and the generalized energy-momentum conservation law.

In order to derive the Friedmann equation for the BVI space, we will write the metric as

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b \\ &= -A_0(t,x)^2dt^2 + dx^2 + A_2(t,x)^2dy^2 + A_3(t,x)^2dz^2. \end{aligned} \tag{10}$$

Here $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, $A_a(t,x) = a_a(t)d_a(x)$ with the spacetime indices a, b, c running from 0 to 3. Note that the Friedmann equation is the tt component of the Einstein equation. Hence, it can only be derived from varying the action with respect to A_0. As a result, the variation of $L(A_0, H_0, I_0, I_0')$ with respect to A_0, H_0, I_0, and I_0' will lead to the Friedmann equation given by

$$\mathcal{D}_F L = L + L_{A_0} - D_t L_{H_0} - D_x L_{I_0} + (D_x)^2 L_{I_0'} = 0. \tag{11}$$

Here $A_0 = 1$ has been reset after the derivation. In addition, we have also defined $H_a = \dot{A}_a/A_a = \dot{a}_a(t)/a_a(t)$ and $I_a = A_a'/A_a = d_a'(x)/d_a(x)$ as the Hubble parameters and spatial expansion rate for later convenience. In addition, overdot $\dot{}$ and prime $'$ denote differentiation with respect to t and x. Furthermore, $L_{A_0} \equiv \delta L/\delta A_0$ and similarly for L_{I_0} and $L_{I_0'}$. Covariant differentiations $D_t \equiv \partial_t + \langle H \rangle$ and $D_x \equiv \partial_x + \langle I \rangle$ with $\langle H \rangle = H_2 + H_3$ and $\langle I \rangle = I_2 + I_3$.

Note also that we can easily show that the field equations (9) and (11) agree with the field equations (5) for the BH model (4). Moreover, it is also easy to show directly that the solution (7) is indeed a solution to the field equations (9) and (11). We remark here that Eq. (11) is the complete and comprehensive and model-independent expression of the anisotropic Friedmann equation for the BVI spaces. We have not dropped any terms irrelevant to the perturbation equation up to this point.

III. ANISOTROPIC METRIC PERTURBATIONS

We will first derive a set of perturbation equations in an arbitrary background metric space specified by $A_m(t) = A_m^0(t)$ given by the BH metric (7). It turns out that all perturbed fields will depend on δH_m instead of δa_m. Therefore, linear perturbation equations will be derived as differential equations of δH_m. Note that the perturbation equations are in general very complicated for the Bianchi type solutions [33]. It is comparably more difficult to prove that a known solution is really a stable solution. This is because we need to prove that it is stable against all possible perturbations. On the other hand, to show that the system is unstable, we only need to find a consistent unstable solution to the perturbation equation.

For our purpose that will be clear in a moment, the perturbation of the field equations will be made against the metric perturbations through the effect of $\delta H_m = k_m \exp[\nu t]$. Here the subindex m runs from 2 to 3 while a runs from 0 to 3. We will show that at least a positive mode with $\nu > 0$ always exists for the perturbations against the expanding BH background solutions. Therefore, this will prove that the BH solutions are always unstable. Note again that the parametrization of the perturbed field as $\delta H_m = k_m \exp[\nu t]$ is made simply for our purpose to find a unstable mode.

We need to perturb both Eqs. (9) and (11) all together in order to obtain all possible perturbation solutions for further stability analysis. First of all, by perturbing the trace equation (9) we are lead to the following equation:

$$[1 + 2(3\alpha + \beta)(\nu^2 + 2r\nu)]\delta R = 0 \tag{12}$$

as the first perturbation equation for the BH model. Note that the differential equations can be restored easily by replacing ν with ∂_t. There are two independent solutions to the above equation:

$$\text{(i)} \quad \nu = \omega_\pm = -r \pm \sqrt{r^2 - 1/(3\alpha + \beta)} \tag{13}$$

as the solution to the equation $\nu^2 + 2r\nu + \frac{1}{2(3\alpha+\beta)} = 0$, and

$$\text{(ii)} \quad \delta R = 2[(\nu + 3r - s)\delta H_2 + (\nu + 3r + s)\delta H_3] = 0. \tag{14}$$

The first solution is a trivial solution of less interest to the stability analysis. The second solution indicates, however, that the perturbation δR vanishes when $\nu \neq \omega_\pm$ for any physical perturbation following the Hubble parameters $\delta H_m = k_m \exp[\nu t]$.

We will focus on the perturbation solution under the condition $\delta R = 0$ throughout the rest of this paper. As a result, we can freely ignore the perturbations of any function of R when perturbations are applied to the Friedmann equation. This follows directly from the fact that $\delta f(R) = [\delta f(R)/\delta R]\delta R = 0$ if $\delta R = 0$ for any function of scalar curvature $f(R)$. Note that convenience also derived from the fact that $R = 4\Lambda$ on the background metric (7).

In addition, we can also remove all terms that are proportional to $(\nu + 3r - s)\delta H_2 + (\nu + 3r + s)\delta H_3 = 0$ in the derivation of the second perturbation equation derived from perturbing the Friedmann equation (11).

By doing so, it greatly simplifies the complicated expression of the second perturbation equation derived from the quadratic terms.

In order to classify and simplify the complicate derivation, let us define the following variables: $F_0 = \mathcal{D}_F R$, $F_\alpha = \mathcal{D}_F(R^2)$, and $F_\beta = \mathcal{D}_F(R^b_a R^a_b)$ for convenience. As a result, the Friedmann equation can be written as

$$F_0 + \alpha F_\alpha + \beta F_\beta = 0. \tag{15}$$

In addition, we can show that

$$F_0 = -2[G^t_t + \Lambda], \tag{16}$$

$$F_\alpha = -R^2 + 2RF_0 + 4\langle H\rangle \dot{R} \tag{17}$$

directly from the formal expression of the Friedmann equation, see the Appendix for details. Therefore we can show that

$$\delta F_\alpha = 2R\delta F_0. \tag{18}$$

Note that the constants r, s, a, h and the cosmological constant Λ are related to each other by the following identities:

$$1 + 2[R, -2(r^2 + s^2 - a^2 - a^2h^2)] = 0, \tag{19}$$

$$R = 4\Lambda = 2(3r^2 + s^2 - 3a^2 - a^2h^2) \tag{20}$$

in the BH metric background. These relations follow directly from the definition of r^2 and a^2 given in Eq. (8). The complicated relations of these constraints also reflect the forthcoming difficulty of the stability analysis.

IV. STABILITY CONDITIONS

Therefore the perturbation equation of the Friedmann equation reduces to the following form:

$$\delta F_\beta - 4(r^2 + s^2 - a^2 - a^2h^2)\delta F_0 = 0 \tag{21}$$

independent of the α-dependent term. This means that the R^2 term will not affect the stability of the BH solutions once the effect of $\delta R = 0$ is included. This is a quite remarkable and useful result for the quadratic models. It is also related to the fact that the BH solution does not exist in the limit $\beta \to 0$. The perturbations of δF_0 and δF_β can be derived straightforwardly. The results are, see the Appendix for details,

$$\delta F_0 = 2(r + s)\delta H_2 + 2(r - s)\delta H_3, \tag{22}$$

$$\begin{aligned}\delta F_\beta &- 4(r^2 + s^2 - a^2 - a^2h^2)\delta F_0 \\ &= [A_2\nu^2 + B_2\nu + C_2]\delta H_2 + [A_3\nu^2 + B_3\nu + C_3]\delta H_3 \\ &= 0.\end{aligned} \tag{23}$$

Note also that we have used the condition $\delta R = 0$ in deriving the first equation. Here A_m, B_m, and C_m are polynomial functions of r, s, a, and h defined as

$$A_2(s, h) = 2(3r - s) \tag{24}$$

$$B_2(s, h) = 4[3r^2 - 2rs - s^2 + a^2 - a^2h] \tag{25}$$

$$C_2(s, h) = 8[-2r^3 + sr^2 - 3s^2r + 2s^3 + a^2s + a^2rh^2]. \tag{26}$$

In addition, we can also show that $A_3(s, h) = A_2(-s, -h)$, $B_3(s, h) = B_2(-s, -h)$, and $C_3(s, h) = C_2(-s, -h)$. For convenience, we have only written s, h explicitly in defining the polynomial functions A_m and B_m that also depend on r and a. Note also that all δH_2-related coefficients and δH_3-related coefficients in the complete perturbation equation are related by the transformation [$s \to -s$ and $h \to -h$]. These symmetric relations follow directly from the fact that the BH background solutions (7) also obey the same symmetry. Finally, we can write the perturbation equations as

$$\begin{aligned}M\delta H &\equiv \begin{pmatrix} A_2\nu^2 + B_2\nu + C_2 & A_3\nu^2 + B_3\nu + C_3 \\ \nu + 3r - s & \nu + 3r + s \end{pmatrix}\begin{pmatrix}\delta H_2 \\ \delta H_3\end{pmatrix} \\ &= 0.\end{aligned} \tag{27}$$

Note the second row equation represents the equation $\delta R = 0$. It can be shown that the above perturbation equations can also be derived directly from perturbing the complete set of the field equation (5). As a result, a nontrivial solution exists only when $\det M \equiv -sF(\nu) = 0$. The function $F(\nu)$ is a polynomial function of degree 3 given by

$$F(\nu) = \nu^3 + 2A\nu^2 + B\nu - 4C \tag{28}$$

with the coefficients A, B, and C given by

$$A = 2r + a^2h/s \tag{29}$$

$$B = r^2 - 3s^2 - 3a^2 + 3a^2rh/s \tag{30}$$

$$C = r^3 + 3rs^2 + a^2h^2 + 3ra^2 > 0. \tag{31}$$

Note that C is always positive for all positive r which stands for an expanding solution. Therefore, we end up with a polynomial equation

$$F(\nu) = 0 \tag{32}$$

for the perturbation equation.

Any polynomial equation of degree 3 can always be solved by using the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ after some proper reparametrization of the coefficients. The results are very difficult to interpret due to the complicated structure of the solution.

We can show, however, that there always exists at least a positive mode $\nu = \nu_+$ solution if $C > 0$. This result follows directly from two observations of the equation $F(\nu) = 0$: (a) $F(\nu) \to \infty$ as $\nu \to \infty$, and (b) $F(0) = -4C < 0$. Therefore, the one-dimensional continuous curve $F(\nu)$ and the positive ν axis must have at least a point of intersection on the ν-$F(\nu)$ plane. The intersection point K with coordinate $(\nu = \nu_+, F(\nu) = 0)$ represents a positive mode solution $\nu = \nu_+$ to the polynomial equation $F(\nu) = 0$. Therefore the existence of a positive mode ν_+ shows that the perturbation against the BH background space is always unstable provided that $C > 0$.

As shown clearly earlier that $C > 0$ represents an expanding solution. Therefore, we prove that the expanding solutions found in the Bianchi type VI quadratic models are inevitably unstable.

Note that if no positive mode can be found in the above perturbation approach only indicates that an unstable mode does not exist in the perturbation along the considered directions δH_i. Unstable modes could also exist if we perturb the expanding solutions in the directions prescribed by the inhomogeneous spaces. Unstable modes could also exist by perturbing extra fields or extra terms that are not considered here. It is clear, however, that the expanding solution is unstable once an unstable mode is found in any direction of perturbation. Therefore, the simple method, identifying the possible existence of any positive mode, shown here in this paper could be useful to providing clues of stability analysis even if the stability equations are too complicated to be analyzed.

V. CONCLUSIONS

We have derived the Friedmann equation and trace equation as the base of our stability analysis in an expanding BVI metric space. These new sets of equations are shown to agree with the $H_{tt} = 0$ and $H_{11} = 0$ components of Eq. (5) for the quadratic curvature model (4). It can also be verified directly that BH solutions (7) are also solutions to these new equations. In addition, we have introduced a useful method to induce anisotropic perturbations to the BH solution in the quadratic models.

Anisotropically perturbations are obtained by perturbing these two field equations against the BVI background metric. A complete set of perturbation equations against the BH background metric solutions (7) is therefore obtained directly. As a result, we derive a polynomial equation of degree 3 as the perturbation equation.

We also show that the unstable mode always exists from a simple observation of this stability equation. Therefore, there is no need to analyze the property of the mode solutions with a complicated format. In addition, the $\omega_\pm$ modes become unstable if $3\alpha + \beta < 0$. Therefore the system can only remain stable for a brief moment before the unstable mode $\nu = \nu_+$ dominant the expanding process even if the prescribed energy conditions DEC and SEC are both violated. Consequently, we show that the BH solution is always unstable against these anisotropic perturbations in favor of the no-hair conjecture for the Einstein gravity. It appears that the energy conditions to be held for the no-hair theorem can be further relaxed for many existing models. Hopefully, the result shown in this paper will be helpful to the relative analysis of the no-hair theorem.

In addition, the perturbation equation $\delta R = 0$ could also be used to simplify the derivation of the full set of perturbation equations in different Bianchi spaces. For example, the anisotropically expanding solutions for the action (4) were found to be [20]

$$d^2 = \frac{11 + 8\Lambda(11\alpha + 3\beta)}{30\beta}, \qquad b^2 = \frac{8\Lambda(\alpha + 3\beta) + 1}{30\beta} \tag{33}$$

in the Bianchi type II space represented by the metric:

$$ds^2 = -dt^2 + a_1^2(t)dr^2 + g_{mn}dx^m dx^n \tag{34}$$

with $(x^0, x^1, x^2, x^3) = (t, r, z, \phi)$, $a_1^2(t) = \exp[bt]/d^2$, $a_2^2(t) = \exp[2bt]/d^2$, and

$$g_{mn} = \begin{pmatrix} a_2^2(t) & ra_2^2(t) \\ ra_2^2(t) & a_1^2(t) + r^2 a_2^2(t) \end{pmatrix}. \tag{35}$$

Here d and b are some constant functions of α, β, and Λ. We will also write $x = d^2/b^2$ for convenience. Note that these solutions also follow the relation $R = 4\Lambda = (11b^2 - d^2)/2$. Therefore the perturbation equation of the trace equation also gives $\delta R = 0$ that can be shown to give

$$\delta K = 2(2\nu^2 + 5\nu + x)\delta D_1 + (2\nu^2 + 6\nu - x)\delta D_2 = 0 \tag{36}$$

with $a_i(t) = \exp[bD_i(t)]/d$ and $\delta D_i = k_i \exp[b\nu t]$ defining the perturbation of D_i.

In addition, the perturbation equation of the Friedmann equation can be shown to be [26,34,35]

$$\begin{aligned} &\alpha\delta K_\alpha + \beta\delta K_\beta \\ &\equiv \alpha\{2[16\nu^3 + 28\nu^2 + (8x - 30)\nu - 6x]\delta D_1 \\ &\quad + [16\nu^3 + 36\nu^2 - (8x + 36)\nu + 6x]\delta D_2\} \\ &\quad + \beta\{2[5\nu^3 + 9\nu^2 + (9x - 8)\nu - 3x]\delta D_1 \\ &\quad + [6\nu^3 + 13\nu^2 + (6x - 10)\nu + 3x]\delta D_2\} = 0. \end{aligned} \tag{37}$$

The α-dependent term δK_α can be factorized as $\delta K_\alpha = (8\nu - 6)\delta K$. Hence, we can use the equation $\delta K = 0$ to eliminate the α-dependent term δK_α. As a result, we can write the perturbation equation as

$$\mathcal{D}\,\delta D \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 2\delta D_1 \\ \delta D_2 \end{pmatrix} = 0 \tag{38}$$

with

$$C_{11} = 2\nu^2 + 5\nu + x, \tag{39}$$

$$C_{12} = 2\nu^2 + 6\nu - x, \tag{40}$$

$$C_{21} = 5\nu^3 + 9\nu^2 + (9x - 8)\nu - 3x, \tag{41}$$

$$C_{22} = 6\nu^3 + 13\nu^2 + (6x - 10)\nu + 3x. \tag{42}$$

Therefore, nontrivial solutions for δD_i exist only when $\det\mathcal{D} = C_{11}C_{22} - C_{12}C_{21} = 0$. This equation can be shown to be a polynomial equation of degree 4:

$$J(\nu) = 2\nu^4 + 8\nu^3 + (5x + 7)\nu^2 + 2(5x - 1)\nu + 15x(x + 1) = 0. \tag{43}$$

As a result, $J(\nu) = 0$ can be solved to obtain the following four different solutions:

$$\nu = \nu_\pm \equiv -1 + \tfrac{1}{2}\Big[5 - 5x \pm \sqrt{1 - 130x - 95x^2}\Big]^{1/2}, \tag{44}$$

$$\nu = \tilde{\nu}_\pm \equiv -1 - \tfrac{1}{2}\Big[5 - 5x \pm \sqrt{1 - 130x - 95x^2}\Big]^{1/2}. \tag{45}$$

This result shows that the $\delta R = 0$ equation can help reduce the labor in deriving the perturbation equation in different types of Bianchi spaces. Note that a similar result also applies to the case in the Bianchi type I space [27].

ACKNOWLEDGMENTS

This research is supported in part by NSC of Taiwan under Contract No. NSC-98-2112-M-009-02-MY3.

APPENDIX A: FRIEDMANN EQUATION

We can show that all nonvanishing components of the Riemann curvature tensor are given by

$$R^{t1}{}_{t1} = \frac{1}{A_0^2}[\dot{H}_1 + H_1^2 - H_0H_1] - \frac{1}{A_1^2}[-I_0I_1 + I_0' + I_0^2], \tag{A1}$$

$$R^{tm}{}_{tm} = \frac{1}{A_0^2}[\dot{H}_m + H_m^2 - H_0H_m] - \frac{1}{A_1^2}I_0I_m, \tag{A2}$$

$$R^{1m}{}_{1m} = \frac{1}{A_0^2}H_1H_m - \frac{1}{A_1^2}[I_m' + I_m^2 - I_1Im], \tag{A3}$$

$$R^{tm}{}_{1m} = \frac{1}{A_0^2}[I_m(H_m - H_1) - H_mI_0], \tag{A4}$$

$$R^{23}{}_{23} = \frac{1}{A_0^2}H_2H_3 - \frac{1}{A_1^2}I_2I_3 \tag{A5}$$

for the BVI metric, Eq. (34),

$$ds^2 = g_{ab}dx^a dx^b = -A_0(t,x)^2dt^2 + dx^2 + A_2(t,x)^2dy^2 + A_3(t,x)^2dz^2.$$

Here the Riemann curvature tensor is defined by the differential 2-form $R^a{}_b \equiv R^a{}_{bcd}dx^c \wedge dx^d/2 = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ with the connection 1-form $\omega^a{}_b$ related to the spin connection Γ^a_{bc} by the relation $\omega^a{}_b \equiv \Gamma^a_{bc}dx^c$. In addition, the Ricci tensor and scalar curvature are defined as $R^a{}_b \equiv R^{ac}{}_{bc}$ and $R \equiv R^a{}_a$. Note that any Lagrangian derived from the Riemann curvature tensor given above will become a functional of H_i and I_i once we set $A_0 = 1$.

We can show that the variation of the Lagrangian L with respect to A_0, I_0, and I_0' can be replaced by the variation of H_i and $\dot{H}_i$. This is because, with $L_i \equiv \delta L/\delta H_i$, $L^i \equiv \delta L/\delta\dot{H}_i$,

$$\frac{\delta L}{\delta A_0} = -H_iL_i - 2\dot{H}_iL^i, \tag{A6}$$

$$\frac{\delta L}{\delta I_0'} = -L^1, \tag{A7}$$

$$\frac{\delta L}{\delta I_0} = -I_iL^i + J, \tag{A8}$$

from an observation that relative terms always appear in the Riemann curvature tensor in a coherent way. For example, I_0' is always there when $-\dot{H}_1$ appears in $R^{t1}{}_{t1}$. Note that in the above equations we have set $A_0 = A_1 = 1$ after the substitution is done so that the above equations become helpful for our stability analysis free from the irrelevant parts related to the effect of A_1. In addition, the term J is not written explicitly because there is an I_0-dependent term in a combination of H_mI_0. The effect of this term can not be replaced by the variation of the other combination in a simple manner. We will leave it here and derive the explicit form of J when we apply this equation to the quadratic interaction. In fact, it will only show up in $-2(R^t{}_1)^2$ as a part of the Lagrangian $R^b{}_aR^a{}_b$. In particular, this term does not contribute to the Lagrangian $R + \alpha R^2$.

We would like to draw a few remarks concerning the simplified version of the Friedmann equation derived above. First of all, we will drop all terms that have no contribution to the perturbation equation δF_β. This includes all terms independent of δH_m. By doing so we can drop irrelevant terms and make the derivation a lot easier. Second of all, the Friedmann equation derived above does not have an isotropic limit except the trivial limit $H_m = 0$. This is simply because of the fact that BH solutions set the scale factor $a_1 = 1$ for all time. Therefore, all terms related to the effect of a_1 cannot be restored by all means. This is a very special effect of the BH solutions. We cannot, however, throw away the a_1 terms from the very beginning. The reason is obvious from a simple observation at the H_1 dependence of the Riemann curvature tensors. For example,

$$R^{t1}{}_{t1} = \frac{1}{A_0^2}[\dot{H}_1 + H_1^2 - H_0H_1] - \frac{1}{A_1^2}[-I_0I_1 + I_0' + I_0^2] \to 0 \tag{A9}$$

when we set $H_1 = H_0 = I_0 = I_1 = 0$. The variational contribution from $\delta R^{t1}{}_{t1}/\delta \dot{H}_1$ and $\delta R^{t1}{}_{t1}/\delta I'_0$ does not vanish at all. Therefore, we have to keep track of these terms before we set $H_1 = H_0 = I_0 = I_1 = 0$ in order to derive the correct formula for the Friedmann equation.

As a result, we can show that the Friedmann equation, Eq. (11),

$$\mathcal{D}_F L = L + L_{A_0} - D_t L_{H_0} - D_x L_{I_0} + (D_x)^2 L_{I'_0} = 0,$$

can be written as

$$\mathcal{D}_F L = L + H_i D_t L^i - H_i L_i - \dot{H}_i L^i - [D_x]^2 L^1 + I_i D_x L^i - \langle I \rangle J = 0. \tag{A10}$$

Here $J = 4\beta \langle H^2 \rangle R^t_1$ for the quadratic Lagrangian $\beta R^b{}_a R^a{}_b$. In addition, $\langle H^2 \rangle \equiv H_2^2 + H_3^2$. Summation over repeated indices is understood in the above equation. The contribution of $i = 1$ terms will be mostly irrelevant to the stability analysis of our problem after we set $A_1 = 1$. It is written here not only for a complete presentation of the Friedmann equation. In fact, there is a contribution from $[R^t{}_t]^1 = \delta R^t{}_t/\delta \dot{H}_1$ to the Friedmann equation before we set $A_1 = 1$. This complete version of the Friedmann equation will be helpful to the model-independent study for all BVI metric spaces.

We can easily show that the Friedmann equation shown above is identical to the *tt* component of the field equations in Eq. (5) in the presence of the metric (34). It also agrees with the equation derived directly from Eq. (11).

In particular, we can show that, upon removing all terms that vanish when BH background metric is inserted,

$$F_\beta = -R^b{}_a R^a{}_b + 2\langle H \rangle \dot{R}^t{}_t + 4H_2 H_3 R^t{}_t + 2H_2 \dot{R}^2{}_2 + 2H_3 \dot{R}^3{}_3 - 4\langle I \rangle \langle H \rangle R^t{}_1 + \text{irrelevant terms} \tag{A11}$$

for the quadratic part $L_\beta = R^b{}_a R^a{}_b$. There are also additional terms in F_β that are irrelevant to the perturbation of δH_m. These terms are in fact very complicated in structure. In addition, we also drop any term that is proportional to R^2 which will not contribute to the perturbation equation under the constraint $\delta R = 0$. We will simply drop these terms for convenience. In fact, relative derivation becomes a lot easier by ignoring these terms. As a result, we can derive the perturbation equation for the β term:

$$\begin{aligned}\delta F_\beta = {} & 2[(3r - s)\nu^2 + 2(3r^2 - 2rs - s^2 + a^2 - a^2 h)\nu \\ & - 4(a^2 r + a^2 s h^2 + r^3 - 2sr^2 + 2rs^2 - 3s^3)]\delta H_2 \\ & 2[(3r + s)\nu^2 + 2(3r^2 + 2rs - s^2 + a^2 + a^2 h)\nu \\ & - 4(a^2 r - a^2 s h^2 + r^3 + 2sr^2 + 2rs^2 + 3s^3)]\delta H_3.\end{aligned} \tag{A12}$$

Consequently, the perturbation equation of the Friedmann equation, (23), can be derived from the above equation together with Eq. (21).

[1] A. Guth, Phys. Rev. D **23**, 347 (1981); A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[2] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[3] S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).
[4] R. Wald, Phys. Rev. D **28**, 2118 (1983).
[5] J. D. Barrow, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, 1983), p. 267.
[6] W. Boucher and G. W. Gibbons, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, 1983), p. 273.
[7] A. A. Starobinskii, Sov. Phys. JETP Lett. **37**, 66 (1983), http://www.jetpletters.ac.ru/ps/1488/article_22720.pdf.
[8] L. G. Jensen and J. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987).
[9] J. D. Barrow, Phys. Lett. B **187**, 12 (1987).
[10] J. D. Barrow, Phys. Lett. B **235**, 40 (1990); J. D. Barrow and P. Saich, Phys. Lett. B **249**, 406 (1990); J. D. Barrow and A. R. Liddle, Phys. Rev. D **47**, R5219 (1993); A. D. Rendall, Classical Quantum Gravity **22**, 1655 (2005).
[11] J. D. Barrow, Nucl. Phys. **B296**, 697 (1988).
[12] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).
[13] K. I. Maeda, Phys. Rev. D **39**, 3159 (1989).
[14] S. Kanno, M. Watanabe, and J. Soda, Phys. Rev. Lett. **102**, 191302 (2009).
[15] S. Kanno, J. Soda, and M. Watanabe, J. Cosmol. Astropart. Phys. 12 (2010) 024.
[16] J. D. Barrow, Phys. Lett. B **180**, 335 (1986).
[17] J. D. Barrow, Nucl. Phys. **B310**, 743 (1988).
[18] J. D. Barrow, Phys. Lett. B **183**, 285 (1987).
[19] E. Weber, J. Math. Phys. (N.Y.) **25**, 3279 (1984); H. H. Soleng, Classical Quantum Gravity **6**, 1387 (1989); N. Kaloper, Phys. Rev. D **44**, 2380 (1991).
[20] J. D. Barrow and Sigbjorn Hervik, Phys. Rev. D **73**, 023007 (2006).
[21] A. Berkin, Phys. Rev. D **44**, 1020 (1991).
[22] E. Bruning, D. Coule, and C. Xu, Gen. Relativ. Gravit. **26**, 1197 (1994).
[23] H.-J. Schmidt, arXiv:gr-qc/0407095.
[24] A. D. Linde, Phys. Lett. B **175**, 395 (1986); A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).
[25] J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Oxford University Press, Oxford, 1986).

[26] W. F. Kao and I. C. Lin, J. Cosmol. Astropart. Phys. 01 (2009) 022.
[27] J. D. Barrow and Sigbjorn Hervik, Phys. Rev. D **74**, 124017 (2006).
[28] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. (N.Y.) **17**, 986 (1976).
[29] J. Lauret, Math. Ann. **319**, 715 (2001); Quart. J. Math. **52**, 463 (2001); Math. Z. **241**, 83 (2002); Diff. Geom. Appl. **18**, 177 (2003).
[30] S. Hervik, arXiv:1002.2016v2.
[31] E. E. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996).
[32] A. A. StarobinskyA, Phys. Lett. **91B**, 99 (1980).
[33] M. Tanimoto, V. Moncrief, and K. Yasuno, Classical Quantum Gravity **20**, 1879 (2003).
[34] W. F. Kao, Phys. Rev. D **79**, 043001 (2009).
[35] W. F. Kao, Eur. Phys. J. C **65**, 555 (2009).