PHYSICAL REVIEW D **83,** 056011 (2011)

Higher-loop corrections to the infrared evolution of a gauge theory with fermions

Thomas A. Ryttov and Robert Shrock

C. N. Yang Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794, USA

(Received 19 November 2010; revised manuscript received 9 February 2011; published 17 March 2011)

We consider a vectorial, asymptotically free gauge theory and analyze the effect of higher-loop corrections to the beta function on the evolution of the theory from the ultraviolet to the infrared. We study the case in which the theory contains N_f copies of a fermion transforming according to the fundamental representation and several higher-dimensional representations of the gauge group. We also calculate higher-loop values of the anomalous dimension of the mass, γ_m of $\bar{\psi}\psi$ at the infrared zero of the beta function. We find that for a given theory, the values of γ_m calculated to three- and four-loop order, and evaluated at the infrared zero computed to the same order, tend to be somewhat smaller than the value calculated to two-loop order. The results are compared with recent lattice simulations.

DOI: 10.1103/PhysRevD.83.056011 PACS numbers: 11.15.−q, 11.10.Hi, 12.60.−i

I. INTRODUCTION

In this paper we investigate how higher-loop corrections to the beta function affect the evolution of a vectorial, asymptotically free gauge theory [in (3 + 1) dimensions, at zero temperature] from the ultraviolet to the infrared. We assume that the theory contains a certain number, N_f, of massless Dirac fermions ψ transforming according to a representation R of the gauge group. We consider cases where R is the fundamental, adjoint, and rank-2 symmetric or antisymmetric tensor representation. We also study the effect of higher-loop corrections to the anomalous dimension γ_m of the fermion mass. This work yields more complete information on the nature of the evolution of the theory from the ultraviolet to the infrared, in particular, on the determination of the infrared zero of the beta function and the scaling behavior of the $\bar{\psi}\psi$ operator in the vicinity of this zero. We will give a number of results for a general gauge group G but will focus on the case $G = \mathrm{SU}(N)$.

We denote the running gauge coupling of the theory as $g(\mu)$, with $\alpha(\mu) = g(\mu)^2/(4\pi)$, where μ is the Euclidean energy/momentum scale (which will often be suppressed in the notation). The property that the SU(N) gauge interaction is asymptotically free means that $\lim_{\mu\to\infty}\alpha(\mu) = 0$, and, since the beta function is negative for small α, it follows that, as the energy/momentum scale μ decreases from large values, α increases. As μ decreases and the theory evolves into the infrared, two different types of behavior may occur, depending on the fermion content. In a theory with a sufficiently small number of fermions in small enough representations R, as μ decreases through a scale Λ, the coupling α exceeds a critical value $\alpha_{R,\mathrm{cr}}$, depending on R, for the formation of bilinear fermion condensates, and these condensates are produced. This may or may not be associated with an infrared (IR) zero of the two-loop beta function at a value $\alpha = \alpha_{\mathrm{IR}}$; if the two-loop ($2\ell$) beta function does have an IR zero, $\alpha_{\mathrm{IR},2\ell}$, then this type of behavior requires that $\alpha_{\mathrm{IR},2\ell} > \alpha_{R,\mathrm{cr}}$ [1,2]. As μ decreases toward Λ and α increases toward the IR zero of the beta function, the increase of α as a function of decreasing μ is reduced. This gives rise to an α that is of order unity, but varies slowly as a function of μ. This behavior is interestingly different from the behavior of the gauge coupling in quantum chromodynamics (QCD). As the condensates form, the fermions gain dynamical masses of order Λ so that, in the low-energy effective theory applicable at scales $\mu < \Lambda$, they are integrated out, and the further evolution of the theory into the infrared is controlled by the $N_f = 0$ beta function.

Alternatively, if the theory has a sufficiently large number N_f of fermions and/or if these fermions are in a large enough representation R (as bounded above by the requirement of asymptotic freedom), then the IR zero of the beta function occurs at a value smaller than $\alpha_{R,\mathrm{cr}}$, so that as the scale μ decreases from large values, the theory evolves into the infrared without ever spontaneously breaking chiral symmetry. In this latter case, the IR zero of the beta function is an exact IR fixed point (IRFP), approached from below as $\mu \to 0$. More complicated behavior occurs in theories containing fermions in several different representations [3]; here we restrict to the case of fermions in a single representation. For a given asymptotically free theory that features an IR fixed point at $\alpha = \alpha_{\mathrm{IR}}$, the value of this IRFP decreases as a function of N_f. There is thus a critical value of N_f, denoted $N_{f,\mathrm{cr}}$, depending on R, at which α_{IR} decreases below $\alpha_{R,\mathrm{cr}}$. This value serves as the boundary, as a function of N_f, between the interval of nonzero $1 \le N_f < N_{f,\mathrm{cr}}$, where the theory evolves into the infrared in a manner that involves fermion condensate formation and associated spontaneous chiral symmetry breaking (SχSB), and the interval $N_{f,\mathrm{cr}} < N_f < N_{f,\mathrm{max}}$, where the theory evolves into the infrared without this condensate formation and chiral symmetry breaking, with $N_{f,\mathrm{max}}$ denoting the maximal value of N_f consistent with the requirement of asymptotic freedom.

The anomalous dimension γ_m contains important information about the scaling behavior of the operator $\bar{\psi}\psi$ for which m is the coefficient, as probed on different momentum scales. In a theory with an $\alpha_{\rm IR} \sim O(1)$, it follows that γ_m may also be $O(1)$, which can produce significant enhancement of dynamically generated fermion masses due to the renormalization-group factor

$$\eta = \exp\left[\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} \gamma_m(\alpha(\mu))\right]. \tag{1.1}$$

In a phase where no dynamical mass is generated, γ_m simply describes the scaling behavior of the $\bar{\psi}\psi$ operator.

There are several motivations for the study of higher-order corrections to this evolution of an asymptotically free theory into the infrared. First, the critical coupling $\alpha_{R,{\rm cr}}$ is generically of order unity, and hence there is a need to have a quantitative assessment of the importance of higher-loop corrections to the evolution of the theory. Second, besides being of fundamental field-theoretic interest, a knowledge of this evolution plays an important role in modern technicolor (TC) models with dynamical electroweak symmetry breaking, in which the slow running of the coupling associated with an approximate infrared zero of the beta function provides necessary enhancement of quark and lepton masses [1,2] (recent reviews include [4–6]), and can reduce technicolor corrections to precision electroweak quantities [7,8]. In addition to the fundamental representation, fermions in higher-dimensional representations have been studied in the context of technicolor [6,9]. Fermions in higher-dimensional representations of chiral gauge groups have long played a valuable role in studies of extended technicolor (ETC) models that were reasonably ultraviolet-complete and explicitly worked out the details of the sequential breaking of the ETC chiral gauge symmetries down to the TC group [10]. Recently, there has been a considerable amount of effort devoted to lattice studies of gauge coupling evolution and condensate formation in vectorial SU(N) gauge theories as a function of N_f, for fermions in both the fundamental representation [8,11–15] and higher representations [16–23] (a recent review is [24]). Thus, another important motivation for the present work is to provide higher-order calculations that can be compared with these lattice studies.

II. GENERAL THEORETICAL FRAMEWORK

A. Beta function

The beta function of the theory is denoted $\beta = dg/dt$, where $dt = d\ln\mu$. In terms of the variable

$$a \equiv \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi}, \tag{2.1}$$

the beta function can be written equivalently as $\beta_\alpha \equiv d\alpha/dt$, expressed as a series

$$\frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell, \tag{2.2}$$

where ℓ denotes the number of loops involved in the calculation of b_ℓ and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. Although this series and series for other quantities in quantum field theories do not have finite radii of convergence but are only asymptotic, experience shows that in situations where the effective expansion parameter [here, (α/π) times appropriate group invariants] is not too large, the first few terms can provide both qualitative and quantitative insight into the physics. The first two coefficients in the expansion (2.2), which are scheme-independent, are [25]

$$b_1 = \frac{1}{3}(11C_A - 4T_f N_f) \tag{2.3}$$

and [26]

$$b_2 = \frac{1}{3}[34C_A^2 - 4(5C_A + 3C_f)T_f N_f]. \tag{2.4}$$

Here $C_f \equiv C_2(R)$ is the quadratic Casimir invariant for the representation R to which the N_f fermions belong, $C_A \equiv C_2(G)$ is the quadratic Casimir invariant for the adjoint representation, and $T_f \equiv T(R)$ is the trace invariant for the fermion representation R. Higher-order coefficients, which are scheme-dependent [27], have been calculated up to four-loop order [28,29]. Some further details are given in Appendix A. Values of $\bar{b}_\ell$ for $1 \le N \le 4$ and relevant ranges of N_f are given in Table I.

B. Anomalous dimension of the $\bar{\psi}\psi$ operator

The anomalous dimension γ_m for the fermion bilinear $\bar{\psi}\psi$ describes the scaling properties of this operator and can be expressed as a series in a or equivalently, α,

$$\gamma_m = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell, \tag{2.5}$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop series coefficient. Via Eq. (1.1), the anomalous dimension γ_m governs the running of a dynamically generated fermion mass. The coefficients c_ℓ have been calculated to four-loop order [30]. The first two are

$$c_1 = 6C_f \tag{2.6}$$

and

$$c_2 = 2C_f\left[\frac{3}{2}C_f + \frac{97}{6}C_A - \frac{10}{3}T_f N_f\right]. \tag{2.7}$$

For reference, the coefficient c_3 is listed in Appendix A. Since as N_f approaches $N_{f,{\rm max}}$ from below, $b_1 \to 0$ with nonzero b_2 and hence $\alpha_{\rm IR} \to 0$, and since the perturbative calculation expresses γ_m in a power series in α, it follows that as $\gamma_m \to 0$ as N_f approaches $N_{f,{\rm max}}$ from below. We note that a conjectured beta function that directly relates β to γ has been proposed [31].

TABLE I. Values of the ℓ-loop beta function coefficients $\bar{b}_\ell$ defined in Eq. (2.2) in the SU(N) gauge theory with N_f fermions transforming according to the fundamental representation, as functions of N and N_f, for the range (3.1) where the theory is asymptotically free.

N	N_f	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$
2	0	0.584	0.287	0.213	0.268
2	1	0.5305	0.235	0.154	0.191
2	2	0.477	0.184	0.099	0.127
2	3	0.424	0.132	0.047	0.078
2	4	0.371	0.080	−0.0003	0.044
2	5	0.318	0.0285	−0.044	0.024
2	6	0.265	−0.023	−0.084	0.020
2	7	0.212	−0.075	−0.120	0.030
2	8	0.159	−0.127	−0.152	0.057
2	9	0.106	−0.178	−0.180	0.099
2	10	0.053	−0.230	−0.205	0.156
3	0	0.875	0.646	0.720	1.173
3	1	0.822	0.566	0.582	0.910
3	2	0.769	0.485	0.450	0.681
3	3	0.716	0.405	0.324	0.485
3	4	0.663	0.325	0.205	0.322
3	5	0.610	0.245	0.091	0.194
3	6	0.557	0.165	−0.016	0.099
3	7	0.504	0.084	−0.118	0.039
3	8	0.451	0.004	−0.213	0.015
3	9	0.398	−0.076	−0.303	0.025
3	10	0.345	−0.156	−0.386	0.072
3	11	0.292	−0.236	−0.463	0.154
3	12	0.239	−0.317	−0.534	0.273
3	13	0.186	−0.397	−0.599	0.429
3	14	0.133	−0.477	−0.658	0.622
3	15	0.080	−0.557	−0.711	0.852
3	16	0.0265	−0.637	−0.758	1.121
4	0	1.17	1.15	1.71	3.50
4	1	1.11	1.04	1.46	2.88
4	2	1.06	0.932	1.22	2.31
4	3	1.01	0.824	0.986	1.80
4	4	0.955	0.716	0.762	1.36
4	5	0.902	0.607	0.546	0.972
4	6	0.849	0.499	0.339	0.647
4	7	0.796	0.391	0.140	0.385
4	8	0.743	0.283	−0.051	0.184
4	9	0.690	0.175	−0.234	0.046
4	10	0.637	0.066	−0.409	−0.029
4	11	0.584	−0.042	−0.575	−0.040
4	12	0.531	−0.150	−0.733	0.013
4	13	0.477	−0.258	−0.883	0.131
4	14	0.424	−0.366	−1.025	0.314
4	15	0.371	−0.474	−1.16	0.562
4	16	0.318	−0.583	−1.28	0.877
4	17	0.265	−0.691	−1.40	1.26
4	18	0.212	−0.799	−1.51	1.71
4	19	0.159	−0.907	−1.61	2.22
4	20	0.106	−1.015	−1.70	2.81
4	21	0.053	−1.124	−1.79	3.46

III. PROPERTIES OF BETA FUNCTION COEFFICIENTS AND APPLICATION TO FUNDAMENTAL REPRESENTATION

In this section we discuss some general properties of the beta function coefficients as functions of N_f, and give particular results for the case of fermions in the fundamental representation. In later sections we consider fermions in two-index representations.

A. b_1

Since we restrict our considerations to an asymptotically free theory, we require that, with our sign conventions, $b_1 > 0$. This, in turn, implies that

$$N_f < N_{f,\max}, \tag{3.1}$$

where

$$N_{f,\max} = \frac{11C_A}{4T_f}. \tag{3.2}$$

Thus, for fermions in the fundamental representation, $N_{f,\max,\mathrm{fund}} = (11/2)N$.

B. b_2 and condition for infrared zero of β

We next proceed to characterize the behavior of the higher-loop coefficients of the beta function, b_ℓ with $\ell = 2, 3, 4$, and the resultant zero(s) of the beta function, in terms of their dependence on N_f. The two-loop results are well-known and are included here so that the discussion will be self-contained. Since only the first two coefficients of the beta function are scheme-independent, it follows that, to the extent that one is in a momentum regime where one can reliably use the perturbative beta function, the zeros obtained from these first two coefficients should be sufficient to characterize the physics at least qualitatively. When one includes higher-loop contributions to the beta function, one expects shifts of zeros, and there are, indeed, generically substantial shifts if zeros of the two-loop beta function occur at $\alpha \sim O(1)$. However, if inclusion of three- and/or higher-loop contributions to β leads to a qualitative change in behavior, relative to the behavior obtained from the two-loop β function, then the results cannot be considered fully reliable, since they are scheme-dependent. For example, for a given gauge group G and fermion content, if the two-loop beta function does not have an infrared zero but the three-loop beta function does, one could not conclude reliably that this is a physical prediction of the theory. Moreover, it should be noted that even if there is no zero of the two-loop beta function away from the origin, i.e., a perturbative IRFP, the beta function may exhibit a nonperturbative slowing of the running associated with the fact that at energy scales below the confinement scale, the physics is not accurately described in terms of the Lagrangian degrees of freedom (fermions and gluons) [32–34].

Another general comment is that the expression of the beta function in Eq. (2.2) is semiperturbative and does not incorporate certain nonperturbative properties of the physics, such as instantons, whose contributions involve essential zeros of the form $\exp(-\kappa\pi/\alpha)$, where κ is a numerical constant. These instanton effects are absent to any order of the perturbative expansion in Eq. (2.2) but play an important role in the theory. For example, they break the global U(1)A symmetry [35] and also enhance spontaneous chiral symmetry breaking [36–39]. Estimates of the effects of instantons on the running of α in quantum chromodynamics have found that they increase this running, i.e., they make β more negative in the region of small to moderate α values [37]. If one were to model the effect of instantons crudely via a modification of β such as

$$\beta_\alpha = \frac{d\alpha}{dt} = -2\alpha^2\left[\sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^{\ell-1} + \lambda \exp\left(-\frac{\kappa\pi}{\alpha}\right)\right], \quad (3.3)$$

then, since $\lambda > 0$, this would have the effect of increasing the value of the smallest (nonzero, positive) IR zero $\alpha_{\rm IR}$ of β. For a given minimal value of $\alpha_{{\rm cr},R}$ for condensate formation and spontaneous chiral symmetry breaking, since at least at the perturbative level $\alpha_{\rm IR}$ is a decreasing function of N_f, it would follow that incorporating instanton effects would increase the value of $N_{f,{\rm cr}}$, i.e., would increase the interval in N_f where there is SχSB. Furthermore, since instantons enhance chiral symmetry breaking, they would tend to reduce the value of $\alpha_{{\rm cr},R}$, which also has the same effect of increasing $N_{f,{\rm cr}}$. We shall comment below on how, although the semiperturbative one-gluon exchange approximation to the Dyson-Schwinger (DS) equation does not directly include effects of confinement or instantons, it may nevertheless yield an approximately correct value of $N_{f,{\rm cr}}$ because of another approximation involved that has the opposite effect on the estimate.

If one knows the beta function calculated to a maximal loop order $\ell_{\rm max}$, then the equation for the zeros of the beta function, aside from the zero at $a = 0$, is

$$\sum_{\ell=1}^{\ell_{\rm max}} b_\ell a^{\ell-1} = b_1\left[1 + \sum_{\ell=2}^{\ell_{\rm max}} \left(\frac{b_\ell}{b_1}\right) a^{\ell-1}\right] = 0. \quad (3.4)$$

As is clear from Eq. (3.4), the zeros of β away from the origin depend only on the $\ell_{\rm max} - 1$ ratios b_ℓ/b_1 for $2 \le \ell \le \ell_{\rm max}$.

The coefficients b_1 and b_2 are linear functions of N_f, while b_3 and b_4 are, respectively, quadratic and cubic functions of N_f. With our sign convention in which an overall minus sign is extracted in Eq. (2.2), each of these coefficients is positive for $N_f = 0$. The coefficients b_1 and b_2 are both monotonically and linearly decreasing functions of N_f. As N_f increases sufficiently, b_2 thus reverses sign, from positive to negative, vanishing at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}. \quad (3.5)$$

(The subscript $b\ell z$ stands for the condition that b_ℓ is *zero*). Since

$$N_{f,{\rm max}} - N_{f,b2z} = \frac{3C_A(11C_f + 7C_A)}{4T_f(3C_f + 5C_A)} > 0, \quad (3.6)$$

i.e., $N_{f,{\rm max}} > N_{f,b2z}$, it follows that there is always a nonvacuous interval in the variable N_f, where the theory is asymptotically free and the two-loop (2ℓ) beta function has an infrared zero, namely

$$N_{f,b2z} < N_f < N_{f,{\rm max}}. \quad (3.7)$$

This zero occurs at

$$\alpha_{{\rm IR},2\ell} = -\frac{4\pi b_1}{b_2} \quad (3.8)$$

and is physical for $b_2 < 0$. Explicitly, for the fundamental representation,

$$N_{f,b2z,{\rm fund}} = \frac{34N^3}{13N^2 - 3} \quad (3.9)$$

and

$$\alpha_{{\rm IR},2\ell,{\rm fund}} = \frac{4\pi(11N - 2N_f)}{-34N^2 + N_f(13N - 3N^{-1})}. \quad (3.10)$$

Illustrative values of $N_{f,b2z,{\rm fund}}$ are given in Table II. The sizes of ℓ-loop contributions are determined by $(\alpha/\pi)^\ell$ multiplied by corresponding powers of various group invariants. Illustrative values of $\alpha_{{\rm IR},2\ell,{\rm fund}}$ are given in Table III for $N = 2, 3, 4$ and the subset of the interval (3.7) for which $\alpha_{{\rm IR},2\ell,{\rm fund}}$ is not so large as to render the two-loop perturbative calculation obviously unreliable. Here and below, when α and γ values are listed without an explicit R, it is understood that they refer to the fundamental representation. Examples of cases that we do not include in the table because the two-loop result cannot be considered reliable include the following (with formal values of $\alpha_{{\rm IR},2\ell,{\rm fund}}$ listed): $N = 2$, $N_f = 5$, where $\alpha_{{\rm IR},2\ell,{\rm fund}} = 11.4$; $N = 3$, $N_f = 9$, where $\alpha_{{\rm IR},2\ell,{\rm fund}} = 5.2$; and $N = 4$, $N_f = 11, 12$, where $\alpha_{{\rm IR},2\ell,{\rm fund}} = 14, 3.5$.

For reference, the estimate in Eq. (B1) of $\alpha_{\rm cr}$ from the analysis of the Dyson-Schwinger equation for the fermion propagator, in the one-gluon exchange approximation, takes the form in Eq. (B1) for a fermion in the fundamental

TABLE II. Values of $N_{f,b2z}$, $N_{f,b3z,\pm}$, and $N_{f,b4z,j}$, $i = 2, 3$, for SU(N) with N_f fermions in the fundamental representation. We only list physical, i.e., real, non-negative values. Thus, since $N_{f,bz4,1} < 0$, is not included.

N	$N_{f,{\rm max}}$	$N_{f,b2z}$	$(N_{f,b3z,-}, N_{f,b3z,+})$	$(N_{f,b4z,2}, N_{f,b4z,3})$
2	11	5.55	(3.99, 27.6)	none
3	16.5	8.05	(5.84, 40.6)	none
4	22	10.61	(7.73, 53.8)	(9.51, 11.83)

TABLE III. Values of the (approximate or exact) IR zeros in α of the SU(N) beta function with N_f fermions in the fundamental representation, for $N = 2, 3, 4$, calculated at n-loop order, and denoted as $\alpha_{\mathrm{IR},n\ell}$. For each N, we only give results for the integral N_f values in the range (3.7), where the theory is asymptotically free and the two-loop beta function has an infrared zero. For the four-loop beta function, the cubic equation (3.32) has three zeros, one of which is negative, one of which is $\alpha_{\mathrm{IR},4\ell}$, and the third of which is positive but farther from the origin. We include the latter, denoted as $\alpha_{4\ell,u}$. We also list zeros from the [1,2] and [2,1] Padé approximants to the four-loop beta function.

N	N_f	$\alpha_{\mathrm{IR},2\ell}$	$\alpha_{\mathrm{IR},3\ell}$	$\alpha_{\mathrm{IR},4\ell}$	$\alpha_{\mathrm{IR},4\ell,[1,2]}$	$\alpha_{\mathrm{IR},4\ell,[2,1]}$	$\alpha_{4\ell,u}$
2	7	2.83	1.05	1.21	2.30	1.16	4.12
2	8	1.26	0.688	0.760	0.952	0.741	3.11
2	9	0.595	0.418	0.444	0.475	0.438	2.395
2	10	0.231	0.196	0.200	0.202	0.200	1.97
3	10	2.21	0.764	0.815	1.47	0.807	5.62
3	11	1.23	0.578	0.626	0.871	0.616	3.29
3	12	0.754	0.435	0.470	0.561	0.462	2.295
3	13	0.468	0.317	0.337	0.367	0.333	1.78
3	14	0.278	0.215	0.224	0.231	0.222	1.48
3	15	0.143	0.123	0.126	0.127	0.125	1.29
3	16	0.0416	0.0397	0.0398	0.0398	0.0398	1.15
4	13	1.85	0.604	0.628	1.14	0.625	6.94
4	14	1.16	0.489	0.521	0.776	0.516	3.49
4	15	0.783	0.397	0.428	0.556	0.422	2.30
4	16	0.546	0.320	0.345	0.407	0.340	1.73
4	17	0.384	0.254	0.271	0.298	0.267	1.40
4	18	0.266	0.194	0.205	0.215	0.203	1.19
4	19	0.175	0.140	0.145	0.149	0.145	1.05
4	20	0.105	0.091	0.092	0.0930	0.0921	0.947
4	21	0.0472	0.044	0.044	0.0444	0.0443	0.870

representation. As listed in Table IV, this has the respective values 1.4, 0.79, and 0.56 for $N = 2, 3, 4$, respectively (where we quote the results to two significant figures but do not mean to imply that they have such a high degree of accuracy). Setting $\alpha_{\mathrm{cr}} = \alpha_{\mathrm{IR},2\ell}$ yields the resultant estimates of $N_{f,\mathrm{cr}}$ which, rounded to the nearest integers, are 8, 12, and 16 for these values of N. We denote these as βDS estimates since they combine a calculation of α_{IR} from the perturbative two-loop β function with the (one-gluon exchange approximation to the) Dyson-Schwinger equation.

TABLE IV. Estimates of $\alpha_{\mathrm{cr},R}$ from the one-gluon exchange approximation to the Dyson-Schwinger equation for the fermion propagator. Values are listed for SU(N) with $2 \le N \le 6$ and the representations $R =$ (i) fundamental (fund), (ii) adjoint (adj), (iii) symmetric rank-2 tensor (S2), and (iv) antisymmetric rank-2 (A2).

N	$\alpha_{\mathrm{cr,fund}}$	$\alpha_{\mathrm{cr,adj}}$	$\alpha_{\mathrm{cr,S2}}$	$\alpha_{\mathrm{cr,A2}}$
2	1.40	0.52	0.52	–
3	0.79	0.35	0.31	0.79
4	0.56	0.26	0.23	0.42
5	0.44	0.21	0.19	0.29
6	0.36	0.17	0.16	0.22

As N_f approaches its maximum value $N_{f,\max}$ allowed by the constraint that the theory be asymptotically free, b_2 reaches its most negative value, namely $b_2 = -C_A(7C_A + 11C_f)$. Clearly, for N_f values such that b_2 is only negative by a small amount and $\alpha_{\mathrm{IR},2\ell}$ is large, the perturbative calculation is not reliable. As N_f increases further in the range (3.1) and $\alpha_{\mathrm{IR},2\ell}$ decreases, the calculation becomes more reliable. In Table II we list the numerical values of $N_{f,b2z}$ for some illustrative values of N. At the two-loop level, depending on whether $\alpha_{\mathrm{IR},2\ell}$ is smaller or larger than a critical value for fermion condensation, this is an exact or approximate infrared fixed point of the renormalization group for the gauge coupling. The existence of such an IRFP is of fundamental importance in determining how the theory evolves from the ultraviolet to the infrared [40]. In particular, as mentioned above, this determines whether, as the scale μ decreases sufficiently to a scale Λ (depending on the group G and the fermion content), α grows to a large enough size to produce fermion condensates or, on the contrary, the coupling never gets this large and the theory evolves into the infrared in a chirally symmetric manner, without ever producing such fermion condensates. Note that in the former case, the fermions involved in the condensates get dynamical masses of order Λ and are integrated out of the effective low-energy field theory applicable for scales $\mu < \Lambda$, so that the further evolution into the infrared is governed by a different beta function.

It is useful to observe how rapidly the numbers $N_{f,b2z}$ approach their large-N values. The number $N_{f,b2z,\mathrm{fund}}$ has the large-N expansion

$$N_{f,b2z,\mathrm{fund}} = N\left[\frac{34}{13} + \frac{102}{(13N)^2} + \frac{306}{(13)^3N^4} + O\left(\frac{1}{N^6}\right)\right] = N\left[2.615 + \frac{0.603\,55}{N^2} + \frac{0.1393}{N^4} + O\left(\frac{1}{N^6}\right)\right]. \quad (3.11)$$

As is evident from Table II, the values of $N_{f,b2z,\mathrm{fund}}$ approach the leading asymptotic form for moderate N, as a result of the fact that the subleading term in Eq. (3.11) is suppressed by $1/N^2$.

It is of interest to consider the 't Hooft large-N limit, where

$$N \to \infty \quad \text{with} \quad \alpha N \quad \text{fixed.} \quad (3.12)$$

In a theory with fermions in the fundamental representation, in order for them to have a non-negligible effect in this limit, one considers the simultaneous Veneziano limit

$$N_f \to \infty \quad \text{with} \quad r \equiv \frac{N_f}{N} \quad \text{fixed.} \quad (3.13)$$

In the combined limit of Eqs. (3.12) and (3.13), the range of r satisfying the requirement of asymptotic freedom and the condition that $b_2 < 0$ so that the two-loop beta function has an IR zero is [41]

$$\frac{34}{13} < r < \frac{11}{2}, \quad \text{i.e.,} \quad 2.615 < r < 5.5. \quad (3.14)$$

C. Coefficient b_3 and three-loop behavior of the beta function

The three-loop beta-function coefficient b_3 is a quadratic function of N_f with positive coefficients of its N_f^0 and N_f^2 terms and a negative coefficient of its N_f term. Hence, regarded as a function of the formal real variable N_f, it is positive for large negative and positive N_f, and positive at $N_f = 0$. The derivative of b_3 with respect to N_f is

$$\frac{db_3}{dN_f} = T_f\left[-\frac{1415}{27}C_A^2 - \frac{205}{9}C_AC_f + 2C_f^2 + T_fN_f\left(\frac{88}{9}C_f + \frac{316}{27}C_A\right)\right]. \quad (3.15)$$

For the fermion representations R that we consider here, for small values of N_f, this derivative db_3/dN_f is negative, so that in this region of N_f, b_3 decreases from its positive value at $N_f = 0$ as N_f increases. Because b_3 is a quadratic polynomial in N_f, the condition that it vanishes gives two formal solutions for N_f, namely

$$N_{f,b3z,\pm} = \frac{(1415C_A^2 + 615C_AC_f - 54C_f^2 \pm 3\sqrt{F_{Rb3}})}{4T_f(79C_A + 66C_f)}, \quad (3.16)$$

and

$$F_{Rb3} = 122\,157C_A^4 + 109\,578C_A^3C_f + 25\,045C_A^2C_f^2 - 7380C_AC_f^3 + 324C_f^4. \quad (3.17)$$

Given that $F_{Rb3} > 0$, as is the case here, so that the values $N_{f,b3z,j}$ are real, it follows that b_3 is positive in the intervals $N_f < N_{f,b3z,-}$ and $N_f > N_{f,b3z,+}$ and negative in the interval $N_{f,b3z,-} < N_f < N_{f,b3z,+}$. The value $N_{f,b3z,+}$ and the neighborhood of N_f values in the vicinity of $N_{f,b3z,+}$ are not of interest here because they are larger than the maximal value $N_{f,\max}$ allowed by the requirement of asymptotic freedom,

$$N_{f,b3z,+} > N_{f,\max}. \quad (3.18)$$

Thus, b_3 only changes sign once for N_f in the asymptotically free interval $0 \le N_f < N_{f,\max}$. As N_f approaches $N_{f,\max}$ from below, b_3 decreases to a negative value given by

$$(b_3)_{N_f=N_{f,\max}} = -\frac{C_A}{24}[1127C_A^2 + 44C_f(14C_A - 3C_f)]. \quad (3.19)$$

For fermions in the fundamental representation, this is

$$(b_3)_{N_f=N_{f,\max,\mathrm{fund}}} = -\frac{701}{12}N_c^3 + \frac{121}{12}N_c + \frac{11}{8N_c}. \quad (3.20)$$

As is clear from Table II, for this case

$$N_{f,b3z,1} < N_{f,b2z}. \quad (3.21)$$

We noted above that any physically reliable zero of the beta function must be present already at the level of the two-loop beta function, since this is the maximal scheme-independent part of this function. Hence, in analyzing such a zero for the case under consideration where the fermions transform according to the fundamental representation of SU(N), we only consider the interval (3.7). Combining this fact with our results (3.21) and (3.18), it follows that b_3 is negative throughout all of the interval (3.7) of interest here. For this fundamental-representation case, the $N_{f,b3z,j}$ with $j = 1, 2$ have the large-N expansions

$$N_{f,b3z,1} = N\left[1.911 + \frac{0.3244}{N^2} + \frac{0.06844}{N^4} + O\left(\frac{1}{N^6}\right)\right] \quad (3.22)$$

and

$$N_{f,b3z,2} = N\left[13.348 + \frac{1.667}{N^2} + \frac{0.3978}{N^4} + O\left(\frac{1}{N^6}\right)\right]. \quad (3.23)$$

Here, $N_{f,\max} = 5.5N$.

At three-loop order, the equation for the zeros of the beta function, aside from the zero at $a = 0$, is $b_1 + b_2a + b_3a^2 = 0$. Formally, this equation has two solutions for a and hence for α, namely

$$\alpha_{\beta z,3\ell,\pm} = \frac{2\pi}{b_3}\left[-b_2 \pm \sqrt{b_2^2 - 4b_1b_3}\right]. \tag{3.24}$$

Since b_2 must be negative in order for the beta function to have a scheme-independent infrared zero, and since for fermions in the fundamental representation we have shown that $b_3 < 0$ in the relevant interval (3.7), we can rewrite this equivalently as

$$\alpha_{\beta z,3\ell,\pm} = \frac{2\pi}{|b_3|}\left[-|b_2| \mp \sqrt{b_2^2 + 4b_1|b_3|}\right]. \tag{3.25}$$

In order for a given solution to be physical, it must be real and positive. As is evident from Eq. (3.25), the solution corresponding to the $+$ sign in Eq. (3.24) [i.e., the $-$ sign in Eq. (3.25)] is negative and hence unphysical. Thus, there is a unique physical solution for the IR zero of the beta function to three-loop order, namely

$$\alpha_{\mathrm{IR},3\ell} = \alpha_{\beta z,3\ell,-}. \tag{3.26}$$

Illustrative values for this IR zero of the beta function at three-loop order are listed in Table IV.

For an arbitrary fermion representation for which β has a two-loop IR zero, we observe that the value of this zero decreases when one calculates it to three-loop order, i.e.,

$$\alpha_{\mathrm{IR},3\ell} < \alpha_{\mathrm{IR},2\ell}. \tag{3.27}$$

This can be proved as follows. We have

$$\alpha_{\mathrm{IR},2\ell} - \alpha_{\mathrm{IR},3\ell} = \frac{2\pi}{|b_2b_3|}\left[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\right]. \tag{3.28}$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0. \tag{3.29}$$

But the difference in (3.29) is equal to the positive-definite quantity $b_1^2b_3^2$, which proves the inequality (3.27). This inequality is evident in Table IV.

D. Coefficient b_4 and four-loop behavior of β

The four-loop beta-function coefficient b_4 was calculated in Ref. [29]. We next analyze its behavior as a function of N_f and the result four-loop IR zero of the beta function. The coefficient b_4 is a cubic polynomial in N_f which has positive coefficients of its N_f^0, and N_f^3 terms. Hence, regarded as a function of the formal real variable N_f, b_4 is negative for large negative N_f, positive for $N_f = 0$, and also positive for large positive N_f. For fermions in the fundamental representation, the derivative at $N_f = 0$ is

$$\left(\frac{db_4}{dN_f}\right)_{N_f=0} = -\left(\frac{485513}{1944} + \frac{20}{9}\zeta(3)\right)N^3 + \left(\frac{58583}{1944} - \frac{548}{9}\zeta(3)\right)N + \left(-\frac{2341}{216} + \frac{44}{9}\zeta(3)\right)N^{-1} - \frac{23}{8}N^{-3}, \tag{3.30}$$

where $\zeta(z)$ is the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \tag{3.31}$$

and $\zeta(3) = 1.202\,056\,90\ldots$. This derivative is negative for all N. (In the complex N plane, it has six zeros at three complex-conjugate pairs of N values.) It follows that, again as a function of the formal real variable N_f, b_4 has a local maximum at a negative value of N_f and then decreases through positive values as N_f increases toward 0 and passes through 0 into the interval of physical values.

The detailed behavior of b_4 in the physical asymptotically free interval $0 \le N_f \le N_{f,\mathrm{max}}$ depends on N. In particular, one may determine the value of N_f where b_4 has a minimum and whether b_4 has any zeros for positive N_f. For SU(2), b_4 decreases to a minimum positive value as N_f ascends through the approximate value $N_f = 5.8$, and then increases monotonically for larger N_f, so that it is positive-definite for all non-negative N_f, in particular, the asymptotically free region $0 \le N_f < 11$. For SU(3), b_4 is also positive-definite for all non-negative N_f, reaching a local minimum as N_f ascends through a value of approximately 8.2 and then increasing monotonically for larger N_f. However, for SU(4), b_4 is positive for $0 \le N_f \le 9.51$, negative for the interval $9.51 \le N_f \le 11.83$, and positive again for $N_f > 11.83$, with zeros at $N_f \simeq 9.51$ and $N_f \simeq 11.83$. We list these zeros of b_4 as a function of N_f in Table II. (Again, we recall that the physical values of N_f are, of course, restricted to non-negative integers.) Thus, this reversal of sign occurs in the interval of interest here, $0 \le N_f < 22$, where the SU(4) theory is asymptotically free. For SU(5), b_4 behaves in a manner qualitatively similar to the SU(4) case; it is positive for $0 \le N_f \le 11.18$, negative in the interval $11.18 \le N_f \le 15.18$, and positive for larger values of N_f, vanishing at $N_f \simeq 11.18$ and $N_f \simeq 15.18$. Thus, again, b_4 reverses sign in the region $0 \le N_f < 22.5$ where the SU(5) theory is asymptotically free. Thus, in contrast with b_2 and b_3, which are negative throughout the interval of N_f of interest (and b_1, which is positive), b_4 can, for $N \ge 4$, vanish and reverse sign in this interval.

At the four-loop level, the equation for the zeros of the beta function, aside from $a = 0$, is the cubic equation

$$b_1 + b_2a + b_3a^2 + b_4a^3 = 0. \tag{3.32}$$

This equation has three solutions for a and hence for α, which will be denoted $\alpha_{\beta z,4\ell,j}$, $j = 1, 2, 3$. Since the coefficients b_ℓ are real, there are two generic possibilities for these three roots, namely, that they are all real, or that one is real and the other two form a complex-conjugate pair. The properties of the roots are further restricted by the asymptotic freedom condition that $b_1 > 0$, the existence of a two-loop IR zero, which requires that $b_2 < 0$, and the fact that, as we have shown, for the relevant range (3.7) of N_f, where these conditions are met, $b_3 < 0$. As is evident in Table IV, we find that for the values of N and N_f that we consider, the roots of Eq. (3.32) are real. For all of the values of N and N_f where there is a reliable two-loop value for an IR zero of the beta function (i.e., where it does not occur at such a large value of α as to render the perturbative calculation untrustworthy), one of these roots is negative and hence not physical, one of them, namely, the minimal positive one, is the physical IR zero, which we will denote $a_{\mathrm{IR},4\ell} = \alpha_{\mathrm{IR},4\ell}/(4\pi)$, and there is a third root at a larger positive value. This third root, denoted $a_{4\ell,u} = \alpha_{4\ell,u}/(4\pi)$, is not relevant for our analysis, since the initial value of α at a high-energy scale μ is assumed to be close to zero, so that as the scale μ decreases, α increases and approaches the (positive) zero of the beta function closest to the origin, namely $\alpha_{\mathrm{IR},4\ell}$ [42].

It is straightforward to display the analytic expressions for the root $\alpha_{\mathrm{IR},4\ell}$, but we shall not need this for our analysis. We list numerical values for $\alpha_{\mathrm{IR},4\ell}$ for various values of N and N_f in Table IV. For completeness, we note the specific sets (N, N_f) where $\alpha_{\mathrm{IR},2\ell}$ is so large that we consider the analysis via the perturbative beta function unreliable: these are $(N, N_f) = (2, 6)$, (3,9), (4,11), and (4,12).

E. Estimates of zeros of the four-loop beta function via Padé approximants

For the beta function, or more conveniently, the reduced function with the prefactor removed, $\sum_{j=0}^{\ell_{\max}-1} b_j a^{j-1}$, it is useful to calculate and analyze Padé approximants, since these provide closed-form expressions that, by construction, agree with the series to the maximal order to which it is calculated. The expansion for $\bar{\beta}_\alpha$ to $\ell = 4$ loop order can be used in two ways. First, one can simply solve the cubic equation $\bar{\beta}_\alpha = b_1 + b_2 a + b_3 a^2 + b_4 a^3 = 0$ and obtain the three roots, one of which is the root of interest, giving the IR zero. Secondly, one can calculate Padé approximants, e.g., the [2,1] and [1,2] approximants, and determine their zeros. The [1,2] Padé approximant has a single zero at

$$a_{\beta z,4\ell,[1,2]} = \frac{\alpha_{\mathrm{IR},4\ell,[1,2]}}{4\pi} = \frac{b_1(b_1 b_3 - b_2^2)}{b_2^3 - 2b_1 b_2 b_3 + b_1^2 b_4)}. \quad (3.33)$$

Taking into account the fact that b_2 and b_3 are negative in the relevant interval (3.7), this can be rewritten as

TABLE V. Values of the ℓ-loop coefficients $\bar{c}_\ell$ in the series expansion (2.5) for the anomalous dimension γ_m, as functions of N and N_f, for the range (3.1), where the theory is asymptotically free.

N	N_f	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$
2	0	0.358	0.318	0.310	0.329
2	1	0.358	0.302	0.254	0.234
2	2	0.358	0.286	0.195	0.143
2	3	0.358	0.270	0.134	0.0577
2	4	0.358	0.254	0.0712	−0.0218
2	5	0.358	0.239	0.006 56	−0.0952
2	6	0.358	0.223	−0.0601	−0.162
2	7	0.358	0.207	−0.129	−0.222
2	8	0.358	0.191	−0.199	−0.274
2	9	0.358	0.175	−0.272	−0.319
2	10	0.358	0.1595	−0.346	−0.355
3	0	0.637	0.853	1.26	2.03
3	1	0.637	0.825	1.11	1.64
3	2	0.637	0.796	0.957	1.27
3	3	0.637	0.768	0.801	0.909
3	4	0.637	0.740	0.642	0.561
3	5	0.637	0.712	0.479	0.227
3	6	0.637	0.684	0.312	−0.0926
3	7	0.637	0.656	0.142	−0.396
3	8	0.637	0.628	−0.0313	−0.683
3	9	0.637	0.599	−0.208	−0.953
3	10	0.637	0.571	−0.389	−1.21
3	11	0.637	0.543	−0.573	−1.44
3	12	0.637	0.515	−0.760	−1.65
3	13	0.637	0.487	−0.951	−1.85
3	14	0.637	0.459	−1.145	−2.02
3	15	0.637	0.431	−1.34	−2.18
3	16	0.637	0.402	−1.54	−2.31
4	0	0.895	1.60	3.17	6.86
4	1	0.895	1.56	2.89	5.88
4	2	0.895	1.52	2.61	4.93
4	3	0.895	1.48	2.32	4.00
4	4	0.895	1.44	2.03	3.09
4	5	0.895	1.40	1.73	2.21
4	6	0.895	1.36	1.43	1.36
4	7	0.895	1.33	1.12	0.526
4	8	0.895	1.29	0.808	−0.275
4	9	0.895	1.25	0.492	−1.05
4	10	0.895	1.21	0.170	−1.79
4	11	0.895	1.17	−0.157	−2.50
4	12	0.895	1.13	−0.488	−3.18
4	13	0.895	1.09	−0.825	−3.83
4	14	0.895	1.05	−1.17	−4.45
4	15	0.895	1.01	−1.51	−5.025
4	16	0.895	0.969	−1.86	−5.57
4	17	0.895	0.930	−2.22	−6.08
4	18	0.895	0.890	−2.58	−6.54
4	19	0.895	0.850	−2.95	−6.97
4	20	0.895	0.811	−3.32	−7.37
4	21	0.895	0.771	−3.69	−7.71

$$a_{\beta z,4\ell,[1,2]} = \frac{\alpha_{\mathrm{IR},4\ell,[1,2]}}{4\pi} = \frac{b_1(b_1|b_3| + b_2^2)}{|b_2|^3 + 2b_1|b_2||b_3| - b_1^2 b_4)}. \tag{3.34}$$

The two zeros from the [2,1] approximant are

$$a_{\beta z,4\ell,[2,1],\pm} = \frac{b_2 b_3 - b_1 b_4 \pm [(b_2 b_3 - b_1 b_4)^2 - 4b_1 b_3(b_3^2 - b_2 b_4)]^{1/2}}{2(b_2 b_4 - b_3^2)}. \tag{3.35}$$

Taking account of the fact that b_2 and b_3 are negative in the relevant interval (3.7), this can be rewritten as

$$a_{\beta z,4\ell,[2,1],\pm} = \frac{b_1 b_4 - |b_2||b_3| \mp [(|b_2||b_3| - b_1 b_4)^2 + 4b_1|b_3|(b_3^2 + |b_2|b_4)]^{1/2}}{2(|b_2|b_4 + b_3^2)}. \tag{3.36}$$

The expression in Eq. (3.36) with the $-$ sign in front of the square root is negative and unphysical, while the expression with the $+$ sign in front of the square root yields the estimate of the IR fixed point, as $\alpha_{\mathrm{IR},4\ell,[2,1]} = 4\pi a_{\beta z,4\ell,[2,1]}$. As is evident from Eqs. (3.33) and (3.35), the zeros of the [1,2] and [2,1] Padé approximants incorporate information on β up to four loops. One readily verifies that in the limit $b_4 \to 0$, the zero of the [1,2] Padé reduces to the two-loop result $a = -b_1/b_2$, and the two zeros of the [2,1] Padé reduce to those obtained from the three-loop beta function (3.36). We list the values of α_{IR} obtained from the zeros of the [1,2] and [2,1] Padé approximants to the four-loop beta function for the case of fermions in the fundamental representation in Table IV.

From our calculations of α_{IR} at the three- and four-loop level for SU(N) with fermions in the fundamental representation, we can make several remarks. Although n-loop calculations of the beta function for $n \geq 3$ loops are scheme-dependent, the results obtained with the present $\overline{\mathrm{MS}}$ scheme provide a quantitative measure of the accuracy of the scheme-independent two-loop result. For a given N, as N_f increases above the minimal value $N_{f,b2z}$, where the IR zero first appears, and as the resultant $\alpha_{\mathrm{IR},2\ell}$ decreases to values $\lesssim 1$, the difference between $\alpha_{\mathrm{IR},2\ell}$ and the higher-loop values $\alpha_{\mathrm{IR},n\ell}$ for $n = 3, 4$ decrease. As is evident from Table IV, the value of $\alpha_{\mathrm{IR},n\ell}$ generically decreases as one goes from $n = 2$ to $n = 3$ loops and then increases by a smaller amount as one goes from $n = 3$ to $n = 4$ loops, so that $\alpha_{\mathrm{IR},4\ell}$ is smaller than $\alpha_{\mathrm{IR},2\ell}$. In the same region of N_f values such that $\alpha_{\mathrm{IR},2\ell}$ is reasonably small, the values obtained via the [1,2] and [2,1] Padé approximants to the four-loop beta function are close to those obtained from the zeros of this beta function itself.

IV. EVALUATION OF THE ANOMALOUS DIMENSION γ_m AT THE INFRARED ZERO OF β

In this section we evaluate the anomalous dimension of $\gamma \equiv \gamma_m$, calculated to the n-loop order in perturbation theory, at the (approximate or exact) IR zero of the beta function to this order, $\alpha_{\mathrm{IR},n\ell}$, for $n = 2, 3, 4$. We denote these as $\gamma_{n\ell}(\alpha_{\mathrm{IR},n\ell})$. We focus here on general results and their application to the case of fermions in the fundamental representation, and discuss higher-dimensional representations in subsequent sections. In general, this anomalous dimension must be positive to avoid unphysical singularities in fermion correlation functions. The coefficients $\bar{c}_\ell$ that enter in Eq. (2.5) used in this calculation are listed in Table V.

A running fermion mass $\Sigma(k)$, that is dynamically generated at a scale Λ, decays with Euclidean momentum $k > \Lambda$ like

$$\Sigma(k) \sim \Lambda\left(\frac{\Lambda}{k}\right)^{2-\gamma_m} \tag{4.1}$$

up to logs. Since, for $k > \Lambda$, the running coupling α is smaller than the critical value $\alpha_{R,\mathrm{cr}}$ and there is no spontaneous chiral symmetry breaking, it follows that $\Sigma(k)$ must decrease toward zero as $k/\Lambda \to \infty$. In turn, this implies that $\gamma_m < 2$. Hence, a physical value of γ_m must lie in the range

$$0 < \gamma_m < 2. \tag{4.2}$$

For values of N_f such that the theory evolves into the infrared in a chirally symmetric manner, so that the IR zero of the beta function is exact, the same upper bound follows from a related unitarity consideration [43].

Using the two-loop result for γ and evaluating it at the two-loop value of the IR zero of the beta function, we have

$$\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell}) = \frac{C_f(11C_A - 4T_f N_f)(455C_A^2 + 99C_A C_f + (180C_f - 248C_A)T_f N_f + 80T_f^2 N_f^2)}{12(-17C_A^2 + (10C_A + 6C_f)T_f N_f)^2}. \tag{4.3}$$

For the fundamental representation, this is

$$\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell}) = \frac{(N^2-1)(11N-2N_f)(1009N^3-99N-(158N^2+90)N_f+40NN_f^2)}{12(-34N^3+(13N^2-3)N_f)^2}. \tag{4.4}$$

We list numerical values of $\gamma(\alpha_{\mathrm{IR},2\ell})$ in Table VI for the illustrative values $N = 2, 3, 4$ and, for each N, a set of N_f values in the range (3.7). For sufficiently small $N_f > N_{f,b2z}$ in each N case, $\alpha_{\mathrm{IR},2\ell}$ is so large that the formal value of $\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell})$ is larger than 2 and hence unphysical; we enclose these values in parentheses to indicate that they are unphysical artifacts of a perturbative calculation at an excessively large value of α.

In the large-N, large-N_f limit of Eqs. (3.12) and (3.13) with $r \equiv N_f/N$, Eq. (4.4) reduces to

$$\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell}) = \frac{(11-2r)(1009-158r+40r^2)}{12(-34+13r)^2} + O\left(\frac{1}{N^2}\right). \tag{4.5}$$

TABLE VI. Values of the anomalous dimension in the SU(N) theory with N_f fermions in the fundamental representation γ_m, calculated to the n-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, $\alpha_{\mathrm{IR},n\ell}$, for $\ell = 2, 3, 4$. We denote these as $\gamma_{n\ell}(\alpha_{\mathrm{IR},n\ell})$. For sufficiently small $N_f > N_{f,b2z}$ in each N case, $\alpha_{\mathrm{IR},2\ell}$ is so large that the formal value of $\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell})$ is larger than 2 and hence unphysical; we indicate this by placing these values in parentheses.

N	N_f	$\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell})$	$\gamma_{3\ell}(\alpha_{\mathrm{IR},3\ell})$	$\gamma_{4\ell}(\alpha_{\mathrm{IR},4\ell})$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259
4	13	(5.38)	0.755	0.192
4	14	(2.45)	0.552	0.259
4	15	1.32	0.420	0.281
4	16	0.778	0.325	0.269
4	17	0.481	0.251	0.234
4	18	0.301	0.189	0.187
4	19	0.183	0.134	0.136
4	20	0.102	0.0854	0.0865
4	21	0.0440	0.0407	0.0409

For $r = 4$ corresponding to the asymptotic value of $N_{f,\mathrm{cr,fund}}$ in Eq. (B2), $\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell}) = 113/144 \simeq 0.785$, which is the same as the large-N limit of Eq. (4.7).

One may evaluate $\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell})$ at N_f equal to the value $N_{f,\mathrm{cr,fund}}$ predicted by the one-gluon exchange (ladder) approximation to the Dyson-Schwinger equation for the fermion propagator, given in Eq. (B2). This is somewhat formal, since these values of $N_{f,\mathrm{cr,fund}}$ are not, in general, integers and hence not actually physical; for example, $N_{f,\mathrm{cr,fund}} = 7.86, 11.91, 15.94$ for $N = 2, 3, 4$). This procedure yields the result

$$\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell}; N_{f,\mathrm{cr,fund}}) = \frac{565N^4 - 706N^2 + 225}{144(N^2-1)(5N^2-3)}. \tag{4.6}$$

For the illustrative cases $N = 2, 3, 4$, this anomalous dimension takes the values 0.88, 0.82, and 0.80, respectively. As $N \to \infty$, Eq. (4.6) has the expansion

$$\gamma_{2\ell}(\alpha_{\mathrm{cr,fund}}) = \frac{113}{144} + \frac{11}{40N^2} + O\left(\frac{1}{N^4}\right). \tag{4.7}$$

Since the estimate (B2) is close to $4N$ even for the smallest value, $N = 2$, and asymptotically approaches $4N$ as $N \to \infty$, it is worthwhile to compare the above values of γ, viz., 0.88, 0.82, and 0.80 for $N = 2, 3, 4$, with $\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell})$ evaluated at the nearest physical, integer values of N_f, namely $N_f = 8, 12, 16$ for $N = 2, 3, 4$. This procedure yields $\gamma_{2\ell}(\alpha_{\mathrm{IR},2\ell}) = 0.75, 0.77, 0.78$, as recorded in Table VI. To within the strong-coupling theoretical uncertainties of these calculations, these values are mutually consistent.

A closely related approach is to evaluate the two-loop expression for γ_m at $\alpha = \alpha_{\mathrm{cr},R}$, where $\alpha_{\mathrm{cr},R}$ is the estimate of the critical coupling for fermion condensation obtained from the one-gluon exchange approximation to the Dyson-Schwinger equation, and then substitute $N_f = N_{f,\mathrm{cr}}$ from the βDS analysis (see Appendix B). This yields the result

$$\gamma_{2\ell}(\alpha_{\mathrm{cr},R}; N_f = N_{f,\mathrm{cr},R}) = \frac{21C_A^2 + 128C_AC_f + 225C_f^2}{144C_f(C_A+3C_f)}. \tag{4.8}$$

For the fundamental representation, this reduces to the same result as was obtained in Eq. (4.6).

We have evaluated the three-loop result for γ at the three-loop value of the IR zero of the beta function, which we denote as $\gamma_{3\ell}(\alpha_{\mathrm{IR},3\ell})$, and the four-loop result for γ at the four-loop value of the IRFP, which we denote as $\gamma_{4\ell}(\alpha_{IR,4\ell})$. We list the resultant values in Table VI. From our calculations of γ_m for the case of fermions in the fundamental representation, we can make several observations. Although computations of $\alpha_{IR,n\ell}$ and $\gamma_{n\ell}(\alpha_{IR,n\ell})$ are scheme-dependent for $n \geq 3$ loops, they provide a useful measure of the accuracy of the lowest-order results. As was the case with the position of $\alpha_{IR,n\ell}$ itself, we find that, for a given N and for N_f reasonably well above $N_{f,b2z}$ so that the perturbative calculation of $\alpha_{IR,n\ell}$ is not too large, the value of $\gamma_{n\ell}(\alpha_{IR,n\ell})$ generically decreases as one goes from $n = 2$ to $n = 3$ loops. Some of this decrease can be ascribed to the decrease in $\alpha_{IR,n\ell}$ going from ($n = 2$)-loop to ($n = 3$)-loop order. At the four-loop level, $\gamma_{4\ell}(\alpha_{IR,4\ell})$ tends to be smaller than $\gamma_{3\ell}(\alpha_{IR,3\ell})$ for values of N_f from $N_{f,b2z}$ to values of N_f slightly above the middle of the range (3.7), while for values of N_f in the upper end of this range, $\gamma_{4\ell}(\alpha_{IR,4\ell})$ is slightly larger than $\gamma_{3\ell}(\alpha_{IR,3\ell})$. In general, for the values of N_f where α_{IR} is sufficiently small that the calculation may be trustworthy, the value of the anomalous dimension evaluated at the IR zero of the beta function (both calculated to n-loop order) $\gamma_{n\ell}(\alpha_{IR,n\ell})$, is somewhat smaller than unity.

Several recent high-statistics lattice simulations have been carried out on an SU(3) gauge theory with a varying number N_f of fermions in the fundamental representation in the range $6 \leq N_f \leq 12$ [8,11–15,24]. This work has yielded evidence for a regime of slowly running gauge couplings for $N_f \lesssim 12$, consistent with the presence of an IR zero of the beta function, in agreement with the earlier continuum estimates in Ref. [1]. Ref. [11] also found a considerable enhancement of $\langle\bar{\psi}\psi\rangle/f_P^3$ in the SU(3) theory with $N_f = 6$. Further lattice simulations and analysis of data should yield values of γ_m that can be compared with our higher-loop calculations in this paper. A preliminary study of the SU(2) theory with $N_f = 6$ fermions has also been reported [21].

V. ADJOINT REPRESENTATION

In this section we analyze the SU(N) theory with N_f copies of a Dirac fermion, or equivalently, $2N_f$ copies of a Majorana fermion, in the adjoint representation. For this case, the general expression for the maximal value of N_f allowed by the requirement of asymptotic freedom, Eq. (3.2), reduces to

$$N_{f,\mathrm{max,adj}} = \frac{11}{4}, \tag{5.1}$$

i.e., restricting N_f to the integers $N_{f,\mathrm{max}} = 2$. The general expression in Eq. (3.5) for the value of N_f at which b_2 changes sign from positive to negative with increasing N_f reduces to

$$N_{f,b2z,\mathrm{adj}} = \frac{17}{16} = 1.0625. \tag{5.2}$$

Hence there is only one (integer) value of N_f, namely $N_f = 2$ Dirac fermions (equivalently, $N_f = 4$ Majorana fermions), for which the theory is asymptotically free and has an IR zero of the two-loop beta function. This zero occurs at

$$\alpha_{IR,2\ell,\mathrm{adj}} = \frac{2\pi}{5N} \simeq \frac{1.257}{N} \quad \text{for } N_f = 2. \tag{5.3}$$

Specializing the general formula for the critical coupling $\alpha_{cr,R}$ from the one-gluon exchange approximation to the Dyson-Schwinger equation, Eq. (B1) (see Appendix B) for the present case where R is the adjoint representation, one obtains $\alpha_{\mathrm{cr,adj}} = \pi/(3N)$. Formally setting $\alpha_{IR,2\ell,\mathrm{adj}} = \alpha_{\mathrm{adj,cr}}$ yields the corresponding estimate for the critical number $N_{f,\mathrm{cr}} = 83/40 = 2.075$. This may be rounded off to the nearest integer, giving $N_{f,\mathrm{cr}}$ for the adjoint representation. In view of the theoretical uncertainty in such an estimate, due to the strong-coupling nature of the physics involved, an SU(N) gauge theory with $N_f = 2$ adjoint fermions could be either slightly inside the chirally broken, confined side of $N_{f,\mathrm{cr}}$ or slightly on the other side, where the theory is chirally symmetric and the evolution into the infrared is governed by an exact conformal IR fixed point.

For the present case of $N_f = 2$ fermions in the adjoint representation of SU(N), the coefficients of the beta function are $b_1 = N$, $b_2 = -10N^2$, $b_3 = -(101/2)N^3$, and

$$b_4 = N^2\left[\frac{1843}{18}N^2 - 312\right] - 4\zeta(3)N^2(N^2 + 72). \tag{5.4}$$

At the four-loop level, the beta function has three zeros away from the origin, one of which is the four-loop IR zero, denoted $\alpha_{IR,4\ell,\mathrm{adj}}$. For $N = 2$, the others form an unphysical complex-conjugate pair, while for the other values of N that we consider, the others consist of a negative one and a another, denoted $\alpha_{4\ell,u}$, which is not relevant to our study, since it is not reached by evolution of the coupling starting at small α for large μ. We list the numerical values of these zeros in Table VII.

The coefficients $\bar{c}_\ell$ in Eq. (2.5) for γ for this case are $\bar{c}_1 = 3N/(2\pi)$, $\bar{c}_2 = (11N^2)/(8\pi^2)$, and $\bar{c}_3 = -N^3/(2\pi^3)$, with $\bar{c}_4$ given by

$$\pi^4\bar{c}_4 = \frac{N^2}{8}\left[9 - \frac{5395}{192}N^2\right] + \frac{15}{16}\zeta(3)N^2(N^2 - 9). \tag{5.5}$$

[The term in $\bar{c}_3$ proportional to $\zeta(3)$ and the terms in $\bar{c}_4$ proportional to $\zeta(4)$ and $\zeta(5)$ vanish for the adjoint representation for arbitrary N_f.]

Evaluating the two-loop expression in Eq. (4.3) for γ_m at the IR zero of the beta function, also calculated at the two-loop level, $\alpha_{IR,2\ell,\mathrm{adj}}$, we obtain

TABLE VII. Values of the (approximate or exact) IR zeros in α of the SU(N) beta function with $N_f = 2$ fermions in the adjoint representation, for $N = 2, 3, 4$, calculated at n-loop order, and denoted as $\alpha_{\mathrm{IR},n\ell,\mathrm{adj}}$. For the four-loop beta function, the cubic Eq. (3.32) has three zeros, one of which is $\alpha_{\mathrm{IR},4\ell,\mathrm{adj}}$. Depending on N, there may be another real zero, denoted $\alpha_{4\ell,u,\mathrm{adj}}$, at a larger value of α. We also list zeros from the [1,2] and [2,1] Padé approximants to the four-loop beta function.

N	$\alpha_{\mathrm{IR},2\ell,\mathrm{adj}}$	$\alpha_{\mathrm{IR},3\ell,\mathrm{adj}}$	$\alpha_{\mathrm{IR},4\ell,\mathrm{adj}}$	$\alpha_{\mathrm{IR},4\ell,[1,2],\mathrm{adj}}$	$\alpha_{\mathrm{IR},4\ell,[2,1],\mathrm{adj}}$	$\alpha_{4\ell,u,\mathrm{adj}}$
2	0.628	0.459	0.450	0.455	0.449	-
3	0.419	0.306	0.308	0.317	0.308	9.38
4	0.314	0.2295	0.234	0.242	0.233	3.29

$$\gamma_{2\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},2\ell,\mathrm{adj}}) = \frac{(11-4N_f)(277-34N_f+40N_f^2)}{6(-17+16N_f)^2}, \quad (5.6)$$

so that for the $N_f = 2$ case of interest here,

$$\gamma_{2\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},2\ell,\mathrm{adj}}) = \frac{41}{50} = 0.820 \quad \text{for } N_f = 2. \quad (5.7)$$

It is also of interest to evaluate the two-loop γ_m at the value of α_{cr} from the one-gluon exchange (ladder) approximation to the Dyson-Schwinger equation. With $N_f = 2$, this yields

$$\gamma_{2\ell}(\alpha_{\mathrm{cr,adj}}) = \frac{47}{72} \simeq 0.653. \quad (5.8)$$

Evaluating the three-loop result for γ_m at the IR zero of the beta function calculated at the three-loop level, $\alpha_{\mathrm{IR},3\ell,\mathrm{adj}}$, for the $N_f = 2$ case of interest, we obtain

$$\gamma_{3\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},3\ell,\mathrm{adj}}) = 0.543 \quad \text{for } N_f = 2, \quad (5.9)$$

which is again independent of N. At the four-loop level, the value of $\gamma_{4\ell,\mathrm{adj}}(\alpha_{IR,4\ell,\mathrm{adj}})$ does depend slightly on N. We list the values of these anomalous dimensions in Table VIII. The most recent simulations of a lattice gauge theory with SU(2) gauge group and $N_f = 2$ fermions in the adjoint representation report $\gamma_m = 0.49 \pm 0.13$ [23]. This is in agreement with the calculations of γ_m here at the three- and four-loop level, to within the uncertainties of the respective calculations.

VI. SYMMETRIC AND ANTISYMMETRIC RANK-2 TENSOR REPRESENTATIONS

In this section we consider the SU(N) theory with N_f fermions in the symmetric or antisymmetric rank-2 representation, denoted S2 and A2. Since a number of formulas are similar for these two cases, we will often give these in a unified way for both cases, denoted T2 (for rank-2 tensor representation), with $\pm$ signs distinguishing them. For S2, our analysis applies for any N, while for A2, we restrict to $N \geq 4$, since the A2 representation is the singlet for SU(2) and is equivalent to the conjugate fundamental representation for SU(3). Note that for SU(4), the A2 representation is self-conjugate. Also, since for SU(2) the S2 representation is the same as the adjoint representation, which has already been analyzed, we only consider the illustrative values $N = 3, 4$.

For the two T2 cases, the general expression for the maximal value of N_f allowed by the requirement of asymptotic freedom, Eq. (3.2), reduces to

$$N_{f,\mathrm{max,T2}} = \frac{11N}{2(N \pm 2)}, \quad (6.1)$$

where the $\pm$ refers to S2 and A2, respectively. As N increases from 2 to ∞, $N_{f,\mathrm{max,S2}}$ increases monotonically from 2.75 to $11/2 = 4.5$, and as N increases from 3 to ∞, $N_{f,\mathrm{max,A2}}$ decreases monotonically from 16.5 to the same limit, 4.5. The physical values of $N_{f,\mathrm{max}}$ in both cases are the greatest integral parts of these rational numbers.

For these representations, the general expression in Eq. (3.5) for the value of N_f at which the beta function coefficient b_2 changes sign from positive to negative with increasing N_f takes the form

TABLE VIII. Values of the anomalous dimension γ_m in an SU(N) gauge theory with $N_f = 2$ (Dirac) fermions in the adjoint representation, calculated to the n-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $n = 2, 3, 4$. We denote these as $\gamma_{n\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},n\ell,\mathrm{adj}})$. We also list the value of $\gamma_{2\ell,\mathrm{adj}}$ evaluated at α equal to the βDS estimate, Eq. (B1), for $\alpha_{\mathrm{cr,adj}}$.

N	$\gamma_{2\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},2\ell,\mathrm{adj}})$	$\gamma_{3\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},3\ell,\mathrm{adj}})$	$\gamma_{4\ell,\mathrm{adj}}(\alpha_{\mathrm{IR},4\ell,\mathrm{adj}})$	$\gamma_{2\ell,\mathrm{adj}}(\alpha_{\mathrm{cr,adj}})$
2	0.820	0.543	0.500	0.653
3	0.820	0.543	0.523	0.653
4	0.820	0.543	0.532	0.653

$$N_{f,b2z,\mathrm{T2}} = \frac{17N^2}{(N \pm 2)(8N \pm 3 - 6N^{-1})}. \quad (6.2)$$

As a consequence of the general inequality (3.6), it follows that $N_{f,b2z,\mathrm{S2}} < N_{f,\max,\mathrm{S2}}$ and $N_{f,b2z,\mathrm{A2}} < N_{f,\max,\mathrm{A2}}$. For $N = 2$, the S2 representation is just the adjoint representation, so we only consider the illustrative values $N = 3, 4$. The respective intervals $N_{f,b2z,\mathrm{S2}} < N_f < N_{f,\max,\mathrm{S2}}$ for which the SU(N) gauge theory is asymptotically free and has an IR zero of β are $1.06 < N_f < 2.75$ for $N = 3$ and $1.22 < N_f < 3.30$ for $N = 4$. These ranges imply that the only physical integral values of N_f satisfying these conditions are $N_f = 2, 3$ for both SU(3) and SU(4).

For large N, $N_{f,b2z,\mathrm{T2}}$ has the series expansion

$$N_{f,b2z,\mathrm{T2}} = \frac{17}{2^3} \mp \frac{323}{2^6 N} + \frac{6137}{2^9 N^2} \mp \frac{103\,547}{2^{12} N^3} + O\left(\frac{1}{N^4}\right). \quad (6.3)$$

As N increases from 2 to ∞, $N_{f,b2z,\mathrm{S2}}$ increases monotonically from $17/16 = 1.0625$ to $17/8 = 2.125$, and as N increases from 3 to ∞, $N_{f,b2z,\mathrm{A2}}$ decreases monotonically from 8.05 to the same limit, 2.125. This limit is twice the (N-independent) value of $N_{f,b2z,\mathrm{adj}} = 17/16$ for the adjoint representation. Thus, for large N, the range (3.7) where the SU(N) theory with N_f fermions in the S2 or A2 representation is asymptotically the same for both, namely, $17/8 < N_f < 11/2$; restricting N_f to physical, integer values, this range consists of the three values $N_f = 3, 4, 5$.

For our further discussion we assume that N_f is in the range $N_{f,b2z,\mathrm{T2}} < N_f < N_{f,\max,\mathrm{T2}}$ where the theory is asymptotically free and the two-loop beta function has an IR zero, for the respective cases S2 and A2. This zero occurs at the value

$$\alpha_{\mathrm{IR},2\ell,\mathrm{T2}} = \frac{2\pi(11N - 2N_f(N \pm 2)}{-17N^2 + N_f(8N^2 \pm 19N \mp 12N^{-1})}. \quad (6.4)$$

TABLE IX. Values of the (approximate or exact) IR zero in α of the SU(N) beta function with $N_f = 2$ fermions in the symmetric rank-2 (i.e., S2) representation, for $N = 3, 4$, calculated at n-loop order, and denoted as $\alpha_{\mathrm{IR},n\ell,\mathrm{S2}}$.

N	N_f	$\alpha_{\mathrm{IR},2\ell,\mathrm{S2}}$	$\alpha_{\mathrm{IR},3\ell,\mathrm{S2}}$	$\alpha_{\mathrm{IR},4\ell,\mathrm{S2}}$
3	2	0.842	0.500	0.470
3	3	0.085	0.079	0.079
4	2	0.967	0.485	0.440
4	3	0.152	0.129	0.131

TABLE X. Values of the (approximate or exact) IR zero in α of the SU(4) beta function with N_f fermions in the antisymmetric rank-2 (i.e., A2) representation, for the range $5 \le N_f \le 10$ where the theory is asymptotically free and has an IR zero of the beta function, calculated at n-loop order, and denoted as $\alpha_{\mathrm{IR},n\ell,\mathrm{A2}}$.

N	N_f	$\alpha_{\mathrm{IR},2\ell,\mathrm{A2}}$	$\alpha_{\mathrm{IR},3\ell,\mathrm{A2}}$	$\alpha_{\mathrm{IR},4\ell,\mathrm{A2}}$
4	6	2.17	0.664	0.770
4	7	0.890	0.437	0.502
4	8	0.449	0.287	0.319
4	9	0.225	0.174	0.184
4	10	0.090	0.080	0.082

We have calculated $\alpha_{\mathrm{IR},n\ell,\mathrm{S2}}$ and $\alpha_{\mathrm{IR},n\ell,\mathrm{A2}}$ up to $n = 4$ loops and list the results in Tables IX and X. The resultant βDS estimates for $N_{f,\mathrm{cr}}$ in the case of the S2 representation and $N = 2, 3, 4$ are $N_{f,\mathrm{cr,S2}} = 2.1, 2.5, 2.8$, respectively. For the A2 representation with $N = 4$, one has $N_{f,\mathrm{cr,A2}} = 8.1$.

The two-loop expression for the anomalous dimension, evaluated at $\alpha = \alpha_{\mathrm{IR},2\ell,\mathrm{T2}}$, is

$$\gamma_{2\ell,\mathrm{T2}}(\alpha_{\mathrm{IR},2\ell,\mathrm{T2}}) = \frac{(N \pm 2)(N \mp 1)[11N - 2(N \pm 2)N_f][N(554N^2 \pm 99N - 198) + (-34N^3 \pm 22N^2 \mp 360)N_f + 20N(N \pm 2)^2 N_f^2]}{12[-17N^3 + (N \pm 2)(8N^2 \pm 3N - 6)N_f]^2}. \quad (6.5)$$

We list values of $\gamma_{2\ell,\mathrm{S2}}(\alpha_{\mathrm{IR},2\ell,\mathrm{S2}})$ for $N = 2, 3, 4$ in Table XI and values of $\gamma_{2\ell,\mathrm{A2}}(\alpha_{\mathrm{IR},2\ell,\mathrm{A2}})$ for $N = 4$ in Table XII with $\ell = 2, 3$.

It is also of interest to evaluate the two-loop expression for γ at the estimated $\alpha = \alpha_{\mathrm{cr,T2}}$. This yields

$$\gamma_{2\ell,\mathrm{T2}}(\alpha_{\mathrm{cr,T2}}) = \frac{322N^2 \pm 225N - 450 - 10N(N \pm 2)N_f}{432(N \pm 2)(N \mp 1)}. \quad (6.6)$$

We list these values in Tables XI and XII.

Evaluating the two-loop anomalous dimensions at the two-loop IR zero of the beta function, $\gamma_{2\ell,\mathrm{T2}}(\alpha_{\mathrm{IR},2\ell,\mathrm{T2}})$, for N_f equal to the respective βDS-estimated critical values, we obtain (again with T2 and the $\pm$ signs referring, respectively, to S2 and A2)

$$\gamma_{2\ell,\mathrm{T2}}(\alpha_{\mathrm{IR},2\ell,\mathrm{T2}})|_{N_f = N_{f,\mathrm{cr,T2}}} = \frac{374N^4 \pm 578N^3 - 931N^2 \mp 900N + 900}{144(N \pm 2)(N \mp 1)(4N^2 \pm 3N - 6)}. \quad (6.7)$$

This has the large-N expansion

TABLE XI. Values of γ_m in an SU(N) gauge theory with N_f fermions in the symmetric rank-2 tensor representation S2, calculated to the n-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $n = 2, 3, 4$. We denote these as $\gamma_{n\ell,S2}(\alpha_{IR,n\ell,S2})$. We also list $\gamma_{2\ell,S2}$ evaluated at α equal to the estimate Eq. (B1) for $\alpha_{cr,S2}$.

N	N_f	$\gamma_{2\ell,S2}(\alpha_{IR,2\ell,S2})$	$\gamma_{3\ell,S2}(\alpha_{IR,3\ell,S2})$	$\gamma_{4\ell,S2}(\alpha_{IR,4\ell,S2})$	$\gamma_{2\ell,S2}(\alpha_{cr,S2})$
3	2	(2.44)	1.28	1.12	0.653
3	3	0.144	0.133	0.133	0.619
4	2	(4.82)	(2.08)	1.79	0.659
4	3	0.381	0.313	0.315	0.629

TABLE XII. Values of γ_m in an SU(N) gauge theory with N_f fermions in the antisymmetric rank-2 tensor representation A2, calculated to the n-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $N = 4$ and $n = 2, 3, 4$. We denote these as $\gamma_{n\ell,A2}(\alpha_{IR,n\ell,A2})$. We also list $\gamma_{2\ell,A2}$ evaluated at α equal to the estimate Eq. (B1) for $\alpha_{cr,A2}$.

N	N_f	$\gamma_{2\ell,A2}(\alpha_{IR,2\ell,A2})$	$\gamma_{3\ell,A2}(\alpha_{IR,3\ell,A2})$	$\gamma_{4\ell,A2}(\alpha_{IR,4\ell,A2})$	$\gamma_{2\ell,A2}(\alpha_{cr,A2})$
4	6	(9.78)	1.38	0.293	0.769
4	7	(2.19)	0.695	0.435	0.750
4	8	0.802	0.402	0.368	0.732
4	9	0.331	0.228	0.232	0.713
4	10	0.117	0.101	0.103	0.695

$$\gamma_{2\ell,T2}(\alpha_{IR,2\ell,T2})|_{N_f=N_{f,cr,T2}} = \frac{187}{288} \mp \frac{17}{128N} + O\left(\frac{1}{N^2}\right). \quad (6.8)$$

The leading term has the value $187/288 \simeq 0.649$.

From a lattice study of SU(3) gauge theory with $N_f = 2$ fermions in the S2 (sextet) representation, Ref. [18] found that this theory is characterized by slow running behavior consistent with an (exact or approximate) IR fixed point, and further reported that $\gamma_m < 0.6$ where it was measured. For SU(3), the estimate of $\alpha_{cr,S2}$ in Eq. (B1) gives $\alpha_{cr,S2} = \pi/10 = 0.31$. Our results for the IR zero of β and the value of γ_m at this zero for $N = 3$ and $N_f = 2$ are listed in Tables IX and XI. We find that $\alpha_{IR,n\ell,S2}$ is approximately 0.84 at $n = 2$-loop level and decreases somewhat to 0.50 at three-loop level. The two-loop result for γ_m is unphysically large, while the three-loop value of γ_m at the corresponding three-loop IR zeros of β is about 1.3. These are somewhat larger than the values reported in Ref. [18], although in assessing this comparison, one must take account of the significant strong-coupling uncertainties in our calculation stemming from the fact that $\alpha_{IR,S2} \sim O(1)$. Our evaluation of the two-loop expression for γ_m at the ladder-Dyson-Schwinger estimate of $\alpha_{cr,S2}$, is 0.65.

VII. EFFECTS OF NONZERO FERMION MASSES

The global chiral symmetry that is operative if the fermions are massless, and the way that it is broken by fermion condensates, is well-known, and we do not review it here. However, it is worthwhile to comment on the situation in which some fermion masses are nonzero. In this paper we generally assume that the fermions have zero intrinsic masses in the Lagrangian describing the high-scale physics, and the only masses that they acquire arise dynamically if they are involved in condensates that form as the gauge interaction becomes sufficiently strongly coupled in the infrared. This is a well-motivated assumption if the vectorial gauge theory arises as a low-energy effective field theory from an ultraviolet completion which is a chiral gauge theory. In turn, this is natural if the latter theory becomes strongly coupled, since it can then form fermion condensates that self-break it down to the vectorial subgroup symmetry. However, one may also choose to focus on the vectorial gauge theory as an ultraviolet-complete theory in itself. In a vectorial gauge theory, an intrinsic (bare) mass term for a fermion ψ, $\mathcal{L}_m = -m\bar{\psi}\psi$, is allowed by the gauge invariance. Hence, one may consider a more general situation in which the fermions may have such intrinsic (hard) masses in the high-scale Lagrangian [44]. In this case, as the reference scale μ decreases below the value of the hard mass of some fermion m_f, the beta function changes from one that includes this to one that excludes this fermion. If the hard fermion masses are small compared with the scale Λ in the situation where the theory confines and breaks chiral symmetry spontaneously, then these hard masses have only a small effect. However, if some of the hard fermion masses are sufficiently large, then as μ decreases below their scale and the corresponding fermions are integrated out of the low-energy theory below this scale, this can significantly change the infrared properties of the resultant theory.

In applications of slowly running gauge theories to technicolor theories, at the scale $\Lambda_{\rm TC}$ where the SU($N_{\rm TC}$) gauge coupling grows to $O(1)$ and is influenced by the presence of an approximate IR zero of the TC beta function, there can also be non-negligible effects due to four-fermion operators arising from the higher-lying extended technicolor dynamics [4–6,10,45,46], and these can affect the scaling properties of $\bar\psi\psi$. Similar comments apply for topcolor-assisted technicolor [4,47].

VIII. CONCLUSIONS

In this paper we have studied the evolution of an asymptotically free vectorial SU(N) gauge theory from high scales to the infrared taking account of higher-loop corrections to the beta function and the anomalous dimension γ_m for fermions in the fundamental, adjoint, and rank-2 symmetric and antisymmetric representations S2 and A2. We have compared our results with lower-order calculations. We have shown that, for fixed N and N_f, in the range for which the two-loop beta function has an IR zero, the value of this zero decreases as one goes from the two-loop to the three-loop calculations, and we have determined this decrease quantitatively. Going further, we have shown that there is a smaller fractional increase in the value of this IR zero when calculated to four-loop accuracy, with the final four-loop result still smaller than the two-loop value. We have analyzed instanton effects and have demonstrated that they tend to increase the value of the IR zero of the beta function somewhat. A major part of our work has been the evaluation of the anomalous dimension γ_m of $\bar\psi\psi$ at the IR zero of the beta function at the $\ell = 2, 3, 4$-loop levels. This zero is approximate or exact, depending on whether for a given N, the value of N_f is below or above the critical value $N_{f,\rm cr}$ below which there is spontaneous chiral symmetry breaking associated with the formation of a fermion condensate. We have found that this γ_m at the (approximate or exact) IR zero of the beta function decreases as one goes from two-loop to three-loop order, and that the four-loop values also tend to be somewhat less than those at the two-loop level. The values that we have calculated for γ_m at the IR zero of the beta function tend to be somewhat smaller than unity. We have compared our higher-loop calculations with results from recent lattice simulations and have found general agreement. We believe that the higher-loop calculations reported here should provide a useful reference for comparison with ongoing and future lattice measurements.

ACKNOWLEDGMENTS

This research was partially supported by Grant No. NSF-PHY-09-69739. After this work was completed, we received the related Ref. [50]. We thank F. Sannino for sending us this preprint and S. Di Chiara for a helpful comment.

APPENDIX A

For the reader's convenience, we list the three-loop beta function coefficient, in the $\overline{\rm MS}$ scheme [28],

$$b_3 = \frac{2857}{54}C_A^3 + T_f N_f\Big[2C_f^2 - \frac{205}{9}C_A C_f - \frac{1415}{27}C_A^2\Big] + (T_f N_f)^2\Big[\frac{44}{9}C_f + \frac{158}{27}C_A\Big]. \tag{A1}$$

The four-loop coefficient is given in Ref. [29] and is a cubic polynomial in N_f. We note that the coefficients of the N_f^0 (which is independent of the fermion representation) is positive, and the coefficient of the N_f^3 term is positive for an arbitrary fermion representation.

Our normalizations for the quadratic Casimir and trace invariants of a Lie group are standard. The quadratic Casimir invariant $C_2(R)$ for the representation R is given by $\sum_{a=1}^{o(G)}\sum_{j=1}^{\dim(R)}[D_R(T_a)]_{ij}[D_R(T_a)]_{jk} = C_2(R)\delta_{ik}$, where a, b are group indices, $o(G)$ is the order of the group, T_a are the generators of the associated Lie algebra, and $D_R(T_a)$ is the matrix form of the T_a in the representation R. The trace invariant $T(R)$ is defined by $\sum_{i,j=1}^{\dim(R)}[D_R(T_a)]_{ij}\times[D_R(T_b)]_{ji} = T(R)\delta_{ab}$.

From the calculations of the coefficients of the perturbative expansion of the anomalous dimension γ_m in the $\overline{\rm MS}$ scheme to four-loop order in Ref. [30], we record the three-loop coefficient

$$c_3 = 2C_f\Big[\frac{129}{2}C_f^2 - \frac{129}{4}C_f C_A + \frac{11\,413}{108}C_A^2 + C_f T_f N_f(-46 + 48\zeta(3)) - C_A T_f N_f\Big(\frac{556}{27} + 48\zeta(3)\Big) - \frac{140}{27}T_f^2 N_f^2\Big] \tag{A2}$$

We have used the four-loop coefficient c_4 from Ref. [30] for our calculations, but it is too lengthy to reproduce here.

APPENDIX B: BETA-DYSON-SCHWINGER ESTIMATE OF $N_{f,\rm cr}$

In this appendix we briefly review the βDS estimate of $N_{f,\rm cr}$. In the one-gluon exchange (also called ladder) approximation to the Dyson-Schwinger equation for the fermion propagator with an initially massless fermion in the representation R of the gauge group, one finds a solution with a dynamically generated, nonzero fermion mass if the coupling $\alpha(\mu)$ exceeds a critical value $\alpha_{{\rm cr},R}$ given by [1,2,48]

$$\alpha_{{\rm cr},R} = \frac{\pi}{3C_f}. \tag{B1}$$

Setting this equal to the two-loop expression for the IR zero of β then yields an estimate for $N_{f,\rm cr}$ to this order, namely

$$N_{f,\mathrm{cr}} = \frac{C_A(66C_f + 17C_A)}{10T_f(C_A + 3C_f)}. \tag{B2}$$

We call this the βDS estimate of $N_{f,\mathrm{cr}}$ since it combines a calculation of α_{IR} from the β function with the estimate of $\alpha_{\mathrm{cr},R}$ from the ladder approximation to the Dyson-Schwinger equation for the fermion propagator. In the same ladder approximation, one finds $\gamma_m = 1$ at $\alpha = \alpha_{\mathrm{cr},R}$ [1] (which also holds for the DS analysis at a UV-stable fixed point [49]). For the gauge group SU(N) with the illustrative values of N used for the tables, namely $N = 2, 3, 4$, $N_{f,\mathrm{cr,fund}}$ is equal to 7.9, 11.9, and 15.9, respectively, with the large-N form $N_f \sim 4N$. For S2, the symmetric rank-2 tensor representation, $N = 2, 3, 4$, $N_{f,\mathrm{cr,S2}}$ is equal to 2.075, 2.5, and 2.9, increasing toward the limit $11/2 = 5.5$ in the large-N limit. In the case of A2, the antisymmetric rank-2 tensor reprepresentation, for $N = 3$, the result is the same as for the fundamental representation, while for $N = 4$, one has $N_{f,\mathrm{crit,A2}} \simeq 8.1$, and as $N \to \infty$, $N_{f,\mathrm{crit,A2}}$ decreases toward the limit $11/2$.

One understands that *a priori* there could be significant uncertainty in these estimates because of the strong-coupling nature of the physics involved and the one-gluon approximation used for the solution of the Dyson-Schwinger equation. Moreover, the DS equation analysis is semiperturbative in the sense that it contains polynomial dependence on α, and it neglects nonperturbative effects associated with confinement and instantons. However, corrections to the one-gluon exchange approximation have been analyzed and found not to be too large [2]. Recent lattice simulations for SU(3) are in broad agreement, to within the uncertainties, with the above prediction of $N_{f,\mathrm{cr}} \sim 12$ [11–15,24]. Some of the success of the βDS prediction for $N_{f,\mathrm{cr}}$ may arise from the fact that two major physical effects that it ignores, namely, confinement and instantons, would shift $N_{f,\mathrm{cr}}$ in opposite directions and hence tend to cancel each other out [34].

APPENDIX C: PADÉ RESULTS

In this appendix we collect some relevant results on Padé approximants. Given a Taylor (or asymptotic) series expansion around $z = 0$ for the function $f(z)$

$$f(z) = \sum_{n=0}^{n_{\max}} f_n z^n, \tag{C1}$$

one can construct a set of $[p, q]$ Padé approximants, namely, rational functions comprised of a numerator polynomial of degree p and a denominator polynomial of degree q, such that $p + q = n_{\max} - 1$, of the form $(\sum_{j=0}^{p} p_j z^j)/(\sum_{k=0}^{q} q_k z^k)$. Without loss of generality, one can divide numerator and denominator by q_0, so that, after redefinition of the coefficients, one has

$$[p, q]_f(z) = \frac{\sum_{j=0}^{p} p_j z^j}{1 + \sum_{k=1}^{q} q_k z^k}. \tag{C2}$$

The $p + q + 1$ coefficients p_j with $0 \le j \le p$ and q_k with $1 \le k \le q$ are uniquely determined in terms of the f_n coefficients with $0 \le n \le n_{\max}$ by expanding the $[p, q]$ Padé approximant in a Taylor series around $z = 0$ and solving the set of $n_{\max}$ linear equations.

[1] T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D **35**, 774 (1987); **36**, 568 (1987); T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996).

[2] T. Appelquist, K. Lane, and U. Mahanta, Phys. Rev. Lett. **61**, 1553 (1988).

[3] T. A. Ryttov and R. Shrock, Phys. Rev. D **81**, 116003 (2010); **82**, 059903(E) (2010).

[4] C. T. Hill and E. H. Simmons, Phys. Rep. **381**, 235 (2003).

[5] *Proceedings of the Workshop on Dynamical Electroweak Symmetry Breaking, Southern Denmark University, 2008*, http://hep.sdu.dk/dewsb (unpublished).

[6] F. Sannino, Acta Phys. Pol. B **40**, 3533 (2009).

[7] T. Appelquist and F. Sannino, Phys. Rev. D **59**, 067702 (1999); M. Harada, M. Kurachi, and K. Yamawaki, Prog. Theor. Phys. **115**, 765 (2006); M. Kurachi and R. Shrock, Phys. Rev. D **74**, 056003 (2006).

[8] T. Appelquist *et al.*, arXiv:1009.5967.

[9] K. Lane and E. Eichten, Phys. Lett. B **222**, 274 (1989); D. Hong, S. Hsu, and F. Sannino, Phys. Lett. B **597**, 89 (2004); F. Sannino and K. Tuominen, Phys. Rev. D **71**, 051901(R) (2005); N. D. Christensen and R. Shrock, Phys. Lett. B **632**, 92 (2006); D. D. Dietrich and F. Sannino, Phys. Rev. D **75**, 085018 (2007); R. Foadi, M. T. Frandsen, T. A. Ryttov, and F. Sannino, Phys. Rev. D **76**, 055005 (2007); T. A. Ryttov and F. Sannino, Phys. Rev. D **76**, 105004 (2007); M. Mojaza, C. Pica, and F. Sannino, Phys. Rev. D **82**, 116009 (2010).

[10] T. Appelquist and J. Terning, Phys. Rev. D **50**, 2116 (1994); T. Appelquist and R. Shrock, Phys. Lett. B **548**, 204 (2002); Phys. Rev. Lett. **90**, 201801 (2003); T. Appelquist, M. Piai, and R. Shrock, Phys. Rev. D **69**, 015002 (2004); N. C. Christensen and R. Shrock, Phys. Rev. D **74**, 015004 (2006); T. A. Ryttov and R. Shrock, Phys. Rev. D **81**, 115013 (2010).

[11] T. Appelquist, G. Fleming, and E. Neil, Phys. Rev. Lett. **100**, 171607 (2008); Phys. Rev. D **79**, 076010 (2009); T. Appelquist, Prog. Theor. Phys. **180**, 72 (2009); T.

Appelquist *et al.*, Phys. Rev. Lett. **104**, 071601 (2010); arXiv:1009.5967.
[12] A. Deuzeman, M. P. Lombardo, and E. Pallante, Phys. Lett. B **670**, 41 (2008); Proc. Sci., LAT2008 (**2008**) 060.
[13] X.-Y. Jin and R. Mawhinney, Proc. Sci., LAT2008 (2008) 059.
[14] A. Hasenfratz, Phys. Rev. D **80**, 034505 (2009).
[15] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B **681**, 353 (2009); Proc. Sci., LAT2009 (**2009**) 055.
[16] S. Catterall, J. Giedt, F. Sannino, and J. Schneible, J. High Energy Phys. 11 (2008) 009.
[17] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D **79**, 034501 (2009).
[18] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D **82**, 054503 (2010).
[19] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B **681**, 353 (2009); J. High Energy Phys. 11 (2009) 103.
[20] A. J. Hietanen, K. Rummukainen, and K. Tuominen, Phys. Rev. D **80**, 094504 (2009).
[21] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, Phys. Rev. D **80**, 074507 (2009); F. Bursa, L. Del Debbio, L. Keegan, C. Pica, and T. Pickup, Phys. Rev. D **81**, 014505 (2010).
[22] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **81**, 114507 (2010).
[23] S. Catterall, L. Del Debbio, J. Giedt, and L. Keegan, Proc. Sci., LAT2010 (**2010**) 057.
[24] G. Fleming, *Proceedings of the International Conference on High Energy Physics, ICHEP, Paris, 2010*, http://www.ichep2010.fr.
[25] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973); G. 't Hooft (unpublished).
[26] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974); D. R. T. Jones, Nucl. Phys. **B75**, 531 (1974).
[27] D. J. Gross and F. Wilczek, Phys. Rev. D **8**, 3633 (1973); D. J. Gross, in *Methods in Field Theory*, Proceedings of the Les Houches Summer School, edited by R. Balian and J. Zinn-Justin (North Holland, Amsterdam, 1976).
[28] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. **93B**, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **303**, 334 (1993).
[29] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B **400**, 379 (1997).
[30] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B **405**, 327 (1997).
[31] T. A. Ryttov and F. Sannino, Phys. Rev. D **78**, 065001 (2008).
[32] S. J. Brodsky and G. F. de Téramond, in *The Origin of Mass and Strong Coupling Gauge Theories, SCGT2006*, edited by M. Harada, M. Tanabashi, and K. Yamawaki (World Scientific, Singapore, 2008), p. 31; S. J. Brodsky, G. F. de Téramond, and A. Deur, Phys. Rev. D **81**, 096010 (2010).
[33] M. Creutz (unpublished).
[34] S. Brodsky and R. Shrock, Phys. Lett. B **666**, 95 (2008).
[35] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976); **18**, 2199(E) (1978).
[36] D. Caldi, Phys. Rev. Lett. **39**, 121 (1977).
[37] C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Rev. D **17**, 2717 (1978); **20**, 3279 (1979).
[38] T. Appelquist and S. Selipsky, Phys. Lett. B **400**, 364 (1997).
[39] E. Shuryak and M. Velkovsky, Phys. Lett. B **437**, 398 (1998).
[40] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).
[41] Here and elsewhere, when expressions are given for N_f that formally evaluate to nonintegral real values, it is understood implicitly that one infers an appropriate integral value of N_f from them, either the greatest integral part or the nearest integer, depending on the context.
[42] Parenthetically, we observe that if (i) the exact beta function of a theory were to have a zero at a (nonzero, positive) value α_1 with $d\beta/d\alpha > 0$ at α_1, and (ii) another zero at a larger value, α_2 with $d\beta/d\alpha < 0$ at α_2 with (iii) $\beta > 0$ for $\alpha_1 < \alpha < \alpha_2$, and if (iv) the initial condition in the deep ultraviolet is that as $\mu \to \infty$, $\alpha(\mu)$ approaches α_2 from below, then as the scale μ decreases, α would decrease from the UV fixed point α_2 and approach the IR fixed point α_1 from above as $\mu \to 0$. This type of behavior is not relevant to our theory with the initial condition on α that we assume in the ultraviolet.
[43] G. Mack, Commun. Math. Phys. **55**, 1 (1977); M. Flato and C. Fronsdal, Lett. Math. Phys. **8**, 159 (1984); V. K. Dobrev and V. B. Petkova, Phys. Lett. **162B**, 127 (1985).
[44] Here a hard fermion mass is a mass that would remain in the hypothetical limit in which one turned off the SU(N) interactions. If these masses are dynamically generated, then they are actually soft on the higher scale where they arise, as discussed in N. D. Christensen and R. Shrock, Phys. Rev. Lett. **94**, 241801 (2005).
[45] T. Appelquist, M. Soldate, T. Takeuchi, and L. C. R. Wijewardhana, in *Proceedings of the Johns Hopkins Workshop on Current Problems in Particle Theory*, edited by G. Domokos and S. Kovesi-Domokos (World Scientific, Singapore, 1988); K. I. Kondo, H. Mino, and K. Yamawaki, Phys. Rev. D **39**, 2430 (1989).
[46] M. Kurachi, R. Shrock, and K. Yamawaki, Phys. Rev. D **76**, 035003 (2007); H. S. Fukano and F. Sannino, Phys. Rev. D **82**, 035021 (2010).
[47] A. Martin and K. Lane, Phys. Rev. D **71**, 015011 (2005); F. Braam, M. Flossdorf, R. S. Chivukula, S. Di Chiara, and E. H. Simmons, Phys. Rev. D **77**, 055005 (2008); T. A. Ryttov and R. Shrock, Phys. Rev. D **82**, 055012 (2010).
[48] K. Lane, Phys. Rev. D **10**, 2605 (1974).
[49] B. Holdom, Phys. Rev. Lett. **60**, 1233 (1988); K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. **56**, 1335 (1986); V. Miransky and K. Yamawaki, Phys. Rev. D **55**, 5051 (1997); Phys. Rev. D **56**, 3768(E) (1997).
[50] C. Pica and F. Sannino, arXiv:1011.3832.