Model-independent constraints on the shape parameters of dilepton angular distributions

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The coefficients determining the dilepton decay angular distribution of vector particles obey certain positivity constraints and a rotation-invariant identity. These relations are a direct consequence of the covariance properties of angular momentum eigenstates and are independent of the production mechanism. The Lam-Tung relation can be derived as a particular case, simply recognizing that the Drell-Yan dilepton is always produced transversely polarized with respect to one or more quantization axes. The dilepton angular distribution continues to be characterized by a frame-independent identity also when the Lam-Tung relation is violated. Moreover, the violation can be easily characterized by measuring a one-dimensional distribution depending on one shape coefficient.

I. INTRODUCTION

Dilepton decay angular distributions directly reflect the average angular momentum composition of the decaying state. Their measurements place strong constraints on the characteristics and topology of the participating production processes and can thus provide key information for the understanding of the mechanisms of fundamental interactions. In this paper we show how rotation covariance implies the existence of completely general constraints on the coefficients of the dilepton decay angular distribution of a $J = 1$ particle. These constraints are valid for any superposition of production mechanisms and are independent of the chosen polarization frame. In particular, as first noted in Ref. [1], the parameters characterizing the polar and azimuthal anisotropies of the distribution satisfy a frame-independent identity, directly reflecting a basic rotational property of $J = 1$ angular momentum eigenstates. The well-known Lam-Tung relation [2], a result specific to Drell-Yan production in perturbative QCD, can be derived as a particular case of this identity by simply noting that all subprocesses, up to $O(\alpha_s)$ contributions, produce transversely polarized dileptons, albeit with respect to different quantization axes. This result allows us to discern what in this relation embodies the dynamical content of the specific processes involved and what reflects completely general kinematic properties. The existence of a frame-independent identity can be seen as a generalization of the Lam-Tung relation. In fact, it is always possible to define a frame-independent polarization observable, even when the Lam-Tung relation is violated (or for processes different from Drell-Yan production). We also show that the value of this observable (and, hence, possible violations of the Lam-Tung relation) can be measured by simply determining a single-variable angular distribution. As an illustration of how simple and powerful the application of the frame-independent formalism can be, we consider the significant violations of the Lam-Tung identity measured in pion-nucleus experiments. The intensively-studied possibility that these effects are caused by higher-order corrections in perturbative QCD is generally agreed to have been ruled out by detailed calculations [3,4]. The same conclusion can be reached in a much simpler way by considering rotational invariance and symmetry properties.

II. ANGULAR DISTRIBUTION OF DILEPTON DECAYS OF VECTOR STATES

We start by expressing the observable dilepton angular distribution in a form that keeps track of the angular momentum composition of the decaying state. We study first the case of a single production "subprocess", here defined as a process where the considered vector state V is formed as a given superposition of the three $J = 1$ eigenstates, $J_z = +1, -1, 0$ with respect to a chosen polarization axis z:

$$|V\rangle = b_{+1}|+1\rangle + b_{-1}|-1\rangle + b_0|0\rangle. \quad (1)$$

The calculations are performed in the V rest frame, where the common direction of the two leptons define the reference axis z', oriented conventionally along the direction of the positive lepton. The adopted notations for axes, angles and angular momentum states are illustrated in Fig. 1. We assume helicity conservation at the dilepton vertex, in the limit of vanishing lepton masses. The dilepton system has thus angular momentum projection ± 1 along z', i.e. it is an eigenstate of $J_{z'}$, $|\ell^+\ell^-; 1, l'\rangle$, with $l' = +1$ or -1. This state can also be expressed as a superposition of eigenstates of J_z, $|\ell^+\ell^-; 1, l\rangle$ with $l = 0, \pm 1$, as

$$|\ell^+\ell^-; 1, l'\rangle = \sum_{l=0,\pm 1} \mathcal{D}_{ll'}^1(\vartheta, \varphi)|\ell^+\ell^-; 1, l\rangle, \quad (2)$$

where $\mathcal{D}_{ll'}^1$ are complex coefficients describing the rotation of a $J = 1$ state from the set of axes (x, y, z) to the set (x', y', z') [5],
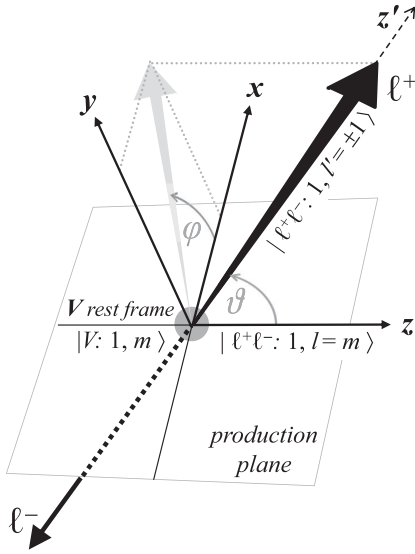
FIG. 1.   Sketch of the decay $V \to \ell^+\ell^-$, showing the notations we use for axes, angles and angular momentum states. The y and z' axes are oriented towards the reader.

$$\mathcal{D}^1_{ll'}(\vartheta, \varphi) = e^{i(l'-l)\varphi} d^1_{ll'}(\vartheta), \qquad (3)$$

with

$$d^1_{0,\pm 1} = \pm \sin\vartheta/\sqrt{2},$$
$$d^1_{\pm 1,\pm 1} = (1 + \cos\vartheta)/2, \qquad (4)$$
$$d^1_{\pm 1,\mp 1} = (1 - \cos\vartheta)/2.$$

The amplitude of the partial process $V(m) \to \ell^+\ell^-(l')$ represented in Fig. 1 is

$$B_{ml'} = \sum_{l=0,\pm 1} \mathcal{D}^{1*}_{ll'}(\vartheta, \varphi)\langle \ell^+\ell^-; 1, l|\mathcal{B}|V; 1, m\rangle$$
$$= B\mathcal{D}^{1*}_{ml'}(\vartheta, \varphi), \qquad (5)$$

where we imposed that the transition operator $\mathcal{B}$ is of the form $\langle \ell^+\ell^-; 1, l|\mathcal{B}|V; 1, m\rangle = B\delta_{ml}$ because of angular momentum conservation, with B independent of m (for rotational invariance). The total amplitude for $V \to \ell^+\ell^-(l')$, where V is given by the superposition written in Eq. (1), is

$$B_{l'} = \sum_{m=0,\pm 1} b_m B\mathcal{D}^{1*}_{ml'}(\vartheta, \varphi) = \sum_{m=0,\pm 1} a_m \mathcal{D}^{1*}_{ml'}(\vartheta, \varphi). \quad (6)$$

The probability of the transition is obtained by squaring Eq. (6) and summing over the (unobserved) spin alignments ($l' = \pm 1$) of the dilepton system, with equal weights attributed, for parity conservation, to the two configurations. Using Eqs. (3) and (4) one finally obtains the angular distribution

$$W(\cos\vartheta, \varphi) \propto \sum_{l'=\pm 1} |B_{l'}|^2$$
$$\propto \frac{\mathcal{N}}{(3 + \lambda_\vartheta)}(1 + \lambda_\vartheta \cos^2\vartheta + \lambda_\varphi \sin^2\vartheta \cos 2\varphi$$
$$+ \lambda_{\vartheta\varphi} \sin 2\vartheta \cos\varphi + \lambda^\perp_\varphi \sin^2\vartheta \sin 2\varphi$$
$$+ \lambda^\perp_{\vartheta\varphi} \sin 2\vartheta \sin\varphi), \qquad (7)$$

with $\mathcal{N} = |a_0|^2 + |a_{+1}|^2 + |a_{-1}|^2$ and

$$\lambda_\vartheta = \frac{\mathcal{N} - 3|a_0|^2}{\mathcal{N} + |a_0|^2},$$
$$\lambda_\varphi = \frac{2\,\mathrm{Re}[a^*_{+1}a_{-1}]}{\mathcal{N} + |a_0|^2},$$
$$\lambda_{\vartheta\varphi} = \frac{\sqrt{2}\,\mathrm{Re}[a^*_0(a_{+1} - a_{-1})]}{\mathcal{N} + |a_0|^2}, \qquad (8)$$
$$\lambda^\perp_\varphi = \frac{2\,\mathrm{Im}[a^*_{+1}a_{-1}]}{\mathcal{N} + |a_0|^2},$$
$$\lambda^\perp_{\vartheta\varphi} = \frac{-\sqrt{2}\,\mathrm{Im}[a^*_0(a_{+1} + a_{-1})]}{\mathcal{N} + |a_0|^2}.$$

In this paper we consider inclusive production. Therefore, for all of the popular choices of frame, the xz plane coincides with the production plane, containing the directions of the colliding particles and of the decaying particle itself. The last two terms in Eq. (7) introduce an asymmetry of the distribution by reflection with respect to the production plane, an asymmetry which is not forbidden in individual events. In hadronic collisions, due to the intrinsic parton transverse momenta, for example, the "natural" polarization plane does *not* coincide event-by-event with the experimental production plane. However, the symmetry by reflection must be a property of the observed *event distribution*, integrating over many events, when only parity-conserving processes contribute. Indeed, the terms in $\sin^2\vartheta \sin 2\varphi$ and $\sin 2\vartheta \sin\varphi$ are unobservable because they vanish on average. In the presence of n contributing production processes with weights $f^{(i)}$, the most general *observable* distribution can be written as

$$W(\cos\vartheta, \varphi) = \sum_{i=1}^n f^{(i)} W^{(i)}(\cos\vartheta, \varphi)$$
$$\propto \frac{1}{(3 + \lambda_\vartheta)}(1 + \lambda_\vartheta \cos^2\vartheta + \lambda_\varphi \sin^2\vartheta \cos 2\varphi$$
$$+ \lambda_{\vartheta\varphi} \sin 2\vartheta \cos\varphi), \qquad (9)$$

where $W^{(i)}(\cos\vartheta, \varphi)$ is the "elementary" decay distribution corresponding to a single subprocess [given by Eqs. (7) and (8), adding the index (i) to the decay parameters]. Each of the three observable shape parameters, $X = \lambda_\vartheta, \lambda_\varphi$ and $\lambda_{\vartheta\varphi}$, is a weighted average of the corresponding parameters, $X^{(i)}$, characterizing the single subprocesses,

$$X = \frac{\sum_{i=1}^{n} g^{(i)} X^{(i)}}{\sum_{i=1}^{n} g^{(i)}}, \tag{10}$$

with $g^{(i)} = f^{(i)} \mathcal{N}^{(i)}/(3 + \lambda_{\vartheta}^{(i)})$.

III. POSITIVITY CONSTRAINTS

Equation (8) implies the relations

$$1 \pm \lambda_{\varphi}^{(i)} = (|a_{+1}^{(i)} \pm a_{-1}^{(i)}|^2 + 2|a_0^{(i)}|^2)/(\mathcal{N}^{(i)} + |a_0^{(i)}|^2),$$

$$\lambda_{\vartheta}^{(i)} \pm \lambda_{\varphi}^{(i)} = (|a_{+1}^{(i)} \pm a_{-1}^{(i)}|^2 - 2|a_0^{(i)}|^2)/(\mathcal{N}^{(i)} + |a_0^{(i)}|^2),$$

$$|\lambda_{\vartheta\varphi}^{(i)}| \le \sqrt{2}|a_0^{(i)}||a_{+1}^{(i)} - a_{-1}^{(i)}|/(\mathcal{N}^{(i)} + |a_0^{(i)}|^2),$$

$$|\lambda_{\vartheta\varphi}^{\perp(i)}| \le \sqrt{2}|a_0^{(i)}||a_{+1}^{(i)} + a_{-1}^{(i)}|/(\mathcal{N}^{(i)} + |a_0^{(i)}|^2), \tag{11}$$

where the index (i) now explicitly denotes the single-subprocess quantities. Equation (11) implies the following relations between the coefficients of the angular distribution:

$$(1 - \lambda_{\varphi}^{(i)})^2 - (\lambda_{\vartheta}^{(i)} - \lambda_{\varphi}^{(i)})^2 \ge 4\lambda_{\vartheta\varphi}^{(i)2},$$

$$(1 + \lambda_{\varphi}^{(i)})^2 - (\lambda_{\vartheta}^{(i)} + \lambda_{\varphi}^{(i)})^2 \ge 4\lambda_{\vartheta\varphi}^{\perp(i)2}. \tag{12}$$

From these expressions we finally reach the following set of inequalities:

$$|\lambda_{\varphi}| \le \frac{1}{2}(1 + \lambda_{\vartheta}), \quad \lambda_{\vartheta}^2 + 2\lambda_{\vartheta\varphi}^2 \le 1, \quad |\lambda_{\vartheta\varphi}| \le \frac{1}{2}(1 - \lambda_{\varphi}),$$

$$(1 + 2\lambda_{\varphi})^2 + 2\lambda_{\vartheta\varphi}^2 \le 1 \text{ for } \lambda_{\varphi} < -1/3. \tag{13}$$

Here we have dropped the index (i) because these relations are completely general and valid for any superposition of production processes, as can be verified using Eq. (10) (being $g^{(i)} > 0$) and, for the two quadratic relations, the Schwarz inequality,

$$\left(\frac{\sum_{i=1}^{n} g^{(i)} X^{(i)}}{\sum_{i=1}^{n} g^{(i)}} \right)^2 \le \frac{\sum_{i=1}^{n} g^{(i)} X^{(i)2}}{\sum_{i=1}^{n} g^{(i)}}. \tag{14}$$

Equation (13) implies, for example, $|\lambda_{\varphi}| \le 1$, $|\lambda_{\vartheta\varphi}| \le \sqrt{2}/2$, $|\lambda_{\varphi}| \le 0.5$ for $\lambda_{\vartheta} = 0$ and $\lambda_{\varphi} \to 0$ for $\lambda_{\vartheta} \to -1$. There is an alternative notation, widespread in the literature, where the coefficients λ, $\nu/2$ and μ replace, respectively, λ_{ϑ}, λ_{φ} and $\lambda_{\vartheta\varphi}$. In that case, hence, we have $|\nu| \le 2$. The most general domain for the three angular parameters is represented in Fig. 2. The upper plot also illustrates the meaning of specific points of the λ_{ϑ}, λ_{φ} plane in terms of angular momentum state of the decaying particle. The six points indicated on the border of the triangle are the combinations of observable parameters corresponding to pure eigenstates of J_x, J_y and J_z with eigenvalues 0 or ± 1. In particular, the three vertices represent *univocally* the well-defined cases in which *all* contributing production processes lead to the same, fully *longitudinal* polarization along the x, y or z axes. The three points lying on the sides of the triangle, however, can either be the result of purely *transverse* polarizations along the x,
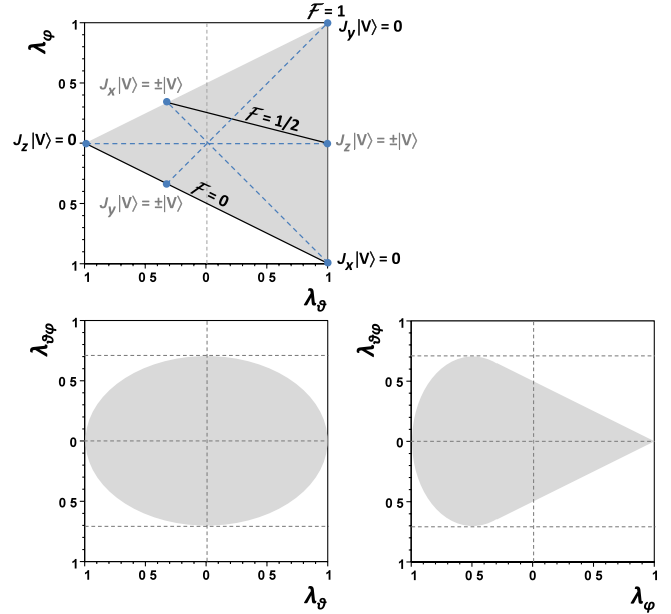

FIG. 2 (color online). Allowed regions for the decay angular parameters (shaded areas). The upper plot also indicates the points corresponding to pure angular momentum configurations and to specific values of the rotation-invariant observable $\mathcal{F}$, introduced in Sec. IV.

y and z axes, or of suitable mixtures of angular momentum eigenstates and/or superpositions of different processes, polarized along different axes.

IV. POLARIZATION-FRAME-INDEPENDENT OBSERVABLE

The rotation-covariance properties of the generic $J = 1$ state defined in Eq. (1) imply two propositions.

(i) *Proposition 1*: The amplitude combination $b_{+1} + b_{-1}$ is invariant by rotation around the y axis.

(ii) *Proposition 2*: There exists a quantization axis $z^\star$ with respect to which $b_0^\star = 0$; if b_0, b_{+1} and b_{-1} are real, $z^\star$ belongs to the xz plane.

In fact, for successive rotations about, respectively, the z and y axes by angles φ and ϑ, a pure $J = 1$, J_z angular momentum eigenstate $|m\rangle$ transforms according to the relation [analogous to Eq. (2), but describing the inverse rotation]

$$|m\rangle = \sum_{m'=0,\pm 1} \mathcal{D}_{mm'}^{1*}(\vartheta, \varphi)|m'\rangle. \tag{15}$$

In the basis of the rotated eigenspace, the state in Eq. (1) has components

$$b_k' = \sum_{m=0,\pm 1} b_m \mathcal{D}_{mk}^{1*}(\vartheta, \varphi). \tag{16}$$

For a rotation in the production plane (about y: $\varphi = 0$),

$$b'_{+1} + b'_{-1} = \sum_{m=0,\pm 1} b_m [d^1_{m,+1}(\vartheta) + d^1_{m,-1}(\vartheta)]$$

$$= b_{+1} + b_{-1}, \qquad (17)$$

where we have used (Eq. (4)) $d^1_{\pm 1,+1}(\vartheta) + d^1_{\pm 1,-1}(\vartheta) = 1$, $d^1_{0,+1}(\vartheta) + d^1_{0,-1}(\vartheta) = 0$. This proves Proposition 1. We now address Proposition 2 taking $|V^{(i)}\rangle$ defined with real b_0 (always possible). After a generic rotation, the zero-helicity component becomes [Eqs. (16), (3), and (4)]

$$b'_0(\vartheta, \varphi) = b_0 \cos\vartheta - \frac{1}{\sqrt{2}}(b_{+1} e^{i\varphi} - b_{-1} e^{-i\varphi}) \sin\vartheta. \quad (18)$$

It can be verified explicitly that the equation $b'_0(\vartheta, \varphi) = 0$ has always a solution, given by

$$\cos\vartheta^\star = \frac{R_+ R_- + I_+ I_-}{\sqrt{2b_0^2(R_+^2 + I_-^2) + (R_+ R_- + I_+ I_-)^2}},$$

$$\cos\varphi^\star = \frac{R_+}{\sqrt{R_+^2 + I_-^2}}, \qquad (19)$$

$$\sin\varphi^\star = -\frac{I_-}{\sqrt{R_+^2 + I_-^2}},$$

where $R_\pm = \text{Re}(b_{+1} \pm b_{-1})$ and $I_\pm = \text{Im}(b_{+1} \pm b_{-1})$. If all three amplitudes are real, then $\varphi^\star = 0$ and the rotation is around the y axis.

We remind that the decay amplitudes a_m are simply proportional to the angular momentum components b_m. Therefore, Proposition 1 and the obvious rotation invariance of $|a_0|^2 + |a_{+1}|^2 + |a_{-1}|^2$ imply that, for each subprocess (i), the quantity

$$\mathcal{F}^{(i)} = \frac{1}{2} \frac{|a^{(i)}_{+1} + a^{(i)}_{-1}|^2}{|a^{(i)}_0|^2 + |a^{(i)}_{+1}|^2 + |a^{(i)}_{-1}|^2} \qquad (20)$$

(included between 0 and 1) is independent of the chosen frame. Using also Eqs. (8) and (10), we find that the following combination of observable parameters of the dilepton decay distribution is frame independent (invariant by rotation about the y axis):

$$\mathcal{F} = \frac{\sum_{i=1}^n f^{(i)} \mathcal{N}^{(i)} \mathcal{F}^{(i)}}{\sum_{i=1}^n f^{(i)} \mathcal{N}^{(i)}} = \frac{1 + \lambda_\vartheta + 2\lambda_\varphi}{3 + \lambda_\vartheta}. \qquad (21)$$

The upper plot in Fig. 2 shows the loci of points in the λ_ϑ, λ_φ plane corresponding to $\mathcal{F} = 0$, $\mathcal{F} = 1/2$ and $\mathcal{F} = 1$. The $\mathcal{F} = 0$ and $\mathcal{F} = 1/2$ lines include the cases of, respectively, full longitudinal and full transverse polarizations with respect to any axis belonging to the production plane. The uniquely defined $\mathcal{F} = 1$ point corresponds to the theoretical case of a full longitudinal polarization along the y axis.

We mention, for completeness, that the quantity

$$\mathcal{G} = \frac{\sum_{i=1}^n f^{(i)} \mathcal{N}^{(i)} \mathcal{G}^{(i)}}{\sum_{i=1}^n f^{(i)} \mathcal{N}^{(i)}} = \frac{1 + \lambda_\vartheta - 2\lambda_\varphi}{3 + \lambda_\vartheta}, \qquad (22)$$

with

$$\mathcal{G}^{(i)} = \frac{1}{2} \frac{|a^{(i)}_{+1} - a^{(i)}_{-1}|^2}{|a^{(i)}_0|^2 + |a^{(i)}_{+1}|^2 + |a^{(i)}_{-1}|^2}, \qquad (23)$$

is invariant by rotation about the x axis. Finally, the parameter λ_ϑ itself is invariant by rotation about z.

V. POLARIZATION-FRAME-INDEPENDENT ANGULAR DISTRIBUTION

Clearly, we can determine the frame-invariant polarization observable $\mathcal{F}$ through the measurement of the two-dimensional, three-parameters angular distribution of Eq. (9). This procedure is particularly useful when performed in two sufficiently different reference frames, to probe systematic effects caused by experimental biases [6], since different values of λ_ϑ and λ_φ, but identical values of $\mathcal{F}$, are expected in each frame. However, it may be convenient to determine $\mathcal{F}$ directly from a one-dimensional, single-parameter angular distribution. The distribution itself must be, like $\mathcal{F}$, invariant by rotation about the y axis. This restricts the possibilities for the definition of the corresponding angular variable to

$$\cos\alpha = \sin\vartheta \sin\varphi, \qquad (24)$$

where α is the angle formed by the lepton with the y axis. The $\cos\alpha$ distribution must be of the form

$$w(\cos\alpha) \propto 1 + \lambda_\alpha \cos^2\alpha, \qquad (25)$$

as any parity-conserving distribution of the angle formed with respect to an axis, when only $J = 1$ wave functions are involved. The relation of λ_α to λ_ϑ, λ_φ and $\mathcal{F}$ can be found by imposing the condition

$$\langle \cos^2\alpha \rangle = \int_{-1}^{+1} \cos^2\alpha \, w(\cos\alpha) d(\cos\alpha)$$

$$= \int_0^{2\pi} \int_{-1}^{+1} (\sin\vartheta \sin\varphi)^2 W(\cos\vartheta, \varphi) d(\cos\vartheta) d\varphi, \qquad (26)$$

with the result

$$\lambda_\alpha = -\frac{\lambda_\vartheta + 3\lambda_\varphi}{2 + \lambda_\vartheta + \lambda_\varphi} = \frac{1 - 3\mathcal{F}}{1 + \mathcal{F}}. \qquad (27)$$

VI. THE LAM-TUNG RELATION AS A PARTICULAR CASE

It has been noticed long ago that, in the case of Drell-Yan production, the shape parameters λ_ϑ and λ_φ obey the frame-independent expression $\lambda_\vartheta + 4\lambda_\varphi = 1$, commonly known as the "Lam-Tung relation" [2]. Although the dilepton production cross section is substantially modified by QCD corrections, the *relation* between the different helicity contributions to this cross section remains unchanged up to $O(\alpha_s)$, a seemingly surprising feature.

Relatively small corrections affect the angular distribution when subsequent orders in α_s are taken into account [3,4]. Given its robustness within perturbative QCD, deviations from the Lam-Tung relation have been considered as a signal of higher twist contributions [7] or nonperturbative effects caused by intrinsic parton k_T [8] or even parton saturation [9].

Actually, the Lam-Tung relation is a particular case of the more general invariant relation presented in Eq. (21). Indeed, in Drell-Yan production up to $O(\alpha_s)$, neglecting parton transverse momenta, the topology of each contributing subprocess (quark-antiquark annihilation without or with single gluon emission, Compton-like quark-gluon scattering, etc.) is characterized by one reaction plane, coinciding with the experimental production plane. Therefore, as mentioned in Sec. II, we can set $\lambda_\varphi^{\perp(i)} = \lambda_{\vartheta\varphi}^{\perp(i)} = 0$ for each single subprocess, (i). Imposing this condition in Eq. (8), we find that the three partial decay amplitudes, $a_m^{(i)}$, and, therefore, the corresponding angular momentum components, $b_m^{(i)}$, have the same complex phase. Proposition 2 implies, then, that the observed dilepton distribution is a superposition of subdistributions characterized by

$$\lambda_\vartheta^{(i)\star} = +1, \qquad \lambda_\varphi^{(i)\star} = 2\mathcal{F}^{(i)} - 1, \qquad \lambda_{\vartheta\varphi}^{(i)\star} = 0, \quad (28)$$

each one referred to a specific polarization axis $z^{(i)\star}$ belonging to the production plane. Assuming helicity conservation at the production vertex (i.e., that the participating quarks are massless), the zero-order quark-antiquark annihilation process in Drell-Yan production, Fig. 3(a), leads to a decay anisotropy of the kind $1 + \cos^2\vartheta$ with respect to the direction of the relative momentum between quark and antiquark, experimentally approximated by the Collins-Soper (CS) frame [10]. In the $O(\alpha_s)$ processes, on the other hand, the photon couples to one real quark and to the intermediate virtual quark, this latter having a well-defined momentum. Also in this case helicity conservation leads to a decay anisotropy of the kind $1 + \cos^2\vartheta$ but now with respect to the direction of the relative momentum between the real and virtual quarks. Experimentally, this quantization axis corresponds to the Gottfried-Jackson [(GJ) [11]] axis for the processes presented in Fig. 3(b) and 3(c) and to the helicity axis for the process shown in Fig. 3(d).
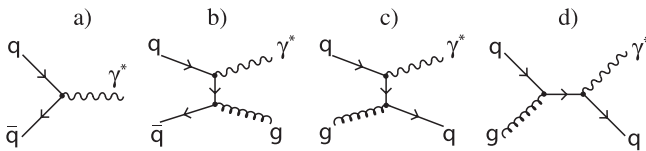

FIG. 3. $O(\alpha_s^0)$ and $O(\alpha_s^1)$ processes for Drell-Yan production, giving rise to transverse dilepton polarizations along different quantization axes: Collins-Soper (a), Gottfried-Jackson (b, c) and helicity (d).

Effectively, therefore, all subprocesses contributing to Drell-Yan production up to $O(\alpha_s)$ lead individually to the same kind of *fully transverse, purely polar* decay anisotropy, even if *with respect to three different natural axes*, $z^{(i)\star}$. In each case $\lambda_\varphi^{(i)\star} = 0$, meaning that $\mathcal{F}^{(i)} = 1/2$ for all subprocesses [Eq. (28)] and, thus, implying $\mathcal{F} = 1/2$. This latter equation coincides with the Lam-Tung relation. In other words, we have shown that the *frame independence* of the Lam-Tung relation is a simple *kinematic* consequence of the rotational properties of the $J = 1$ angular momentum eigenstates [leading, in general, to Eq. (21)], while its specific *form* ($\mathcal{F} = 1/2$) derives from the *dynamical* input that all contributing subprocesses produce *transversely* polarized states.

Deviations from the Lam-Tung relation are often parametrized in terms of the quantity $\Delta = \lambda_\vartheta + 4\lambda_\varphi - 1$ in the experimental and theoretical literature. However, the correspondingly assumed relation $\lambda_\vartheta + 4\lambda_\varphi = 1 + \Delta$ is not frame invariant [it cannot be rewritten in the form of Eq. (21) for a certain value of $\mathcal{F}$], and, hence, the "violation level" expressed by Δ depends on the frames used in the analyses. We propose that future searches for violations of the Lam-Tung relation evaluate the (frame invariant) deviation of $\mathcal{F}$ from $1/2$.

Following the considerations of Sec. V, tests of the Lam-Tung relation can be performed by simply determining the $\cos\alpha$ distribution and measuring the deviation of λ_α from the value $-1/3$. Vice versa, in regimes where the validity of the Lam-Tung relation (or, more generally, the intrinsic transverseness of the polarization) can be considered as a characterizing feature of the physical process under study, the event distribution

$$w(\cos\alpha) \propto 1 - \frac{1}{3}\cos^2\alpha \qquad (29)$$

can be used to check the purity of the selected signal sample and/or to provide an event-by-event criterium for signal-background discrimination.

VII. VIOLATION OF THE LAM-TUNG RELATION IN PION-NUCLEUS DATA

Violations of the Lam-Tung relation in Drell-Yan angular distributions were searched for in several experimental conditions. "Anomalous" effects were evidenced in pion-nucleus experiments, which measured large azimuthal anisotropies increasing with p_T and a strong reduction of the transverse polarization at high x_1 (momentum fraction of the annihilating antiquark in the beam pion). The largest effects, measured by E615 [12], are shown in Fig. 4. Figure 5 shows the values of $\mathcal{F}$ derived from the angular distributions measured by E615, as well as by NA10 [13], as a function of x_1 (a), and of the dimuon mass M (b) and transverse momentum p_T (c). The condition $\mathcal{F} = 1/2$ is increasingly violated with increasing p_T, while there is no significant dependence on x_1 or M. The panel (d) shows the
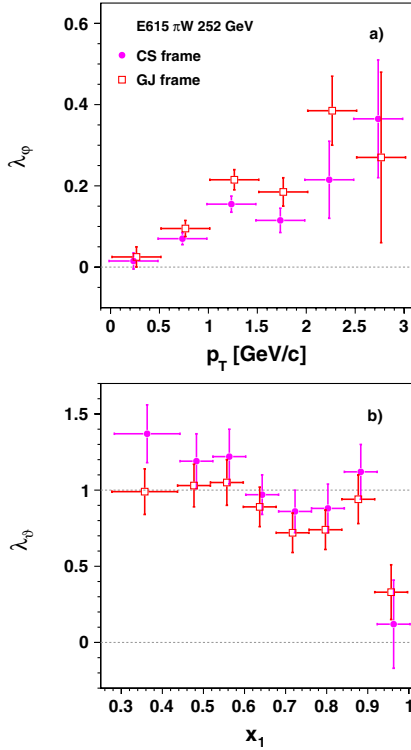
FIG. 4 (color online).   The E615 measurements of the Drell-Yan azimuthal anisotropy as a function of p_T (a) and of the polar anisotropy as a function of x_1 (b). The points are slightly displaced in the horizontal axis for improved visibility.

E866 results, obtained in pp and pd interactions at 800 GeV [14], perfectly consistent with the Lam-Tung expectation. The most significant deviations from purely transverse dilepton polarization are measured by E615 for $1 < p_T < 1.5$ GeV/c and $\langle x_1 \rangle \simeq 0.6$, in π-W collisions at 252 GeV, and by NA10 for $1.5 < p_T < 2$ GeV/c and $\langle x_1 \rangle \simeq 0.4$, in π-W and πd collisions at 286 GeV. The corresponding values of $\mathcal{F} - 1/2$ are, respectively, 0.109 ± 0.015 and 0.058 ± 0.018.

Contrary to what the strong p_T dependence might suggest, we can easily exclude that the enhancement of $\mathcal{F}$ with respect to $1/2$ is the result of an event-by-event tilt between the polarization axis used in the experimental analysis and the natural axis, caused by the intrinsic transverse momenta of the partons. This observation can be easily understood by considering the leading-order $1 + \cos^2 \vartheta_{CS}$ distribution and two extreme categories of events, in both of which the instantaneous natural axis (direction of the collision between partons) is significantly tilted with respect to the Collins-Soper axis, but towards two orthogonal directions: in one case the natural axis belongs to the plane of the colliding hadrons, in the other case it belongs to the perpendicular plane. The first type of events has a distribution characterized by $\lambda_\vartheta < 1$ and $\lambda_\varphi > 0$, while $\mathcal{F}$, unaffected by rotations around the axis perpendicular to the production plane, remains $1/2$.
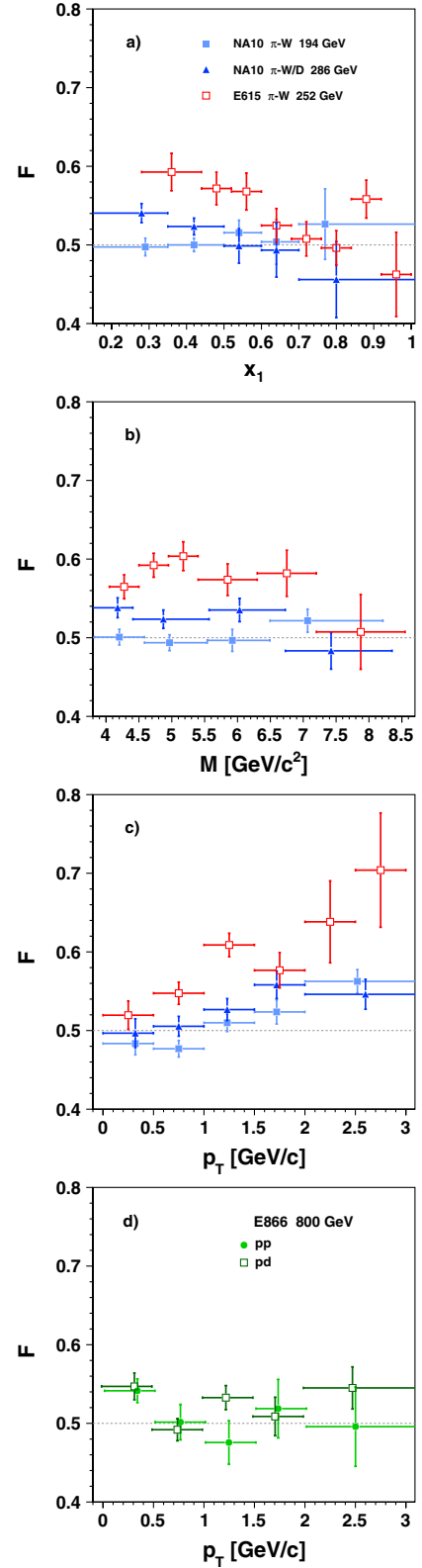

FIG. 5 (color online).   The frame-invariant parameter $\mathcal{F}$ as a function of dilepton kinematic variables, derived from Drell-Yan measurements obtained with pion (a–c) and proton (d) beams. The E866 data points are slightly displaced in the horizontal axis for improved visibility.

The second event distribution is rotated with respect to the previous one by an angle $\pi/2$ about the z axis, implying that λ_φ, the coefficient of the term $\sin^2\vartheta\cos2\varphi$, becomes negative, given that $\cos2(\varphi\pm\pi/2)=-\cos2\varphi$. On the other hand, λ_ϑ remains unchanged (and smaller than 1). Therefore, $\mathcal{F}<1/2$. In conclusion, parton transverse momenta lead on average to a reduction, rather than an enhancement, of the overall observable anisotropy and, hence, of $\mathcal{F}$. The anomalous experimental observations cannot, therefore, be the geometrical consequence of a deviation of the experimental axis from the quantization axis of the elementary processes. They must reflect intrinsic properties of the production mechanism, not properly described considering only lowest-order perturbative processes.

Furthermore, we can also say that these anomalies cannot be caused by relatively rare subprocesses, contributing as higher-order "perturbations" in a standard QCD approach to the study of Drell-Yan production. We derive this observation in the next lines, as a good example of the usefulness of the formalism presented in this paper. The maximum deviation of $\mathcal{F}$ from $1/2$ measured by E615, 0.109 ± 0.015, allows us to deduce, using Eq. (21), that the fraction of dilepton events violating the condition $\mathcal{F}^{(i)}\le 1/2$ is *at least* as large as 0.22 ± 0.03. Such a fraction would already have to be considered a very large contribution to the Drell-Yan production yield, certainly not a "perturbation". However, the real value must be even larger because this lower limit corresponds to an extreme hypothesis: $\mathcal{F}^{(i)}$ is always either $1/2$, in the case of standard $O(\alpha_s^0)$ and $O(\alpha_s^1)$ processes, or 1, in the case of the anomalous (hypothetical) "higher-order processes" violating the Lam-Tung relation. In reality, the standard processes should have $\mathcal{F}^{(i)}<1/2$, accounting for the parton transverse momenta effect mentioned above, and the anomalous processes should have $\mathcal{F}^{(i)}$ values smaller than the extreme limit of $\mathcal{F}^{(i)}=1$. In these more realistic conditions, in order to reproduce the E615 measurement the fraction of Drell-Yan dileptons produced by the anomalous processes would need to be comparable to the contribution of the "lowest-order" processes represented in Fig. 3. Furthermore, such additional mechanisms, characterized by $\mathcal{F}^{(i)}$ values approaching 1, would produce the dilepton in a rather uncommon angular momentum state, where the $m=+1$ and $m=-1$ component amplitudes have comparable magnitudes and interfere constructively, so as to maximize $|a_{+1}^{(i)}+a_{-1}^{(i)}|$. In particular, the limiting

case $\mathcal{F}^{(i)}=1$ corresponds to the angular momentum state $\frac{1}{\sqrt{2}}|+1\rangle+\frac{1}{\sqrt{2}}|-1\rangle$, which is invariant by rotation around the y axis. As mentioned in Sec. III, this specific point of the phase space must be univocally attributed to the decay of a pure eigenstate of J_y with eigenvalue 0 (top panel in Fig. 2), and it is impossible to reproduce it with a superposition of different states. In conclusion, attributing the violation of the Lam-Tung relation in pion-nucleus data to the existence of anomalous "higher-order" processes in a perturbative-QCD approach is equivalent to say that a *very large* fraction of Drell-Yan dileptons is produced in a fully (longitudinally) polarized state *with respect to the quantization axis perpendicular to the production plane*. This is an extremely peculiar and unrealistic scenario, requiring the existence of a production mechanism that would lead, essentially in each and every event, to a very exotic configuration of the dilepton spin.

VIII. SUMMARY

The average angular momentum composition of a vector state is reflected in the shape of its dilepton decay angular distribution [Eqs. (7)–(10)]. The parameters of the distribution can only take values inside a well-defined domain [Eq. (13)]. For a specific mixture of production processes in a given kinematic condition, there always exists a polarization observable $\mathcal{F}$ [Eq. (21)] independent of the choice of the quantization axis (belonging to the production plane). The Lam-Tung relation represents the particular case $\mathcal{F}=1/2$ (independent of production kinematics), meaning that all subprocesses produce transversely polarized difermions with respect to any polarization axis belonging to the plane of the colliding hadrons. $\mathcal{F}$ can be determined from a single-variable distribution (Eqs. (24)–(27)), facilitating, in particular, measurements of the violation of the Lam-Tung relation. The significant violations of this relation found in pion-nucleus experiments cannot be ascribed to the contribution of anomalous higher-order processes because rotational invariance and topological symmetry properties rule out a "perturbative" interpretation of the phenomenon.

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