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Low-energy phenomenology of scalarless standard-model extensions with high-energy Lorentz violation

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We consider renormalizable standard model extensions that violate Lorentz symmetry at high energies, but preserve CPT, and do not contain elementary scalar fields. A Nambu–Jona-Lasinio mechanism gives masses to fermions and gauge bosons and generates composite Higgs fields at low energies. We study the effective potential at the leading order of the large-N_c expansion, prove that there exists a broken phase, and study the phase space. In general, the minimum may break invariance under boosts, rotations, and CPT, but we give evidence that there exists a Lorentz invariant phase. We study the spectrum of composite bosons and the low-energy theory in the Lorentz phase. Our approach predicts relations among the parameters of the low-energy theory. We find that such relations are compatible with the experimental data within theoretical errors. We also study the mixing among generations, the emergence of the CKM matrix, and neutrino oscillations.

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I. INTRODUCTION

Lorentz symmetry is a basic ingredient of the standard model of particle physics and one of the best tested symmetries in nature [1]. From the theoretical viewpoint, we often take for granted that Lorentz symmetry must be exact. However, our present knowledge cannot exclude that it may just be approximate. Specifically, it could be violated at very high energies or very large distances. Both possibilities have motivated several authors to investigate the new physics that would emerge. Although no sign of Lorentz violation has been found so far, these kinds of investigations are useful because, after comparison with experiments, they allow us to put bounds on the parameters of the violation. Among the most relevant reference works, we mention Refs. [2,3], together with the data tables of [1]. For a recent update on the state of the art see Ref. [4].

In quantum field theory, if we assume that Lorentz symmetry is explicitly violated at high energies we can turn nonrenormalizable interactions into renormalizable ones [5]. In flat space, and in the realm of perturbation theory, it is possible to construct gauge theories [6,7] and extensions of the standard model [8,9], without violating physical principles. The Lorentz-violating models can contain several types of terms of higher dimensions. They are multiplied by inverse powers of a scale Λ_L, which is interpreted as the scale of Lorentz violation. Renormalizability holds because the theory includes quadratic terms of higher dimensions that contain higher-space derivatives. The modified dispersion relations generate propagators with improved ultraviolet behaviors. A "weighted" power counting, according to which space and time have different weights, controls the ultraviolet divergences of Feynman diagrams and allows us to determine which vertices are compatible with each set of quadratic terms. No terms containing higher time derivatives (which would spoil unitarity) are present nor generated back by renormalization. Lorentz symmetry is recovered at energies much smaller than Λ_L. It is not necessary to assume that CPT is also violated to achieve these results.

The theories formulated using these tools are not meant to be just effective field theories, but can be regarded as fundamental theories, in the sense that, very much like the standard model, in principle they can describe nature at arbitrarily high energies (when gravity is switched off). Some standard model extensions contain the vertex $(LH)^2/\Lambda_L$ at the fundamental level, therefore giving neutrinos Majorana masses after symmetry breaking. No right-handed neutrinos, nor other extra fields, are necessary to achieve this goal. Some extensions contain four-fermion vertices $(\bar{\psi}\psi)^2/\Lambda_L^2$ at the fundamental level. Such vertices can explain proton decay and trigger a Nambu–Jona-Lasinio mechanism, which generates fermion masses, gauge-boson masses, and composite bosons. Finally, some of our models can be phenomenologically viable even if they do not contain elementary scalars.

Because of renormalizability, these extensions contain a finite set of independent parameters, so they are to some extent predictive. It is important to check that they can reproduce the known physics at low energies. For example, it is not obvious that the models containing no elementary scalars are able to fully reproduce the standard model at low energies.

In this paper we study the low-energy phenomenology of the scalarless models and compare predictions with

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experimental data. Although our approach has large theoretical errors, the predictions are still meaningful, because they can be falsified by data. We give enough evidence that our models do reproduce the known low-energy physics. We study the effective potential in detail, prove that there exists a broken phase, study the phase space, and give evidence that there exists a Lorentz phase. We investigate the mixing among generations and show how the Cabibbo-Kobayashi-Maskawa (CKM) matrix emerges. We study the spectrum of composite bosons that propagate at low energies, the low-energy effective action, and the compatibility with experimental constraints. Finally, we discuss neutrino oscillations.

The search for consistent standard model alternatives that do not contain elementary scalar fields has a long history, from Technicolor [10] to the more recent extra-dimensional Higgsless models [11]. Worth mentioning are also the asymptotic-safety approach of Ref. [12] and the standard perturbative approach of Ref. [13]. In this respect, the violation of Lorentz symmetry offers, among the other things, a new guideline and source of insight and in our opinion deserves the utmost attention.

The paper is organized as follows. In Sec. II we review the minimal scalarless standard model extension that we are going to study. In Sec. III we describe the dynamical symmetry-breaking mechanism and calculate the effective potential to the leading order. In Sec. IV we prove that there exists a domain in parameter space where the dynamical symmetry breaking takes place, namely the effective potential has a nontrivial absolute minimum. We also investigate the phase space. In Sec. V we reconsider the case of one generation treated in Ref. [9] and prove some new results. In Sec. VI we study the case of three generations and show how the CKM matrix emerges. We also show that, in general, the Lorentz violation predicts a more severe mixing among generations besides the CKM matrix. In Sec. VII we show that there exist CPT violating local minima. In Sec. VIII we derive and study the low-energy effective action in the Lorentz phase, in the case of one generation, and compare predictions with data. In Sec. IX we show that the minimal model cannot generate (Majorana) masses for left-handed neutrinos. Nevertheless, it is possible that neutrino oscillations are explained in a different way. Section X contains conclusions and outlook.

II. THE MODEL

We assume that CPT and invariance under rotations are preserved. The (minimal) scalarless model we are going to study reads

$$\mathcal{L}_{\mathrm{noH}} = \mathcal{L}_Q + \mathcal{L}_{\mathrm{kin}f} - \sum_{I=1}^{5} \frac{1}{\Lambda_L^2} g \bar{D} \bar{F} (\bar{\chi}_I \bar{\gamma} \chi_I) + \frac{Y_f}{\Lambda_L^2} \bar{\chi}\chi\bar{\chi}\chi - \frac{g}{\Lambda_L^2} \bar{F}^3, \qquad (2.1)$$

where

$$\mathcal{L}_{\mathrm{kin}f} = \sum_{a,b=1}^{3} \sum_{I=1}^{5} \bar{\chi}_I^a i \Big(\delta^{ab} \gamma^0 D_0 - \frac{b_0^{Iab}}{\Lambda_L^2} \bar{\not{D}}^3 + b_1^{Iab} \bar{\not{D}} \Big) \chi_I^b,$$

$$\mathcal{L}_Q = \frac{1}{4} \sum_G (2 F_{0i}^G F_{0i}^G - F_{ij}^G \tau^G(\bar{\Upsilon}) F_{ij}^G) \qquad (2.2)$$

are the quadratic terms of fermions and gauge fields, respectively, and Λ_L is the scale of Lorentz violation. Bars are used to denote space components and $\bar{F}$ denotes the "magnetic" components F_{ij} of the field strengths. Moreover, $\chi_1^a = L^a = (\nu_L^a, \ell_L^a)$, $\chi_2^a = Q_L^a = (u_L^a, d_L^a)$, $\chi_3^a = \ell_R^a$, $\chi_4^a = u_R^a$, and $\chi_5^a = d_R^a$, $\nu^a = (\nu_e, \nu_\mu, \nu_\tau)$, $\ell^a = (e, \mu, \tau)$, $u^a = (u, c, t)$, and $d^a = (d, s, b)$. The sum $\sum_G$ is over the gauge groups $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$. The last three terms of (2.1) are symbolic. Finally, $\bar{\Upsilon} \equiv -\bar{D}^2/\Lambda_L^2$ and τ^G are polynomials of degree 2. Gauge anomalies cancel out exactly as in the standard model [8].

The model is "minimal" in the sense that it contains the minimal set of elementary fields. It contains fewer fields than the minimal standard model, because we have suppressed the elementary scalars. No right-handed neutrinos, nor other extra fields, are included.

The model is renormalizable by weighted power counting in two "weighted dimensions". This means that at high energies renormalizability is governed by a power counting that resembles the one of a two dimensional field theory, where energy has weight one, and the three space coordinates altogether have weight one, therefore each of them separately has weight 1/3.

The weights of fields and couplings are determined so that each Lagrangian term has weight 2. Gauge couplings g have weight 1/3, so they are super-renormalizable. For this reason, at very high energies gauge fields become free and decouple, so the theory (2.1) becomes a four-fermion model in two weighted dimensions, described by the Lagrangian

$$\mathcal{L}_{4\mathrm{f}} = \sum_{a,b=1}^{3} \sum_{I=1}^{5} \bar{\chi}_I^a i \Big(\delta^{ab} \gamma^0 \partial_0 + b_1^{Iab} \bar{\not{\partial}} - \frac{b_0^{Iab}}{\Lambda_L^2} \bar{\not{\partial}}^3 \Big) \chi_I^b + \frac{Y_f}{\Lambda_L^2} \bar{\chi}\chi\bar{\chi}\chi. \qquad (2.3)$$

We have kept also the terms multiplied by b_1^{Iab}, since they are necessary to recover Lorentz invariance at low energies.

Our purpose is to investigate whether (2.1) can describe the known low-energy physics by means of a dynamical symmetry-breaking mechanism triggered by four-fermion vertices.

The low-energy limit is the limit $\Lambda_L \to \infty$. From the point of view of renormalization, powerlike and logarithmic divergences in Λ_L appear in this limit and add to the divergences already present in the high-energy theory. The Λ_L divergences make the difference between the renormalization of the high-energy theory and the one of the low-energy theory, which are controlled by weighted power counting and ordinary power counting, respectively. When no symmetry-breaking mechanism takes place, Lorentz symmetry can always be restored at low energies fine-tuning the parameters of the low-energy Lagrangian. In Ref. [14] these aspects of the low-energy limit have been studied in the QED subsector of (2.1). However, taking the low-energy limit in the full model (2.1) is more involved.

Because of the dynamical symmetry-breaking mechanism, the symmetries of the low-energy theory depend on the vacuum. In turn, the vacuum depends on the coefficients of the four-fermion vertices and the other free parameters of the theory. The absolute minimum of the effective potential may break boosts and even rotations and CPT. If that happens, it is impossible to recover Lorentz invariance at low energies and have compatibility with experimental data. Thus, it is important to show that there exists a phase (namely a range in parameter space) where the minimum is Lorentz invariant, so that Lorentz symmetry can be restored at low energies. One of the purposes of this paper is to provide evidence that such a phase exists. This is the phase where the standard model lives, and we call it the *Lorentz phase*.

We proceed according to the following high-energy → low-energy pattern. It is useful to first switch gauge interactions off and switch them back on later. Normally, this is just a trick to simplify the presentation, but in our model it has a more physical justification, because, as explained above, gauge fields decouple at very high energies, where the complete model (2.1) reduces to the four-fermion model (2.3) plus free fields. We show that the model (2.3) exhibits a dynamical symmetry-breaking mechanism in the large N_c expansion. Under suitable assumptions, we argue that the effective potential has a Lorentz invariant minimum. The minimum produces fermion condensates $\langle \bar{q}q \rangle$ and gives masses to the fermions. Massive bound states (composite Higgs bosons) emerge together with Goldstone bosons. When gauge interactions are finally switched back on, the Goldstone bosons associated with the breaking of $SU(2)_L \times U(1)_Y$ to $U(1)_Q$ are "eaten" by the $W^\pm$ and Z bosons, which then become massive.

When we study the compatibility of our predictions with experimental data we set the scale of Lorentz violation Λ_L to 10^{14} GeV. This value was suggested in Ref. [8] assuming that neutrino masses are due to the vertex

$$\frac{1}{\Lambda_L}(LH)^2. \tag{2.4}$$

However, in the minimal model (2.1) this vertex is absent, both at the fundamental and effective levels, and neutrino oscillations must be explained in a different way (see Sec. IX). Still, a number of considerations suggest that $\sim 10^{14-15}$ GeV are meaningful values for the scale of Lorentz violation. They can be thought of as the smallest values allowed by data. For example, they also agree with existing bounds on proton decay, derived from four-fermion vertices $(\bar{\psi}\psi)^2/\Lambda_L^2$: if we assume that the dimensionless coefficients multiplying such vertices are of order one, we obtain $\Lambda_L \gtrsim 10^{15}$ GeV [15]. For other, recent considerations on the magnitude of Λ_L and compatibility with ultrahigh-energy cosmic rays, see [16].

III. DYNAMICAL SYMMETRY-BREAKING MECHANISM

In this section we describe the dynamical symmetry-breaking mechanism in the model (2.3) and calculate the effective potential to the leading order of the $1/N_c$ expansion.

The most general four-fermion vertices can be expressed using auxiliary fields that we call M, N, a quadratic potential V_2, and Yukawa terms:

$$V_2(M,N) + \sum_{\alpha\beta ABIJ} [M^{AB}_{\alpha\beta,IJ}\bar{\psi}^{\alpha A}_I \psi^{\beta B}_J + (N^{AB}_{\alpha\beta,IJ}\psi^{\alpha A}_I \psi^{\beta B}_J + \text{H.c.})].$$

Here α, β are spinor indices, I, J are indices that denote the type of fermions, A, B are $SU(N_c) \times SU(2)_L$ indices. $V_2(M,N)$ is the most general quadratic potential that is invariant under $SU(N_c) \times SU(2)_L \times U(1)_Y$ and CPT. The Yukawa terms are made symmetric assigning suitable transformation properties to the fields M and N.

The four-fermion vertices are obtained integrating out the auxiliary fields M and N. Several combinations of auxiliary fields may produce the same four-fermion vertices. We do not need to select a minimal set of auxiliary fields here. Actually, we include the maximal set of auxiliary fields, because we want to study all possible intermediate channels. Some components of the fields M and N become propagating at low energies (composite bosons), others remain nonpropagating also after the symmetry breaking.

Large N_c expansion — The Nambu–Jona-Lasinio dynamical symmetry-breaking mechanism is not perturbative in the usual sense, so we need to have a form of control on it. We use a large N_c expansion. A rough estimate of the error due to the large N_c expansion can be obtained summing all powers of $1/N_c$ with opposite signs, assuming that higher order contributions are of the same magnitude (apart from the powers of $1/N_c$ in front of them). Thus, calling "1" a generic quantity, its corrections are

$$\pm \sum_{k=1}^{\infty} \frac{1}{N_c^k} = \pm \frac{1}{N_c - 1}. \tag{3.1}$$

For the purposes of this paper, we just need to consider the leading order of the $1/N_c$ expansion. For $N_c = 3$, formula (3.1) tells us that we have a $\pm 50\%$ of error. Even if this error is large, some of our predictions are enough precise to be possibly ruled out.

We cannot exclude that other symmetry-breaking mechanisms may take place in the exact model, but we do not consider such possibilities here, because we do not have a form of control on them such as the one provided by the large N_c expansion.

The leading order of the $1/N_c$ expansion receives contributions only from color-singlet fermion bilinears, on which we focus for the moment. We consider the Yukawa terms

$$\mathcal{L}_{\rm Y} = -\sum_{abmn}[S^{ab}_{mn}(\bar{Q}^{am}_R Q^{bn}_L) + \bar{S}^{ba}_{nm}(\bar{Q}^{am}_L Q^{bn}_R) + H^{ab}_{\mu,mn}(\bar{Q}^{am}_L\gamma^\mu Q^{bn}_L) + K^{ab}_{\mu,mn}(\bar{Q}^{am}_R\gamma^\mu Q^{bn}_R) + L^{ab}_{\mu\nu,mn}(\bar{Q}^{am}_R\sigma^{\mu\nu}Q^{bn}_L) + \bar{L}^{ba}_{\mu\nu,nm}(\bar{Q}^{am}_L\sigma^{\mu\nu}Q^{bn}_R)], \tag{3.2}$$

where $Q^a_R = (u^a_R, d^a_R)$ and m, n are both $SU(2)_L$ and $SU(2)_R$ indices, depending on the case. The Yukawa terms are $U(2)_L \times U(2)_R$ invariant, and so is the leading-order correction to the effective potential. The (contracted) $SU(N_c)$ indices are not shown. The fields S and $L_{\mu\nu}$ are CPT even, while the fields H_μ and K_μ are CPT odd. The matrices H_μ and K_μ are Hermitian.

Lagrangian of the high-energy model and effective potential — As usual, we first switch the gauge fields off, because they decouple at high energies. We will turn them back on later. The fermionic kinetic terms are

$$\mathcal{L}_{\rm kf} = \sum_{abm}\bar{Q}^{am}_L i\left(\delta^{ab}\gamma^0\partial_0 + b^{ab}_{1L}\bar{\partial} - \frac{b^{ab}_{0L}}{\Lambda_L^2}\bar{\partial}^3\right)Q^{bm}_L + \bar{Q}^{am}_R i\left(\delta^{ab}\gamma^0\partial_0 + b^{abm}_{1R}\bar{\partial} - \frac{b^{abm}_{0R}}{\Lambda_L^2}\bar{\partial}^3\right)Q^{bm}_R,$$

where $b^{ab}_{0,1L}$ and $b^{abm}_{0,1R}$ are Hermitian matrices for every m. The total Lagrangian reads

$$\mathcal{L} = \mathcal{L}_{\rm kf} + \mathcal{L}_{\rm Y} + \mathcal{L}'_{\rm Y} + W'_2(S, H, K, L, N'),$$

where $\mathcal{L}'_{\rm Y}$ and N' denote all other Yukawa terms and auxiliary fields, respectively. The potential W'_2 is the most general quadratic form compatible with the symmetries of the theory. We can eliminate the off-diagonal terms SN', HN', KN', and LN' translating the fields N'. Calling N the translated fields, we get a quadratic potential of the form

$$W'_2 = W_2(S, H, K, L) + W''_2(N).$$

The leading-order correction to the potential depends only on S, H, K, and L. The fields N have vanishing expectation values, so the fields N' can have nontrivial expectation values because of the translation from N' to N. For the moment we can ignore the N sector and focus on W_2.

By CPT and rotational invariance, the potential W_2 has the symbolic structure

$$W_2(S, H, K, L) \sim SS^\dagger + H_0^2 + H_i^2 + H_0K_0 + H_iK_i + K_0^2 + K_i^2 + L_{0j}L^\dagger_{0j} + L_{ij}L^\dagger_{ij}.$$

The indices not shown explicitly in this formula are contracted with constant tensors, compatibly with the symmetries of the theory.

To study the effective potential we need to consider the Lagrangian

$$\mathcal{L}_q = \sum_{a,b=1}^{3}\bar{\Psi}^a\left(i\Gamma^0\mathbb{1}\partial_t + i\bar{\Gamma}\cdot\bar{\partial}\left(B_1 - \frac{\bar{\partial}^2}{\Lambda_L^2}B_0\right) - M\right)^{ab}\Psi^b - W_2(M), \tag{3.3}$$

where

$$B_{0,1} = \begin{pmatrix} b^{ab}_{0,1L}\delta^{mn} & 0 \\ 0 & b^{abp}_{0,1R}\delta^{pq}\end{pmatrix}, \qquad M = \begin{pmatrix} S + L_{\mu\nu}\hat{\sigma}^{\mu\nu} & K_\mu\sigma^\mu \\ H_\mu\bar{\sigma}^\mu & S^\dagger + L^\dagger_{\mu\nu}\check{\sigma}^{\mu\nu}\end{pmatrix},$$

and

$$(\Gamma^\mu)^{ab} = \delta^{ab}\begin{pmatrix} 0 & \Sigma^\mu \\ \bar{\Sigma}^\mu & 0\end{pmatrix}, \qquad \Psi^a = \begin{pmatrix} Q^{am}_L \\ Q^{ap}_R\end{pmatrix},$$

$(\Sigma^\mu)^{mp} = \delta^{mp}\sigma^\mu$, $(\bar{\Sigma}^\mu)^{pm} = \delta^{mp}\bar{\sigma}^\mu$, $\sigma^\mu = (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, $\check{\sigma}^{\mu\nu} = -i(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)/2$, $\hat{\sigma}^{\mu\nu} = -i(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)/2$.

The leading-order effective potential reads

$$W(M) = W_2(M) + \mathcal{V}(M),$$

where $\mathcal{V}(M)$ is calculated integrating over the fermions. It is the renormalized version of

$$\mathcal{V}_{\rm div}(M) \equiv -N_c\int^\Lambda \frac{d^4p}{(2\pi)^4}\ln\det(P - \Gamma^0 M), \qquad P \equiv i\mathbb{1}p_4 + \Gamma^0\bar{\Gamma}\cdot\bar{p}\left(B_1 + \frac{\bar{p}^2}{\Lambda_L^2}B_0\right).$$

The integral has already been rotated to the Euclidean space. We regulate the ultraviolet divergences with a cutoff Λ and subtract them expanding in M around $M = 0$. The lowest order in M is a constant, while the first order in M is proportional to the integral of $\mathrm{tr}[P^{-1}\Gamma^0 M]$, which is odd in momentum, so it vanishes. The second order is logarithmically divergent, while all other orders are convergent. We have

$$\mathcal{V}(M) = -N_c\int\frac{d^4p}{(2\pi)^4}\left(\ln\det(\mathbb{1} - P^{-1}\Gamma^0 M) + \frac{1}{2}\mathrm{tr}[P^{-1}\Gamma^0 MP^{-1}\Gamma^0 M]\right). \tag{3.4}$$

Observe that $\mathcal{V}(M)$ is regular in the infrared.

IV. EXISTENCE OF A NON-TRIVIAL ABSOLUTE MINIMUM

In this section we prove that there exists a phase where the dynamical symmetry-breaking mechanism takes place. Precisely, the potential has a nontrivial absolute minimum if some parameters contained in $W_2(M)$ satisfy certain bounds and B_1 is in the neighborhood of the identity. The assumption $B_1 \sim 1$ is not only useful to simplify the calculations, but also justified by all known experimental data [1].

It is sufficient to work at $B_1 = 1$, because the result, once proved for $B_1 = 1$, extends to the neighborhood of the identity by continuity. On the other hand, for the moment we keep the matrix B_0 free, because its entries can differ from one another by several orders of magnitude.

We first prove that the potential grows for large M, in all directions. This result allows us to conclude that there exists an absolute minimum. Indeed, since the function $W(M)$ is continuous the extreme value theorem ensures that it has absolute maxima and absolute minima in an arbitrary sphere $|M| \leq R$. If we take R large enough $W(M)$ grows outside the sphere. Then the absolute minima inside the sphere are absolute minima of the function.

Later we show that, in a suitable domain $\mathcal{D}$ of parameter space, the point $M = 0$, which is stationary, is not a minimum. This proves that the absolute minimum of $W(M)$ is nontrivial in $\mathcal{D}$. Along with the proof, we derive the bounds that define $\mathcal{D}$.

To study $W(M)$ for large M, we rescale M by a factor λ and then let λ tend to infinity. It is useful to rescale also p_4 by a factor λ and $\bar{p}$ by a factor $\lambda^{1/3}$. We get

$$W(\lambda M) = -\frac{N_c \lambda^2}{2} \int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} \times \int_{\mathrm{IR}} \frac{d^3\bar{p}}{(2\pi)^3} \mathrm{tr}[\hat{P}^{-1}\Gamma^0 M \hat{P}^{-1}\Gamma^0 M] + \mathcal{O}(\lambda^2), \tag{4.1}$$

where $\hat{P}$ is the same as P, but with $B_1 \to B_1/\lambda^{2/3}$. The subscript IR means that the $\bar{p}$ integral is restricted to the IR region. It gives contributions proportional to $\lambda^2 \ln\lambda^2$.

Formula (4.1) is proved as follows. If we factor out a λ^2 and take λ to infinity inside the integrand of (3.4), we notice that the integral remains convergent in the ultraviolet region, but becomes divergent in the infrared region. Thus, when $\lambda \to \infty$ the infrared region provides dominant contributions that grow faster than λ^2. The first term of (3.4) does not give dominant contributions; indeed, in the IR region it is safe to take $\lambda \to \infty$ inside the logarithm. Instead, it is not safe to do the same in the second term of (3.4). This explains formula (4.1).

Now we calculate the dominant contributions. It is convenient to work in the basis where the matrix B_0 is diagonal:

$$B_0^{ab} = \delta^{ab}\, \mathrm{diag}(b_L^a, b_L^a, b_{uR}^a, b_{dR}^a). \tag{4.2}$$

Here the indices u, d refer to the "up" and "down" quarks of the family labeled by the index a (so they mean c, s and t, b for $a = 2$ and 3, respectively). In this basis the propagator is diagonal in a, b. The trace is invariant under rotations, so it can be calculated orienting $\bar{p}$ along the z direction and rewriting the result as a scalar. With this choice, the propagator is diagonal in all indices, and the trace can be easily calculated. We obtain a linear combination of integrals of the form

$$\int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} \int_{\mathrm{IR}} \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{ip_4 + X} \frac{1}{ip_4 + Y}.$$

The integral in p_4 can be calculated using the residue theorem. The $\bar{p}$ integrand, which is quadratic in M, is at most quadratic in the components of $\bar{p}$ and can be symmetrized using

$$\bar{p}^{\,i}\bar{p}^{j} \to \frac{\delta_{ij}}{3}\bar{p}^2.$$

We obtain a linear combination of $\bar{p}$ integrals of the form

$$\int_{\mathrm{IR}} \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{|\bar{p}|(\frac{1}{\lambda^{2/3}} + \frac{\bar{p}^2}{\Lambda_L^2} b_{xy})} \sim \frac{\Lambda_L^2}{3(2\pi)^2 b_{xy}} \ln\lambda^2,$$

where b_{xy} is the sum of two entries of the matrix B_0. The result is a linear combination of contributions of the form

$$N_c \Lambda_L^2 c_{xy} \frac{|(\Gamma^0 M)_{xy}|^2}{b_{xy}} \lambda^2 \ln\lambda^2, \tag{4.3}$$

where c_{xy} is a non-negative numerical factor. Converting the result to a generic basis, where B_0 is not necessarily diagonal, we find

$$W(\lambda M) = W_{\mathrm{dom}}(\lambda M) + \mathcal{O}(\lambda^2),$$

with

$$W_{\mathrm{dom}}(\lambda M) = \frac{N_c \Lambda_L^2}{6(2\pi)^2} \lambda^2 \ln\lambda^2 \int_0^\infty d\xi\, \mathrm{tr}\Big[\mathcal{S} e^{-\xi B_0} \mathcal{S} e^{-\xi B_0} + \frac{2}{3}\mathcal{H}_i e^{-\xi B_0} \mathcal{H}_i e^{-\xi B_0} + \frac{2}{3}\mathcal{K}_i e^{-\xi B_0}\mathcal{K}_i e^{-\xi B_0} + \frac{1}{3}\mathcal{G}_i e^{-\xi B_0}\mathcal{G}_i e^{-\xi B_0}\Big], \tag{4.4}$$

where $\mathcal{S}$, $\mathcal{H}_i$, $\mathcal{K}_i$, $\mathcal{G}_i$ are matrices obtained from $\Gamma^0 M$ dropping all entries that are not S, H_i, K_i, and $G_i \equiv 2iL_{0i} - \varepsilon_{ijk}L_{jk}$, respectively.

The dominant contribution (4.4) of $W(\lambda M)$ is positive definite in the M entries that it contains. Indeed, recalling that B_0 and $\Gamma^0 M$ are Hermitian, the integrand is the sum of terms of the form

$$\mathrm{tr}\,[(e^{-\xi B_0/2}\mathcal{M} e^{-\xi B_0/2})(e^{-\xi B_0/2}\mathcal{M} e^{-\xi B_0/2})^\dagger],$$

which are positive definite. Thus, the effective potential grows in all directions on which $W_{\mathrm{dom}}(\lambda M)$ depends.

However, $W_{\mathrm{dom}}(\lambda M)$ does not depend on all M entries. Precisely, it does not contain H_0, K_0, and L_{0i} (in the basis L_{0i}-G_i). Thus, the dominant contributions of $\mathcal{V}(\lambda M)$ that depend on such entries are at most of order λ^2, as are the contributions coming from the tree-level potential $W_2(\lambda M)$. Now, $\mathcal{V}(\lambda M)$ is uniquely determined, while $W_2(\lambda M)$ contains free parameters. If we assume that the $W_2(\lambda M)$ coefficients that multiply the terms containing H_0, K_0, and L_{0i} satisfy suitable inequalities, which define a certain domain $\mathcal{D}'$ in parameter space, the total leading-order potential $W(\lambda M)$ grows in all directions. Then, by continuity, it must have a minimum somewhere. This is not the end of our argument, since the minimum could still be trivial.

Let us investigate the point $M = 0$. It is certainly a stationary point, since the first derivatives of both $W_2(M)$ and $\mathcal{V}(M)$ vanish at $M = 0$. Moreover, the second derivatives of $\mathcal{V}(M)$ vanish at $M = 0$ by construction, so the second derivatives of $W(M)$ at $M = 0$ coincide with those of $W_2(M)$. Thus, choosing some free parameters of $W_2(M)$ to be negative, or smaller that certain bounds, we can define a domain $\mathcal{D}''$ in parameter space where the origin $M = 0$ is not a local minimum.

The domains $\mathcal{D}'$ and $\mathcal{D}''$ have a nonempty intersection $\mathcal{D}$. Indeed, it is sufficient to choose a $\mathcal{D}''$ region defined by bounds on the $W_2(M)$ parameters that are unrelated to H_0, K_0, and L_{0i}.

In the domain $\mathcal{D}$, the potential $W(M)$ grows in every direction for large M; therefore it has a minimum. Moreover, the minimum cannot be the origin, but it is located somewhere at $M \neq 0$. This means that the symmetry-breaking mechanism necessarily takes place in $\mathcal{D}$, as we wanted to prove.

Phase diagram — Varying the parameters contained in W_2, the absolute minimum moves around and we can study the phase diagram of the theory.

So far, we have rigorously proved that the theory has an unbroken phase and a broken phase. We still do not know much about the minimum of the broken phase. To make contact with experiments it is necessary to prove that there exists a broken phase that (i) preserves rotations and CPT and (ii) allows us to recover Lorentz symmetry at low energies. In this Lorentz phase only the fields S may have nontrivial expectation values, while H_μ, K_μ, and $L_{\mu\nu}$ must vanish at the minimum.

A number of technical difficulties prevent us from rigorously proving that the Lorentz phase exists in the most general case. However, we give a number of results providing evidence that it does exist in several particular cases of interest.

Let us assume for the moment that tuning the $W_2(M)$ parameters we can obtain every configuration of expectation values we want. Then the theory has a rich phase diagram. Besides the unbroken phase and the Lorentz phase, we have broken phases where Lorentz symmetry is violated also at low energies, namely some vector fields or tensor fields acquire nontrivial expectation values. Among these phases, we have (i) a phase where invariance under rotations is preserved, but CPT is broken, if $H_i = K_i = L_{\mu\nu} = 0$ at the minimum and H_0, K_0 have nontrivial expectation values; (ii) a phase where rotational invariance is broken, but CPT is preserved, if $H_\mu = K_\mu = 0$, but $L_{\mu\nu} \neq 0$; (iii) a phase where rotational invariance and CPT are both broken, if H_i, K_i have nontrivial expectation values. Note that there is no Lorentz-violating phase where CPT and invariance under rotations are both preserved.

At the leading order of the $1/N_c$ expansion it is consistent to project onto the scalar sector putting $H_\mu = K_\mu = L_{\mu\nu} \equiv 0$, because such fields are generated back by renormalization only at subleading orders. Equivalently, adding quadratic terms proportional to H^2, K^2, and L^2 to the tree-level potential $W_2(M)$, multiplied by arbitrarily large positive coefficients, it is possible to freeze the vector and tensor directions at the leading order. Then, the expectation values of H_μ, K_μ, and $L_{\mu\nu}$ become arbitrarily small and may be assumed to be zero for all practical purposes. This argument can partially justify the existence of the Lorentz phase and the projection onto the scalar subsector, which we advocate in the next sections. However, we stress that it works only at the leading order of the $1/N_c$ expansion.

Lorentz invariant local minimum

We begin to study the Lorentz phase investigating when the point

$$S = S_0 \neq 0, \qquad H_\mu = K_\mu = L_{\mu\nu} = 0 \tag{4.5}$$

is a local minimum. Again, we consider a neighborhood of $B_1 = 1$ (which allows us to work at precisely $B_1 = 1$, by continuity) and restrict the tree-level couplings of $W_2(M)$ to a suitable domain in parameter space.

Consider the first derivatives $\partial W/\partial M$ calculated at (4.5). Clearly, both $\partial W/\partial H$ and $\partial W/\partial K$ vanish, since they are CPT odd, and $\partial W/\partial L_{0i}$ and $\partial W/\partial L_{ij}$ vanish by invariance under rotations. Instead, $\partial W/\partial S = \partial W_2/\partial S + \partial \mathcal{V}/\partial S$ can be made to vanish adjusting the free parameters that multiply the S-$\bar{S}$-quadratic terms contained in W_2. Observe that all other W_2 parameters remain arbitrary, a fact that will be useful in a moment.

Now we study the second derivatives $\partial^2 W/\partial M^2$ at the point (4.5). Assume that the matrix

$$\begin{pmatrix} \frac{\partial^2 W}{\partial S^2} & \frac{\partial^2 W}{\partial S \partial \bar{S}} \\ \frac{\partial^2 W}{\partial \bar{S} \partial S} & \frac{\partial^2 W}{\partial \bar{S}^2} \end{pmatrix} \tag{4.6}$$

is positive definite at the minimum. The derivatives $\partial^2 W/(\partial H \partial S)$ and $\partial^2 W/(\partial K \partial S)$ vanish, since they are CPT odd. The derivatives $\partial^2 W/(\partial S \partial L_{0i})$ and $\partial^2 W/(\partial S \partial L_{ij})$ vanish by rotational invariance. The matrix

$\partial^2 W/\partial M^2$ is then block diagonal. One block is (4.6), and the second block does not contain derivatives with respect to S. The second block can be made positive definite assuming that the W_2 parameters that have remained arbitrary satisfy suitable inequalities.

We still have to prove that (4.6) is positive definite. This calculation is rather involved in a generic situation. We study this problem in a number of special cases.

V. THE CASE OF ONE GENERATION REVISITED

While experiments tell us that the matrix B_1 is close to the identity, we have no such information about the matrix B_0. Actually, its entries could differ from one another by several orders of magnitude, so in principle the matrix B_0 should be kept generic. However, calculations with a generic B_0 are rather involved, so we have to make some simplifying assumptions. In this section we reconsider the case of one generation (which we assume to be the third one, for future use) in the scalar sector with $B_0 = B_1 = 1$ [9]. We also prove some statements that were not proved in [9], for example, that the minimum is absolute and unique in the scalar sector.

In the scalar sector, $H_\mu = K_\mu = L_{\mu\nu} = 0$. We have

$$\Gamma^0 M = \begin{pmatrix} 0 & \tau^\dagger \\ \tau & 0 \end{pmatrix},$$

where τ is a 2×2 matrix, with indices of $SU(2)_L$ to the right and indices of $SU(2)_R$ to the left. The fermions are organized as $\Psi = (Q_L, Q_R)$ and $Q = (t, b)$.

If we assume the axial symmetry $U(1)_A$, besides $SU(2)_L$ and $U(1)_Y$, the leading-order potential is

$$W(M) = \Lambda_L^2 \operatorname{tr}[\tau\tau^\dagger C] - N_c \int \frac{d^4p}{(2\pi)^4}\Big(\ln\det(\mathbb{1} - P^{-1}\Gamma^0 M) + \frac{1}{2}\operatorname{tr}[P^{-1}\Gamma^0 M P^{-1}\Gamma^0 M]\Big), \tag{5.1}$$

where C is a diagonal constant matrix, $C = \operatorname{diag}(c_t, c_b)$.

We use the "polar" decomposition (A2) to write

$$\tau = \tilde{U}_R D U_L, \qquad D = \begin{pmatrix} d_t & 0 \\ 0 & d_b \end{pmatrix},$$

and the diagonalization (A5) for $N = \Gamma^0 M$. See Appendix A for notation and details. At $B_0 = B_1 = 1$, the one-loop correction to the potential does not depend on the diagonalizing matrix U of (A5), but only on the entries d_t, d_b of D. It is useful to define the four-vector

$$p' = \Big(p^0, \bar{p}\Big(1 + \frac{\bar{p}^2}{\Lambda_L^2}\Big)\Big), \tag{5.2}$$

because the integrand of (5.1) is "Lorentz invariant" in this four-vector, therefore, it can be calculated at $\bar{p} = 0$. Writing

$$\tilde{U}_R = \sqrt{1 - |u|^2} + iu\sigma_+ + i\bar{u}\sigma_-,$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, $|u| \le 1$, we obtain the potential

$$W(M) = \Lambda_L^2(d_t^2 c_t + d_b^2 c_b) - \Lambda_L^2 |u|^2 (d_t^2 - d_b^2)(c_t - c_b) + 2V(d_t^2) + 2V(d_b^2),$$

where

$$V(r) \equiv -N_c \int \frac{d^4p}{(2\pi)^4}\Big(\ln\Big(1 + \frac{r}{(p')^2}\Big) - \frac{r}{(p')^2}\Big).$$

This function is non-negative, monotonically increasing and convex. Indeed, for $r > 0$,

$$V'(r) = rN_c \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p')^2((p')^2 + r)} > 0,$$

$$V''(r) = N_c \int \frac{d^4p}{(2\pi)^4} \frac{1}{((p')^2 + r)^2} > 0.$$

Moreover, $V(0) = V'(0) = 0$ and $V''(0) = +\infty$.

Let us find the stationary points of $W(M)$ and study the Hessians there. We denote the values of $d_{t,b}$ at the stationary point with $m_{t,b}$ and identify them with the top and bottom masses, respectively.

We assume $c_t \neq c_b$, because, as we prove below, the case $c_t = c_b$ is not physically interesting. Then we find the following stationary points:

(1) $u = 0$, while $m_{t,b}$ do not vanish and solve the gap equation

$$\Lambda_L^2 c_i + 2V'(m_i^2) = 0; \tag{5.3}$$

(2) $u = 0$, while one of $m_{t,b}$ vanishes and the other one solves the gap Eq. (5.3);

(3) $|u|^2 = 1/2$ and $m_t = m_b \neq 0$ solve

$$0 = \Lambda_L^2(c_t + c_b) + 4V'(m_t^2); \tag{5.4}$$

(4) $m_t = m_b = 0$.

Now we analyze the Hessian at each stationary point.

(1) Because of (5.3) and $c_t \neq c_b$, and since V' is monotonic, m_t and m_b cannot coincide. The Hessian is diagonal and strictly positive:

$$\frac{\partial^2 W}{\partial d_i^2}\Big|_{\min} = 8m_i^2 V''(m_i^2) > 0,$$

$$\frac{\partial^2 W}{\partial |u|^2}\Big|_{\min} = 2(m_t^2 - m_b^2)(V'(m_t^2) - V'(m_b^2)) > 0.$$

This point is a local minimum. It exists if and only if the gap Eq. (5.3) has a solution, which occurs if and only if both c_t and c_b are negative.

(2) If m_b vanishes then the Hessian is diagonal and

$$\frac{\partial^2 W}{\partial d_t^2}\Big|_{\min} = 8m_t^2 V''(m_t^2),$$
$$\frac{\partial^2 W}{\partial d_b^2}\Big|_{\min} = 2\Lambda_L^2 c_b,$$
$$\frac{\partial^2 W}{\partial |u|^2}\Big|_{\min} = \Lambda_L^2 m_t^2 (c_b - c_t).$$

This point is a local minimum if and only if

$$c_b > 0, \qquad c_b > c_t.$$

(3) The determinant of the Hessian is negative,

$$\det H = -32\Lambda_L^4 m^4 (c_t - c_b)^2 V''(m^2),$$

so this point cannot be the minimum.

(4) From the analysis of the previous section, we already know that the origin is a local minimum if and only if both c_i's are positive.

The physically interesting case is clearly (1). Since both c_t and c_b are negative, we may assume

$$c_t < c_b < 0. \tag{5.5}$$

Then point (1) is the unique local minimum in the scalar sector. The theorem proved in the previous section (existence of the absolute minimum) allows us to conclude that point (1) is also the absolute minimum of $W(M)$ in the scalar sector. Moreover, the argument of the previous section about the existence of a local minimum ensures that if the other tree-level couplings of $W_2(M)$ belong to a suitable region in parameter space, point (1) is also a local minimum in the full M space.

Note that these arguments still do not prove that there exists a phase where point (1) is the absolute minimum in the full M space.

Because of the symmetries of the potential, its minimum is not just a point, but a geometric locus of points. By means of a $SU(2)_L \times U(1)_Y \times U(1)_A$ transformation, we can choose the physical minimum

$$\tau_0 = \begin{pmatrix} m_t & 0 \\ 0 & m_b \end{pmatrix}, \tag{5.6}$$

which preserves $U(1)_Q$.

The other cases are not physically interesting. For example, if either c_t or c_b vanish or are positive the absolute minimum is either point (2) or the origin $M = 0$. Then at least one mass vanishes. Instead, if $c_t = c_b$, the theory is invariant under the custodial symmetry $SU(2)_R$ and $m_{t,b}$ either vanish or solve the gap Eq. (5.3). Using $SU(2)_R \times SU(2)_L \times U(1)_Y \times U(1)_A$, we can always make the minimum have the form (5.6), but either some masses vanish or coincide.

We conclude that there is a (unique, up to exchange of m_t and m_b) phase such that $W(M)$ has the absolute minimum (5.6) in the scalar sector and point (5.6) is also a local minimum in the full M space.

VI. THREE GENERATIONS

Now we study the case of three generations, focusing again on the scalar sector and still assuming $B_0 = B_1 = 1$. We look for evidence that the Lorentz phase exists. Assuming again axial symmetry, the potential $W(M) = W_2(M) + \mathcal{V}(M)$ has

$$W_2(M) = \Lambda_L^2 \sum_{mnabcd} S_{mn}^{ab} \bar{S}_{mn}^{cd} C_m^{abcd}, \qquad \mathcal{V}(M) = 2\sum_i V(d_i^2), \tag{6.1}$$

where C_m^{abcd} are constants. The correction $\mathcal{V}(M)$ is calculated using the polar decomposition (A5) for $N = \Gamma^0 M$ and noting that the integrand is independent of U. Moreover, it is Lorentz invariant in the four-vector (5.2), so it can be easily calculated at $\bar{p} = 0$ and later rewritten in covariant form.

As before, $\mathcal{V}(M)$ is positive definite, monotonically increasing, and convex. Its minimum is $M = 0$, so the minimum of $W(M)$ is determined by the free parameters C_m^{abcd} contained in $W_2(M)$.

Illustrative example — To begin with, it is worth considering the simple case

$$C_m^{abcd} = H^{bd} C_m^{ca}, \tag{6.2}$$

where H and C_m are Hermitian matrices.

Define the matrices $\mathcal{H}_{nn'}^{ab} = \delta_{nn'} H^{ab}$, $\mathcal{C}_{mm'}^{ab} = \delta_{mm'} C_m^{ab}$. Using the polar decomposition (A2), we write

$$V(SS^\dagger) = \tilde{U}_R \,\mathrm{diag}(V(d_1^2), \cdots, V(d_n^2)) \tilde{U}_R^\dagger,$$
$$V(S^\dagger S) = U_L^\dagger \,\mathrm{diag}(V(d_1^2), \cdots, V(d_n^2)) U_L.$$

The potential reads

$$W(M) = \mathrm{tr}[\Lambda_L^2 \mathcal{H} S^\dagger \mathcal{C} S + 2V(S^\dagger S)] = \mathrm{tr}[\Lambda_L^2 \mathcal{C} S \mathcal{H} S^\dagger + 2V(SS^\dagger)]. \tag{6.3}$$

The stationary points must satisfy

$$\frac{\partial W(M)}{\partial S} = \Lambda_L^2 \mathcal{H} S^\dagger \mathcal{C} + 2V'(S^\dagger S) S^\dagger = 0, \qquad \frac{\partial W(M)}{\partial S^\dagger} = \Lambda_L^2 \mathcal{C} S \mathcal{H} + 2V(SS^\dagger) S = 0. \tag{6.4}$$

We may assume that S is nonsingular at the minimum. Indeed, it is not difficult to prove, following the example treated before, that, if the free parameters contained in W_2 satisfy suitable inequalities, the singular configurations can be stationary points, but not minima.

Defining

$$H_\Delta = U_L \mathcal{H} U_L^\dagger, \qquad C_\Delta = \tilde{U}_R^\dagger \mathcal{C} \tilde{U}_R,$$

Eq. (6.4) becomes

$$-\Lambda_L^2 H_\Delta D C_\Delta = -\Lambda_L^2 C_\Delta D H_\Delta = 2V'(D^2)D = \text{diagonal}.$$

We see that the matrices $\tilde{H}_D = \sqrt{D} H_\Delta \sqrt{D}$ and $\tilde{C}_D = \sqrt{D} C_\Delta \sqrt{D}$ are Hermitian and commute with each other, so they can be simultaneously diagonalized with a unitary transformation. Moreover, their product $\tilde{H}_D \tilde{C}_D$ is itself diagonal. This means that both $\tilde{H}_D$ and $\tilde{C}_D$ are already diagonal. In turn, also H_Δ and C_Δ are diagonal, so U_L and $\tilde{U}_R$ must be matrices that diagonalize $\mathcal{H}$ and $\mathcal{C}$, respectively. The most general such matrices are

$$U_L = \begin{pmatrix} U_L' & 0 \\ 0 & U_L' \end{pmatrix} U_2, \qquad \tilde{U}_R = \begin{pmatrix} \tilde{U}_{Ru} & 0 \\ 0 & \tilde{U}_{Rd} \end{pmatrix}, \tag{6.5}$$

where $U_L' \in U(3)$ and $\tilde{U}_{Ru}, \tilde{U}_{Rd} \in \tilde{U}_\ell(3)$ are unitary matrices that rotate the generations, but are inert on the $SU(2)_R$ and $SU(2)_L$ indices m and n, while $U_2 \in SU(2)$ acts on the indices m, n, but is inert on the generations. The reason why U_L has this factor U_2 is that $\mathcal{H}$ has two coinciding diagonal blocks which can be freely rotated. We could factor out the unitary diagonal matrices that multiply U_L' to the left, as we do for the unitary diagonal matrices that multiply $\tilde{U}_{Ru}$ and $\tilde{U}_{Rd}$ to the right, but we do not need to.

We conclude that the nonsingular stationary points have the form

$$S_{\min} = \begin{pmatrix} \tilde{U}_{Ru} & 0 \\ 0 & \tilde{U}_{Rd} \end{pmatrix} D \begin{pmatrix} U_L' & 0 \\ 0 & U_L' \end{pmatrix} U_2. \tag{6.6}$$

Arguing as before, these points are also global minima in the scalar sector and local minima in the full M space.

Now, observe that the kinetic and Yukawa terms of the action are invariant under $G_S \equiv U(3)_L \times U(3)_{Ru} \times U(3)_{Rd}$, if the auxiliary fields are transformed appropriately. The leading-order correction $\mathcal{V}(M)$ to the potential is also invariant under G_S, while the tree-level potential $W_2(M)$ breaks G_S explicitly. The G_S and $SU(2)_L \times U(1)_Y$ transformations allow us to turn the minimum (6.6) into the diagonal form $S_{\min} = D$, which preserves $U(1)_Q$. Once we have done this, the diagonal entries of D are the quark masses. However, we discover that the CKM matrix is trivial, namely there is no mixing among generations. Thus, our assumption (6.2) is phenomenologically too restrictive.

In the special case

$$C_m^{abcd} = c_m \delta^{ac} \delta^{bd}, \tag{6.7}$$

the theory is completely invariant under the global symmetry G_S, which is also preserved by renormalization. The minimum of the effective potential does break this symmetry (because it is diagonal in the space of generations, but not proportional to the identity). However, with the choice (6.7) the model predicts only two different quark masses, since W_2 contains only two free parameters, c_t and c_b.

The results obtained in this example generalize immediately to an arbitrary number of generations; with a choice like (6.2) the minimum can always be put into a diagonal form, with no mixing among generations.

A source of mixing among generations is provided by the matrix B_0, which was taken to be proportional to the identity in this section. Now we show that there is enough room for a nontrivial CKM matrix *even if* we still assume $B_0 = 1$. Indeed, it is sufficient to take a less symmetric tree-level potential $W_2(M)$.

Emergence of the CKM matrix and mixing among generations — Now we show that the emergence of the CKM matrix can be explained taking

$$C_m^{abcd} = H_m^{bd} C_m^{ca}, \tag{6.8}$$

where H_m and C_m are again Hermitian matrices. We still assume $B_0 = B_1 = 1$. Defining the matrices $\mathcal{H}_{1nn'}^{ab} = \delta_{nn'} H_1^{ab}$, $\mathcal{H}_{2nn'}^{ab} = \delta_{nn'} H_2^{ab}$, $\mathcal{C}_{1mm'}^{ab} = \delta_{m1}\delta_{m'1} C_1^{ab}$, $\mathcal{C}_{2mm'}^{ab} = \delta_{m2}\delta_{m'2} C_2^{ab}$, now the potential reads

$$W(M) = \mathrm{tr}[\Lambda_L^2 S \mathcal{H}_1 S^\dagger \mathcal{C}_1 + \Lambda_L^2 S \mathcal{H}_2 S^\dagger \mathcal{C}_2 + 2V(S^\dagger S)]. \tag{6.9}$$

The stationary points are the solutions of

$$\begin{aligned} \tilde{H}_{1D}\tilde{C}_{1D} + \tilde{H}_{2D}\tilde{C}_{2D} &= \tilde{C}_{1D}\tilde{H}_{1D} + \tilde{C}_{2D}\tilde{H}_{2D} \\ &= -\frac{2}{\Lambda_L^2} V'(D^2) D^2 \\ &= \text{diagonal}, \end{aligned}$$

where

$$\begin{aligned} \tilde{H}_{mD} &= \sqrt{D} U_L \mathcal{H}_m U_L^\dagger \sqrt{D}, \\ \tilde{C}_{mD} &= \sqrt{D} \tilde{U}_R^\dagger \mathcal{C}_m \tilde{U}_R \sqrt{D}, \qquad m = 1, 2. \end{aligned}$$

If we search for a solution of the form

$$S_{\min} = \begin{pmatrix} \tilde{U}_{Ru} & 0 \\ 0 & \tilde{U}_{Rd} \end{pmatrix} D \begin{pmatrix} U_{Lu} & 0 \\ 0 & U_{Ld} \end{pmatrix} U_2 \tag{6.10}$$

and argue as before, we find that $U_{Lu}, U_{Ld} \in U(3)$, $\tilde{U}_{Ru}$, $\tilde{U}_{Rd} \in \tilde{U}_\ell(3)$ must be matrices that diagonalize H_1, H_2, C_1, C_2, respectively.

At this point we can proceed as usual; the invariance of the rest of the action under phase transformations and $SU(2)_L \times U(1)_Y \times G_S$ allows us to turn the minimum into the form

$$S'_{\min} = \begin{pmatrix} 1 & 0 \\ 0 & C_{\mathrm{KM}} \end{pmatrix} D, \tag{6.11}$$

which preserves $U(1)_Q$, where C_{KM} is the CKM matrix. This stationary point can describe the properties of the standard model at low energies.

We have only proved that (6.11) belongs to the set of extremal points of the potential. Strictly speaking, there could be other extrema that are not block diagonal and therefore spontaneously break also charge conservation.

If we take the most general potential (6.1) every minimum that preserves $U(1)_Q$ can be cast into the form (6.11). Indeed, $U(1)_Q$ conservation means that the charged S entries, which are S_{12}^{ab} and S_{21}^{ab}, vanish; therefore the minimum is block-diagonal. Then it can be turned to the form (6.11) arguing as before, namely using invariance under phase transformations and $SU(2)_L \times U(1)_Y \times G_S$.

Finally, let us comment about the case $B_0 \neq 1$. If B_0 is not diagonal it can be diagonalized using $SU(3)_L \times SU(3)_{Ru} \times SU(3)_{Rd}$. Then we cannot use such transformations to turn (6.10) into the form (6.11). We can only simplify (6.10) by means of (eight) phase transformations. So, the Lorentz violation predicts more mixing among generations besides the CKM matrix. It also predicts mixing among leptons. If leptons have a nondiagonal matrix $B_{0\ell}$, we can use the freedom we have to diagonalize it, but then the lepton mass matrix remains nondiagonal.

If both B_0 and B_1 are different from the identity, we can diagonalize only one of them for each particle.

VII. *CPT* VIOLATING LOCAL MINIMA

In this section we want to show that the effective potential may also give nontrivial expectation values to the vector and tensor fields H_μ, K_μ, $L_{\mu\nu}$. For simplicity, we assume $B_0 = B_1 = 1$ and concentrate on the vector H_μ.

The most general tree-level potential with one generation is

$$W_2(M) = \Lambda_L^2 (c_1 \operatorname{tr}[H_0]^2 + c_2 \operatorname{tr}[H_i]^2 + c_3 \operatorname{tr}[H_0^2] + c_4 \operatorname{tr}[H_i^2]),$$

where $c_{1\text{-}4}$ are constants. After simple manipulations, the one-loop correction can be expressed in the form

$$\mathcal{V}(M) = -N_c \int \frac{d^4 p}{(2\pi)^4} \left[\ln \det(A + \sigma_i B_i) - \frac{4 \operatorname{tr}[H_i^2](\bar{p}')^2}{3(p'^2)^2} \right],$$

where

$$A = 1 + \frac{1}{(p')^2}(i p_4 H_0 - p_i' H_i),$$

$$B_i = \frac{1}{(p')^2}(-i p_4 H_i + p_i' H_0 - i p_j' H_k \varepsilon_{ijk}).$$

However, since H_0 and H_i are 2×2 matrices, it is still difficult to evaluate $\mathcal{V}(M)$ explicitly. If we restrict to the case of a single fermion, we can perform the calculation to the end. We find

$$W_2(M) = \Lambda_L^2 (c_1' H_0^2 + c_2' H_i^2),$$

$$\mathcal{V}(M) = \frac{N_c \Lambda_L^4}{7560 h \pi^2} [630 h^3 \ln(v^2 + 1) - v^3 (140 v^6 + 360 v^4 - 630 h v^3 + 252 v^2 - 945 h v + 1260 h^2)], \tag{7.1}$$

with

$$v = \frac{2^{1/3} \Delta^{2/3} - 2 \cdot 3^{1/3}}{6^{2/3} \Delta^{1/3}},$$

$$\Delta = \sqrt{12 + 81 h^2} + 9h,$$

$$h = \sqrt{\frac{H_i^2}{\Lambda_L^2}}.$$

The one-loop correction $\mathcal{V}(M)$ does not depend on H_0, so to have a minimum we must assume $c_1' > 0$. As a function of h, $\mathcal{V}(M)$ is monotonic and convex, and $\mathcal{V}(M) = \mathcal{O}(H^4)$ in a neighborhood of the origin. Thus, we have two phases:

(1) the unbroken phase has $c_1' > 0$, $c_2' > 0$;

(2) the broken phase has

$$c_1' > 0, \qquad c_2' < 0,$$

where H has a nontrivial expectation value. Here the minimum of the effective potential spontaneously breaks invariance under boosts, rotations, and *CPT*.

In the simple example just studied, the potential $\mathcal{V}(M)$ does not depend on H_0. The reason is that H_0 can be reabsorbed with an imaginary translation of p_4. Observe that H_i cannot be reabsorbed away. Indeed, although the integrand depends only on the sum $p_i' + H_i$, we cannot translate p_i', because the integral is in p_i not in p_i'. On the other hand, only one p_4 translation is available, so we expect that with more fermions, where H_0 is a matrix, there exist broken phases where $H_i = 0$ but some entries of the H_0-matrix get nontrivial expectation values. In such phases *CPT* and boosts are broken, but rotations are preserved.

VIII. LOW-ENERGY EFFECTIVE ACTION

In this section we study the low-energy effective action in the Lorentz phase. We work at the leading order of the $1/N_c$ expansion, at $B_0 = B_1 = 1$, and focus on the third generation. As usual, we first turn the gauge-field interactions off and turn them back on at a second stage. We study the spectrum of composite bosons, derive a number predictions, and show that the model is compatible with the experimental data. For the moment, we can concentrate on the scalar sector.

To keep the presentation readable, at first we assume not only invariance under $SU(2)_L \times U(1)_Y \times U(1)_B$, but also the axial symmetry $U(1)_A$. With this assumption, however, the low-energy model is ruled out by experimental data.

It is straightforward to relax the assumption of axial symmetry at a second stage. We show that once $U(1)_A$ is explicitly broken full compatibility with data is achieved.

We refer to Sec. V for the notation. The total four-fermion Lagrangian is $\mathcal{L}_{\rm tot} = \mathcal{L}_q + \mathcal{L}_\ell$, where the quark and lepton contributions are

$$\mathcal{L}_q = \bar{\Psi}\left(i\Gamma^0 \mathbb{1}\partial_t + i\bar{\Gamma}\cdot\bar{\partial}\left(1-\frac{\bar{\partial}^2}{\Lambda_L^2}\right) - M\right)\Psi - \Lambda_L^2 \,\mathrm{tr}[\tau\tau^\dagger C] \qquad (8.1)$$

$$\mathcal{L}_\ell = \mathcal{L}_{\ell\rm kin} - \sum_{ab}(y^{ab}\tau_{2n}\bar{\ell}_R^a L_n^b + \bar{y}^{ba}\bar{L}_n^a \ell_R^b \bar{\tau}_{2n}), \qquad (8.2)$$

y^{ab} being constants, while $\Psi = ((t_L, b_L), (t_R, b_R))$. The form of $\mathcal{L}_\ell$ is justified as follows.

Since we are working in the leading order of the $1/N_c$ expansion, we have to calculate one-loop diagrams with circulating quarks. Thus, we can focus on four-fermion vertices that contain two quarks q and two leptons ℓ, or four quarks, and ignore the vertices that contain four leptons. Introducing auxiliary scalar fields τ and σ, as usual, we get Yukawa and potential terms of the form

$$-\tau q q - \sigma\ell\ell - \frac{a}{2}\tau^2 - b\tau\sigma - \frac{c}{2}\sigma^2.$$

The leading-order correction $\mathcal{V}$ to the potential depends only on τ, so the effective potential has the form

$$\mathcal{W}(\tau,\sigma) = \frac{a}{2}\tau^2 + b\tau\sigma + \frac{c}{2}\sigma^2 + \mathcal{V}(\tau).$$

Its extrema can also be found replacing σ with the solution $\sigma = -b\tau/c$ of its field equation, namely working with $\mathcal{W}(\tau, -b\tau/c)$. Therefore, we do not need to multiply the lepton bilinears $\ell\ell$ by independent auxiliary scalars σ. We can just multiply them by entries of τ and free parameters. Because of the symmetries we have assumed, (8.2) is the only form that is allowed. Moreover, using the polar decomposition on y^{ab} and performing unitary transformations on L^a and ℓ_R^a, we can diagonalize the matrices y^{ab}. Thus, from now on, we take $y^{ab} = \delta^{ab}\,\mathrm{diag}(y^a)$, with y^a real.

We expand around the minimum (5.6), writing $\tau = \tau_0 + \eta$. We first recall the leading contributions to the quadratic effective action Γ_2 [9], namely

$$\Gamma_2 = -N_c\sum_{ij}\eta_{ij}(\partial^2 + 2m_j^2)f_{ij}\bar{\eta}_{ij} - N_c\sum_{ij} m_i m_j f_{ij}(\eta_{ij}\eta_{ji} + \bar{\eta}_{ij}\bar{\eta}_{ji}) \qquad (8.3)$$

(the constants f_{ij} being defined in Appendix B and the integration over spacetime being understood), which gives the following propagating fields: (i) two neutral massive scalars $\varphi_{1,2}$ and a charged massive scalar φ,

$$\varphi_1 = \sqrt{2N_c f_{tt}}\,\mathrm{Re}\,\eta_{tt},$$
$$\varphi_2 = \sqrt{2N_c f_{bb}}\,\mathrm{Re}\,\eta_{bb},$$
$$\varphi = \sqrt{\frac{N_c f_{tb}}{m_t^2 + m_b^2}}(m_b\eta_{tb} + m_t\bar{\eta}_{bt}),$$

with squared masses

$$m_1^2 = 4m_t^2, \qquad m_2^2 = 4m_b^2, \qquad m^2 = 2(m_t^2 + m_b^2),$$

respectively; (ii) the Goldstone bosons associated with the spontaneously broken generators of $SU(2)_L \times U(1)_Y$, which are

$$\phi^+ = i\sqrt{\frac{N_c}{2f_W}}f_{tb}(m_t\eta_{tb} - m_b\bar{\eta}_{bt}),$$

$$\phi^0 = \sqrt{\frac{N_c}{f_Z}}(m_b f_{bb}\,\mathrm{Im}\,\eta_{bb} - m_t f_{tt}\,\mathrm{Im}\,\eta_{tt}),$$

and $\phi^- = \bar{\phi}^+$, where

$$f_W = \frac{f_{tb}}{2}(m_t^2 + m_b^2), \qquad f_Z = \frac{1}{2}(m_t^2 f_{tt} + m_b^2 f_{bb});$$

iii) a Goldstone boson

$$\tilde{\phi}^0 = \sqrt{\frac{N_c f_{bb} f_{tt}}{f_Z}}(m_b\,\mathrm{Im}\,\eta_{tt} + m_t\,\mathrm{Im}\,\eta_{bb}),$$

associated with the broken axial symmetry.

When gauge interactions are switched back on, the Goldstone bosons $\phi^{\pm,0}$ are eaten by the gauge fields. Then the gauge fields acquire squared masses

$$m_W^2 = N_c g^2 f_W, \qquad m_Z^2 = N_c\tilde{g}^2 f_Z. \qquad (8.4)$$

Including the covariant derivatives for $U(1)_Q$, the quadratic effective action Γ_2 becomes

$$\begin{aligned}\Gamma_2 = {} & \frac{1}{2}\sum_{i=1}^{2}[(\partial_\mu\varphi_i)(\partial^\mu\varphi_i) - m_i^2\varphi_i^2] \\ & + (\partial_\mu\bar{\varphi} - ieA_\mu\bar{\varphi})(\partial^\mu\varphi + ieA^\mu\varphi) \\ & - m^2\bar{\varphi}\varphi + \frac{1}{2}\partial_\mu\tilde{\phi}^0\partial^\mu\tilde{\phi}^0 \\ & + (\partial_\mu\phi^+ - m_W W_\mu^+)(\partial^\mu\phi^- - m_W W^{\mu-}) \\ & + \frac{1}{2}(\partial_\mu\phi^0 - m_Z Z_\mu)(\partial^\mu\phi^0 - m_Z Z^\mu),\end{aligned}$$

and it is invariant under the linearized gauge transformations

$$\begin{aligned}\delta W_\mu^\pm &= \partial_\mu C^\pm, \qquad \delta Z_\mu = \partial_\mu C^0, \\ \delta\phi^\pm &= m_W C^\pm, \qquad \delta\phi^0 = m_Z C^0.\end{aligned} \qquad (8.5)$$

Now we calculate the three-leg and four-leg terms Γ_3 and Γ_4 of the effective action. We focus on the terms proportional to factors of the form $\ln(\Lambda_L^2/m^2)$, where m

is a function of the masses, because they are numerically more important, in our approximation. We find (again, refer to Appendix B for the notation)

$$\Gamma_3 + \Gamma_4 = -2N_c \sum_{ijk} m_i f_{ijk}(\eta_{ij}\bar{\eta}_{kj}\eta_{ki} + \bar{\eta}_{ij}\eta_{kj}\bar{\eta}_{ki}) - N_c \sum_{ijkl} f_{ijkl}\eta_{ij}\bar{\eta}_{kj}\eta_{kl}\bar{\eta}_{il}. \quad (8.6)$$

Writing (8.3) and (8.6), we have omitted some terms that are numerically negligible. Basically, they do not contain the enhancing factor $\sim \ln \Lambda_L^2$. Examples of such terms are

$$\frac{N_c}{24\pi^2}(\partial_\mu \mathrm{Re}\eta_{tt})(\partial^\mu \mathrm{Re}\eta_{tt}), \qquad \frac{N_c m_t}{3(4\pi)^2}\eta_{tt}^3, \qquad \frac{N_c m_b}{3(4\pi)^2}\eta_{bb}^3, \qquad \frac{2N_c m_b}{(4\pi)^2}\eta_{tt}\eta_{tb}\eta_{bt}, \quad (8.7)$$

(using $m_t \gg m_b$). We can compare them with the smallest cubic term in (8.6), which is

$$-\frac{2N_c}{(4\pi)^2} m_b(\eta_{bj}\bar{\eta}_{kj}\eta_{kb} + \bar{\eta}_{bj}\eta_{kj}\bar{\eta}_{kb}) \ln\frac{\Lambda_L^2}{m_t^2}. \quad (8.8)$$

Numerically, with $\Lambda_L = 10^{14}$ GeV and using $m_t = 171.2$ GeV, $m_b = 4.2$ GeV, we find that the coefficient of the second term of (8.7) is about 13% of the coefficient of (8.8). All other terms of type (8.7) are suppressed by a factor $1/\ln(\Lambda_L^2/m_t^2)$, which is a 2%. In any case, these contributions are below our errors. Moreover, since $\ln(\Lambda_L^2/m_t^2)$ and $\ln(\Lambda_L^2/m_b^2)$ differ only by a 14%, we can also neglect their difference and replace m_b with m_t inside the logarithms. Finally, the recurring factor

$$\mathcal{N} \equiv \frac{N_c}{(4\pi)^2}\ln\frac{\Lambda_L^2}{m_t^2}$$

can be approximated to one up to a negligible 3%. However, we continue to write it down explicitly, to keep track of the Λ_L dependence.

Collecting Γ_2, Γ_3, and Γ_4, we get the low-energy scalar effective action

$$\Gamma \sim \mathcal{N}\mathrm{tr}[\partial_\mu\tau\partial^\mu\tau^\dagger + 2\tau_0^2\tau\tau^\dagger - \tau\tau^\dagger\tau\tau^\dagger], \quad (8.9)$$

which is a type II two Higgs doublet model (2HDM), namely a model with two Higgs doublets, where one doublet couples only to top quarks, while the other doublet couples only to bottom quarks and leptons.

Because of the assumed axial symmetry $U(1)_A$, the scenario explored so far is ruled out by data. Indeed, it predicts very light neutral Higgs bosons, such as the field φ_2 of mass $\sim 2m_b$ and the massless $U(1)_A$ Goldstone boson. These fields violate the present experimental lower bound on the mass of neutral Higgs bosons, which is 114 GeV [17]. This bound, established through the process $Z \to Zh \to Z\bar{b}b$, applies to our model. Indeed, take, for example, the field φ_2 as the Higgs-boson h. It is easy to check that although the vertex ZZh is suppressed by a factor m_b/m_t, the Yukawa coupling $h\bar{b}b$ is enhanced by the reciprocal factor m_t/m_b, so the process $Z \to Zh \to Z\bar{b}b$ is not suppressed with respect to one predicted by the minimal standard model.

Compatibility with data can be obtained breaking $U(1)_A$ explicitly.

Low-energy model compatible with data — It is easy to see that, because of $SU(2)_L \times U(1)_Y$ invariance, the $U(1)_A$ symmetry can be explicitly broken in a unique way by four-fermion vertices. Indeed, only one term can be added to the tree-level potential W_2, namely

$$\Delta W_2 = \tilde{m}_{12}^2 \mathrm{tr}[\tau\epsilon\tau^T\epsilon] + \tilde{m}_{12}^{*2}\mathrm{tr}[\tau^\dagger\epsilon\tau^*\epsilon], \quad (8.10)$$

where T denotes transposition, $\epsilon_{tt} = \epsilon_{bb} = 0$, $\epsilon_{tb} = -\epsilon_{bt} = 1$, and $\tilde{m}_{12}$ is a complex constant. The one-loop correction $\mathcal{V}$ is unaffected, therefore, still $U(1)_A$ symmetric. The term (8.10) displaces the minimum and changes the mass spectrum.

For simplicity, we take $\tilde{m}_{12}$ real. To bring the displaced minimum back to the form (5.6), we also modify the term

$$2\tau_0^2\tau\tau^\dagger$$

of (8.9) replacing τ_0^2 with a different diagonal matrix. With our approximations we find the low-energy type II 2HDM Lagrangian

$$\Gamma = \mathcal{N}\,\mathrm{tr}\Big[\partial_\mu\tau\partial^\mu\tau^\dagger + 2\tau_0^2\tau\tau^\dagger - \tau\tau^\dagger\tau\tau^\dagger - \frac{m_{12}^2 m_t m_b}{2(m_t^2 + m_b^2)}(\tau\epsilon\tau^T\epsilon + \tau^\dagger\epsilon\tau^*\epsilon - 2\epsilon\tau_0\epsilon\tau_0^{-1}\tau\tau^\dagger)\Big]. \quad (8.11)$$

Expanding τ as $\tau_0 + \eta$, we can first check that the minimum is still τ_0 and then work out the new spectrum. We find that, using $m_b \ll m_t$,

(i) the three Goldstone bosons $\phi^{\pm,0}$ associated with the $SU(2)_L \times U(1)_Y$ symmetry are unaffected,

(ii) the mass of the charged composite Higgs boson φ becomes

$$m_\varphi \sim \sqrt{2m_t^2 + m_{12}^2},$$

(iii) assuming also $m_b \ll m_{12}$, the masses of the neutral Higgs bosons φ_1 and φ_2 become

$$m_{12}, \qquad 2m_t,$$

which is which depending on whether $m_{12} > 2m_t$ or $m_{12} < 2m_t$,

(iv) the field $\tilde{\phi}^0$ acquires a mass equal to m_{12},

(v) the neutral fields (φ_1, φ_2) are rotated by an angle α, while all other fields preserve the expressions they had before.

Since four-fermion vertices are multiplied by $1/\Lambda_L^2$, the tree-level potential terms, such as (8.10), are proportional to Λ_L^2, which means that m_{12} is large. For m_{12} large the masses of all particles become compatible with data. Taking into account of our errors ($\pm$ 50%), even a Higgs mass predicted to be around $2m_t$ could in the end be more close to m_t, which is contained in the present mass range for Higgs boson.

Moreover, because of (i) the gauge-boson masses are unaffected and formulas (8.4) still hold. The Fermi constant and the parameter ρ are given by the relations [9]

$$\frac{1}{G_F} = \frac{N_c m_t^2}{4\pi^2\sqrt{2}} \ln\frac{\Lambda_L^2}{m_t^2}, \qquad \rho = \frac{\tilde{g}^2 m_W^2}{g^2 m_Z^2} \sim 1. \tag{8.12}$$

Formulas (8.12) provide two important checks of our model. The standard model provides no analogue of the first formula. At $\Lambda_L = 10^{14}$ GeV the first prediction turns out to be very precise. As far as ρ is concerned, the standard model predicts $\rho = 1$ up to radiative corrections, which matches experimental data very well. Our approach is consistent with this, but cannot be equally precise, because our theoretical errors are large.

So far, we have focused on the scalar sector and ignored the fields H_μ, K_μ, and $L_{\mu\nu}$. It is easy to prove, computing their two-point functions in the low-energy limit, that such fields do become propagating at some point. Moreover, the dominant contributions to their kinetic terms, namely the contributions proportional to $\ln\Lambda_L^2$, are Lorentz invariant. Thus, our model also predicts composite vectors and tensors at low energies. Nevertheless, it is unable to predict their masses, whose values can be changed at will adding quadratic terms proportional to H^2, K^2, and L^2 to the tree-level potential $W_2(M)$, multiplied by coefficients proportional to Λ_L^2. The basic reason is that in the Lorentz phase H_μ, K_μ, and $L_{\mu\nu}$ have trivial gap equations. Thus, we are free to assume that the masses of these fields are sufficiently large, in which case this subsector of our model is also compatible with data.

The limit — $m_{12} \to \infty$ The limit $m_{12} \to \infty$ is particularly interesting, because it gives the usual one-doublet model. The coefficient of m_{12}^2 in (8.11) must vanish in the limit, which requires

$$\tau = u\begin{pmatrix} H_2 & -H_1 \\ \kappa\bar{H}_1 & \kappa\bar{H}_2 \end{pmatrix}, \qquad \kappa = \frac{m_b}{m_t}, \qquad u^{-2} = (1+\kappa^2)\mathcal{N}. \tag{8.13}$$

Then we find a particular case of the usual Higgs Lagrangian, namely (using again $m_t \gg m_b$)

$$\Gamma_H = \partial_\mu H^\dagger \partial^\mu H - V(H), \qquad V(H) = 2m_t^2 H^\dagger H - u^2(H^\dagger H)^2. \tag{8.14}$$

From this formula, we can read: (i) the Higgs vacuum expectation value ($|H|_{\min} = v/\sqrt{2}$), which is (with $\Lambda_L = 10^{14}$ GeV)

$$v = \frac{m_t}{u}\sqrt{2} \sim 247 \text{ GeV},$$

(ii) the constant

$$\lambda = u^2 \sim 1,$$

and, consequently, (iii) the Higgs-boson mass, which is $2m_t$.

The Yukawa couplings are automatically correct. We have

$$\mathcal{L}_{\text{Yukawa}} = -\frac{m_t}{v}\sqrt{2}(\bar{t}_R\tilde{H}Q_L + \bar{Q}_L t_R \tilde{H}^\dagger) - \frac{m_b}{v}\sqrt{2}(H^\dagger \bar{b}_R Q_L + \bar{Q}_L b_R H) - \frac{\sqrt{2}}{v}\sum_{a=1}^{3} m_\ell^a (H^\dagger \bar{\ell}_R^a L^a + \bar{L}^a \ell_R^a H), \tag{8.15}$$

where $\tilde{H}_n = \varepsilon_{nq}H_q$ and $m_\ell^a = m_b y^a$. The lepton mass terms do not give new predictions, but just determine the Yukawa parameters y^a.

The one-doublet model (8.14) was already considered in [9], but not fully justified there (it was presented as a subsector of the model with $m_{12} = 0$). The limit $m_{12} \to \infty$ provides the missing justification for (8.14).

IX. NEUTRINO MASSES AND NEUTRINO OSCILLATIONS

Among the compatibility checks we can make, we mention neutrino oscillations. In this section we show that the minimal versions of our models cannot give masses to neutrinos and discuss alternative ways to explain neutrino oscillations.

First, we prove that the vertex

$$\frac{1}{\Lambda_L}(LH)^2 = \frac{1}{\Lambda_L}\sum_{a,b=1}^{3} Y_{ab}(L_m^{\alpha a}\varepsilon_{\alpha\beta}L_p^{\beta b})\varepsilon_{mn}H_n\varepsilon_{pq}H_q + \text{H.c.}, \tag{9.1}$$

which gives Majorana masses to the neutrinos when H is replaced by its expectation value, cannot be generated.

The vertex (9.1) breaks the conservation of $B - L$ by two units. However, the vacuum we are considering does not break $B - L$ spontaneously. Moreover, the global $B - L$ symmetry is anomaly-free in our model. The reason is that it is anomaly-free in the minimal standard model [18], and anomalies are unaffected by the Lorentz violation (see [8]). Finally, the $B - L$ symmetry cannot be explicitly violated in the model (2.1), because

Theorem 1 — all CPT invariant four-fermion vertices constructed with the fields of the minimal standard model preserve $B - L$.

This theorem is a simple generalization of a well-known property stating the same conclusion about Lorentz invariant four-fermion vertices [19]. We stress here that it is not necessary to assume Lorentz symmetry, because CPT is sufficient. The theorem can be proved writing down all four-fermion vertices that are invariant under $SU(2)_L \times U(1)_Y$ and using a property proved in Ref. [20] stating that all four-fermion vertices of the form $\ell\ell\ell\ell$ and $\ell\ell\ell^*\ell^*$ are CPT invariant and all four-fermion vertices of the form $\ell\ell\ell\ell^*$ are CPT violating, ℓ denoting a left-handed fermion.

For the sake of completeness, we write the structures of four-fermion vertices with nonvanishing $\Delta B = \Delta L$. They are

$$LQ_L^3, \qquad Q_L^2 u_R \ell_R, \qquad LQ_L u_R d_R, \qquad u_R^2 d_R \ell_R, \tag{9.2}$$

plus their Hermitian conjugates. They all have $|\Delta B| = |\Delta L| = 1$. Such vertices do not affect the effective potential at the leading order of the $1/N_c$ expansion.

The $B - L$ symmetry could be spontaneously broken at subleading orders. However, we are not going to explore this possibility here.

Were it present, the vertex (9.1) could explain neutrino masses with a scale Λ_L around 10^{14}–10^{15} GeV. However, it has been speculated [21,22] that in Lorentz-violating models neutrino masses may not be necessary to explain neutrino oscillations. We make some observations about this fact in the realm of our models.

In the minimal model (2.1), the energies of neutrinos with given momentum p are the eigenvalues of the matrix

$$\mathcal{H} = p\left(b_{\nu 1} + b_{\nu 0}\frac{p^2}{\Lambda_L^2}\right),$$

where $b_{\nu 1}$ and $b_{\nu 0}$ are constant Hermitian matrices. In the simple case of two generations, the mixing probability after traveling a distance ℓ is

$$P_{\text{mixing}} = \left(1 - \frac{(\text{tr}[\Delta\mathcal{H}\sigma_z])^2}{\Omega^2}\right)\sin^2\left(\frac{\ell\Omega}{2}\right),$$
$$\text{where } \Omega = \sqrt{2\,\text{tr}[\Delta\mathcal{H}^2] - (\text{tr}[\Delta\mathcal{H}])^2},$$

where $\Delta\mathcal{H}$ is $\mathcal{H}$ minus any contribution proportional to the identity matrix.

If was shown in Ref. [23] that several existing data about neutrino oscillations can be accounted for by the matrix $b_{\nu 1} - 1$. The values of its entries were determined to be around 10^{-17}–10^{-22}, which are compatible with our approach. A different class of massless models (with five parameters) was considered in Ref. [24] and shown to be unable to explain all combined data about neutrino oscillations. The models considered in Ref. [24] explore a region of parameter space that is absent in our approach, because they contain four CPT-violating parameters out of five. At present, the problem to construct massless Lorentz-violating models that are globally compatible with data is still open and challenging. We suggest that it may be considered in a fully CPT invariant framework first.

Higher-derivative corrections do not appear to be helpful here. If we wanted to explain neutrino oscillations using only $b_{\nu 0}$ (setting $b_{\nu 1} = 1$), we would find $b_{\nu 0} \gg 1$ by several orders of magnitude. We expect that large $b_{\nu 0}$ values are unlikely. The matrices b_0 have been studied in other sectors of the model, particularly quantum electrodynamics [16] and found to be small or at most of order 1. Thus, the effects of terms containing higher-space derivatives are expected to be negligible for neutrino oscillations. Nonminimal versions of our model can be considered and certainly have the chance to account for all data. Nevertheless, there is still hope that neutrino oscillations can be fully accounted for by the sole matrix $b_{\nu 1} - 1$ in the minimal scalarless model.

X. CONCLUSIONS AND OUTLOOK

In this paper we have studied the low-energy phenomenology of renormalizable CPT invariant standard model extensions that violate Lorentz symmetry at high energies. These models include operators of higher dimensions, in particular four-fermion vertices, and contain no elementary scalar fields. At the leading order of the large N_c expansion, a dynamical symmetry-breaking mechanism gives masses to fermions and gauge bosons and generates composite scalars. We have studied the effective potential and the phase diagram. A broken phase always exists. In general, it may break boosts, rotations, and CPT. We have given evidence that there exists a Lorentz phase, described the mixing among generations and the emergence of the CKM matrix.

The low-energy effective action in the Lorentz phase looks like a standard model with one or more Higgs doublets and possibly very heavy composite vectors and tensors. Not all parameters are free, but some are related by formulas induced by the high-energy model. For example, our approach gives a formula relating the Fermi constant, the top mass and the scale of Lorentz violation Λ_L. So far, our predictions are compatible with present data, within theoretical errors.

We have considered the minimal version of our Lorentz-violating standard model extensions and made certain assumptions to simplify calculations (such as $B_1 = B_0 = 1$). When such assumptions are relaxed new effects appear, such as lepton mixing and a more severe quark mixing. It would be interesting to explore these aspects further and study the low-energy Lagrangian with B_0 generic. Another topic for future investigations is to explore the lowest energies where we can find remnants of the Lorentz violation, then look for the effects that can be tested in existing or planned experiments. It would also be interesting to explore more general models and include right-handed neutrinos and elementary scalars.

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APPENDIX A: POLAR DECOMPOSITION AND DIAGONALIZATION OF MATRICES

In this appendix we review some definitions and results about the polar decomposition of matrices and their diagonalization. We present them in ways that are useful for the arguments of our paper.

Definition — Let $g \in U(n)$ be a unitary $n \times n$ matrix and h a diagonal unitary matrix, namely an element of the subgroup $U(1)^n \subset U(n)$. Consider the set of left cosets of $U(1)^n$ in $U(n)$, namely the equivalence classes under the equivalence relation: $g \sim g'$ if and only if $g^{-1}g' = h \in U(1)^n$. This set is denoted with $\tilde{U}_\ell(n)$. Its real dimension is $n(n-1)$.

Theorem 2 — Let H be a Hermitian $n \times n$ matrix. There exists a diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$ and a unitary matrix $\tilde{U}$ belonging to $\tilde{U}_\ell(n)$, such that

$$H = \tilde{U} D \tilde{U}^\dagger. \tag{A1}$$

The diagonal unitary matrices of $U(1)^n$ commute with D, so they do not contribute to (A1). The diagonalization (A1) is unique if H does not have degenerate eigenvalues. We can prove this statement checking that the dimensions match: the set of Hermitian matrices has real dimension n^2, which is equal to the sum of the dimension of $\tilde{U}_\ell(n)$, which is $n^2 - n$, plus the dimension of the set of diagonal matrices D, which is n.

Now we consider the polar decomposition of matrices, which we present in a form that is again generically unique.

Theorem 3 — Let S be any invertible complex $n \times n$ matrix. There exists a non-negative diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and matrices U_L, $\tilde{U}_R$ belonging to $U(n)$ and $\tilde{U}_\ell(n)$, respectively, such that[1]

$$S = \tilde{U}_R D U_L. \tag{A2}$$

Proof — Since S is invertible, we can write

$$S = SS^\dagger (S^\dagger)^{-1}. \tag{A3}$$

Now, $SS^\dagger$ is Hermitian, so it can diagonalized with a unitary matrix $\tilde{U}_R \in \tilde{U}_\ell(n)$. Since $SS^\dagger$ is also positive definite, we call its diagonal form D^2 and define D as the positive square root of D^2. We have

$$SS^\dagger = \tilde{U}_R D^2 \tilde{U}_R^\dagger. \tag{A4}$$

Inserting (A4) in (A3), we get (A2) with

$$U_L = D \tilde{U}_R^\dagger (S^\dagger)^{-1}.$$

This matrix is unitary. Indeed,

$$U_L^\dagger U_L = S^{-1} \tilde{U}_R D^2 \tilde{U}_R^\dagger (S^\dagger)^{-1} = 1.$$

Again, the dimensions match, because S, $\tilde{U}_R$, D, and U_L contain $2n^2$, $n^2 - n$, n, and n^2 real parameters, respectively. Thus, if the eigenvalues of $SS^\dagger$ are nondegenerate the decomposition is unique.

Finally, consider the Hermitian matrix

$$N = \begin{pmatrix} 0 & S^\dagger \\ S & 0 \end{pmatrix}.$$

Using (A2), we can diagonalize it with the unitary matrix,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} U_L^\dagger & U_L^\dagger \\ \tilde{U}_R & -\tilde{U}_R \end{pmatrix}.$$

The eigenvalues of N come in pairs of opposite signs and coincide with the diagonal entries of D and their opposites:

$$N = U \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} U^\dagger. \tag{A5}$$

APPENDIX B: MATHEMATICAL DEFINITIONS

Here we collect some mathematical definitions used in the paper. The calculation of our one-loop diagrams gives the functions

$$f_{i_1 \cdots i_n} = \frac{(n-1)!}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-\sum_{k=1}^{n-2} x_k} \times dx_{n-1} \left(\ln \frac{\Lambda_L^2}{M_{n,x}^2} + c_n \right), \tag{B1}$$

where $i_1 \cdots i_n$ can have the values t and b,

$$M_{n,x}^2 = \sum_{k=1}^{n-1} m_{i_k}^2 x_k + m_{i_n}^2 \left(1 - \sum_{k=1}^{n-1} x_k \right)$$

and c_n are constants. The first constants c_n have approximate numerical values

$$c_2 = -2.11371,$$
$$c_3 = -2.61371,$$
$$c_4 = -2.94704.$$

The diagrams are calculated as follows. Using the gap equation, the momentum integrals are convergent for $\Lambda_L < \infty$ and logarithmically divergent when Λ_L is sent to infinity. They can be viewed as regularized by the Lorentz violation. A direct evaluation of Lorentz-violating integrals is very difficult. However, renormalization theory ensures that everything but finite numerical constants (the constants c_n) can be unambiguously calculated with any regularization method. We used an ordinary cutoff. Later,

[1] The reason why "R" stands to the left and "L" stands to the right in formula (A2) is that in this way U_L is attached to left-handed quarks and $\tilde{U}_R$ is attached to right-handed quarks, according to (3.2).

we evaluated the constants c_n taking equal masses in the Lorentz-violating integrals.

With $\Lambda_L = 10^{14}$ GeV and the known values of $m_{t,b}$, we see that the constants c_n are numerically not important for the analysis of our paper.

Clearly, $f_{i_1\cdots i_n}$ is completely symmetric. Using $m_b \ll m_t \ll \Lambda_L$, we have

$$f_{i_1\cdots i_n} \sim \frac{1}{(4\pi)^2}\ln\frac{\Lambda_L^2}{m_t^2},$$

any time at least one index is t. Instead,

$$f_{b\cdots b} \sim \frac{1}{(4\pi)^2}\ln\frac{\Lambda_L^2}{m_b^2}.$$

Note the change of notation with respect to [9], because we have expanded all functions contained in the low-energy effective action in powers of the momentum.

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