RI′/SMOM scheme amplitudes for deep inelastic scattering operators at one loop in QCD

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(Received 13 August 2010; revised manuscript received 23 November 2010; published 18 March 2011)

We compute the amplitudes for the insertion of various operators in a quark 2-point function at one loop in the RI′ symmetric momentum scheme, RI′/SMOM. Specifically we focus on the moments $n = 2$ and 3 of the flavor nonsinglet twist-2 operators used in deep inelastic scattering as these are required for lattice computations.

I. INTRODUCTION

Lattice gauge theories are the main theoretical tool for exploring the nonperturbative régime of the strong nuclear force by simulating the underlying quantum field theory which is the nonabelian gauge theory of quarks and gluons, (QCD). The numerical techniques allow one to explore the infrared region where perturbation theory becomes impractical because the value of the parameter controlling the expansion, which is the coupling constant, becomes large. Briefly, one uses the path integral formalism but with the space-time continuum replaced by a discrete Euclidean grid. Then one can construct Green's functions numerically, representing, for example, bound state particle spectra, and make measurements of the masses. While overlooking many of the technical aspects of this procedure at this point, it is a remarkable achievement that the formalism is in more than solid agreement with nature. Aside from determining particle spectra, one of the current main problems is to measure matrix elements of operators in the nonperturbative region. These are important in, for instance, understanding the structure of nucleons if one focuses on the operators relating to deep inelastic scattering. These were introduced originally in [1] which subsequently produced an intense industry to determine the operator anomalous dimensions for arbitrary moments to eventually three loops in the $\overline{\text{MS}}$ renormalization scheme, [2–7]. Indeed there has been a large degree of progress in the area of measuring matrix elements of quark bilinear currents and operators and the associated renormalization constants by the QCDSF collaboration, [8–14], and others [15–24]. However, in order to make reliable measurements and hence accurate predictions one has to overcome various theoretical difficulties. Aside from those relating specifically to the lattice, there is the problem of ensuring the results match what would be expected at high energies. In other words, the Green's function depends on some reference momentum value and the numerical simulations, in principle, make measurements not only at low but also at high energies. In the latter case perturbation theory is actually valid there and hence is a reliable complementary tool. Therefore, if the same Green's function is computed

to several loop orders then it ought to be the case that the numerical measurements will overlap at large energies. Given this, it is sometimes the situation when there are accurate large loop order results that the continuum estimate is used to assist with normalizing the lattice measurements.

This brief overview clouds some of the more technical aspects of the overall procedure. For instance, all the operators of interest undergo renormalization. While this is not a major problem, since the formalism to carry out a renormalization has been established for many years now, there is the problem of the relation of a continuum renormalization to what is performed in practice on the lattice. For instance, the standard practice in high energy problems is to dimensionally regularize QCD and then to subtract the resulting divergences, which are manifested as poles in the regularizing parameter ϵ, in a minimal or modified minimal way. The latter scheme, $\overline{\text{MS}}$, is the main procedure primarily as convergence is improved by removing a specific finite part in addition to the basic poles. The main advantage of this mass independent renormalization scheme is that one can compute to very high loop order when the quarks are massless and even to a reasonable order in some cases when there are massive quarks. While this provides accurate results for the lattice to match, there is a technical problem to be overcome which is that the lattice computations are invariably in a nonminimal renormalization scheme. So to make a proper comparison for matching at high energy one has to convert the results to the *same* scheme. One of the more widely used lattice schemes is the RI′ scheme, [25,26], which denotes the modified regularization invariant scheme. It is a modification of the RI scheme, [25,26], where the essential difference is in the way the quark wave function renormalization constant is defined. Briefly, the difference between the RI′ and RI schemes is in a differentiation of the quark 2-point function with respect to the momentum. As the derivative has a financial cost for the lattice, the RI′ scheme is more efficient and hence is the default scheme in this respect, [25,26]. Both these lattice schemes are mass dependent renormalization schemes and are a hybrid of $\overline{\text{MS}}$ and MOM schemes. By this we mean that in the main 2-point

functions are renormalized according to a MOM type subtraction while three and higher point functions are renormalized using $\overline{\text{MS}}$. This ensures, for example, that the RI and RI′ scheme coupling constants are the same as that of the $\overline{\text{MS}}$ scheme, [25,26]. While introduced in [25,26], the renormalization of QCD in the continuum has been studied in the Landau gauge in [27] and later in a general linear covariant gauge in [28].

Consequently, with interest in measuring matrix elements on the lattice relating to nucleon structure there has been a need to carry out the continuum RI′ renormalization of the same Green's functions. These involve the low moment flavor nonsinglet twist-2 Wilson operators which arise in the deep inelastic scattering formalism. Indeed three loop results are available in RI′ in [28–30]. As these renormalization constants are determined by inserting the operator into a quark 2-point function with massless quarks then one could apply standard algorithms such as MINCER, [31,32], to achieve high loop orders. However, from a technical point of view this renormalization is carried out at a point of exceptional momentum since the operator is inserted at zero momentum. In this configuration one is effectively dealing with a reduced 2-point function. It has recently been pointed out, [33], that for matrix elements relating to quark masses, one could have infrared issues in extracting reliable results in the nonperturbative region as a consequence of this momentum configuration. To circumvent this technicality an alternative scheme has been developed which is called the RI′/SMOM scheme, [33]. It differs from the RI′ scheme in that 3-point functions are not subtracted at an exceptional point but instead at a symmetric point. This means that none of the external momenta are nullified, so that the potential infrared singularity embedded within the logarithms of the Green's function are bypassed, [33]. While the one loop computations were performed in [33], this has recently been extended to two loops in [34,35] for the scalar and tensor currents. However, given that there is recent interest in measuring twist-2 flavor nonsinglet operator matrix elements on the lattice for low moments, [23], it is the purpose of this article to present the first one loop computations for the moments $n = 2$ and 3. Although lattice computations focus on the Landau gauge, partly because that gauge is less complicated to fix numerically, the Green's function with the operator insertion is gauge dependent. So in developing another renormalization scheme to overcome one technical problem there are potential numerical errors in measurements from gauge fixing issues. Instead, all our results will be in a general linear covariant gauge. While the renormalization of these Wilson operators has already been carried out for the RI′ scheme, [28–30], it may appear to be rather simple to follow that procedure for the latest scheme. This is not the case for an elementary reason. This has to do with the fact that the Wilson operators mix under renormalization,

[36]. It is widely accepted that the flavor singlet Wilson operators mix among themselves and the flavor nonsinglet ones do not, [1–3]. Indeed in the original context of [1–3] this is the situation. However, that is only the case for the latter set if the Green's function containing the operator does not have a momentum flowing out of the operator itself. If there is a net momenta flow through the operator then they mix into a set of total derivative operators. Given that the RI′/SMOM scheme is at nonexceptional momenta then this mixing cannot be ignored. It has been studied to three loops in a practical situation in [36] where a similar problem for the lattice was examined but in a context which involves a Green's function which is gauge independent but contains two operators. Indeed the mixing matrix for the two Wilson operators we consider here was calculated to three loops in the $\overline{\text{MS}}$ scheme. Without including the mixing in [36] the operator correlation function did not correctly satisfy the renormalization group equation at two and three loops.

The article is organised as follows. We review the problem and the necessary background in Sec. II. All our results are collected in Sec. III and Sec. IV where we give all the amplitudes as a function of the gauge parameter of an arbitrary linear covariant gauge for both the $\overline{\text{MS}}$ and RI′/SMOM schemes. We conclude in Sec. V and include an appendix. It contains the bases of tensors into which all the Green's functions are decomposed together with the coefficients of the projection tensors which project out each specific amplitude.

II. PRELIMINARIES

In this section, we recall the background to the problem and the calculational setup. First, the various operators we will be considering, which are gauge invariant, are:

$$
\begin{aligned}
S &\equiv \bar{\psi}\psi, \\
V &\equiv \bar{\psi}\gamma^{\mu}\psi, \\
T &\equiv \bar{\psi}\sigma^{\mu\nu}\psi, \\
W_2 &\equiv \mathcal{S}\bar{\psi}\gamma^{\mu}D^{\nu}\psi, \\
\partial W_2 &\equiv \mathcal{S}\partial^{\mu}(\bar{\psi}\gamma^{\nu}\psi), \\
W_3 &\equiv \mathcal{S}\bar{\psi}\gamma^{\mu}D^{\nu}D^{\sigma}\psi \\
\partial W_3 &\equiv \mathcal{S}\partial^{\mu}(\bar{\psi}\gamma^{\nu}D^{\sigma}\psi), \\
\partial\partial W_3 &\equiv \mathcal{S}\partial^{\mu}\partial^{\nu}(\bar{\psi}\gamma^{\sigma}\psi)
\end{aligned}
\tag{2.1}
$$

where the first three operators are included for checking purposes and all derivatives, both ordinary and covariant, act to the right. In (2.1) $\mathcal{S}$ means total symmetrization in the free Lorentz indices and we use the same labelling and notation as [36] for ease of reference. For instance, at certain points in this respect we will refer to the level W_2 or W_3. By this we will mean either the specific operator

with that label or the set of operators within that level which are additionally either ∂W_2 or ∂W_3 and $\partial\partial W_3$ respectively. Like [36], it will be clear from the context which is meant. As discussed already one must include these additional total derivative operators within each level since there is mixing between the operators. Such mixing must be included when there is a momentum flowing through the operator insertion in the Green's function irrespective of the number of such included operators. For the earlier RI′ scheme computations of the anomalous dimensions, the mixing issue was not relevant since the operator was inserted at zero momentum, [28–30]. Given that we are considering operators with free Lorentz indices, whether they relate to deep inelastic scattering or not, we cannot follow the earlier prescription of [1–3]. There the free Lorentz indices of the matrix elements were con-

tracted with a null vector, Δ_μ, which projected out that part of the matrix element containing the divergence. The reason that we have to take a different approach resides in the fact that on the lattice measurements are made for various individual components of the free indices. Therefore, we have to take a more general approach and decompose our Green's functions into a basis of Lorentz tensors which have the *same* symmetry structure as the operator which is inserted into the Green's function. For the tensor current this means that the basis has to be antisymmetric in the two free indices since $\sigma^{\mu\nu} = \frac{1}{2} \times [\gamma^\mu, \gamma^\nu]$. In the case of the two Wilson operators each operator in the respective levels are totally symmetric in the indices and are traceless in d-dimensions, [1]. To be specific we have

$$\mathcal{S}\mathcal{O}_{\mu\nu}^{W_2} = \mathcal{O}_{\mu\nu}^{W_2} + \mathcal{O}_{\nu\mu}^{W_2} - \frac{2}{d}\eta_{\mu\nu}\mathcal{O}_{\sigma}^{W_2\sigma},$$

$$\mathcal{S}\mathcal{O}_{\mu\nu\sigma}^{W_3} = \mathcal{O}_{S\mu\nu\sigma}^{W_3} - \frac{1}{(d+2)}[\eta_{\mu\nu}\mathcal{O}_{S\sigma\rho}^{W_3\rho} + \eta_{\nu\sigma}\mathcal{O}_{S\mu\rho}^{W_3\rho} + \eta_{\sigma\mu}\mathcal{O}_{S\nu\rho}^{W_3\rho}] \tag{2.2}$$

with

$$\mathcal{O}_{S\mu\nu\sigma}^{W_3} = \frac{1}{6}[\mathcal{O}_{\mu\nu\sigma}^{W_3} + \mathcal{O}_{\nu\sigma\mu}^{W_3} + \mathcal{O}_{\sigma\mu\nu}^{W_3} + \mathcal{O}_{\mu\sigma\nu}^{W_3} + \mathcal{O}_{\sigma\nu\mu}^{W_3} + \mathcal{O}_{\nu\mu\sigma}^{W_3}] \tag{2.3}$$

where the basic operators are

$$\mathcal{O}_{\mu\nu}^{W_2} = \bar{\psi}\gamma_\mu D_\nu \psi,$$

$$\mathcal{O}_{\mu\nu\sigma}^{W_3} = \bar{\psi}\gamma_\mu D_\nu D_\sigma \psi \tag{2.4}$$

and D_μ is the usual covariant derivative. These are the same definitions as used in earlier computations, [28–30]. The total derivative operators in the same respective levels satisfy these same template definitions. (We have suppressed the free flavor indices but note that these are flavor nonsinglet operators.)

Next we recall the key points concerning the mixing of the operators in levels W_2 and W_3. First, as we are working with massless quarks there are no lower dimensional operators to be included and there is no mixing between levels. Next the particular choice of operators, (2.1), means that while there is mixing the mixing matrix of renormalization constants is upper triangular and given by, [36],

$$Z_{ij}^{W_2} = \begin{pmatrix} Z_{11}^{W_2} & Z_{12}^{W_2} \\ 0 & Z_{22}^{W_2} \end{pmatrix} \tag{2.5}$$

and

$$Z_{ij}^{W_3} = \begin{pmatrix} Z_{11}^{W_3} & Z_{12}^{W_3} & Z_{13}^{W_3} \\ 0 & Z_{22}^{W_3} & Z_{23}^{W_3} \\ 0 & 0 & Z_{33}^{W_3} \end{pmatrix}. \tag{2.6}$$

Again we avoid a clumsy index on the matrix elements by using a numerical map to the respective sets $\{W_2, \partial W_2\}$ and $\{W_3, \partial W_3, \partial\partial W_3\}$ respectively. Here the superscript indicates the level. Once these have been determined in a specific renormalization scheme, then the anomalous dimension matrix is deduced from

$$\gamma_{ij}^{\mathcal{O}} = \mu\frac{d}{d\mu}\ln Z_{ij}^{\mathcal{O}} \tag{2.7}$$

with

$$\mu\frac{d}{d\mu} = \beta(a)\frac{\partial}{\partial a} + \alpha\gamma_\alpha(a, \alpha)\frac{\partial}{\partial\alpha} \tag{2.8}$$

where α is the gauge parameter of the canonical linear covariant gauge and $\alpha = 0$ corresponds to the Landau gauge. Although the operators we are considering are gauge invariant, in a mass dependent renormalization scheme, such as RI′ or RI′/SMOM, the anomalous dimensions can depend on the gauge. This is why we have included the second term on the right side of (2.8). However, for the one loop computation here the leading term is scheme independent so that there is no gauge dependence at this order. In order to compare with the structure of our RI′/SMOM results later, we recall the three loop $\overline{\text{MS}}$ scheme anomalous dimension mixing matrices, [36], are

$$\gamma_{11}^{W_2}(a) = \frac{8}{3}C_F a + \frac{1}{27}[376C_A C_F - 112C_F^2 - 128C_F T_F N_f]a^2$$

$$+ \frac{1}{243}[(5184\zeta(3) + 20920)C_A^2 C_F - (15552\zeta(3) + 8528)C_A C_F^2$$

$$- (10368\zeta(3) + 6256)C_A C_F T_F N_f + (10368\zeta(3) - 560)C_F^3$$

$$+ (10368\zeta(3) - 6824)C_F^2 T_F N_f - 896C_F T_F^2 N_f^2]a^3 + O(a^4),$$

$$\gamma_{12}^{W_2}(a) = -\frac{4}{3}C_F a + \frac{1}{27}[56C_F^2 - 188C_A C_F + 64C_F T_F N_f]a^2$$

$$+ \frac{1}{243}[(7776\zeta(3) + 4264)C_A C_F^2 - (2592\zeta(3) + 10460)C_A^2 C_F$$

$$+ (5184\zeta(3) + 3128)C_A C_F T_F N_f - (5184\zeta(3) - 280)C_F^3$$

$$- (5184\zeta(3) - 3412)C_F^2 T_F N_f + 448C_F T_F^2 N_f^2]a^3 + O(a^4),$$

$$\gamma_{22}^{W_2}(a) = O(a^4) \tag{2.9}$$

and

$$\gamma_{11}^{W_3}(a) = \frac{25}{6}C_F a + \frac{1}{432}[8560C_A C_F - 2035C_F^2 - 3320C_F T_F N_f]a^2$$

$$+ \frac{1}{15552}[(285120\zeta(3) + 1778866)C_A^2 C_F - (855360\zeta(3) + 311213)C_A C_F^2$$

$$- (1036800\zeta(3) + 497992)C_A C_F T_F N_f + (570240\zeta(3) - 244505)C_F^3$$

$$+ (1036800\zeta(3) - 814508)C_F^2 T_F N_f - 82208C_F T_F^2 N_f^2]a^3 + O(a^4)$$

$$\gamma_{12}^{W_3}(a) = -\frac{3}{2}C_F a + \frac{1}{144}[81C_F^2 - 848C_A C_F + 424C_F T_F N_f]a^2 + O(a^3),$$

$$\gamma_{13}^{W_3}(a) = -\frac{1}{2}C_F a + \frac{1}{144}[103C_F^2 - 388C_A C_F + 104C_F T_F N_f]a^2 + O(a^3), \; \gamma_{22}^{W_3}(a)$$

$$= \frac{8}{3}C_F a + \frac{1}{27}[376C_A C_F - 112C_F^2 - 128C_F T_F N_f]a^2$$

$$+ \frac{1}{243}[(5184\zeta(3) + 20920)C_A^2 C_F - (15552\zeta(3) + 8528)C_A C_F^2$$

$$- (10368\zeta(3) + 6256)C_A C_F T_F N_f + (10368\zeta(3) - 560)C_F^3$$

$$+ (10368\zeta(3) - 6824)C_F^2 T_F N_f - 896C_F T_F^2 N_f^2]a^3 + O(a^4),$$

$$\gamma_{23}^{W_3}(a) = -\frac{4}{3}C_F a + \frac{1}{27}[56C_F^2 - 188C_A C_F + 64C_F T_F N_f]a^2$$

$$+ \frac{1}{243}[(7776\zeta(3) + 4264)C_A C_F^2 - (2592\zeta(3) + 10460)C_A^2 C_F$$

$$+ (5184\zeta(3) + 3128)C_A C_F T_F N_f - (5184\zeta(3) - 280)C_F^3$$

$$- (5184\zeta(3) - 3412)C_F^2 T_F N_f + 448C_F T_F^2 N_f^2]a^3 + O(a^4),$$

$$\gamma_{33}^{W_3}(a) = O(a^4) \tag{2.10}$$

where $\zeta(z)$ is the Riemann zeta function, $a = g^2/(16\pi^2)$ and N_f is the number of massless quarks. The group Casimirs are defined by

$$T^a T^a = C_F, \qquad \text{Tr}(T^a T^b) = T_F \delta^{ab},$$

$$f^{acd} f^{bcd} = C_A \delta^{ab} \tag{2.11}$$

where T^a are the Lie group generators with structure functions f^{abc} and $1 \le a \le N_A$ where N_A is the dimension of the adjoint representation.

We turn now to the setup for the particular Green's function we are interested in which is $\langle \psi(p) \mathcal{O}_{\mu_1 \ldots \mu_{n_i}}^i (-p - q) \bar{\psi}(q) \rangle$ and is illustrated in Fig. 1. The independent external momenta we use are p and q and are the momenta flowing into the external quark legs. Thus there is a
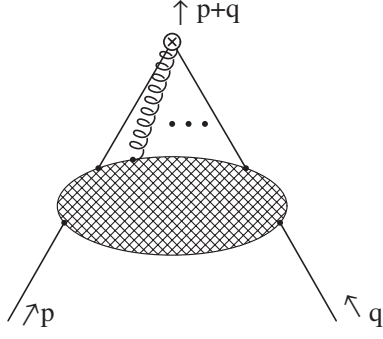
FIG. 1. Graphical illustration of the Green's function, $\langle \psi(p) \mathcal{O}^i_{\mu_1 \ldots \mu_{n_i}}(-p-q)\bar{\psi}(q)\rangle$, used to renormalize operators in the RI′/SMOM scheme.

momentum of $(p+q)$ flowing out through the operator insertion whose location is indicated by the circle containing a cross. In order to determine the renormalization constants for the basic operators in the $\overline{\text{MS}}$ and RI′ schemes one chooses $q = -p$. However, for the RI′/SMOM scheme the two momenta are left unconstrained. Instead to define the symmetric point of subtraction for the renormalization the square of the momenta satisfy, [33–35],

$$p^2 = q^2 = (p+q)^2 = -\mu^2 \qquad (2.12)$$

which imply

$$pq = \frac{1}{2}\mu^2 \qquad (2.13)$$

where μ is the renormalization scale introduced to ensure the coupling constant is dimensionless in d-dimensions. Given this, the Green's function is decomposed into a basis of independent Lorentz tensors, $\mathcal{P}^i_{(k)\mu_1 \ldots \mu_{n_i}}(p,q)$, with associated amplitude, $\Sigma^{\mathcal{O}^i}_{(k)}(p,q)$, which is the value we will compute at the symmetric subtraction point,

$$\langle \psi(p) \mathcal{O}^i_{\mu_1 \ldots \mu_{n_i}}(-p-q)\bar{\psi}(q)\rangle|_{p^2=q^2=-\mu^2}$$

$$= \sum_{k=1}^{n_i} \mathcal{P}^i_{(k)\mu_1 \ldots \mu_{n_i}}(p,q)\Sigma^{\mathcal{O}^i}_{(k)}(p,q). \qquad (2.14)$$

Here the bracketed subscript k labels the tensor of the basis and the superscript i is the operator level label, (2.1). The explicit tensors for each level are given in Appendix A together with the method which allows one to compute the amplitude itself via a projection onto the Green's function with free indices. The same tensor basis and projection is used for each level. The total number of projectors, n_i, is different for each level and recorded in Table I. It is worth noting that the basis of projection tensors which we use for each level is not unique. They are constructed from the basic momentum vectors, γ-matrices and metric tensors available, in such a way that each final tensor has the same symmetry structure as its associated operator insertion, as well as being traceless in d-dimensions. However, to construct all the tensors in each set of projectors we have isolated all the basic one loop tensor integrals within each computation. These are then decomposed into their own tensor basis with their associated scalar integrals. For instance,

$$\int_k \frac{k_\mu k_\nu}{k^2(k-p)^2(k+q)^2}\bigg|_{p^2=q^2=-\mu^2} = \frac{4}{[d-2]}\bigg[\eta_{\mu\nu}\bigg[\frac{1}{4}I_1 + \frac{1}{3}I_2 + \frac{1}{6}I_3 + \frac{1}{6}I_4 + \frac{1}{3}I_5\bigg]$$

$$+ \frac{p_\mu p_\nu}{\mu^2}\bigg[\frac{1}{3}I_1 + \frac{4}{9}(d-1)I_2 + \frac{2}{9}(d-1)I_3 + \frac{2}{9}(d-1)I_4 + \frac{1}{9}(d+2)I_5\bigg]$$

$$+ \frac{p_\mu q_\nu}{\mu^2}\bigg[\frac{1}{6}I_1 + \frac{2}{9}(d-1)I_2 + \frac{1}{9}(4d-7)I_3 + \frac{1}{9}(d-1)I_4 + \frac{2}{9}(d-1)I_5\bigg]$$

$$+ \frac{q_\mu p_\nu}{\mu^2}\bigg[\frac{1}{6}I_1 + \frac{2}{9}(d-1)I_2 + \frac{1}{9}(d-1)I_3 + \frac{1}{9}(4d-7)I_4 + \frac{2}{9}(d-1)I_5\bigg]$$

$$+ \frac{q_\mu q_\nu}{\mu^2}\bigg[\frac{1}{3}I_1 + \frac{1}{9}(d+2)I_2 + \frac{2}{9}(d-1)I_3 + \frac{2}{9}(d-1)I_4 + \frac{4}{9}(d-1)I_5\bigg]\bigg] \qquad (2.15)$$

where $\int_k = \int \frac{d^d k}{(2\pi)^d}$ and

$$I_1 = \int_k \frac{1}{(k-p)^2(k+q)^2}\bigg|_{p^2=q^2=-\mu^2},$$

$$I_2 = \int_k \frac{(kp)^2}{k^2(k-p)^2(k+q)^2}\bigg|_{p^2=q^2=-\mu^2},$$

$$I_3 = I_4 = \int_k \frac{kpkq}{k^2(k-p)^2(k+q)^2}\bigg|_{p^2=q^2=-\mu^2},$$

$$I_5 = \int_k \frac{(kq)^2}{k^2(k-p)^2(k+q)^2}\bigg|_{p^2=q^2=-\mu^2}, \qquad (2.16)$$

are the basic scalar integrals. This together with the other relevant tensor integrals is substituted back into the main computation and the index contractions are performed. This ensures that no basis tensor is overlooked in the procedure of constructing the basis. Once the basis is

TABLE I. Number of projectors for each operator insertion.

i	S	V	T	W_2	W_3
n_i	2	6	8	10	14

established we determine the explicit projection of Appendix A. This is used to check the final results. Moreover, having found the tensor basis at one loop it can then be applied directly in a two loop calculation.

In addition, for the γ-algebra we use the generalized basis of γ-matrices which has been introduced in [37–39] with more explicit details of their properties given in [40,41]. Briefly, we define these new matrices, $\Gamma_{(n)}^{\mu_1\ldots\mu_n}$, by

$$\Gamma_{(n)}^{\mu_1\ldots\mu_n} = \gamma^{[\mu_1}\ldots\gamma^{\mu_n]} \qquad (2.17)$$

which is totally antisymmetric in the Lorentz indices for $n \geq 1$ and the square bracket notation includes the overall factor of $1/n!$. So, for instance, $\Gamma_{(0)}$ is the unit matrix, $\Gamma_{(0)} = I$, and $\sigma^{\mu\nu} = \Gamma_{(2)}^{\mu\nu}$. Though we will invariably retain γ^μ for $n = 1$. One advantage of this basis is that

$$\text{tr}\,(\Gamma_{(m)}^{\mu_1\ldots\mu_m}\Gamma_{(n)}^{\nu_1\ldots\nu_n}) \propto \delta_{mn}I^{\mu_1\ldots\mu_m\nu_1\ldots\nu_n} \qquad (2.18)$$

where $I^{\mu_1\ldots\mu_m\nu_1\ldots\nu_n}$ is the unit matrix in this Γ-space, [37–41]. As a result of this there is a partitioning of the projection matrix into sectors with different $\Gamma_{(n)}$ as is evident in the examples in Appendix A. Two final points concerning the projectors are worth emphasizing. First, our choice of basis tensors, $\mathcal{P}_{(k)\mu_1\ldots\mu_{n_i}}^i(p,q)$, only has meaning strictly at the symmetric subtraction point. Away from this point there will be a bigger basis set of tensors since then we would have $p^2 \neq q^2$, $p^2 \neq (p+q)^2$ and $q^2 \neq (p+q)^2$ as is evident from the explicit forms given in Appendix A. Second, at the symmetric point the number of independent tensors in each case is clearly larger than that of the asymmetric forward subtraction point considered in the RI$'$ scheme. For instance, in the case of the scalar there are two tensors in the basis unlike the unique one of the RI$'$ scheme. In this case the additional basis element is $\Gamma_{(2)}^{pq}$ where we use the convention that the appearance of a momentum vector in place of a Lorentz index indicates the contraction of that index with that momentum. Clearly this object vanishes if either momentum is zero or one is proportional to the other, corresponding to the RI$'$ scheme momentum configuration. As regards the tensor basis for each of the operators, the use of the generalized matrices $\Gamma_{(n)}^{\mu_1\ldots\mu_n}$ is important in regarding the basis as complete. This is because they span spinor space in d-dimensions, [37,39], with n being any positive integer. Therefore, it is natural to make use of them in dimensionally regularized computations. Though for the operators considered here n never exceeds 4. If there were more than two independent momenta then obviously a larger value of n would be necessary. As $\Gamma_{(n)}^{\mu_1\ldots\mu_n}$ clearly form the basis of the spinor vector space of the tensor basis decomposition, the Lorentz vector space part of the overall basis for each operator is then made complete by building Lorentz tensors from combinations of elements of the set $\{\eta^{\mu\nu}, p^\mu, q^\mu, \Gamma_{(n)}^{\mu_1\ldots\mu_n}\}$. These have, of course, to be consistent with the symmetries of the Lorentz indices of the operator inserted in the Green's function. From this it is clear that our basis is complete for each operator, as there is no room to build additional tensors from the basic structures of the Lorentz and spinor vector spaces.

Given that there is more than one amplitude for each operator insertion, we have to be careful in defining the renormalization constant in the RI$'$/SMOM scheme. For all the cases we consider here the ultraviolet divergence resides in a subset of the amplitudes which in fact contains at least one element except for the special case of the vector current. For V the RI$'$/SMOM scheme renormalization has to be treated separately. If, for the moment, we denote this representative basis tensor by the label 0 then we define the renormalization constant for the operator $\mathcal{O}$, $Z_\mathcal{O}^{\text{RI}'/\text{SMOM}}$, by the condition

$$\lim_{\epsilon \to 0}[Z_\psi^{\text{RI}'}Z_\mathcal{O}^{\text{RI}'/\text{SMOM}}\Sigma_{(0)}^\mathcal{O}(p,q)]|_{p^2=q^2=-\mu^2} = 1 \qquad (2.19)$$

where $Z_\psi^{\text{RI}'}$ is the quark wave function renormalization constant in the RI$'$ scheme which is given in [27,28]. The reason why the value in the original RI$'$ scheme is used has been discussed in [33–35]. In determining the final renormalization constant $Z_\mathcal{O}^{\text{RI}'/\text{SMOM}}$, we follow the procedure of [42] for automatic Feynman diagram computations. In other words we compute all diagrams in terms of their bare quantities which here are essentially the coupling constant and the gauge parameter. Then the renormalized parameters are introduced by rescaling with the already determined coupling constant and gauge parameter renormalization constants. Although the latter should be taken to be in the RI$'$ scheme to the one loop order we are working, any scheme effect will not show up until two loops. While this is a standard procedure for introducing counterterms, the main issue here is that this rescaling from bare to renormalized quantities must also include the mixing of the operators. Therefore, in constructing our amplitudes, which are recorded in sections three and four, the matrices (2.5) and (2.6) have been included. In practical terms this means that the renormalization constants are found by first fixing those in the last row of each matrix. Then those in the next row are determined and repeated until the ultimate row is found. This is similar to the method used in [36] to deduce (2.9) and (2.10). The basic reason for this literal bottom up approach is that the counterterms to be determined are intertwined due to the triangularity of the matrix and this is the systematic way to disentangle them. However, in defining the RI$'$/SMOM scheme here in the general terms indicated in (2.19) one has to be careful in any situation involving the vector current due to the underlying Slavnov-Taylor identity which affects this operator. We will discuss this caveat in more detail later as it arises not just in the case of the vector current itself but is embedded in each set W_2 and W_3.

Having concentrated on the general quantum field theoretic formalism that is used, we now comment on the practicalities of the calculation. We use standard tools for this. All the algebra is carried out with the symbolic manipulation language FORM, [43]. The three one loop Feynman diagrams are generated in electronic form by the QGRAF package, [44], with the output converted into FORM input notation. This procedure appends indices and labels to all the fields. Various FORM modules were then used to insert the basic Feynman rules for the propagators and vertices before those of the particular operator of interest. The n_i amplitudes were then projected out by the theory given in Appendix A for successive Green's functions. These are first determined by constructing all the basic tensor integral decompositions akin to (2.15). Once determined we construct the tensor basis and repeat the computation by use of the projections given in Appendix A. The final part is to actually substitute the explicit values of the master one loop scalar integrals. This may have required integration by parts but for the most part the resulting integrals are one loop with only two propagators such as I_1. These bubbles are simple to determine. The remaining integral is

$$\int_k \frac{1}{k^2(k-p)^2(k+q)^2}\bigg|_{p^2=q^2=-\mu^2} = \frac{9s_2}{\mu^2} + O(\epsilon) \quad (2.20)$$

where $s_2 = (2\sqrt{3}/9)\text{Cl}_2(2\pi/3)$ and $\text{Cl}_2(x)$ is the Clausen function which was evaluated in [45]. We use dimensional regularization in $d = 4 - 2\epsilon$ dimensions. Given these ingredients we have been able to determine all the one loop amplitudes for the set of operators (2.1) where we have repeated the calculation for S, V and T as an elementary check on our programmes. We correctly reproduced all the one loop expressions given in [33–35] for both anomalous dimensions and amplitudes.

III. QUARK CURRENTS.

Having concentrated on describing the background to the problem and the methodology of the computations, we turn to the mundane task of recording the explicit amplitudes. In this section we do this successively for the scalar, vector and tensor currents. In [33–35] the anomalous dimensions were computed at one and two loops for the scalar and tensor cases in the RI′/SMOM schemes. In addition the one loop amplitudes in the $\overline{\text{MS}}$ scheme were given for each of the three operators in [33]. In the appendix we give the explicit mapping between our amplitudes, $\Sigma^{\mathcal{O}}_{(k)}(p,q)$, for the quark currents and those defined in [33] in order to compare. Though we note that we are in full agreement with the results of [33]. Since lattice computations will measure various directions and then extract estimates for the overall Green's function it seems appropriate to provide the amplitudes for the $\overline{\text{MS}}$ scheme, as well as for RI′/SMOM, as ultimately the former is the reference

scheme one will map to. Equally the explicit $\overline{\text{MS}}$ expressions will be useful for lattice groups who convert to $\overline{\text{MS}}$ first before comparing to the high energy expressions rather than work in the RI′/SMOM directly in order to do the matching. This seems appropriate since it will be noted later that there is not a definitive RI′/SMOM scheme especially when operators have Lorentz indices such as the Wilson operators. Moreover, while providing results for both schemes may appear to introduce a degree of redundancy, because at one loop there is overlap in the actual expressions for the amplitudes between the schemes, this is not preserved at two loops, [46]. Also for the Wilson operators the results in both schemes will be useful to ensure that we have preserved properties of the sets of operators in each level between schemes. For the scalar current we have the $\overline{\text{MS}}$ result at one loop

$$\Sigma^S_{(1)}(p,q)|_{\overline{\text{MS}}} = -1 + C_F\left[\frac{27}{2}s_2 - 4 - 2\alpha + \frac{9}{2}s_2\alpha\right]a$$
$$+ O(a^2),$$
$$\Sigma^S_{(2)}(p,q)|_{\overline{\text{MS}}} = C_F[3s_2\alpha - 3s_2]a + O(a^2). \quad (3.1)$$

To make contact with other work in this area, [33], we note the relations

$$C_0(1) = 9s_2 = \frac{2}{3}\psi'\left(\frac{1}{3}\right) - \left(\frac{2\pi}{3}\right)^2 \quad (3.2)$$

where $\psi(z)$ is the derivative of the logarithm of the Euler Γ-function and $C_0(\omega)$ was the function used in [33] to interpolate between the symmetric and exceptional momenta scheme choices. While this case has already been analyzed in [33–35] we discuss it here as it raises several issues with regard to defining the subsequent RI′/SMOM scheme renormalization constants for the Wilson operators. For instance, it would appear that the definition of the scheme can be given in several ways. Given this form for the Green's function, one could either define the RI′/SMOM scheme operator renormalization constant by absorbing the finite part of the 1 direction. Alternatively one could take the projection of the Green's function by some tensor, such as that of the Born term, and then absorb whatever the finite part emerges into the renormalization constant. For the scalar case this would actually give the same renormalization constant because of (2.18). However, for other operators there appears to be a degree of freedom in how one can actually define the scheme. Though, of course, one would retain the fundamental criterion of (2.19) to ensure there are no corrections beyond the leading order. As these methods are equivalent here we note that we reproduce the scalar current results of [33] for the renormalization constant

$$Z^S = 1 + C_F\left[-\frac{3}{\epsilon} - 4 + \frac{27}{2}s_2 - \alpha + \frac{9}{2}s_2\alpha\right]a + O(a^2) \quad (3.3)$$

where we use the notation that all expressions will be in RI′/SMOM unless explicitly indicated to be in $\overline{\text{MS}}$. The resulting amplitudes are

$$\Sigma^S_{(1)}(p, q) = -1 + O(a^2),$$

$$\Sigma^S_{(2)}(p, q) = C_F[3s_2\alpha - 3s_2]a + O(a^2) \qquad (3.4)$$

$$\Sigma^V_{(1)}(p, q)|_{\overline{\text{MS}}} = -1 + C_F[2 - 3s_2 - 2\alpha + 6s_2\alpha]a + O(a^2),$$

$$\Sigma^V_{(2)}(p, q)|_{\overline{\text{MS}}} = \Sigma^V_{(5)}(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{8}{3} - 6s_2 - \frac{4}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma^V_{(3)}(p, q)|_{\overline{\text{MS}}} = \Sigma^V_{(4)}(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{4}{3} - \frac{2}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma^V_{(6)}(p, q)|_{\overline{\text{MS}}} = -6C_Fs_2a + O(a^2). \qquad (3.5)$$

Here the renormalization constant for the current is

$$Z^V|_{\overline{\text{MS}}} = 1 + O(a^2). \qquad (3.6)$$

This value derives from the fact that the vector current is a physical operator and undergoes no renormalization to all orders in perturbation theory. Moreover this is a consequence of the Slavnov-Taylor identity. Turning to the RI′/SMOM situation, it is tempting to define the renormalization constant in this case by either the projection by the Born tensor or by taking the coefficient of the channel 1 amplitude. This would lead to a renormalization constant in the RI′/SMOM scheme with a nonzero finite part. However, this would be inconsistent with the Slavnov-Taylor identity. In addition as the $\overline{\text{MS}}$ renormalization constant is unity to all orders then this implies, [47], that the RI′/SMOM renormalization constant is already determined and equivalent to (3.6). Therefore, we have for the vector current renormalization

$$Z^V = 1 + O(a^2). \qquad (3.7)$$

In order to see that this is equivalent with the Slavnov-Taylor identity one can contract the Green's function with the vector $(p + q)_\mu$ as this is the momentum flowing through the inserted operator. This procedure corresponds to the renormalization of the divergence of the current. In terms of (3.5) the combination of amplitudes in this contraction proportional to $p\!\!\!/$ is

$$\Sigma^V_{(1)}(p, q) - \frac{1}{2}\Sigma^V_{(2)}(p, q) - \frac{1}{2}\Sigma^V_{(5)}(p, q) = -1 + O(a^2). \qquad (3.8)$$

where channel 2 is the same as the $\overline{\text{MS}}$ expression. However, this will not be the case at two loops, [46].

The situation for the vector current is more involved partly because of the increase in amplitudes with one free Lorentz index. First, the $\overline{\text{MS}}$ amplitudes are

A different combination determines the coefficient for the piece involving $q\!\!\!/$ but with the same result. The fact that there is no $O(a)$ correction is because we have renormalized the quark 2-point functions in an RI′ scheme and the wave function renormalization constant defined in this way leaves the 2-point function as unity to all orders. Hence the Slavnov-Taylor identity is satisfied. We have checked that this is also the situation in the $\overline{\text{MS}}$ case and, moreover, this has been extended to two loops in RI′/SMOM, [46], where it was verified that the Slavnov-Taylor identity was consistent in that case too in keeping with the general argument given in [33]. Crucial to this analysis is knowledge of the full basis of tensors into which the Green's function can be written. Although the channel 6 amplitude appears to be decoupled due to the $\Gamma_{(n)}$ basis we use, omitting it would lead to an inconsistent operator renormalization. Finally, in considering the Slavnov-Taylor identity in this way it would have become clearer if a different combination of tensors was used in the basis rather than the ones given in Appendix A. In other words we could have used a basis where all but one tensor in the basis vanished when contracted by $(p + q)_\mu$ whence the presence of the Slavnov-Taylor identity would have been explicit. Finally, we note the RI′/SMOM scheme amplitudes after renormalization are

$$\Sigma^V_{(1)}(p, q) = -1 + C_F[2 - \alpha - 3s_2 + 6s_2\alpha]a + O(a^2),$$

$$\Sigma^V_{(2)}(p, q) = \Sigma^V_{(5)}(p, q) = -C_F\left[6s_2 - \frac{8}{3} - 6s_2\alpha + \frac{4}{3}\alpha\right]a + O(a^2),$$

$$\Sigma^V_{(3)}(p, q) = \Sigma^V_{(4)}(p, q) = C_F\left[\frac{4}{3} - \frac{2}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma^V_{(6)}(p, q) = -6C_Fs_2a + O(a^2). \qquad (3.9)$$

As this particular Green's function is symmetric under swapping the two independent momenta, then two pairs of the amplitudes are equivalent. The different values for $\Sigma_{(1)}^V(p, q)$ in both schemes is due to the fact that the finite parts of the quark wave function are not the same in each scheme.

For the tensor case we provide the same information as the previous cases. First, the $\overline{\text{MS}}$ amplitudes are

$$\Sigma_{(1)}^T(p, q)|_{\overline{\text{MS}}} = -1 + C_F\left[2 - \frac{15}{2}s_2 - 2\alpha + \frac{15}{2}s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(2)}^T(p, q)|_{\overline{\text{MS}}} = C_F[9s_2 + 3s_2\alpha]a + O(a^2),$$

$$\Sigma_{(3)}^T(p, q)|_{\overline{\text{MS}}} = \Sigma_{(6)}^T(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{4}{3} - 6s_2 - \frac{4}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^T(p, q)|_{\overline{\text{MS}}} = \Sigma_{(5)}^T(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{2}{3} - 3s_2 - \frac{2}{3}\alpha + 3s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^T(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{8}{3} - 12s_2 - \frac{8}{3}\alpha + 12s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(8)}^T(p, q)|_{\overline{\text{MS}}} = -C_F[9s_2 + 3s_2\alpha]a + O(a^2). \tag{3.10}$$

The earlier comments concerning how one actually defines the RI′/SMOM scheme apply here. In [33,35] the finite part of the tensor current renormalization constant was defined by contracting the Green's function with $\Gamma_{(2)}^{\mu\nu}$ and ensuring that the resulting expression had no a dependence after renormalization. Following that with our amplitudes we find exact agreement with the divergent and finite parts of the same renormalization constant as, [33],

$$Z^T = 1 + C_F\left[\frac{1}{\epsilon} + \frac{4}{3} - \frac{9}{2}s_2 - \frac{1}{3}\alpha + \frac{9}{2}s_2\alpha\right]a + O(a^2). \tag{3.11}$$

As in the vector case with this projection to define the renormalization constant the $\Gamma_{(0)}$ and $\Gamma_{(4)}$ sectors do not contribute but are part of the full Green's function at the symmetric subtraction point. Indeed we have checked that the full expression for the two loop amplitude, [46], leads to the same renormalization constant as [35]. However, if one were to follow an alternative way of defining the RI′/SMOM renormalization constant by merely using the coefficient of channel 1 which is where the divergence resides then one would have another expression for Z^T which is

$$Z^T|_{\text{alt RI}'/\text{SMOM}} = 1 + C_F\left[\frac{1}{\epsilon} + 2 - \frac{15}{2}s_2 - \alpha\right.$$
$$\left. + \frac{15}{2}s_2\alpha\right]a + O(a^2). \tag{3.12}$$

This alternative definition is of course dependent on the choice of basis tensors. One feature of it is that the numerical value of the finite part in the Landau gauge is significantly smaller than the corresponding part of (3.11). Indeed with a judicious choice of the projection basis it might be possible to have a rapidly converging series for the conversion function. However, that requires a higher loop analysis, [46]. Returning to the original RI′/SMOM scheme definition of [33–35] the amplitudes are

$$\Sigma_{(1)}^T(p,q) = -1 + C_F\left[\frac{2}{3} - 3s_2 - \frac{2}{3}\alpha + 3s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(2)}^T(p,q) = C_F[9s_2 + 3s_2\alpha]a + O(a^2)$$

$$\Sigma_{(3)}^T(p,q) = \Sigma_{(6)}^T(p,q) = C_F\left[\frac{4}{3} - 6s_2 - \frac{4}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^T(p,q) = \Sigma_{(5)}^T(p,q) = C_F\left[\frac{2}{3} - 3s_2 - \frac{2}{3}\alpha + 3s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^T(p,q) = C_F\left[\frac{8}{3} - 12s_2 - \frac{8}{3}\alpha + 12s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(8)}^T(p,q) = -C_F[9s_2 + 3s_2\alpha]a + O(a^2). \tag{3.13}$$

Similar to the vector case there is a clear symmetry here because the Green's function is symmetric under the interchange of p and q which is responsible for the two pairs of equivalent amplitudes which is clear from the explicit tensor definitions given in Appendix A.

IV. DEEP INELASTIC SCATTERING OPERATORS

The situation for both moments of the flavor nonsinglet twist-2 Wilson operators is more involved due to the operator mixing issue as well as the issue with the Slavnov-Taylor identity noted previously. First, the $\overline{\text{MS}}$ amplitudes for $n = 2$ are

$$\Sigma_{(1)}^{W_2}(p,q)|_{\overline{\text{MS}}} = C_F\left[2s_2 - \frac{11}{6} - \frac{1}{3}\alpha\right]a + O(a^2),$$

$$\Sigma_{(2)}^{W_2}(p,q)|_{\overline{\text{MS}}} = -1 + C_F\left[\frac{23}{6} - 5s_2 - \frac{5}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(3)}^{W_2}(p,q)|_{\overline{\text{MS}}} = C_F\left[\frac{16}{27} + \frac{4}{3}s_2 - \frac{8}{9}\alpha + 4s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{W_2}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{16}{3}s_2 - \frac{71}{27} - 8s_2\alpha + \frac{16}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(5)}^{W_2}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{20}{3}s_2 - \frac{100}{27} - 16s_2\alpha + \frac{26}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(6)}^{W_2}(p,q)|_{\overline{\text{MS}}} = C_F\left[\frac{20}{3}s_2 - \frac{28}{27} - 4s_2\alpha + \frac{14}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^{W_2}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{2}{3}s_2 - \frac{37}{27} - 4s_2\alpha + \frac{2}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(8)}^{W_2}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{40}{3}s_2 - \frac{128}{27} - 8s_2\alpha + \frac{16}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(9)}^{W_2}(p,q)|_{\overline{\text{MS}}} = -\frac{1}{3}a + O(a^2),$$

$$\Sigma_{(10)}^{W_2}(p,q)|_{\overline{\text{MS}}} = C_F\left[\frac{1}{3} - 6s_2\right]a + O(a^2),$$

$$\Sigma_{(1)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(2)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = -1 + C_F[2 - 3s_2 - 2\alpha + 6s_2\alpha]a + O(a^2),$$

$$\Sigma_{(3)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(8)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = -C_F\left[12s_2 - \frac{16}{3} - 12s_2\alpha + \frac{8}{3}\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(7)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = -C_F[6s_2 - 4 - 12s_2\alpha + 2\alpha]a + O(a^2),$$

$$\Sigma_{(5)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(6)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{4}{3}\alpha - \frac{8}{3} - 12s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(9)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(10)}^{\partial W_2}(p,q)|_{\overline{\text{MS}}} = -6C_Fs_2a + O(a^2) \tag{4.1}$$

where the operator superscript label here corresponds to the row of the matrix with that operator on the diagonal. Unlike the previous two cases there is now no symmetry for the Green's function itself when the original operator is inserted. This is because the covariant derivative in the operator only acts on the quark and not the antiquark. So swapping the external momenta in the Green's function is not a symmetric operation. By contrast for the associated total derivative operator this symmetry is still valid which is why there are equivalences between various pairs of amplitudes. Moreover, the actual expressions for the tensors labeled 3 and 5 are proportional to those, respectively, labeled 2 and 3 of the vector case. This is not unexpected because the total derivative operator associated with this moment is effectively the vector current in disguise at the higher level. That the expressions are not precisely the same is due to the fact that one has an extra Lorentz index present at this level so that the projection coefficient into the basis will not have a completely parallel value. Although there is no unique basis for the projection tensors

this agreement at one loop is a check on our calculational setup particularly since with another choice of tensors this proportionality could have been hidden. For the renormalization constants the relation between the divergences has already been noted at three loops in the $\overline{\text{MS}}$ scheme in [36].

Now that we have all the finite parts at the symmetric subtraction point we can define the RI$'$/SMOM scheme renormalization constants. Similar to the vector current this requires care in the case of ∂W_2. This operator is the total derivative of the vector current whose renormalization constant is already determined in all schemes as it is a physical operator. However, only when one takes the contraction of the two free Lorentz indices of ∂W_2 does the divergence of the vector current emerge. Therefore, to have a renormalization consistent with Z^V we have to define $Z_{22}^{W_2}$ to be unity in the RI$'$/SMOM scheme. In the $\overline{\text{MS}}$ case, to contrast with (3.8), the appropriate combination of amplitudes for the piece proportional to $\not{p}$ gives

$$-\frac{[d-2]}{d}\Sigma_{(1)}^{\partial W_2}(p,q)|_{\overline{MS}} - \Sigma_{(2)}^{\partial W_2}(p,q)|_{\overline{MS}}$$

$$+\frac{[d-4]}{4d}\Sigma_{(3)}^{\partial W_2}(p,q)|_{\overline{MS}} + \frac{[d+2]}{2d}\Sigma_{(4)}^{\partial W_2}(p,q)|_{\overline{MS}}$$

$$+\frac{[d-4]}{4d}\Sigma_{(5)}^{\partial W_2}(p,q)|_{\overline{MS}} = \frac{3}{2} + \frac{3}{2}C_F\alpha a + O(a^2) \qquad (4.2)$$

whence it can be recognised that the Slavnov-Taylor identity is preserved. An alternative choice gives the $\not{q}$ contribution but it has the same result as (4.2). This combination is deduced by contracting with $(p+q)_\mu \times (p+q)_\nu$ to ensure each term in the definition of the operator involves the divergence of the vector current. However, for the other operator of level W_2 there is no underlying Slavnov-Taylor identity as the contraction of the two free Lorentz indices is not the divergence of a physical current. Therefore, there is no constraint on how to define the remaining two elements of the W_2 renormalization constant matrix in RI′/SMOM. Indeed in some sense there is an infinite choice. However, for

our purposes here we have chosen to define these renormalization constants by ensuring that there are no $O(a)$ corrections to channels 1 and 2 which both contain the divergences in ϵ. The former contains the off-diagonal counterterm of the mixing matrix while the latter contains both counterterms from the first row of the matrix. Clearly one fixes the off-diagonal one first. Therefore, we have the W_2 matrix of renormalization constants

$$Z_{11}^{W_2} = 1 + C_F\left[\frac{8}{3\epsilon} + \frac{17}{3} - 7s_2 - \frac{1}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$Z_{12}^{W_2} = C_F\left[-\frac{4}{3\epsilon} - \frac{11}{6} + 2s_2 - \frac{1}{3}\alpha\right]a + O(a^2),$$

$$Z_{22}^{W_2} = 1 + O(a^2), \qquad (4.3)$$

where in defining $Z_{22}^{W_2}$ the right hand side of the analogous expression to (4.2) we have ensured that there are no $O(a)$ corrections. Consequently, the RI′/SMOM amplitudes are

$$\Sigma_{(1)}^{W_2}(p,q) = O(a^2), \qquad \Sigma_{(2)}^{W_2}(p,q) = -1 + O(a^2),$$

$$\Sigma_{(3)}^{W_2}(p,q) = C_F\left[\frac{4}{3}s_2 + \frac{16}{27} - \frac{8}{9}\alpha + 4s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{W_2}(p,q) = -C_F\left[\frac{16}{3}s_2 - \frac{71}{27} - 8s_2\alpha + \frac{16}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(5)}^{W_2}(p,q) = -C_F\left[\frac{20}{3}s_2 - \frac{100}{27} - 16s_2\alpha + \frac{26}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(6)}^{W_2}(p,q) = C_F\left[\frac{20}{3}s_2 - \frac{28}{27} - 4s_2\alpha + \frac{14}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^{W_2}(p,q) = -C_F\left[\frac{2}{3}s_2 - \frac{37}{27} - 4s_2\alpha + \frac{2}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(8)}^{W_2}(p,q) = -C_F\left[\frac{40}{3}s_2 - \frac{128}{27} - 8s_2\alpha + \frac{16}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(9)}^{W_2}(p,q) = -\frac{1}{3}a + O(a^2),$$

$$\Sigma_{(10)}^{W_2}(p,q) = C_F\left[\frac{1}{3} - 6s_2\right]a + O(a^2),$$

$$\Sigma_{(1)}^{\partial W_2}(p,q) = \Sigma_{(2)}^{\partial W_2}(p,q) = -1 + C_F[2 - 3s_2 - \alpha + 6s_2\alpha]a + O(a^2),$$

$$\Sigma_{(3)}^{\partial W_2}(p,q) = \Sigma_{(8)}^{\partial W_2}(p,q) = -C_F\left[12s_2 - \frac{16}{3} - 12s_2\alpha + \frac{8}{3}\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{\partial W_2}(p,q) = \Sigma_{(7)}^{\partial W_2}(p,q) = -C_F[6s_2 - 4 - 12s_2\alpha + 2\alpha]a + O(a^2),$$

$$\Sigma_{(5)}^{\partial W_2}(p,q) = \Sigma_{(6)}^{\partial W_2}(p,q) = -C_F\left[\frac{4}{3}\alpha - \frac{8}{3} - 12s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(9)}^{\partial W_2}(p,q) = \Sigma_{(10)}^{\partial W_2}(p,q) = -6C_Fs_2a + O(a^2). \qquad (4.4)$$

At this loop order the finite parts of a substantial number of the amplitudes are the same as their $\overline{MS}$ counterparts. However, we do not expect this to extend necessarily to two loops. Finally, we note that there is a relation between various amplitudes,

$$\Sigma_{(3)}^{\partial W_2}(p, q) - 2\Sigma_{(4)}^{\partial W_2}(p, q) + \Sigma_{(5)}^{\partial W_2}(p, q) = O(a^2), \tag{4.5}$$

which may still be valid at higher loop order.

For the next moment, $n = 3$, the situation is of course more substantial since there are 14 different basis tensors and three operators which mix. First, we record the $\overline{\text{MS}}$ amplitudes to one loop are:

$$\Sigma_{(1)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{1}{108} + \frac{1}{3}s_2 + \frac{11}{54}\alpha - \frac{2}{3}s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(2)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{35}{72} + \frac{5}{27}\alpha - \frac{1}{3}s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(3)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -\frac{1}{3} + C_F\left[\frac{203}{108} - \frac{8}{3}s_2 - \frac{35}{54}\alpha + \frac{8}{3}s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[2s_2 - \frac{7}{9} - \frac{26}{9}s_2\alpha + \frac{58}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(5)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{8}{9}s_2 - \frac{89}{162} - \frac{22}{9}s_2\alpha + \frac{47}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(6)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{32}{9}s_2 - \frac{221}{162} - \frac{44}{9}s_2\alpha + \frac{94}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{20}{3}s_2 - \frac{137}{54} - \frac{100}{9}s_2\alpha + \frac{194}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(8)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{14}{81}\alpha - \frac{1}{6} - \frac{10}{9}s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(9)}^{W_3}(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{2}{9}s_2 + \frac{25}{162} + \frac{2}{9}s_2\alpha + \frac{8}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(10)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{16}{9}s_2 - \frac{133}{162} - \frac{16}{9}s_2\alpha + \frac{17}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(11)}^{W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{28}{3}s_2 - \frac{77}{27} - \frac{44}{9}s_2\alpha + \frac{94}{81}\alpha\right]a + O(a^2),$$

$$\Sigma_{(12)}^{W_3}(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{1}{81} - \frac{2}{9}s_2\right]a + O(a^2),$$

$$\Sigma_{(13)}^{W_3}(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{2}{81} - \frac{4}{9}s_2\right]a + O(a^2),$$

$$\Sigma_{(14)}^{W_3}(p, q)|_{\overline{\text{MS}}} = C_F\left[\frac{19}{81} - \frac{20}{9}s_2\right]a + O(a^2),$$

$$\Sigma_{(1)}^{\partial W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{11}{18} - \frac{2}{3}s_2 + \frac{1}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(2)}^{\partial W_3}(p, q)|_{\overline{\text{MS}}} = -\frac{1}{6} + C_F\left[\frac{1}{3} - \frac{1}{2}s_2 - \frac{1}{3}\alpha + s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(3)}^{\partial W_3}(p, q)|_{\overline{\text{MS}}} = -\frac{1}{3} + C_F\left[\frac{23}{18} - \frac{5}{3}s_2 - \frac{5}{9}\alpha + 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{\partial W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{4}{9}\alpha - \frac{2}{3}s_2 - \frac{8}{27} - 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(5)}^{\partial W_3}(p, q)|_{\overline{\text{MS}}} = -C_F\left[\frac{14}{9}s_2 - \frac{79}{81} - \frac{10}{3}s_2\alpha + \frac{20}{27}\alpha\right]a + O(a^2),$$

$$\Sigma_{(6)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{26}{9}s_2 - \frac{121}{81} - \frac{16}{3}s_2\alpha + \frac{29}{27}\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{10}{3}s_2 - \frac{50}{27} - 8s_2\alpha + \frac{13}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(8)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = C_F\left[\frac{10}{3}s_2 - \frac{14}{27} - 2s_2\alpha + \frac{7}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(9)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = C_F\left[\frac{8}{9}s_2 + \frac{23}{81} + \frac{2}{3}s_2\alpha + \frac{5}{27}\alpha\right]a + O(a^2),$$

$$\Sigma_{(10)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{22}{9}s_2 - \frac{101}{81} - \frac{8}{3}s_2\alpha + \frac{10}{27}\alpha\right]a + O(a^2),$$

$$\Sigma_{(11)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{20}{3}s_2 - \frac{64}{27} - 4s_2\alpha + \frac{8}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(12)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = -\frac{1}{9}a + O(a^2),$$

$$\Sigma_{(13)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_Fs_2a + O(a^2),$$

$$\Sigma_{(14)}^{\partial W_3}(p,q)|_{\overline{\text{MS}}} = C_F\left[\frac{1}{9} - 2s_2\right]a + O(a^2),$$

$$\Sigma_{(1)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -\frac{1}{3} + C_F\left[\frac{2}{3} - s_2 - \frac{2}{3}\alpha + 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(2)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -\frac{1}{3} + C_F\left[\frac{2}{3} - s_2 - \frac{2}{3}\alpha + 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(3)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -\frac{1}{3} + C_F\left[\frac{2}{3} - s_2 - \frac{2}{3}\alpha + 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(11)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[6s_2 - \frac{8}{3} - 6s_2\alpha + \frac{4}{3}\alpha\right]a + O(a^2),$$

$$\Sigma_{(5)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(10)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[4s_2 - \frac{20}{9} - 6s_2\alpha + \frac{10}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(6)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(9)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[2s_2 - \frac{16}{9} - 6s_2\alpha + \frac{8}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(8)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -C_F\left[\frac{2}{3}\alpha - \frac{4}{3} - 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(12)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(13)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = \Sigma_{(14)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} = -2C_Fs_2a + O(a^2). \tag{4.6}$$

While there are more amplitudes, some features are similar to level W_2 such as equivalences between certain amplitudes for $\partial\partial W_3$ and a proportionality with various amplitudes in V and ∂W_2. As this is a level higher there is also a proportionality of certain ∂W_3 amplitudes with W_2. For instance, the channels 2, 3 and 4, respectively, of V, ∂W_2 and $\partial\partial W_3$ are proportional as well as the respective channels 3, 5 and 7. The same is true for W_2 and ∂W_3 for the two respective pairs of channel 3 and 4 as well as 5 and 7. This is of course not unexpected

since the operators are successive total derivatives of the lower level one. However, the renormalization of $\partial\partial W_3$ in the RI′/SMOM scheme is already predetermined for the same reasons as ∂W_2. Though in the case of $\partial\partial W_3$ one has to contract with $(p+q)_\mu(p+q)_\nu(p+q)_\sigma$ to produce the relevant divergence of the vector current. Using $(p+q)_\mu\eta_{\nu\sigma}$ as an alternative nullifies the operator as it is constucted to be traceless and gives nothing nontrivial. With the former contraction this produces the relation between the amplitudes involving $\not{p}$ is

$$\frac{3[d-6]}{4[d+2]}\Sigma^{\partial\partial W_3}_{(1)}(p,q) + \frac{3}{2}\Sigma^{\partial\partial W_3}_{(2)}(p,q) + \frac{3[d-4]}{4[d+2]}\Sigma^{\partial\partial W_3}_{(3)}(p,q) - \frac{[d-10]}{8[d+2]}\Sigma^{\partial\partial W_3}_{(4)}(p,q) - \frac{3}{8}\Sigma^{\partial\partial W_3}_{(5)}(p,q)$$

$$-\frac{3}{8}\Sigma^{\partial\partial W_3}_{(6)}(p,q) - \frac{[d-10]}{8[d+2]}\Sigma^{\partial\partial W_3}_{(7)}(p,q) = -\frac{1}{2} + O(a^2). \tag{4.7}$$

An analogous relation produces the piece involving $\not{q}$ which also has no $O(a)$ piece consistent with the RI$'$ choice for the quark wave function renormalization. Clearly this combination ensures consistency with the Slavnov-Taylor identities.

Again the choice of how to determine the remaining renormalization constants within the RI$'$/SMOM scheme ethos is relatively free. The point of view we take to do this is to build on the previous level. Though there is no reason why one should necessarily make this the definitive choice. By building on the previous level we parallel the way we ensured that the hidden vector current renormalization constant was determined. As ∂W_3 is the total derivative of W_2 we choose that level of the renormalization constant mixing matrix to be the same renormalization constants as the first row of the W_2 matrix. This leaves the first row of W_3 and we then define this by ensuring that after renormalization there are no $O(a)$ corrections in the tensor basis which originally had a divergence in ϵ. As there is mixing within the counterterms we have to determine these in sequence, which was the same as the $\overline{\text{MS}}$ scheme, [36]. Therefore we have

$$Z^{W_3}_{11} = 1 + C_F\left[\frac{25}{6\epsilon} + \frac{307}{36} - 9s_2 - \frac{4}{9}\alpha + 8s_2\alpha\right]a + O(a^2),$$

$$Z^{W_3}_{12} = C_F\left[-\frac{3}{2\epsilon} - \frac{103}{36} + 2s_2 + \frac{1}{9}\alpha - 2s_2\alpha\right]a + O(a^2),$$

$$Z^{W_3}_{13} = C_F\left[-\frac{1}{2\epsilon} - \frac{1}{36} - s_2 - \frac{11}{18}\alpha + 2s_2\alpha\right]a + O(a^2),$$

$$Z^{W_3}_{22} = 1 + C_F\left[\frac{8}{3\epsilon} + \frac{17}{3} - 7s_2 - \frac{1}{3}\alpha + 6s_2\alpha\right]a + O(a^2),$$

$$Z^{W_3}_{23} = C_F\left[-\frac{4}{3\epsilon} - \frac{11}{6} + 2s_2 - \frac{1}{3}\alpha\right]a + O(a^2),$$

$$Z^{W_3}_{33} = 1 + O(a^2). \tag{4.8}$$

Once these have been determined the RI$'$/SMOM scheme amplitudes are

$$\Sigma^{W_3}_{(1)}(p,q) = O(a^2), \quad \Sigma^{W_3}_{(2)}(p,q) = O(a^2),$$

$$\Sigma^{W_3}_{(3)}(p,q) = -\frac{1}{3} + O(a^2),$$

$$\Sigma^{W_3}_{(4)}(p,q) = -C_F\left[2s_2 - \frac{7}{9} - \frac{26}{9}s_2\alpha + \frac{58}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(5)}(p,q) = -C_F\left[\frac{8}{9}s_2 - \frac{89}{162} - \frac{22}{9}s_2\alpha + \frac{47}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(6)}(p,q) = -C_F\left[\frac{32}{9}s_2 - \frac{221}{162} - \frac{44}{9}s_2\alpha + \frac{94}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(7)}(p,q) = -C_F\left[\frac{20}{3}s_2 - \frac{137}{54} - \frac{100}{9}s_2\alpha + \frac{194}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(8)}(p,q) = -C_F\left[\frac{14}{81}\alpha - \frac{1}{6} - \frac{10}{9}s_2\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(9)}(p,q) = C_F\left[\frac{2}{9}s_2 + \frac{25}{162} + \frac{2}{9}s_2\alpha + \frac{8}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(10)}(p,q) = -C_F\left[\frac{16}{9}s_2 - \frac{133}{162} - \frac{16}{9}s_2\alpha + \frac{17}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(11)}(p,q) = -C_F\left[\frac{28}{3}s_2 - \frac{77}{27} - \frac{44}{9}s_2\alpha + \frac{94}{81}\alpha\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(12)}(p,q) = -C_F\left[\frac{2}{9}s_2 - \frac{4}{81}\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(13)}(p,q) = -C_F\left[\frac{4}{9}s_2 - \frac{2}{81}\right]a + O(a^2),$$

$$\Sigma^{W_3}_{(14)}(p,q) = -C_F\left[\frac{20}{9}s_2 - \frac{19}{81}\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(1)}(p,q) = O(a^2), \quad \Sigma^{\partial W_3}_{(2)}(p,q) = -\frac{1}{6} + O(a^2),$$

$$\Sigma^{\partial W_3}_{(3)}(p,q) = -\frac{1}{3} + O(a^2),$$

$$\Sigma^{\partial W_3}_{(4)}(p,q) = -C_F\left[\frac{4}{9}\alpha - \frac{2}{3}s_2 - \frac{8}{27} - 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(5)}(p,q) = -C_F\left[\frac{14}{9}s_2 - \frac{79}{81} - \frac{10}{3}s_2\alpha + \frac{20}{27}\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(6)}(p,q) = -C_F\left[\frac{26}{9}s_2 - \frac{121}{81} - \frac{16}{3}s_2\alpha + \frac{29}{27}\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(7)}(p,q) = -C_F\left[\frac{10}{3}s_2 - \frac{50}{27} - 8s_2\alpha + \frac{13}{9}\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(8)}(p,q) = C_F\left[\frac{10}{3}s_2 - \frac{14}{27} - 2s_2\alpha + \frac{7}{9}\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(9)}(p,q) = C_F\left[\frac{8}{9}s_2 + \frac{23}{81} + \frac{2}{3}s_2\alpha + \frac{5}{27}\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(10)}(p,q) = -C_F\left[\frac{22}{9}s_2 - \frac{101}{81} - \frac{8}{3}s_2\alpha + \frac{10}{27}\alpha\right]a + O(a^2),$$

$$\Sigma^{\partial W_3}_{(11)}(p,q) = -C_F\left[\frac{20}{3}s_2 - \frac{64}{27} - 4s_2\alpha + \frac{8}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(12)}^{\partial W_3}(p,q) = -\frac{1}{9}a + O(a^2),$$

$$\Sigma_{(13)}^{\partial W_3}(p,q) = -C_F s_2 a + O(a^2),$$

$$\Sigma_{(14)}^{\partial W_3}(p,q) = C_F\left[\frac{1}{9} - 2s_2\right]a + O(a^2),$$

$$\Sigma_{(1)}^{\partial\partial W_3}(p,q) = \Sigma_{(2)}^{\partial\partial W_3}(p,q) = \Sigma_{(3)}^{\partial\partial W_3}(p,q)$$
$$= -\frac{1}{3} + C_F\left[\frac{2}{3} - s_2 - \frac{1}{3}\alpha + 2s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(4)}^{\partial\partial W_3}(p,q) = \Sigma_{(11)}^{\partial\partial W_3}(p,q)$$
$$= -C_F\left[6s_2 - \frac{8}{3} - 6s_2\alpha + \frac{4}{3}\alpha\right]a + O(a^2),$$

$$\Sigma_{(5)}^{\partial\partial W_3}(p,q) = \Sigma_{(10)}^{\partial\partial W_3}(p,q)$$
$$= -C_F\left[4s_2 - \frac{20}{9} - 6s_2\alpha + \frac{10}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(6)}^{\partial\partial W_3}(p,q) = \Sigma_{(9)}^{\partial\partial W_3}(p,q)$$
$$= -C_F\left[2s_2 - \frac{16}{9} - 6s_2\alpha + \frac{8}{9}\alpha\right]a + O(a^2),$$

$$\Sigma_{(7)}^{\partial\partial W_3}(p,q) = \Sigma_{(8)}^{\partial\partial W_3}(p,q)$$
$$= -C_F\left[\frac{2}{3}\alpha - \frac{4}{3} - 6s_2\alpha\right]a + O(a^2),$$

$$\Sigma_{(12)}^{\partial\partial W_3}(p,q) = \Sigma_{(13)}^{\partial\partial W_3}(p,q) = \Sigma_{(14)}^{\partial\partial W_3}(p,q)$$
$$= -2C_F s_2 a + O(a^2). \tag{4.9}$$

As with the lower moment there are similar relations to one loop. First, there is an obvious parallel with the vector and total derivative operator for $n = 2$ with the double total derivative of the third moment which was noted in [36]. In addition there are cross connections with the central row of the mixing matrix. Specifically, the amplitudes labeled 6 and 8 of W_2 are proportional to 8 and 11 of ∂W_3 respectively for the same reasons as before. Further, there is another apparent relation within the level since

$$\Sigma_{(5)}^{\partial\partial W_3}(p,q) - 2\Sigma_{(6)}^{\partial\partial W_3}(p,q) + \Sigma_{(7)}^{\partial\partial W_3}(p,q) = O(a^2) \tag{4.10}$$

in addition to the reflection of the one already noted for moment $n = 2$. As an aside we note that in highlighting relations between amplitudes, we have concentrated on those which are true for all values of α. There appears to be other relations which are only valid to one loop in the Landau gauge and since the case for regarding these as significant is diminished because of the apparent gauge dependence we do not draw attention to them.

As our computations have been at one loop the conversion functions that are used to convert from the $\overline{\text{MS}}$ scheme to the RI′/SMOM scheme are in effect the finite parts of the renormalization constants themselves. See, for example, [47]. Therefore, in order to appreciate the magnitude of the

one loop contribution it is a straightforward exercise to evaluate the finite parts numerically. It turns out that the correction increases in value with the operator moment, in parallel, for example, with the numerical value of the one loop anomalous dimension coefficient. If we take the Landau gauge case and compare the finite parts, the moments $n = 2$ and 3 are, respectively, 3.8436 and 6.1839 where we use the (11) element for the latter two or equivalently the diagonal of the W_3 matrix. This is with ignoring the common color factor, C_F, and coupling constant, a. Therefore, for higher moments it would seem that one would require higher loop corrections in order to have a more reliable convergence of the perturbative series for the conversion functions. Comparing with the RI′ scheme results of [28–30] the respective numbers are 3.444 and 5.9444. Unlike the situation with the mass operator the RI′/SMOM scheme corrections for W_2 and W_3 are slightly larger. Though in some sense this is not a fair comparison because of the mixing. The RI′ scheme is based on a specific momentum routing in the Green's function which is blind to the off-diagonal matrix element of the renormalization constant matrix. Equally the convergence of the RI′/SMOM series could be improved by choosing a different combination of the finite parts to define the actual renormalization constant since there appears to be a large degree of freedom in doing this. Only a higher order computation could give insight into this, [46].

V. DISCUSSION

We have determined the renormalization constants at one loop for the twist-2 flavor nonsinglet operators with moments $n = 2$ and 3 in the RI′/SMOM scheme and have taken account of operator mixing. In the Landau gauge the correction increases in value with the operator moment. So it would appear that the series for the conversion function will converge slower at high moment. Of course, a one loop computation is not sufficient to make a definitive statement since the higher loop corrections may produce an improvement. This is currently under investigation, [46]. Equally it would be useful to see how the lattice measurements in this symmetric scheme for the deep inelastic scattering operators compares with the same analysis in the earlier RI′ scheme. Though it may be the case that not all the various tensor projections would give a clear accurate numerical signal. Finally, while the RI′/SMOM scheme is designed in order to circumvent infrared problems, one is still restricted to the Landau gauge due to the fact that the Green's function of Fig. 1 is a gauge dependent quantity. While this is not a problem for the high energy régime, it may be the case that in the infrared there are additional complications with Gribov copies and hence the definitive measurement of operator matrix elements.

ACKNOWLEDGMENTS

APPENDIX A: PROJECTORS

In this appendix we record in succession the basis of projection tensors we have used in each of the various cases. These were denoted earlier by the general notation $\mathcal{P}^i_{(k)\mu_1\ldots\mu_{n_i}}(p,q)$. The matrix, $\mathcal{M}^i_{kl}$, in the second part of each subsection contains the coefficients associated with each basis projection tensor, which allows one to project out the amplitudes $\Sigma^{\mathcal{O}^i}_{(k)}(p,q)$. It is constructed by first determining the matrix

$$\mathcal{N}^i_{kl} = \mathcal{P}^i_{(k)\mu_1\ldots\mu_{n_i}}(p,q)\mathcal{P}^{i\mu_1\ldots\mu_{n_i}}_{(l)}(p,q)|_{p^2=q^2=-\mu^2} \quad \text{(A1)}$$

where there is no summation over the label i, and k and l index the projection tensors. The matrix $\mathcal{N}^i_{kl}$ is symmetric in k and l. Given that the elements are contractions of Lorentz tensors in d-dimensions then the elements of $\mathcal{N}^i_{kl}$ are polynomials in d. Finally, $\mathcal{M}^i_{kl}$ is the inverse of $\mathcal{N}^i_{kl}$ and then

$$\Sigma^{\mathcal{O}^i}_{(k)}(p,q) = \mathcal{M}^i_{kl}\mathcal{P}^{i\mu_1\ldots\mu_{n_i}}_{(l)}(p,q)$$
$$\times \left(\langle\psi(p)\mathcal{O}^i_{\mu_1\ldots\mu_{n_i}}(-p-q)\bar\psi(q)\rangle\right)|_{p^2=q^2=-\mu^2} \quad \text{(A2)}$$

where again there is no summation over the level label i.

1. Scalar

$$\mathcal{P}^S_{(1)}(p,q) = \Gamma_{(0)}, \qquad \mathcal{P}^S_{(2)}(p,q) = \frac{1}{\mu^2}\Gamma^{pq}_{(2)} \quad \text{(A3)}$$

$$\mathcal{M}^S = \frac{1}{12}\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}. \quad \text{(A4)}$$

In order to compare with [33], the mapping between the amplitudes is

$$\Sigma^S_{(1)}(p,q) = -A_S, \qquad \Sigma^S_{(2)}(p,q) = -2C_S \quad \text{(A5)}$$

where the notation of the right hand side is that of [33].

2. Vector

$$\mathcal{P}^V_{(1)\mu}(p,q) = \gamma_\mu, \quad \mathcal{P}^V_{(2)\mu}(p,q) = \frac{p^\mu\not{p}}{\mu^2},$$

$$\mathcal{P}^V_{(3)\mu}(p,q) = \frac{p_\mu\not{q}}{\mu^2}, \quad \mathcal{P}^V_{(4)\mu}(p,q) = \frac{q_\mu\not{p}}{\mu^2},$$

$$\mathcal{P}^V_{(5)\mu}(p,q) = \frac{q_\mu\not{q}}{\mu^2}, \quad \mathcal{P}^V_{(6)\mu}(p,q) = \frac{1}{\mu^2}\Gamma_{(3)\mu pq}. \quad \text{(A6)}$$

$$\mathcal{M}^V = \frac{1}{36(d-2)}\begin{pmatrix} 9 & 12 & 6 & 6 & 12 & 0 \\ 12 & 16(d-1) & 8(d-1) & 8(d-1) & 4(d+2) & 0 \\ 6 & 8(d-1) & 4(4d-7) & 4(d-1) & 8(d-1) & 0 \\ 6 & 8(d-1) & 4(d-1) & 4(4d-7) & 8(d-1) & 0 \\ 12 & 4(d+2) & 8(d-1) & 8(d-1) & 16(d-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{pmatrix}. \quad \text{(A7)}$$

For comparison with [33] the amplitudes are related by

$$\Sigma^V_{(1)}(p,q) = -A_V + 2B_V + \frac{1}{2}C_V + \frac{1}{2}D_V, \qquad \Sigma^V_{(2)}(p,q) = \Sigma^V_{(5)}(p,q) = 2B_V,$$
$$\Sigma^V_{(3)}(p,q) = \Sigma^V_{(4)}(p,q) = -C_V - D_V, \qquad \Sigma^V_{(6)}(p,q) = C_V - D_V. \quad \text{(A8)}$$

3. Tensor

$$\mathcal{P}^T_{(1)\mu\nu}(p,q) = \Gamma_{(2)\mu\nu}, \quad \mathcal{P}^T_{(2)\mu\nu}(p,q) = \frac{1}{\mu^2}[p_\mu q_\nu - p_\nu q_\mu]\Gamma_{(0)}, \quad \mathcal{P}^T_{(3)\mu\nu}(p,q) = \frac{1}{\mu^2}[\Gamma_{(2)\mu p}p_\nu - \Gamma_{(2)\nu p}p_\mu],$$

$$\mathcal{P}^T_{(4)\mu\nu}(p,q) = \frac{1}{\mu^2}[\Gamma_{(2)\mu p}q_\nu - \Gamma_{(2)\nu p}q_\mu], \quad \mathcal{P}^T_{(5)\mu\nu}(p,q) = \frac{1}{\mu^2}[\Gamma_{(2)\mu q}p_\nu - \Gamma_{(2)\nu q}p_\mu], \quad \mathcal{P}^T_{(6)\mu\nu}(p,q) = \frac{1}{\mu^2}[\Gamma_{(2)\mu q}q_\nu - \Gamma_{(2)\nu q}q_\mu],$$

$$\mathcal{P}^T_{(7)\mu\nu}(p,q) = \frac{1}{\mu^4}[\Gamma_{(2)pq}p_\mu q_\nu - \Gamma_{(2)pq}p_\nu q_\mu], \quad \mathcal{P}^T_{(8)\mu\nu}(p,q) = \frac{1}{\mu^2}\Gamma_{(4)\mu\nu pq}. \quad \text{(A9)}$$

To save space for the projection matrix in this case, we have partitioned the 8×8 matrix into four submatrices. We have

$$\mathcal{M}^T = \frac{1}{36(d-2)(d-3)} \begin{pmatrix} \mathcal{M}^T_{11} & \mathcal{M}^T_{12} \\ \mathcal{M}^T_{21} & \mathcal{M}^T_{22} \end{pmatrix}, \qquad (\text{A10})$$

$$\mathcal{M}^T_{11} = \begin{pmatrix} -9 & 0 & -12 & -6 \\ 0 & 6(d-2)(d-3) & 0 & 0 \\ -12 & 0 & -8(d-1) & -4(d-1) \\ -6 & 0 & -4(d-1) & -4(2d-5) \end{pmatrix},$$

$$\mathcal{M}^T_{12} = \begin{pmatrix} -6 & -12 & -12 & -6 \\ 0 & 0 & 0 & 0 \\ -4(d-1) & -2(d+5) & -8(d-1) & 0 \\ -2(d-1) & -4(d-1) & -4(d-1) & 0 \end{pmatrix},$$

$$\mathcal{M}^T_{21} = \begin{pmatrix} -6 & 0 & -4(d-1) & -2(d-1) \\ -12 & 0 & -2(d+5) & -4(d-1) \\ -12 & 0 & -8(d-1) & -4(d-1) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}^T_{22} = \begin{pmatrix} -4(2d-5) & -4(d-1) & -4(d-1) & 0 \\ -4(d-1) & -8(d-1) & -8(d-1) & 0 \\ -4(d-1) & -8(d-1) & -8(d-1)(d-2) & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix}. \qquad (\text{A11})$$

In the tensor case the relations with the amplitudes of [33] are

$$\Sigma^T_{(1)}(p, q) = -A_T + \frac{1}{2} C_T, \qquad \Sigma^T_{(2)}(p, q) = -2B_T + C_T, \qquad \Sigma^T_{(3)}(p, q) = \Sigma^T_{(6)}(p, q) = 2C_T,$$

$$\Sigma^T_{(4)}(p, q) = \Sigma^T_{(5)}(p, q) = -2B_T + C_T, \qquad \Sigma^T_{(8)}(p, q) = 2B_T - C_T. \qquad (\text{A12})$$

4. Wilson 2

$$\mathcal{P}^{W_2}_{(1)\mu\nu}(p, q) = \gamma_\mu p_\nu + \gamma_\nu p_\mu - \frac{2}{d} \not{p} \eta_{\mu\nu}, \qquad \mathcal{P}^{W_2}_{(2)\mu\nu}(p, q) = \gamma_\mu q_\nu + \gamma_\nu q_\mu - \frac{2}{d} \not{q} \eta_{\mu\nu},$$

$$\mathcal{P}^{W_2}_{(3)\mu\nu}(p, q) = \not{p} \left[\frac{1}{\mu^2} p_\mu p_\nu + \frac{1}{d} \eta_{\mu\nu} \right], \qquad \mathcal{P}^{W_2}_{(4)\mu\nu}(p, q) = \not{p} \left[\frac{1}{\mu^2} p_\mu q_\nu + \frac{1}{\mu^2} q_\mu p_\nu - \frac{1}{d} \eta_{\mu\nu} \right],$$

$$\mathcal{P}^{W_2}_{(5)\mu\nu}(p, q) = \not{p} \left[\frac{1}{\mu^2} q_\mu q_\nu + \frac{1}{d} \eta_{\mu\nu} \right], \qquad \mathcal{P}^{W_2}_{(6)\mu\nu}(p, q) = \not{q} \left[\frac{1}{\mu^2} p_\mu p_\nu + \frac{1}{d} \eta_{\mu\nu} \right],$$

$$\mathcal{P}^{W_2}_{(7)\mu\nu}(p, q) = \not{q} \left[\frac{1}{\mu^2} p_\mu q_\nu + \frac{1}{\mu^2} q_\mu p_\nu - \frac{1}{d} \eta_{\mu\nu} \right], \qquad \mathcal{P}^{W_2}_{(8)\mu\nu}(p, q) = \not{q} \left[\frac{1}{\mu^2} q_\mu q_\nu + \frac{1}{d} \eta_{\mu\nu} \right],$$

$$\mathcal{P}^{W_2}_{(9)\mu\nu}(p, q) = \frac{1}{\mu^2} [\Gamma_{(3)\mu pq} p_\nu + \Gamma_{(3)\nu pq} p_\mu], \qquad \mathcal{P}^{W_2}_{(10)\mu\nu}(p, q) = \frac{1}{\mu^2} [\Gamma_{(3)\mu pq} q_\nu + \Gamma_{(3)\nu pq} q_\mu]. \qquad (\text{A13})$$

Similar to the previous case we have subdivided the 10×10 matrix here into four submatrices in order to ease the presentation. We have

$$\mathcal{M}^{W_2} = -\frac{1}{108(d-2)} \begin{pmatrix} \mathcal{M}^{W_2}_{11} & \mathcal{M}^{W_2}_{12} \\ \mathcal{M}^{W_2}_{21} & \mathcal{M}^{W_2}_{22} \end{pmatrix},$$

$$\mathcal{M}^{W_2}_{11} = \begin{pmatrix} 18 & 9 & 48 & 24 & 12 \\ 9 & 18 & 24 & 30 & 24 \\ 48 & 24 & 64(d+1) & 32(d+1) & 16(d+4) \\ 24 & 30 & 32(d+1) & 8(5d-1) & 8(4d+1) \\ 12 & 24 & 16(d+4) & 8(4d+1) & 32(2d-1) \end{pmatrix},$$

$$\mathcal{M}^{W_2}_{12} = \begin{pmatrix} 24 & 30 & 24 & 0 & 0 \\ 12 & 24 & 48 & 0 & 0 \\ 32(d+1) & 16(d+4) & 8(d+10) & 0 & 0 \\ 16(d+1) & 20(d+1) & 16(d+4) & 0 & 0 \\ 8(d+4) & 16(d+1) & 32(d+1) & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_{21}^{W_2} = \begin{pmatrix} 24 & 12 & 32(d+1) & 16(d+1) & 8(d+4) \\ 30 & 24 & 16(d+4) & 20(d+1) & 16(d+1) \\ 24 & 48 & 8(d+10) & 16(d+4) & 32(d+1) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_{22}^{W_2} = \begin{pmatrix} 32(2d-1) & 8(4d+1) & 16(d+4) & 0 & 0 \\ 8(4d+1) & 8(5d-1) & 32(d+1) & 0 & 0 \\ 16(d+4) & 32(d+1) & 64(d+1) & 0 & 0 \\ 0 & 0 & 0 & -24 & -12 \\ 0 & 0 & 0 & -12 & -24 \end{pmatrix}. \tag{A14}$$

5. Wilson 3

$$\mathcal{P}_{(1)\mu\nu\sigma}^{W_3}(p,q) = \frac{1}{\mu^2}[\gamma_\mu p_\nu p_\sigma + \gamma_\nu p_\sigma p_\mu + \gamma_\sigma p_\mu p_\nu] + \frac{1}{[d+2]}[\gamma_\mu \eta_{\nu\sigma} + \gamma_\nu \eta_{\sigma\mu} + \gamma_\sigma \eta_{\mu\nu}]$$
$$- \frac{2\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu],$$

$$\mathcal{P}_{(2)\mu\nu\sigma}^{W_3}(p,q) = \frac{1}{\mu^2}[\gamma_\mu p_\nu q_\sigma + \gamma_\nu p_\sigma q_\mu + \gamma_\sigma p_\mu q_\nu + \gamma_\mu q_\nu p_\sigma + \gamma_\nu q_\sigma p_\mu + \gamma_\sigma q_\mu p_\nu]$$
$$- \frac{1}{[d+2]}[\gamma_\mu \eta_{\nu\sigma} + \gamma_\nu \eta_{\sigma\mu} + \gamma_\sigma \eta_{\mu\nu}] - \frac{2\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} q_\nu]$$
$$- \frac{2\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu],$$

$$\mathcal{P}_{(3)\mu\nu\sigma}^{W_3}(p,q) = \frac{1}{\mu^2}[\gamma_\mu q_\nu q_\sigma + \gamma_\nu q_\sigma q_\mu + \gamma_\sigma q_\mu q_\nu] + \frac{1}{[d+2]}[\gamma_\mu \eta_{\nu\sigma} + \gamma_\nu \eta_{\sigma\mu} + \gamma_\sigma \eta_{\mu\nu}]$$
$$- \frac{2\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} q_\nu],$$

$$\mathcal{P}_{(4)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{p}}{\mu^4} p_\mu p_\nu p_\sigma + \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu],$$

$$\mathcal{P}_{(5)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{p}}{\mu^4}[p_\mu p_\nu q_\sigma + p_\mu q_\nu p_\sigma + q_\mu p_\nu p_\sigma] - \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu - \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu],$$

$$\mathcal{P}_{(6)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{p}}{\mu^4}[p_\mu q_\nu q_\sigma + q_\mu p_\nu q_\sigma + q_\mu q_\nu p_\sigma] + \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu - \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu],$$

$$\mathcal{P}_{(7)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{p}}{\mu^4} q_\mu q_\nu q_\sigma + \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} q_\nu],$$

$$\mathcal{P}_{(8)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{q}}{\mu^4} p_\mu p_\nu p_\sigma + \frac{\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu],$$

$$\mathcal{P}_{(9)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{q}}{\mu^4}[p_\mu p_\nu q_\sigma + p_\mu q_\nu p_\sigma + q_\mu p_\nu p_\sigma] - \frac{\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu - \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu],$$

$$\mathcal{P}_{(10)\mu\nu\sigma}^{W_3}(p,q) = \frac{\not{q}}{\mu^4}[p_\mu q_\nu q_\sigma + q_\mu p_\nu q_\sigma + q_\mu q_\nu p_\sigma] + \frac{\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu - \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu],$$

$$\mathcal{P}^{W_3}_{(11)\mu\nu\sigma}(p,q) = \frac{\slashed{q}}{\mu^4}q_\mu q_\nu q_\sigma + \frac{\slashed{q}}{[d+2]\mu^2}[\eta_{\mu\nu}q_\sigma + \eta_{\nu\sigma}q_\mu + \eta_{\sigma\mu}q_\nu],$$

$$\mathcal{P}^{W_3}_{(12)\mu\nu\sigma}(p,q) = \frac{1}{\mu^4}[\Gamma_{(3)\mu pq}p_\nu p_\sigma + \Gamma_{(3)\nu pq}p_\mu p_\sigma + \Gamma_{(3)\sigma pq}p_\mu p_\nu] + \frac{1}{[d+2]\mu^2}[\Gamma_{(3)\mu pq}\eta_{\nu\sigma} + \Gamma_{(3)\nu pq}\eta_{\mu\sigma} + \Gamma_{(3)\sigma pq}\eta_{\mu\nu}],$$

$$\mathcal{P}^{W_3}_{(13)\mu\nu\sigma}(p,q) = \frac{1}{\mu^4}[\Gamma_{(3)\mu pq}p_\nu q_\sigma + \Gamma_{(3)\nu pq}p_\mu q_\sigma + \Gamma_{(3)\sigma pq}p_\mu q_\nu + \Gamma_{(3)\mu pq}q_\nu p_\sigma + \Gamma_{(3)\nu pq}q_\mu p_\sigma + \Gamma_{(3)\sigma pq}q_\mu p_\nu]$$
$$- \frac{1}{[d+2]\mu^2}[\Gamma_{(3)\mu pq}\eta_{\nu\sigma} + \Gamma_{(3)\nu pq}\eta_{\mu\sigma} + \Gamma_{(3)\sigma pq}\eta_{\mu\nu}],$$

$$\mathcal{P}^{W_3}_{(14)\mu\nu\sigma}(p,q) = \frac{1}{\mu^4}[\Gamma_{(3)\mu pq}q_\nu q_\sigma + \Gamma_{(3)\nu pq}q_\mu q_\sigma + \Gamma_{(3)\sigma pq}q_\mu q_\nu] + \frac{1}{[d+2]\mu^2}[\Gamma_{(3)\mu pq}\eta_{\nu\sigma} + \Gamma_{(3)\nu pq}\eta_{\mu\sigma} + \Gamma_{(3)\sigma pq}\eta_{\mu\nu}].$$

$$(A15)$$

Here the partition of the 14×14 matrix is into ten nonzero submatrices. We have

$$\mathcal{M}^{W_3} = \frac{1}{2106d(d-2)}\begin{pmatrix} \mathcal{M}^{W_3}_{11} & \mathcal{M}^{W_3}_{12} & \mathcal{M}^{W_3}_{13} & 0 \\ \mathcal{M}^{W_3}_{21} & \mathcal{M}^{W_3}_{22} & \mathcal{M}^{W_3}_{23} & 0 \\ \mathcal{M}^{W_3}_{31} & \mathcal{M}^{W_3}_{32} & \mathcal{M}^{W_3}_{33} & 0 \\ 0 & 0 & 0 & \mathcal{M}^{W_3}_{44} \end{pmatrix},$$

where the $\Gamma_{(3)}$ sector corresponds to the lower outer corner submatrix or subspace.

$$\mathcal{M}^{W_3}_{11} = \begin{pmatrix} 312(d+1) & 156(d+1) & 78(d+4) & 1248(d+1) \\ 156(d+1) & 39(5d+2) & 156(d+1) & 624(d+1) \\ 78(d+4) & 156(d+1) & 312(d+1) & 312(d+4) \\ 1248(d+1) & 624(d+1) & 312(d+4) & 1664(d+3)(d+1) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{12} = \begin{pmatrix} 624(d+1) & 312(d+2) & 156(d+4) & 624(d+1) \\ 312(2d+1) & 156(3d+2) & 312(d+1) & 312(d+1) \\ 156(3d+4) & 624(d+1) & 624(d+1) & 156(d+4) \\ 832(d+3)(d+1) & 416(d+6)(d+1) & 208(d+12)(d+1) & 832(d+3)(d+1) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{13} = \begin{pmatrix} 624(d+1) & 156(3d+4) & 312(d+4) \\ 156(3d+2) & 312(2d+1) & 624(d+1) \\ 312(d+2) & 624(d+1) & 1248(d+1) \\ 416(d+6)(d+1) & 208(d+12)(d+1) & 104(d^2+22d+48) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{21} = \begin{pmatrix} 624(d+1) & 312(2d+1) & 156(3d+4) & 832(d+3)(d+1) \\ 312(d+2) & 156(3d+2) & 624(d+1) & 416(d+6)(d+1) \\ 156(d+4) & 312(d+1) & 624(d+1) & 208(d+12)(d+1) \\ 624(d+1) & 312(d+1) & 156(d+4) & 832(d+3)(d+1) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{22} = (d+1)\begin{pmatrix} 416(2d+3) & 624(d+2) & 104\frac{(4d^2+25d+12)}{(d+1)} & 416(d+3) \\ 624(d+2) & 208\frac{(4d^2+7d+6)}{(d+1)} & 416(2d+3) & 208(d+6) \\ 104\frac{(4d^2+25d+12)}{(d+1)} & 416(2d+3) & 416(4d+3) & 104(d+12) \\ 416(d+3) & 208(d+6) & 104(d+12) & 416(4d+3) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{23} = \begin{pmatrix} 416(d+3)(d+1) & 312(d^2+7d+4) & 208(d+12)(d+1) \\ 312(d+2)^2 & 416(d+3)(d+1) & 416(d+6)(d+1) \\ 208(d+6)(d+1) & 416(d+3)(d+1) & 832(d+3)(d+1) \\ 416(2d+3)(d+1) & 104(4d^2+25d+12) & 208(d+12)(d+1) \end{pmatrix},$$

$$\mathcal{M}_{31}^{W_3} = \begin{pmatrix} 624(d+1) & 156(3d+2) & 312(d+2) & 416(d+6)(d+1) \\ 156(3d+4) & 312(2d+1) & 624(d+1) & 208(d+12)(d+1) \\ 312(d+4) & 624(d+1) & 1248(d+1) & 104(d^2+22d+48) \end{pmatrix},$$

$$\mathcal{M}_{32}^{W_3} = \begin{pmatrix} 416(d+3)(d+1) & 312(d+2)^2 & 208(d+6)(d+1) & 416(2d+3)(d+1) \\ 312(d^2+7d+4) & 416(d+3)(d+1) & 416(d+3)(d+1) & 104(4d^2+25d+12) \\ 208(d+12)(d+1) & 416(d+6)(d+1) & 832(d+3)(d+1) & 208(d+12)(d+1) \end{pmatrix},$$

$$\mathcal{M}_{33}^{W_3} = \begin{pmatrix} 208(4d^2+7d+6) & 624(d+2)(d+1) & 416(d+6)(d+1) \\ 624(d+2)(d+1) & 416(2d+3)(d+1) & 832(d+3)(d+1) \\ 416(d+6)(d+1) & 832(d+3)(d+1) & 1664(d+3)(d+1) \end{pmatrix},$$

$$\mathcal{M}_{44}^{W_3} = \begin{pmatrix} -416(d+1) & -208(d+1) & -104(d+4) \\ -208(d+1) & -52(5d+2) & -208(d+1) \\ -104(d+4) & -208(d+1) & -416(d+1) \end{pmatrix}. \tag{A16}$$

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