

Central charge bounds in 4D conformal field theoryRiccardo Rattazzi,¹ Slava Rychkov,² and Alessandro Vichi¹¹*Institut de Théorie des Phénomènes Physiques, EPFL, CH-1015 Lausanne, Switzerland*²*Laboratoire de Physique Théorique, École Normale Supérieure, and Faculté de Physique, Université Pierre et Marie Curie, France*
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We derive model-independent lower bounds on the stress tensor central charge C_T in terms of the operator content of a 4-dimensional conformal field theory. More precisely, C_T is bounded from below by a universal function of the dimensions of the lowest and second-lowest scalars present in the conformal field theory. The method uses the crossing symmetry constraint of the 4-point function, analyzed by means of the conformal block decomposition.

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I. INTRODUCTION

Conformal field theory (CFT) was born to describe fixed points of renormalization group flows. This still remains its main vocation, although it has many other applications as well. In 2D, the constraining power of conformal symmetry is tremendous and often leads to an exact solution of the theory. In this paper we are concerned with the much less constrained 4D case. Presumably, there are lots of interacting 4D CFTs out there, but we do not know much about them. For instance, “conformal windows” of gauge theories should provide lots of examples, but even the spectrum of these theories remains unknown (except for the chiral ring in the supersymmetric case).

In absence of an exact solution, it is natural to look for universal constraints, satisfied all over the “landscape” of CFTs. Two years ago [1,2] we found one such constraint, related to a gap in the spectrum of operator dimensions. Namely, we examined the maximal possible dimension of the lowest-dimension operator appearing in the operator product expansion (OPE) of two scalar operators. We found that if one fixes the dimension of external scalars, the lowest-dimension operator that appears cannot have a dimension above a certain model-independent bound.

Last year a different constraint was presented in [3]: it was found that the OPE coefficient of three scalars cannot exceed a certain universal $\mathcal{O}(1)$ bound, which depends only on their dimensions. One can call this bound a universal limit on the interaction strength.

The above results were obtained by using consistency between OPE and crossing symmetry (also known as OPE associativity) as a constraining principle. The principle itself was first proposed more than 35 years ago by Polyakov [4], but until our work no general results were obtained from it.

We would like to mention a parallel line of development in 2D CFT, where Hellerman [5] and others [6,7] have also studied constraints on the gap in the spectrum, in terms of the central charge. Their main constraining principle is modular invariance, which is limited to 2D, but morally not so different from OPE associativity (both are related to

the change of foliation when quantizing the theory). Also interesting is the role played by all these universal constraints in an ambitious program of exploring the space of CFTs initiated by Douglas [8].

Coming back to 4D, in this paper we will explore the following question: *What can we say about the central charge of the theory, if we know something about the spectrum of its operator dimensions?* More precisely, we will assume that the theory contains a scalar operator of a given dimension. Under this assumption, we will show that the central charge must be bigger than a certain universal lower bound. This is natural, since the central charge “measures” the number of degrees of freedom, and by assumption we know that our theory is not trivial.

In a certain range of scalar dimensions we will be able to show that the central charge is necessarily bigger than that of the free scalar (“an interacting theory has more degrees of freedom”).

As we will explain below, the problem of bounding the central charge *from below* is equivalent to the problem of bounding *from above* the OPE coefficient of the stress tensor in the scalar times scalar OPE. The similarity with the problem analyzed in [3] is then clear, and we will be able to use the method of that paper. To make connection with the results of [1,2] we will also study how the central charge bound improves as a function of the assumed gap in the scalar sector of the OPE.

Everywhere we assume that we are dealing with a unitary theory. In a nonunitary theory, the central charge may well be zero (or negative) without the CFT being trivial, so our question would not even make sense.

II. FORMULATION OF THE PROBLEM

We begin by stating precisely our assumptions and goals. In an arbitrary unitary CFT in $D = 4$ spacetime dimensions, we consider the central charge C_T , defined as the coefficient in the 2-point function of the stress tensor operator $T_{\mu\nu}$:

$$\begin{aligned} \langle T_{\mu\nu}(x)T_{\lambda\sigma}(0) \rangle &= \frac{C_T}{S_D^2} \frac{1}{(x^2)^D} \left[\frac{1}{2} (I_{\mu\lambda}I_{\nu\sigma} + I_{\mu\sigma}I_{\nu\lambda}) \right. \\ &\quad \left. - \frac{1}{D} \delta_{\mu\nu}\delta_{\lambda\sigma} \right], \\ I_{\mu\nu} &= \delta_{\mu\nu} - 2x_\mu x_\nu / x^2, \end{aligned} \quad (1)$$

$[S_D = 2\pi^{D/2}/\Gamma(D/2)]$. It is assumed that the stress tensor is normalized canonically, that is consistently with the Ward identities (written here schematically for scalars):

$$\begin{aligned} \partial_\mu \langle T_{\mu\nu}(x)\phi(x_1)\dots\phi(x_n) \rangle \\ = - \sum_i \delta(x-x_i) \langle \phi(x_1)\dots\partial_\nu\phi(x_i)\dots\phi(x_n) \rangle. \end{aligned} \quad (2)$$

The central charge C_T is an interesting quantity because it provides a certain measure of the number of degrees of freedom in the theory. For example, for a free conformal theory of N_ϕ scalars, N_ψ Dirac fermions, and N_A vectors, we have [9]

$$C_T = \frac{4}{3}N_\phi + 8N_\psi + 16N_A. \quad (3)$$

Moreover, by unitarity $C_T > 0$, and $C_T = 0$ corresponds to a trivial theory. It is well known that C_T is an imperfect measure since, unlike in 2D, it does not in general decrease along the renormalization group flow [10]. The other central charge a , defined in terms of the 4D trace anomaly, fares better in this respect [11], while still remaining imperfect [12]. In this paper, we will stick to C_T since it is the one which we are able to constrain.

Now assume that our theory contains a primary Hermitian¹ scalar operator ϕ of a given dimension d . Our main goal will be to show that under this assumption, the central charge of the theory cannot become arbitrarily small. In other words, there exists a certain universal bound

$$C_T \geq f(d) > 0, \quad (4)$$

where $f(d)$ depends on d but is otherwise model-independent. In this paper we will derive such a bound in the interval $1 \leq d \leq 2$.

III. SOLUTION STRATEGY

A. Conformal blocks

We will approach this problem by imposing the constraint of OPE associativity in the 4-point function of the operator ϕ . Consider all primary Hermitian operators appearing in the OPE $\phi \times \phi$:

$$\begin{aligned} \phi \times \phi \supset \mathbb{1} + S_\Delta + \dots \text{(spin 0)} \\ + T_{\mu\nu} + \dots \text{(spin 2)} \\ + \text{higher spins.} \end{aligned} \quad (5)$$

¹If ϕ is not Hermitian, we can consider its real and imaginary parts.

Here in the first line we included the unit operator and all scalar primaries, starting from a certain dimension $\Delta \geq 1$ and higher. In the second line we have the stress tensor (spin 2, dimension 4 primary) and possibly higher dimension spin 2 fields. The third line contains all higher spin primaries ($\Delta \geq l + 2$ by the unitarity bounds [13]). Note that by the permutation symmetry of the $\phi\phi$ state only even spins can appear in this OPE.

Now, it has been shown by Dolan and Osborn [14] that every primary spin l dimension Δ operator $O_{\Delta,l}$ appearing in the $\phi \times \phi$ OPE with a coefficient $c_{\Delta,l}$ gives a contribution to the 4-point function of ϕ of the following form:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \supset c_{\Delta,l}^2 \frac{g_{\Delta,l}(u,v)}{(x_{12}^2)^d (x_{34}^2)^d}, \quad (6)$$

$$u = x_{12}^2 x_{34}^2 / (x_{13}^2 x_{24}^2), \quad v = x_{14}^2 x_{23}^2 / (x_{13}^2 x_{24}^2),$$

$$g_{\Delta,l}(u,v) = \frac{(-)^l}{2^l} \frac{k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})}{z - \bar{z}},$$

$$k_\beta(x) \equiv x^{\beta/2+1} {}_2F_1(\beta/2, \beta/2, \beta; x),$$

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \quad (7)$$

The functions of the cross ratios $g_{\Delta,l}(u,v)$ are called conformal blocks. They can be thought of as summing up the contributions of the primary $O_{\Delta,l}$ and all its descendants to the 4-point function of ϕ , when applying the OPE in the (12) (34) channel. It is nontrivial that such summation can be performed in closed form. The clear advantage of the representation (6) is that it can be used at finite point separation, unlike the OPE which is only useful in the coincidence limit $x_i \rightarrow x_j$.

B. Normalizations

Equation (6) assumes that both ϕ and O are unit normalized:

$$\langle \phi(x)\phi(0) \rangle = (x^2)^{-d}, \quad (8)$$

$$\begin{aligned} \langle O_{\mu_1\dots\mu_l}(x)O_{\lambda_1\dots\lambda_l}(0) \rangle \\ = \frac{1}{(x^2)^\Delta} \left[\frac{1}{l!} (I_{\mu_1\lambda_1}\dots I_{\mu_l\lambda_l} + \text{perms}) - \text{traces} \right], \end{aligned} \quad (9)$$

and the coefficient $c_{\Delta,l}$ is extracted from the 3-point function

$$\begin{aligned} \langle \phi(x_1)\phi(x_2)O_{\mu_1\dots\mu_l}(0) \rangle \\ = \frac{c_{\Delta,l}}{(x_{12}^2)^{d-(\Delta-l/2)}(x_1^2)^{(\Delta-l/2)}(x_2^2)^{(\Delta-l/2)}} \\ \times (Z_{\mu_1}\dots Z_{\mu_l} - \text{traces}), \end{aligned} \quad (10)$$

$$Z_\mu = x_{1\mu}/x_1^2 - x_{2\mu}/x_2^2. \quad (11)$$

On the other hand, the 3-point function of a canonically normalized stress tensor [see Eq. (1)] is fixed by the Ward identity [9]:

$$\langle \phi(x_1)\phi(x_2)T_{\mu\nu}(0) \rangle = -\frac{Dd}{(D-1)S_D} \frac{1}{(x_{12}^2)^{d-1}x_1^2x_2^2} \times \left(Z_\mu Z_\nu - \frac{1}{D} \delta_{\mu\nu} Z^2 \right). \quad (12)$$

These relations determine the coefficient $c_{4,2}$ appearing in (6) and (10) in terms of the central charge C_T and the dimension of ϕ [14]:

$$c_{4,2} = -\frac{Dd}{D-1} \frac{1}{\sqrt{C_T}}. \quad (13)$$

Via (6), this crucial relation implies that for large C_T , the contribution of the stress tensor to the 4-point function of ϕ decreases as $1/C_T$. This is in accord with what happens, for example, in AdS/CFT [15], where $C_T \sim N^2$ while the stress tensor contribution corresponds to the graviton exchange in the bulk, which is $1/N^2$ suppressed. Theories without a stress tensor (i.e. in which $c_{4,2} = 0$) should be viewed as theories with an infinite central charge. One example of such a theory is the Gaussian scalar field of dimension $d > 1$, see [16] for its conformal block decomposition.

However, in this paper we are interested in constraining the opposite limit of small C_T , in which the stress tensor contribution increases. As we will see below, such an increase eventually becomes inconsistent with crossing symmetry, and this will give us the bound (4).

C. Analytic structure of conformal blocks and crossing symmetry

Let us now discuss the analytic structure of the conformal blocks. The variable z appearing in (7) may look *ad hoc*, but in fact it is the 4D analogue of the usual complex variable of the 2D CFT, see Fig. 1. In the Euclidean signature z is complex and $\bar{z} = z^*$. In this case the conformal blocks are real functions, smooth everywhere away from $z = 0$ and from the $(1, +\infty)$ cut along the real axis. Since the imaginary part of the hypergeometrics is discontinuous across the cut, the conformal blocks have a $1/\text{Im } z$

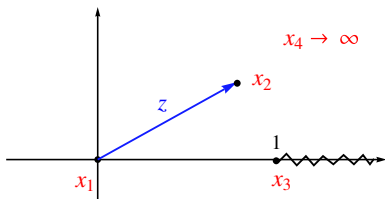


FIG. 1 (color online). Using conformal freedom, any configuration of 4 points can be mapped into the one shown in this figure, in which 3 points are fixed and one (x_2) is moving in a two plane passing through x_1 and x_3 . The complex coordinate of x_2 in this plane is precisely the z of (7), while $\bar{z} = z^*$. The conformal blocks are smooth everywhere in the plane except for $z = 0$ and the shown $(1, +\infty)$ cut along the real axis.

singularity there. Everywhere else on the real axis, and away from it, they are regular.

The asymptotic behavior of conformal blocks as $z \rightarrow 0$ is fixed by the OPE. The singular behavior in the $z \rightarrow 1$ limit, which corresponds to the crossed channel $x_2 \rightarrow x_3$, has no simple physical meaning. However, the sum over all blocks must be crossing symmetric and thus consistent with the OPE in the crossed channel as well. Because of the unphysical singularities, it is not immediately clear how to impose the OPE consistency in the crossed channel. In fact for this reason Polyakov [4] suggested using a different type of expansion into objects he called unitary blocks. However, at the time the explicit and simple expressions (7) were of course not yet known. Armed with these expressions, a different strategy becomes possible.

Namely, we will study the crossing symmetry condition at finite point separation, which can be written as

$$\frac{G(u, v)}{u^d} = \frac{G(v, u)}{v^d}, \quad (14)$$

where we used the fact that crossing $x_1 \leftrightarrow x_3$ corresponds to the interchange of u and v . Here $G(u, v)$ is the sum over all contributing conformal blocks:

$$G(u, v) = 1 + \sum c_{\Delta, l}^2 g_{\Delta, l}(u, v), \quad (15)$$

where $c_{\Delta, l}^2$ are the squares of the OPE coefficients, and we separated the contribution of the unit operator. It will be important that in a unitary theory all $c_{\Delta, l}$ are real, so that their squares are positive [1].

Instead of going straight to the crossed OPE limit $x_2 \rightarrow x_3$, we will study (14) around the democratic configuration² when x_2 is at equal distances from x_1 and x_3 . This corresponds to $z = 1/2$. In fact the same configuration can be mapped conformally to 4 operators inserted at the vertices of a square (Fig. 2). Both sides of (14) are regular around $z = 1/2$. Expanding the crossing condition into a two-dimensional power series around this point, we get an infinite number of linear equations, which have to be satisfied for some positive coefficients $c_{\Delta, l}^2$. Which Δ, l will enter the expansion with nonzero coefficients depends on the CFT. The problem of unphysical singularities, brought up by Polyakov [4], is resolved as follows. The left-hand side of (14) is smooth away from the cut along $(1, +\infty)$, while the right-hand side is smooth away from $(-\infty, 0)$, since the crossing maps $z \rightarrow 1 - z$. Assuming that both sides can be analytically continued from their region of convergence, the cuts *must* cancel when summing over all Δ and l . In other words, imposing that there is no cut should give no additional constraints compared to the

²Similarly, Hellerman [5], in his analysis of the modular invariance constraint, chose to work at the self-dual inverse temperature $\beta = 2\pi$.

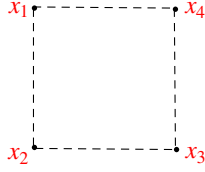


FIG. 2 (color online). This configuration, with 4 points at the vertices of a square, is conformally equivalent to the one in Fig. 1 with $z = 1/2$.

ones that we are using, although it may be a different and perhaps a useful way to package the same information.

D. Method of linear functionals

For further discussion let us rewrite Eq. (14) in the equivalent “sum rule” form:

$$1 = \sum c_{\Delta,l}^2 F_{d,\Delta,l}(u, v), \quad (16)$$

$$F_{d,\Delta,l}(u, v) \equiv \frac{v^d g_{\Delta,l}(u, v) - u^d g_{\Delta,l}(v, u)}{u^d - v^d}. \quad (17)$$

This equation says that the “crossing symmetry deficit” of all the fields in the OPE, normalized to the deficit of the unit operator, has to sum up to 1.

Let us view Eq. (16) as a linear relation in the vector space of functions of two variables $f(u, v)$. Then it can be given the following geometric interpretation. As we keep the CFT spectrum fixed and vary the squared OPE coefficients $c_{\Delta,l}^2 \geq 0$, the vectors in the right-hand side fill in a *convex cone* generated by the functions $F_{d,\Delta,l}$. The sum rule says that the function $f(u, v) \equiv 1$ must belong to this cone (see Fig 3(a)).

If we start imposing restrictions on the CFT spectrum, for example, by demanding that there should be a gap in the scalar sector: $\Delta \geq \Delta_*$ for $l = 0$, this reduces the list of vectors generating the cone, and *a fortiori* the cone itself. It may well happen that the new reduced cone no longer contains the function $f \equiv 1$, Fig 3(b). A spectrum leading to such a cone cannot be realized in any CFT.

If the situation in Fig 3(b) occurs, then, since the cone is convex, one can always find a hyperplane passing through the origin and separating $f \equiv 1$ from the cone, Fig 3(c). In analytical language, this means that there exists a linear functional Λ taking values of the opposite sign on $f \equiv 1$ and on the functions generating the cone:

$$\Lambda[1] \leq 0, \quad \Lambda[F_{d,\Delta,l}] > 0. \quad (18)$$

In practice, the functional may be build up as a linear combination of the partial derivatives with respect to z and \bar{z} at the democratic point $z = \bar{z} = 1/2$.

So far we have described the method used in [1,2] to constrain the maximal allowed gap in the scalar sector. In order to constrain the size of the OPE coefficient $c_{4,2}$, we

proceed as follows [3].³ Let us rewrite the sum rule by transferring a part of the stress tensor contribution into the left-hand side:

$$1 - tF_{d,4,2} = (c_{4,2}^2 - t)F_{d,4,2} + \sum_{(\Delta,l) \neq (4,2)} c_{\Delta,l}^2 F_{d,\Delta,l}. \quad (19)$$

The geometric interpretation of this equation is that the t -dependent vector $1 - tF_{d,4,2}(u, v)$ belongs to the same cone as before as long as $t \leq c_{4,2}^2$. In other words, the maximal allowed value of $c_{4,2}^2$ can be determined as the value $t = t_{\text{cr}}$ for which the curve $1 - tF_{d,4,2}(u, v)$ crosses the cone boundary, Fig. 4. Analytically, we can detect that the crossing happened if there exists a linear functional such that

$$\Lambda[F_{d,\Delta,l}] \geq 0, \quad (20)$$

for all functions generating the cone, and

$$\Lambda[1 - tF_{d,4,2}] = 0. \quad (21)$$

Note that in the present situation the function $f \equiv 1$ must of course belong to the cone, otherwise the CFT simply does not exist and there is no point of discussing an upper bound on the OPE coefficients. Thus, we are assuming from the start $\Lambda[1] \geq 0$, unlike in (18).

Since the functional is linear, Eq. (21) is satisfied for

$$t = \Lambda[1]/\Lambda[F_{d,4,2}], \quad (22)$$

and for larger t the functional will become negative as long as $\Lambda[F_{d,4,2}] > 0$. Thus, we obtain the following result: each functional Λ satisfying (20) gives a bound on the maximal allowed value of $c_{4,2}^2$:

$$\max c_{4,2}^2 \leq \Lambda[1]/\Lambda[F_{d,4,2}]. \quad (23)$$

This bound can be optimized by choosing the functional judiciously.

The method just described was first applied in [3] to constrain the size of the OPE coefficients of scalar operators, while here we will use it to constrain the size of $c_{4,2}$, which via (13) will give us a lower bound on C_7 . Another difference from [3] is that we will study how the bound improves as a function of the assumed gap in the scalar sector of the OPE.

IV. RESULTS

We will now present our numerical results. First of all, let us consider the most general case when we are not making any assumption concerning the gap in the scalar sector of the OPE. This means that the scalar operators appearing in the OPE are allowed to have any dimension $\Delta \geq d$. Operators with lower dimensions are *a priori* excluded if ϕ is the lowest-dimension scalar. Under this

³We choose $c_{4,2}$ for definiteness; the method in fact works for any OPE coefficient.

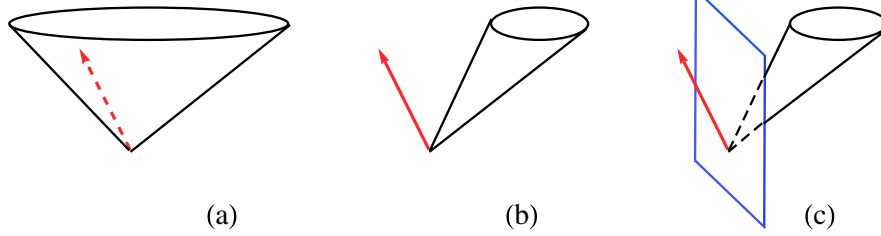


FIG. 3 (color online). Geometric interpretation of the sum rule is shown in the figure: (a) the sum rule has a solution $\Leftrightarrow f \equiv 1$ belongs to the cone; (b) the assumed spectrum is such that the sum rule does not allow for a solution $\Leftrightarrow f \equiv 1$ does not belong to the cone; (c) in the latter situation, a hyperplane (the zero set of a linear functional) can be found separating $f \equiv 1$ from the cone.

assumption, we use the method of linear functionals to bound $c_{4,2}^2$ from above. For this study, we choose linear functionals of the form

$$\Lambda[f] = \sum_{\substack{n,m \text{ even} \\ 0 \leq n+m \leq N}} \frac{\lambda_{n,m}}{n!m!} \partial_a^n \partial_b^m f|_{a=b=0}, \quad (24)$$

$$z = 1/2 + a + b, \quad \bar{z} = 1/2 + a - b. \quad (25)$$

As advertised, we are working around the democratic point $z = \bar{z} = 1/2$. The fact that we are choosing z and \bar{z} as real and independent can be interpreted as a Wick rotation to the Minkowski space [1]. The functional only contains even derivatives because the functions $F_{d,\Delta,l}$ are even in both a and b [1].

We will choose $\lambda_{0,0} = 1$ to have $\Lambda[1] = 1$. Then, to optimize the bound (23), the coefficients of the functional must be chosen so that

$$\Lambda[F_{d,4,2}] \rightarrow \max, \quad (26)$$

subject to the constraints (20), which in our case mean

$$\begin{aligned} \Lambda[F_{d,\Delta,0}] &\geq 0 \quad \text{for all } \Delta \geq d, \\ \Lambda[F_{d,\Delta,l}] &\geq 0 \quad \text{for all } \Delta \geq l + 2, \quad l = 2, 4, \dots \end{aligned} \quad (27)$$

We will consider the functionals with the maximal derivative order up to $N = 16$. Pushing to higher N values is likely to somewhat improve the bound. In principle N as large as 18 were demonstrated feasible in this kind of studies [2].

Equations (26) and (27), define an optimization problem for the coefficients $\lambda_{m,n}$. The constraints are given by linear inequalities, and the cost function is also linear, which

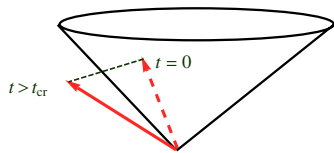


FIG. 4 (color online). Geometric interpretation of Eq. (19) is shown in the figure. As t increases, the vector $1 - tF_{d,4,2}$ eventually leaves the cone.

makes it a *linear programming problem*. Although the number of constraints in (27) is formally infinite, they can be reduced to a finite number by discretizing Δ and truncating at large Δ and l , where the constraints approach a calculable asymptotic form [1]. The reduced problem can be efficiently solved by well-known numerical methods, such as the simplex method. A found solution can be then checked to see if it also solves the full problem. This procedure was developed and successfully used in [1–3].

Using this procedure, we computed a bound on $c_{4,2}^2$ from above, which via (13) translates into a bound on C_T from below. The latter bound is plotted in Fig. 5 as a function of the dimension of ϕ in the range $1 \leq d \leq 2$. We plot our best bound for $N = 16$ and, for comparison, a weaker bound obtained with a smaller value $N = 12$.

Postponing the discussion to the next section, let us now consider what happens with the bound in the presence of a gap in the scalar spectrum. In other words, we now assume that the first scalar operator in the $\phi \times \phi$ OPE has dimension Δ_* *strictly bigger* than d . Technically, this problem is analyzed exactly as the previous one, except that the first set of constraints (27) is replaced by a shorter list:

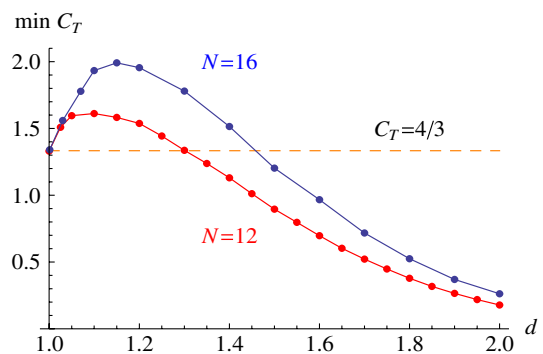


FIG. 5 (color online). The figure shows the lower bound on the central charge C_T in terms of the dimension d of the lowest-dimension scalar primary. The stronger bound (upper blue curve) is obtained with $N = 16$. For comparison we give a weaker bound obtained with $N = 12$ (lower red curve), which corresponds to the horizontal axis $\Delta_* = d$ in the following Fig. 6. The horizontal dashed line $C_T = 4/3$ shows where our bound stays above the free scalar central charge.

$$\Lambda[F_{d,\Delta_0}] \geq 0 \quad \text{for all } \Delta \geq \Delta_*. \quad (28)$$

Because of the considerable computer time involved, we solved this problem by using linear functionals with $N = 12$ only. The bound is given in Fig. 6 as a contour plot in the $d, \Delta_* - d$ plane. On the horizontal axis $\Delta_* = d$ the bound reduces to the $N = 12$ bound from Fig. 5. Naturally, when Δ_* increases, the bound on C_T gets stronger. The white region in upper left corresponds to

$$\Delta_* > 2 + 0.7(d-1)^{1/2} + 2.1(d-1) + 0.43(d-1)^{3/2} \quad (29)$$

and is excluded, since such a large gap cannot be realized in any CFT according to the results of [2].

A text file with the coefficients of linear functionals used to derive the shown bounds can be downloaded [17].

We end this section with a comment concerning the case of 2D conformal field theories. Recall that in [1,2], the maximal allowed gap in the scalar spectrum was studied for the 2D case in parallel with 4D. This was instructive since it allowed us to compare our bounds with the known OPEs in the 2D minimal models. The analysis is feasible because the 2D conformal blocks are known in a form just as simple as (7) (in odd dimensions similarly simple expressions are not available). Analogously, in the course of this project we have looked at the lower bounds on the Virasoro central charge c in the 2D CFTs, in the same d, Δ_* plane ($d \geq 0$ as appropriate in the 2D case). We do not present them here because, in the range that we considered, the found lower bounds were smaller than 1. Since all unitary 2D CFTs with $c < 1$ are classified (these are

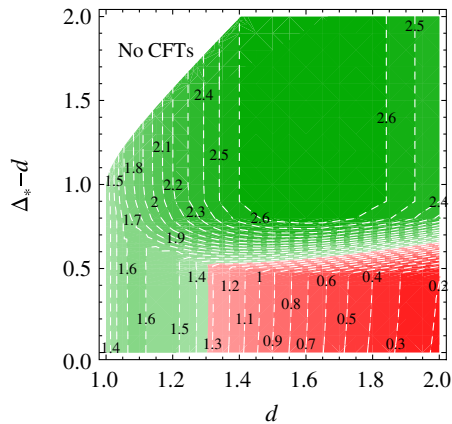


FIG. 6 (color online). The figure shows the contour plot of the C_T lower bound as a function of d and of the gap $\Delta_* - d$, where Δ_* is the dimension of the first scalar in the $\phi \times \phi$ OPE. The gap is nonnegative, since we assume that ϕ is the lowest-dimension scalar. On the horizontal axis the bound reduces to the $N = 12$ curve in Fig. 5. The lighter green color marks the region where the bound is above $C_T^{\text{free}} = 4/3$, while in the darker red region the bound is below this value. As the gap increases, the bound gets stronger, so that a rather weak assumption about the gap is already enough to have $C_T > C_T^{\text{free}}$.

precisely the unitary minimal models [18]), our bounds do not add any new information in this case.

V. DISCUSSION

Figures 5 and 6 contain our advertised main results: universal lower bounds on the stress tensor central charge C_T . Figure 5 gives a bound as a function of the dimension d of the lowest-dimension scalar ϕ present in the CFT. More precisely, the only requirement on ϕ is that the OPE $\phi \times \phi$ not contain any scalar of dimension less than d ; this requirement is trivially satisfied if ϕ is the lowest-dimension scalar.

The first interesting point about this bound is that in the limit $d \rightarrow 1$ it approaches the free scalar central charge value $C_T^{\text{free}} = 4/3$, see Eq. (3). In other words, our method shows that the free theory limit is approached continuously. This is just as in previous work, where we proved that as $d \rightarrow 1$, the first scalar in the $\phi \times \phi$ OPE must have a dimension below 2 [1,2], and the 3-point function $\langle \phi \phi \phi \rangle$ must approach zero [3].

Next, we see that for $1 < d \leq 1.4$ our bound stays above C_T^{free} , thus showing that an interacting theory necessarily has a larger central charge than the free one. This is also rather interesting. Unfortunately, for larger d our bound drops below C_T^{free} . We do not know if this means that there are CFTs with $C_T < C_T^{\text{free}}$. More likely, this indicates that our bound is not the best possible in this range. One could speculate that the best-possible bound should stay above C_T^{free} in the whole range $1 < d < 2$. The fact that it should necessarily come down to C_T^{free} (or lower) for $d = 2$ can be inferred by considering the dimension 2 operator φ^2 in the free scalar theory and its OPE with itself.

Note that we could also derive a bound without using the assumption $\Delta \geq d$, which would be applicable to any scalar, not just the lowest-dimension one. We have in fact derived also such a general bound, although we do not show it here. We found that this general bound differs little from the bound shown in Fig. 5 in the region of small d , say for $d \leq 1.3$. Thus, this general bound could be useful if the lowest-dimension scalar has dimension very close to 1, while the second-lowest is somewhat above 1. However, in the region of larger d , $d \geq 1.7 \div 1.8$, the general bound drops to zero. This happens for the same technical reason that the bounds on the scalar OPE coefficients in [3] were blowing up around this value of the operator dimension. Because of this, in this paper we focused on the lowest-dimension scalar, which allowed us to obtain a nontrivial bound in the full considered range of d .

Now let us discuss Fig. 6, which gives the lower bound on C_T as a function of d and Δ_* . Here Δ_* is the dimension of the lowest-dimension scalar in the OPE $\phi \times \phi$, assumed to be above d . Again, this assumption is trivially satisfied if ϕ is the lowest-dimension scalar present in the theory. Moreover, by the results of [1,2] Δ_* is limited from above by the bound given in Eq. (29).

On the horizontal axis $\Delta_* = d$ the bound in Fig. 6 reduces to the one shown in Fig. 5, while for larger Δ_* it naturally gets stronger. In fact we see that Δ_* , somewhat bigger than d , is already sufficient to raise the bound above C_T^{free} for all d (the lighter green region in the plot). The points with $\Delta_* \sim 2d$ (i.e. with an approximate factorization of operator dimensions) belong to the green region by a big margin.

In summary, we have shown in this work that if a unitary 4D CFT is nontrivial (in that it contains at least one primary scalar operator), then its central charge C_T cannot be arbitrarily low. We presented a universal bound on C_T as a function of the dimensions of the lowest and second-lowest scalar. We hope that these bounds will be helpful in future efforts to chart the landscape of 4D conformal theories.

A relation like the one we derived, viewed from the AdS/CFT perspective (although of course we cannot do it since we are not at large N), would represent a lower bound on the Planck mass. One could then speculate that our result

belongs to the same class of constraints on quantum field theory as the *gravity as the weakest force* conjecture [19]. Unfortunately our result cannot be directly applied to a phenomenon as fascinating and unavoidable as gravity, but it has a non-negligible consolation that it follows from a rigorous mathematical analysis.

We believe more general constraints of the type discussed in this paper lie ahead, ready to be uncovered.

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Note added.—The morning of the day this paper was submitted to arXiv, a nice paper [20] appeared which, among other things, also derives lower bounds on the central charge.

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