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Toward the gravity dual of heterotic small instantons

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The question of what happens when the heterotic $SO(32)$ instanton becomes small was answered sometime back by Witten. The heterotic theory develops an enhanced $Sp(2k)$ gauge symmetry for k small instantons, besides the allowed $SO(32)$ gauge symmetry. An interesting question now is to ask what happens when we take the large k limit. In this paper we argue that in some special cases, where Gauss' law allows the large k limit, the dynamics of the large k small instantons can be captured by a dual gravitational description. For the cases that we elaborate in this paper, the gravity duals are non-Kähler manifolds although in general they could be nongeometric. These small instantons are heterotic five-branes and the duality allows us to study the strongly coupled field theories on these five-branes. We review and elaborate on some of the recent observations pointing towards this duality and argue that in certain cases the gauge-gravity duality may be understood as small instanton transitions under which the instantons smoothen out and consequently lose the $Sp(2k)$ gauge symmetry. This may explain how branes disappear on the dual side and are replaced by fluxes. We analyze the torsion classes before and after the transitions and discuss briefly how the Atiyah-Drinfeld-Hitchin-Manin sigma model and related vector bundles could be studied for these scenarios.

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I. INTRODUCTION

In the full moduli space of string theory, the heterotic theory [1] has always been an important corner where phenomenologically useful models are most easily accessible. Part of its appeal lies in the existence of an anomaly cancelling $SO(32)$ or the $E_8 \times E_8$ vector bundle that is crucial for embedding the standard model in string theory. The existence of a minimal supersymmetric multiplet is also an additional benefit.

On the other hand, the type IIB theory has its own share of advantages. The non-Abelian multiplet in this theory come from nonperturbative branes such that exactly similar physics, as from the heterotic theory, can be studied here using these branes. Additionally, the type IIB theory has a full nonperturbative completion: the so-called F theory [2] where local and nonlocal branes participate to realize the quantum corrections. In fact the F-theory completion of the type IIB theory is directly related to the heterotic theories. Thus various vacua of heterotic theories should be thought of as duals to the various seven-brane configurations in the F theory compactified on Calabi-Yau spaces.

In recent times gauge-gravity dualities have been studied exclusively in type II theories and especially in the type IIB theory. The duality that we are most interested in for IIB is the geometric transition [3], where the strongly coupled far IR dynamics of an $\mathcal{N} = 1$ pure super Yang-Mills theory is studied in terms of a weakly coupled type IIB supergravity on a deformed conifold with three-form fluxes. The far IR theory, on the other hand, is realized as type IIB D5-branes wrapped on the two-cycle of a resolved conifold. Thus this duality is a geometric transition where, under a conifold transition, the wrapped D5-branes *disappear* and are replaced by three-form fluxes on a deformed conifold. The type IIA dual of this in terms of D4- and NS5-branes was understood in [4].

Unfortunately similar dualities have not been addressed in details in the heterotic side. To our knowledge the first attempt to address this issue was done in [5] (see also [6] for a more recent analysis). The difficulty in the heterotic side lies in two things: understanding the vector bundles and solving the Bianchi identity. For example, one would be tempted to realize the geometric transition in the heterotic theory by taking the S dual of the original IIB transition, i.e. replacing the IIB D5-branes with the NS5-branes and interpreting the NS5-branes as heterotic five-branes. However, it is not *a priori* clear whether the dual deformed conifold geometry would indeed solve the Bianchi identity. Additionally, it is not clear how the vector bundles could be pulled across a conifold transition.

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In our earlier papers [5,7] we gave a local description of this duality.[1] Our local analysis reproduced a global configuration that was more general than a deformed conifold with fluxes. In this paper we will show why this is true: The type IIB story cannot be directly dualized to the heterotic side. The Bianchi identity will in fact change the IIB solutions, and so the heterotic duals will not quite be the same as the IIB ones.

In fact the precise duality in the heterotic side can be presented succinctly in the language of small instantons [13]. The large N limit of these small instantons can be captured by a dual gravitational background which is generically nongeometric. The interesting thing about our analysis is the observation that the geometric transition is a small instanton transition where the small instantons become "smooth" on the dual side and therefore lose the $Sp(2N)$ gauge symmetry. This gives a possible explanation of the disappearance of the branes in the dual side. In this paper we will only work with the $SO(32)$ heterotic theory and leave the $E_8 \times E_8$ case for what follows. The $E_8 \times E_8$ case presumably follows a similar path as illustrated in [14]. For earlier studies on small instanton transition, the reader may refer to [15].

The work in this paper is a direct follow-up of our last paper [9] where various supersymmetric duals in the type II and M theories were presented in the geometric transition setup. However, in [9] formal proofs for supersymmetry of the solutions, using, say, torsion classes, were not presented. In this paper we will rectify these shortcomings and start with giving a detailed torsion class analysis of all the type II solutions of [9]. This will help us to state the heterotic duality in a more precise way.

A. Supersymmetric configurations in geometric transitions

The issue of supersymmetry for the intermediate configurations is of course crucial in the geometric transition setup. We have discussed this in some detail in [9]. Here we will elaborate the story a bit more, and new details will be presented in Sec. III.

Our first configuration is in the type IIB theory for wrapped D5-branes on a resolved conifold. The original construction of [10], with a conformally Calabi-Yau metric, is not supersymmetric The supersymmetric configuration is given in [9], where we put a non-Kähler metric on the resolved conifold. Our starting point in [9] is the choice of functions $F_i = F_i(r)$, $i = 1, \dots, 4$, and $F_0 = F_0(r, \theta_1, \theta_2)$, which are used to write the metric for the internal space. Therefore for different choices of F_0 and F_i we get different dual gauge theories. The complete background in type IIB then is (see also [9])

$$\begin{aligned} F_3 &= h \cosh\beta e^{2\phi} * d(e^{-2\phi} J), \\ H_3 &= -h F_0^2 \sinh\beta e^{2\phi} d(e^{-2\phi} J), \\ \phi_{\rm now} &= -\phi, \\ F_5 &= -\tfrac{1}{4}(1 + *) dA_0 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\ ds^2 &= F_0 ds^2_{0123} + ds^2_6, \\ ds^2_6 &= F_1 dr^2 + F_2(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &\quad + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \end{aligned} \tag{1.1}$$

where we have defined ϕ, h, and A_0 using F_0 and a constant "boosting" parameter β in the following way:

$$\begin{aligned} h &= \frac{F_0 \cosh^2\beta}{1 + F_0^2 \sinh^2\beta}, \qquad e^{-\phi} = \frac{F_0^{3/2}\cosh\beta}{\sqrt{1 + F_0^2 \sinh^2\beta}}, \\ A_0 &= (F_0^2 - 1)\tanh\beta \\ &\quad \times \left[1 + \left(\frac{1 - F_0^2}{F_0^2}\right)\mathrm{sech}^2\beta + \left(\frac{1 - F_0^2}{F_0^2}\right)^2 \mathrm{sech}^4\beta\right]. \end{aligned} \tag{1.2}$$

All the coefficients, etc., are also described in more detail in [9]. In Sec. III we will compute all the torsion classes for this background and discuss explicitly how supersymmetry is preserved.

The above background is of course the first step in the chain of dualities associated with IR geometric transitions. To go to the type IIA mirror description using the Strominger, Yau, and Zaslow (SYZ) method [16] we need to make the base very large compared to the fiber. We achieve the final IIA mirror by making the following steps[2]:

(i) Shift of the coordinates (ψ, ϕ_i) using variables $f_i(\theta_i)$. This shifting of the coordinates mixes nontrivially all three isometry directions as described in Eq. (4.24) of [9].

[1] By *local* we mean that the supergravity background is studied around a specific chosen point in the internal six-dimensional space. For example, we choose a point $(r_0, \langle\theta_i\rangle, \langle\phi_i\rangle, \langle\psi\rangle)$ in [5,7] which is away from the $r = 0$ conifold point. This is because the full global picture was hard to construct, and any naive procedure always tends to lead to nonsupersymmetric solutions. In deriving the local metric, we took a simpler model where all the spheres were replaced by tori with periodic coordinates (x, θ_1) and (y, θ_2). The coordinate z formed a nontrivial $U(1)$ fibration over the T^2 base. Here $(r, x, y, z, \theta_1, \theta_2)$ is the coordinate of a point away from $(r_0, \langle\phi_1\rangle, \langle\phi_2\rangle, \langle\psi\rangle, \langle\theta_1\rangle, \langle\theta_2\rangle)$. The replacement of spheres by two tori was directly motivated from the corresponding brane constructions of [8], where noncompact NS5 branes required the existence of tori instead of spheres in the T-dual picture. On the other hand, the term *global* means roughly adding back the curvature, warping, etc., replacing tori by spheres, so that at the end of the day, we have a supersymmetric solution to the equations of motion. In [9] we managed to provide the full global picture of geometric transition. Note also that the only known global solution, i.e. [10], before our work was unfortunately not supersymmetric (see [11,12] for details) although it satisfied the type IIB equations of motion.

[2] These rules have been derived in the local limit in [5,7]. In [9] we have shown how in the global picture these rules could work.

(ii) Shift the metric along ψ direction by the variable ϵ, as given in the second line of Eq. (4.29) of [9]. This variable does not have to be very small in the global limit. The only constraint on ϵ is $\epsilon < 1$ to preserve the signature of the metric.

(iii) Make SYZ transformations along the new shifted directions. Thus the three T dualities are *not* made along the three original isometry directions.

(iv) In the new metric of IIA make a further rotation along the (θ_2, ϕ_2) directions using a 2×2 matrix given as Eq. (4.50) of [9]. The matrix is described using a constant angular variable ψ_0.

(v) Finally in the transformed metric convert ψ_0 to ψ as in Eq. (4.53) of [9].

The final metric after we perform all the above transformations takes the following form:

$$\begin{aligned} ds_{11}^2 = {} & F_0 ds_{0123}^2 + F_1 dr^2 + \frac{\alpha F_2}{\Delta_1 \Delta_2}[d\psi - b_{\psi r} dr - b_{\psi\theta_2} d\theta_2 + \Delta_1 \cos\theta_1 (d\phi_1 - b_{\phi_1\theta_1} d\theta_1 - b_{\phi_1 r} dr) \\ & + \Delta_2 \cos\theta_2 \cos\psi_0 (d\phi_2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr)]^2 + \alpha j_{\phi_2\phi_2}[d\theta_1^2 + (d\phi_1 - b_{\phi_1\theta_1} d\theta_1 - b_{\phi_1 r} dr)^2] \\ & + \alpha j_{\phi_1\phi_1}[d\theta_2^2 + (d\phi_2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr)^2] + 2\alpha j_{\phi_1\phi_2} \cos\psi_0 [d\theta_1 d\theta_2 - (d\phi_1 - b_{\phi_1\theta_1} d\theta_1 \\ & - b_{\phi_1 r} dr)(d\phi_2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr)] + 2\alpha j_{\phi_1\phi_2} \sin\psi_0 [(d\phi_1 - b_{\phi_1\theta_1} d\theta_1 - b_{\phi_1 r} dr) d\theta_2 \\ & + (d\phi_2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr) d\theta_1]. \end{aligned} \tag{1.3}$$

The above metric, which we called the *symmetric* metric in [9], looks very close to the deformed conifold metric. However, due to steps 2, 4, and 5 above, it is not guaranteed that the metric will preserve supersymmetry. Furthermore one might also question whether the SYZ operation itself could preserve supersymmetry.[3] Therefore to verify this we will evaluate all the torsion classes for this background in Sec. III B. Note that in [9] we did not explicitly derive the fluxes in the mirror. In Sec. III B we will be able to determine at least the Neveu-Schwarz (NS) three-form flux that will make the IIA mirror background supersymmetric. This will also help us to fix (f_1, f_2, ϵ).

Once we have the type IIA metric we can lift[4] this to M theory using the one-forms (σ_i, Σ_i) as given in Eqs. (4.47) and (4.48) of [9], respectively. The precise flop transformation of the M-theory manifold is described using a *class* of transformations specified by (a, b) as in Eq. (4.59) of [9]. The final metric after we reduce the flopped metric to type IIA is

$$\begin{aligned} ds_{10}^2 = {} & F_0 ds_{0123}^2 + F_1 dr^2 + e^{2\phi}[d\psi - b_{\psi\mu} dx^\mu + \Delta_1 \cos\theta_1 (d\phi_1 - b_{\phi_1\theta_1} d\theta_1 - b_{\phi_1 r} dr) \\ & + \tilde{\Delta}_2 \cos\theta_2 (d\phi_2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr)]^2 + e^{2\phi/3} a^2 (k^2 G_2 + k G_3 + G_1)[d\theta_1^2 + (d\phi_1^2 \\ & - b_{\phi_1\theta_1} d\theta_1 - b_{\phi_1 r} dr)^2] + e^{2\phi/3} b^2 (\mu^2 G_2 + \mu G_3 + G_1)[d\theta_2^2 + (d\phi_2^2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr)^2] \end{aligned} \tag{1.4}$$

along with the following one-form charge, but no D6-brane sources:

$$\begin{aligned} A = {} & \Delta_1 \cos\theta_1 (d\phi_1 - b_{\phi_1\theta_1} d\theta_1 - b_{\phi_1 r} dr) \\ & - \tilde{\Delta}_2 \cos\theta_2 (d\phi_2 - b_{\phi_2\theta_2} d\theta_2 - b_{\phi_2 r} dr). \end{aligned} \tag{1.5}$$

A torsion class analysis in Sec. III B will help us to fix a particular flop transformation, i.e. fix (a, b) so that the above background remains supersymmetric. Observe again that we have not determined all the flux components in IIA. As before, we expect the torsion class analysis to fix at least the NS three-form. The three-form can then be fixed by equation-of-motion or supersymmetry constraints.

In all the above steps we tried to make duality transformations so that we could get *geometric* manifolds. However, this is not generic. For a more generic choice of the B fields in the original type IIB setup, we could get nongeometric manifolds both before and after flop in IIA. This nongeometric aspect is also reflected in the final type IIB mirror configuration. In fact this tells us that the generic solution spaces we get in type IIB are nongeometric manifolds. For certain choices of parameters (B fields and metric components) we can get geometric manifolds like Klebanov-Strassler [18] or Maldacena-Nunez [19]. This is almost like the parameter space of [20] but now much bigger and allowing *both* geometric and

[3]This is because it is not *a priori* clear whether the fermionic boundary conditions are periodic or antiperiodic along the T-duality circles. Sometime when the cycles degenerate we may need to put in an additional $(-1)^F$ term to preserve supersymmetry (SUSY). An example of this is given in the third reference of [17].

[4]In [9] we lifted the nonsymmetric type IIA metric to M theory. This is more generic than the symmetric one.

nongeometric manifolds that cover various branches of the dual gauge theories.

To make this a little more precise, note that we have analyzed the following two scenarios in [9]:

(i) There are various ways to embed wrapped five-branes on a two-cycle in the internal space that preserve supersymmetry. For a *given* choice of (F_i, F_0, ϵ) we can find the geometry and the fluxes that preserve supersymmetry (see the analysis in Sec. 4 of [9]). In the decoupling limit, this is the gauge theory side of the story. We called this the scenario *before* geometric transition.

(ii) For that particular choice of the background, we followed our duality arguments to give a background *after* geometric transition. We showed that for *generic* choices of the fluxes, the dual gravitational background becomes nongeometric. Therefore the fluxes and the geometry in the brane side of the picture induce nontrivial operators in $\mathcal{N} = 1$ gauge theory that make the dual gravitational background nongeometric.

In this paper we will do an explicit computation to study a *geometrical* dual for the large N small instantons because this case will not be too hard to construct. A similar story was also pointed out in [9]. For example, if we deliberately restrict ourselves to the special case Eq. (4.71) of [9], i.e. make the NS B fields along (ϕ_1, ϕ_2) and (ψ, ϕ_i) directions zero, then the geometric manifold we get has the following metric:

$$\begin{aligned} ds^2 = F_0^2 ds^2_{0,1,2,3} &+ g_{rr}dr^2 + g_{\psi\psi}(\tilde{\mathcal{D}}\psi + \hat{\Delta}_1\tilde{\mathcal{D}}\phi_1 \\ &+ \hat{\Delta}_2\tilde{\mathcal{D}}\phi_2)^2 + g_{\theta_1\theta_1}(d\theta_1^2 + \tilde{\mathcal{D}}\phi_1^2) \\ &+ g_{\theta_2\theta_2}(d\theta_2^2 + \tilde{\mathcal{D}}\phi_2^2) \\ &+ g_{\theta_1\theta_2}(d\theta_1 d\theta_2 + \hat{\Delta}_3\tilde{\mathcal{D}}\phi_1\tilde{\mathcal{D}}\phi_2), \end{aligned} \tag{1.6}$$

which looks surprisingly close to the resolved warped-deformed conifold metric. A torsion class analysis can again be performed for this case (but we will not do so here) that will allow us to put constraints on the parameters from supersymmetry. This way all the intermediate configurations in the cycle of geometric transition will be supersymmetric. In our opinion this is probably the first time where explicit supersymmetric configurations for IR geometric transition in IIB, IIA, and M theories are studied. However, our analysis also revealed the existence of a much bigger picture in the type IIB side where various gauge theory deformations lead to nongeometric duals. Our aim in this paper is to extend this further to the heterotic and type I cases.

B. Organization of the paper

The paper is organized as follows. In Sec. II we will give three pieces of evidence related to the heterotic gauge-gravity duality. Some of these have already appeared in [5], but here we will elaborate them in the global picture. The first evidence, discussed in Sec. II A, will come from taking the orientifold limits of the type IIB duality. The issue of vector bundles, before and after the transition, as well as the Bianchi identity will be discussed therein. This will be elaborated further in Sec. II B, where we will briefly study the Atiyah-Drinfeld-Hitchin-Manin (ADHM) sigma model that captures the physics before the transition. A more direct analysis, using properties of the underlying non-Kähler manifolds, will be discussed in Sec. II C. In Sec. III we will study the supersymmetry of these solutions. We will discuss how torsion classes and supersymmetry put constraints on the warp factors of the background manifolds. In Sec. IV we will give a brief discussion of the interconnections between the torsion classes and the vector bundles both before and after the transition. Details about the heterotic torsions and the torsion-classes are discussed further in the appendixes. We end with a conclusion and some discussions about future directions.

II. THREE ROADS TO HETEROTIC TRANSITIONS

The existence of geometric transition in the heterotic theory was first proposed in [5] using various arguments stemming from U dualities, orientifold actions, and gauge-gravity identifications. However, all these analysis were studied using the so-called *local* geometry. Recently in [9] we have managed to study the complete global picture for type II theories.[5] It is therefore time now to extend the local analysis of [5] for the heterotic case to the full global picture. See Figs. 1 and 2 for more details. In this section we will try to give three pieces of evidence related to geometric transition in the heterotic side. Some of these details have appeared in [5,7,22] for the local case. However, here we will give a somewhat different interpretation for the transition. The configuration before geometric transition will be identified with the heterotic large N small instantons, where N is the number of small instantons or heterotic five-branes. The configuration after geometric transition will be identified to the case where the instantons have all dissolved in the heterotic $SO(32)$ gauge group [in fact the $SO(32)$ group will be broken by Wilson lines; we will discuss this later]. This interpretation is not new, as the small instantons have already been identified to heterotic five-branes by various authors (see [13] and citations therein). What is new is probably the whole *interpretation* of heterotic duality as small instanton transitions for some cases.

[5]Assuming of course that the UV completions should follow a somewhat similar line exemplified in [21] albeit now with more nontrivial UV caps. These UV caps should capture the six-dimensional UV completions of the $\mathcal{N} = 1$ IR gauge theories.

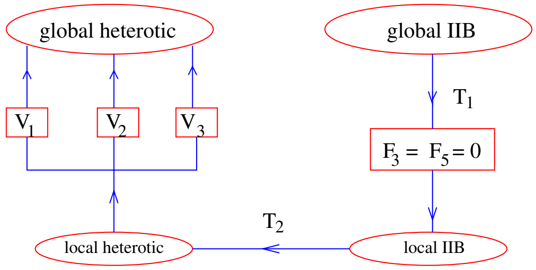

FIG. 1 (color online). A precise flow diagram to illustrate how a global type IIB background can go to a global heterotic background using transformations T_1 and T_2. The transformation T_1 could be an orientifolding operation or something more complicated, as discussed in the text, and similarly T_2 could be T dualities or something more involved. Thus only local geometric are related by T_2 transformations. Many different global completions with various choices of the vector bundles V_i lead to the same local background in the heterotic theory.

Following is the list of steps that could make this duality a bit more precise:

(i) Consider IIB on a resolved conifold with N wrapped five-branes. This is basically the configuration of [9]. We can go to the orientifold limit that keeps the five-branes but generate seven-branes and orientifold seven-planes. Because of this orientifold action the gauge theory on five-branes become $Sp(2N)$. This is basically an embedding in the F-theory setup. We will discuss this a bit more below.

(ii) We T-dualize twice to go to type I where the five-branes remain five-branes but the seven-branes become type I nine-branes. To cancel the nine-brane charges we need orientifold nine-planes. These of course appear from the orientifold seven-planes. These five-branes are small instantons on the nine-branes. So the number of these five-branes is very large.

(iii) We have to avoid the Gauss' law constraint as not all configurations can lead to a *large* number of five-branes in the type I picture. It is crucial that various charge conservation laws are not violated. For a compact scenario one can easily show that the number of five-branes is fixed, and in many other cases Gauss' law will not allow too many five-branes. Only in the case with a sufficient number of noncompact directions can the number of five-branes be made large. Our study will therefore be based on these allowed configurations.

(iv) We S-dualize to heterotic theory where they are the Witten's small instantons and the total gauge symmetry is $\mathcal{G} \times Sp(2N)$ where $\mathcal{G}$ is a subgroup of $SO(32)$. The original $SO(32)$ group will be broken by Wilson lines. These Wilson lines come from the separation of the seven-branes in the type IIB picture.

(v) In IIB we know that there is a geometric transition that takes the wrapped five-branes on the two-cycle of a resolved conifold to fluxes on the three-cycles of a deformed conifold, i.e. the gauge-gravity duality. Embedding this duality in F theory will allow us to introduce fundamental matter via seven-branes. Then the geometric transition will allow us to study the dual geometry in F-theory framework. At the orientifold corner of F theory the seven-brane system can be studied using D7-branes and perturbative orientifold planes along with the wrapped five-branes. In the dual side there would also be an equivalent orientifold corner where we will have fluxes with seven-branes and orientifold seven-planes but no five-branes.

(vi) So in heterotic theory we expect the dual side to have only torsion and no heterotic five-branes, i.e. no small instantons or vector bundles. Therefore these small instantons have smoothed out and have become geometry. The vector bundle before transition will come from the dual of the seven-branes, and the separations between these seven-branes will appear as Wilson lines breaking the $SO(32)$ gauge group to a subgroup. After transition the gauge group is completely broken. The torsion, on the other hand, will appear from the remnants of the F-theory three-form Ramond-Ramond (RR) flux.

The above arguments show us that there is a possibility to understand gauge-gravity duality in the heterotic theory as small instanton transitions where after transition the large N small instantons become torsion. In the following we will try to put together this evidence to form a coherent global picture.

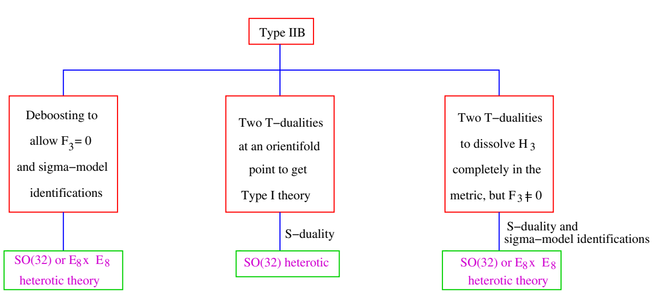

FIG. 2 (color online). Three possible ways to get local heterotic backgrounds from a given type IIB local background. These transformations are basically the transformations depicted as T_1 and T_2 in the previous figure. Note that the three paths are not *generic* and in many cases may not exist at all.

A. Evidence from an orientifold action

For the first step to work we need to go to the orientifold limit. The simplest orientifold action in the type IIB scenario is given by Eq. (4.3) of [9], i.e. $(x, y) \rightarrow (-x, -y)$, where (x, y) are the local coordinates defined in Footnote 1.

This is also the orientifold action discussed in [5]. The proposed local metric before geometric transition in the heterotic theory is given by[6]

$$ds^2 = d_1(dy - b_{y\theta_i}d\theta^i)^2 + d_2(dx - b_{x\theta_j}d\theta^j)^2 + d_3dr^2 - 2d_4(dx - b_{x\theta_j}d\theta^j)(dy - b_{y\theta_i}d\theta^i) + d_5dz^2 + d_6|d\chi_2|^2, \quad (2.1)$$

where d_i are the coefficients. In the above metric one can put nontrivial complex structure on the χ_2 torus also. For the present case we see that the local metric is given by the following nonzero components:

$$G = \begin{pmatrix} G_{xx} & G_{xy} & G_{xz} & G_{x\theta_1} & G_{x\theta_2} \\ G_{xy} & G_{yy} & G_{yz} & G_{y\theta_1} & G_{y\theta_2} \\ G_{xz} & G_{yz} & G_{zz} & G_{z\theta_1} & G_{z\theta_2} \\ G_{x\theta_1} & G_{y\theta_1} & G_{z\theta_1} & G_{\theta_1\theta_1} & G_{\theta_1\theta_2} \\ G_{x\theta_2} & G_{y\theta_2} & G_{z\theta_2} & G_{\theta_1\theta_2} & G_{\theta_2\theta_2} \end{pmatrix} = \begin{pmatrix} d_2 & -d_4 & 0 & s_1 - d_2b_{x\theta_1} & s_2 + d_4b_{y\theta_2} \\ -d_4 & d_1 & 0 & s_3 + d_4b_{x\theta_1} & s_4 - d_1b_{y\theta_2} \\ 0 & 0 & d_5 & 0 & 0 \\ s_1 - d_2b_{x\theta_1} & s_3 + d_4b_{x\theta_1} & 0 & d_6 + \mathcal{A}_1 & -d_4b_{x\theta_1}b_{y\theta_2} - d_4^{-1}s_is_j \\ s_2 + d_4b_{y\theta_2} & s_4 - d_1b_{y\theta_2} & 0 & -d_4b_{x\theta_1}b_{y\theta_2} - d_4^{-1}s_is_j & d_6 + \mathcal{A}_2 \end{pmatrix} \quad (2.2)$$

with the various terms s_i and $\mathcal{A}_i$ defined in terms of the coefficients d_i in the following way:

$$s_1 = d_4b_{y\theta_1}, \qquad s_2 = -d_2b_{x\theta_2}, \qquad s_3 = -d_1b_{y\theta_1},$$
$$s_4 = d_4b_{x\theta_2} \qquad \mathcal{A}_1 = d_1b^2_{y\theta_1} + d_2b^2_{x\theta_1} - 2d_4b_{y\theta_1}b_{x\theta_1},$$
$$\mathcal{A}_2 = d_1b^2_{y\theta_1} + d_2b^2_{x\theta_2} - 2d_4b_{y\theta_2}b_{x\theta_2}, \quad (2.3)$$

and $s_is_j \equiv s_1s_4 - s_2s_4 - s_1s_3$. Recall that the $b_{x\theta_i}$ and $b_{y\theta_i}$ are not the heterotic B fields. The heterotic B fields come from the type IIB F_3 field, which will henceforth be called $\mathcal{H}$. In the presence of a background dilaton ϕ we expect [23,24]

$$d\mathcal{H} = d[e^{2\phi} * d(e^{-2\phi}J)] = \text{sources} \quad (2.4)$$

with J being the usual fundamental form derived from the above metric and the sources are the heterotic five-branes or small instantons. Defining

$$\mathcal{D}x \equiv dx - b_{x\theta_j}d\theta^j, \qquad \mathcal{D}y \equiv dy - b_{y\theta_i}d\theta^i,$$
$$\mathcal{D}z_1 \equiv \mathcal{D}x + \tau_1\mathcal{D}y, \qquad \mathcal{D}z_1 \equiv d\chi_2, \quad (2.5)$$

we see that the local background for the wrapped heterotic five-branes is given by the following metric:

$$ds^2 = d_3dr^2 + d_5(dz + a\cot\langle\theta_1\rangle d\tilde{x} + b\cot\langle\theta_2\rangle d\tilde{y})^2 + d_2|\mathcal{D}z_1|^2 + d_6|\mathcal{D}z_2|^2, \quad (2.6)$$

where we have shifted dz in a suggestive way with (a, b) constants, so that the fibration represents a $U(1)$ fibration over the two two-tori $\mathcal{D}z_1$ and $\mathcal{D}z_2$. We therefore expect the global extension should be

$$(\mathbf{T}^2 \ltimes \mathbf{T}^2) \ltimes \mathbf{S}^1 \to \mathbf{S}^2 \times \mathbf{S}^3 \quad (2.7)$$

with $\ltimes$ representing nontrivial fibration topologically, so that we have heterotic five-branes wrapped on the resolved conifold. Note also that we have used coordinates $\tilde{x}$ and $\tilde{y}$ to denote the $U(1)$ fibration as we expect $d\tilde{x}$ and $d\tilde{y}$ to be nontrivially related to $\mathcal{D}x$, $\mathcal{D}y$, and $d\theta_i$. Therefore globally

$$d_2|\mathcal{D}z_1|^2 + d_6|\mathcal{D}z_2|^2 \to a_1dS_1^2 + a_2dS_2^2,$$
$$a_1 - a_2 = \text{resolution parameter},$$
$$dz + a\cot\langle\theta_1\rangle d\tilde{x} + b\cot\langle\theta_2\rangle d\tilde{y} \to d\psi + a\cos\tilde{\theta}_1d\phi_1 + b\cos\tilde{\theta}_2d\phi_2. \quad (2.8)$$

$S_i \equiv S_i(\tilde{\theta}_i, \phi_i)$ represent squashed spheres with nontrivial complex structures, and a_i are functions of the internal coordinates (including r). Thus the complete global metric is

$$ds^2_{\rm global} = a_3dr^2 + ds^2_{\mathbf{S}^2\times\mathbf{S}^3} \equiv g_{a,b}dR^adR^b, \quad (2.9)$$

where $R^a = (r_2, r_3, r_4, t_2, t_3, t_4)$ are the coordinates and $g_{a,b}$ are a slight variant of the metric components considered in Appendix 1 of [9]. In the following we give a brief description to implicitly describe the coordinate change necessary to compare with the metric written in terms of the usual resolved conifold coordinates $(r, \tilde{\theta}_i, \phi_i, \psi)$. Noting that the $\mathbf{C}^*$ action on the homogeneous coordinates of the resolved conifold identifies (z_1, z_2, z_3, z_4) with $(1, z_2/z_1, z_1z_3, z_1z_4)$, we start with the coordinates (U, Y, λ) of [10],[7] which we see are related to our coordinates by

$$U = z_1z_3, \qquad Y = z_1z_4, \qquad \lambda = -\frac{z_2}{z_1} \quad (2.10)$$

[6] In terms of local geometry the D5-branes wrap the (x, θ_1) direction and are spread along the spacetime x^{0123} directions. The seven-branes are *points* on the xy torus. This configuration is supersymmetric and survives the orientifold action and under T and S dualities leads to the required heterotic configuration.

[7] With $\theta_i \to \tilde{\theta}_i$ therein.

due to a sign convention in [10]. Using Eq. (2.5) of [9] together with (2.10), we can express (U, Y, λ) in terms of (z_2, z_3, z_4) and hence in terms of the real coordinates $(r_2, r_3, r_4, t_2, t_3, t_4)$. The desired change of variables comes from using (2.13) of [10], which expresses (U, Y, λ) instead in terms of the desired six real variables $(r, \psi, \phi_i, \tilde{\theta}_i)$.

An interesting variation of the global scenario is to change the local metric such that the mapping becomes

$$(\mathbf{T}^2 \times \mathbf{T}^2) \ltimes \mathbf{S}^1 \rightarrow \mathbf{S}^2 \times \mathbf{S}^3 \tag{2.11}$$

instead of (2.7). One simple way to decouple the two two-tori is to change the background three-form and the dilaton such that the local metric becomes

$$\begin{aligned} ds^2 = &\frac{d_4}{b_{x\theta_1} b_{y\theta_1}}[b_{x\theta_1}^2 (dy - b_{y\theta_i} d\theta^i)^2 + b_{y\theta_1}^2 (dx - b_{x\theta_j} d\theta^j)^2] \\ &+ d_3 dr^2 - 2d_4 (dx - b_{x\theta_j} d\theta^j)(dy - b_{y\theta_i} d\theta^i) \\ &+ d_5 dz^2 + d_6 |d\chi_2|^2 \end{aligned} \tag{2.12}$$

with an additional constraint that the matrix

$$\begin{pmatrix} b_{x\theta_1} & b_{x\theta_2} \\ b_{y\theta_1} & b_{y\theta_2} \end{pmatrix} \tag{2.13}$$

has a vanishing determinant. This immediately tells us that z_1 torus is decoupled from z_2 torus, with

$$\begin{aligned} \mathcal{D} z_1 \rightarrow dz_1 &\equiv dx + \tau_1 dy, \\ \tau_1 &= -\frac{1 - i\sqrt{3}}{2}\sqrt{\frac{d_4 b_{x\theta_1}}{b_{y\theta_1}}}. \end{aligned} \tag{2.14}$$

The global extension of the above local metric is

$$\begin{aligned} ds^2 &= h^{1/2} e^{\phi} d\tilde{s}_{0123}^2 + h^{-1/2} e^{\phi} ds_6^2, \\ \mathcal{H} &= e^{2\phi} * d(e^{-2\phi} J), \\ d\tilde{s}_{0123}^2 &= F_0 ds_{0123}^2, \\ ds_6^2 &= F_1 dr^2 + F_2 (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &\quad + \sum_{i=1}^{2} F_{2+i} (d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \end{aligned} \tag{2.15}$$

which is of course very close to the configuration that we studied in [9] with (h, ϕ, F_0, F_i) defined as before. Note that we have denoted the global three-form and the dilaton by the earlier notation to avoid clutter. Thus the background (2.15) with $(g_{\mu\nu}, \mathcal{H}, \phi)$ plus a vector bundle $\mathbf{V}$ represents the background for the wrapped heterotic five-branes. We will discuss the vector bundle a little later.

After geometric transition, the local metric takes the following suggestive form (see also [7]):

$$\begin{aligned} ds^2 = &\mathcal{A}_1 (dz + a_1 \cot\langle\theta_1\rangle dx + b_1 \cot\langle\theta_2\rangle dy)^2 \\ &+ \mathcal{A}_2 \left[(dy^2 + d\theta_2^2) + \frac{1}{|\tau|^2}(dx^2 + d\theta_1^2) \right] \\ &- 2\mathcal{A}_2 b_{y\theta_1} [\sin\langle\psi\rangle (dy d\theta_1 + dx d\theta_2) \\ &+ \cos\langle\psi\rangle (d\theta_1 d\theta_2 - dx dy)] + \mathcal{A}_5 dr^2, \end{aligned} \tag{2.16}$$

where $\mathcal{A}_i$ are constants locally, but will become nonconstant when we extend the metric globally, and $i\tau$ is the complex structure of $d\chi_2$ torus (the same one that we discussed above before geometric transition). The B field that allows for this metric is

$$B = b_{y\theta_1} (|\tau|^2 dx \wedge d\theta_2 + dy \wedge d\theta_1), \tag{2.17}$$

and the torsional equation is satisfied with an appropriate dilaton. Note that now we do not expect any five-branes but $\mathcal{H}$ will not be closed. More on this soon.

The global extension of the above local metric is typically of the following form:

$$\begin{aligned} ds_{\rm het}^2 = &ds_{0123}^2 + \mathcal{A}_1 (d\psi + a_1 \cos\theta_1 \mathcal{D}\phi_1 + b_1 \cos\theta_2 \mathcal{D}\phi_2)^2 \\ &+ \mathcal{A}_3 (d\theta_1^2 + \sin^2\theta_1 \mathcal{D}\phi_1^2) + \mathcal{A}_2 (d\theta_2^2 + \sin^2\theta_2 \mathcal{D}\phi_2^2) \\ &- 2\mathcal{A}_2 b_{y\theta_1} [\cos\psi (d\theta_1 d\theta_2 - \sin\theta_1 \sin\theta_2 \mathcal{D}\phi_1 \mathcal{D}\phi_2) \\ &+ \sin\psi (\sin\theta_1 \mathcal{D}\phi_1 d\theta_2 + \sin\theta_2 \mathcal{D}\phi_2 d\theta_1)] + \mathcal{A}_5 dr^2, \end{aligned} \tag{2.18}$$

where $\mathcal{A}_i$ are no longer constants and $\mathcal{D}\phi_i \equiv d\phi_i + f_{ij} dx^j$ with dx^j being the internal coordinates and f_{ij} related to antisymmetric two-form B fields. In this paper we will assume that $f_{ij} \approx 0$; and for a very special choices of these coefficients

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{A}_3 = \frac{\mathcal{A}_5}{4} = \frac{N}{4}, \\ \mathcal{A}_2 &= \frac{N(e^{2g} + a^2)}{4}, \\ a(r) &= -\frac{2r}{\sinh 2r}, \\ e^{2g} &= 4r \coth 2r - \frac{4r^2}{\sinh^2 2r} - 1 \end{aligned} \tag{2.19}$$

with N being the number of wrapped five-branes, and with the following dilaton [25]:

$$e^{2\phi} = \frac{e^{g + 2\phi_0}}{\sinh 2r}, \tag{2.20}$$

the torsion $\mathcal{H}_{\rm MN}$ can be easily computed, and it takes the following form [7]:

$$\begin{aligned}\mathcal{H}_{\rm MN} &\equiv e^{2\phi} * d(e^{-2\phi}J)\\ &= -\frac{Na'}{4}\cos\psi dr\wedge(d\theta_1\wedge d\theta_2 - \sin\theta_1\sin\theta_2 d\phi_1\wedge d\phi_2) - \frac{Na'}{4}\sin\psi dr\wedge(\sin\theta_2 d\theta_1\wedge d\phi_2 - \sin\theta_1 d\theta_2\wedge d\phi_1)\\ &\quad + \frac{Na}{4}\sin\psi d\theta_1\wedge d\theta_2\wedge(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) - \frac{N}{4}(\sin\theta_1\cos\theta_2 - a\cos\psi\cos\theta_1\sin\theta_2)d\theta_1\wedge d\phi_1\wedge d\phi_2\\ &\quad - \frac{N}{4}(\sin\theta_2\cos\theta_1 - a\cos\psi\cos\theta_2\sin\theta_1)d\theta_2\wedge d\phi_1\wedge d\phi_2 - \frac{N}{4}\sin\theta_1 d\theta_1\wedge d\phi_1\wedge d\psi + \frac{N}{4}\sin\theta_2 d\theta_2\wedge d\phi_2\wedge d\psi\\ &\quad - \frac{Na}{4}\cos\psi(\sin\theta_2 d\theta_1\wedge d\phi_2\wedge d\psi - \sin\theta_1 d\theta_2\wedge d\phi_1\wedge d\psi) - \frac{Na}{4}\sin\psi\sin\theta_1\sin\theta_2 d\phi_1\wedge d\phi_2\wedge d\psi \end{aligned} \tag{2.21}$$

with J being the fundamental form. Interestingly, for this $d\mathcal{H}_{\rm MN} = 0$, and we do not expect any five-brane sources. This is a bit subtle now because in the heterotic theory we expect

$$d\mathcal{H} = \alpha'[{\rm tr}\, R_+ \wedge R_+ - {\rm Tr}\, F\wedge F], \tag{2.22}$$

and since ${\rm tr}\, R_+\wedge R_+$ is independent of N, the N dependence of the dissolved small instantons can come either from the torsion $\mathcal{H}$ or/and from the bundle ${\rm Tr}\, F\wedge F$. (Here R_+ is the curvature tensor with modified connection that will be discussed in Sec. IV.) However, the gauge group is completely broken when the small instantons dissolve, and therefore

$${\rm Tr}\; F\wedge F = 0, \tag{2.23}$$

so with $d\mathcal{H} = 0$ the Bianchi identity will be difficult to satisfy. Thus the only way would be to modify the torsion (2.21) by a small amount so that both (2.22) and (2.23) are satisfied with the torsion $\mathcal{H}$ defined as[8]

$$\mathcal{H} = \mathcal{H}_{\rm MN} + \mathcal{H}_{\rm small}, \tag{2.24}$$

where we presented an explicit form for $\mathcal{H}$ in Appendix A. Here $\mathcal{H}_{\rm small}$ is a small N-independent shift of the torsion. For the choice (2.19), the small r limit is the Maldacena-Nunez background [19]. The large r, i.e. the UV limit of the theory, is given in [20]. Comparing (2.21) with (2.17) we see that the coefficient of $d\theta_1\wedge d\phi_2$ has the following terms:

$$\begin{aligned} &-\frac{Na'}{4}\sin\psi\sin\theta_2 dr + \frac{Na}{4}\sin\psi\cos\theta_1 d\theta_1\\ &\quad + \frac{Na}{4}\cos\psi\sin\theta_1 d\psi\\ &\quad - \frac{N}{4}(\sin\theta_2\cos\theta_1 - a\cos\psi\cos\theta_2\sin\theta_1)d\phi_2 \end{aligned} \tag{2.25}$$

and similarly for the coefficient of $d\theta_2\wedge d\phi_1$. This would have been the natural extension of (2.17), but we see that (2.21) has extra terms that are not there in the local limit. For example, there are no such terms like

$$\frac{N}{4}(dx\wedge d\theta_1 - dy\wedge d\theta_2 - a_0 dx\wedge dy)\wedge dz \tag{2.26}$$

with $a_0 \equiv a(r_0)$ in (2.17). Therefore, using all the above arguments, the full global background is a deformation of the background (2.19), (2.20), and (2.21) that satisfies the Bianchi identity (2.22) with the condition (2.23).

B. Evidence from sigma model identification

The second evidence comes from the sigma model identification. For the situation after the transition, one expects a (0, 2) world-sheet sigma model for $\mathcal{N} = 1$ spacetime supersymmetry. The general idea is simple and can be stated as follows.

To develop (0, 2) models in the context of complex structures we start by considering the following world-sheet action for type IIB theory in the presence of $H_{\rm NS}$:

$$S = \frac{1}{8\pi\alpha'}\int d^2\sigma\Big[(g_{ij} + B_{ij})\partial_+ X^i\partial_- X^j + \frac{1}{4}S^g_{\rm fermionic}\Big], \tag{2.27}$$

where $S^g_{\rm fermionic}$ contains the standard kinetic term plus the following interaction part:

$$\begin{aligned} S^g_{\rm int} = \frac{i}{8\pi\alpha'}\int\Big(&\psi^\rho\omega_+^{ab}\sigma_{ab}^{\rho\sigma}\psi^\sigma + \psi^{\dot\rho}\omega_+^{ab}\sigma_{ab}^{\dot\rho\dot\sigma}\psi^{\dot\sigma}\\ &- \frac{i}{2}\mathcal{R}_{ijkl}\sigma^{ij}_{\dot\rho\dot\sigma}\sigma^{kl}_{\kappa\gamma}\psi^{\dot\rho}\psi^{\dot\sigma}\psi^\kappa\psi^\gamma\Big), \end{aligned} \tag{2.28}$$

where $\mathcal{R}_{ijkl}$ is the background Riemann tensor. In this action we have the freedom to add noninteracting fields. This ruins the carefully balanced (2, 2) supersymmetry of this model. We can use this to our advantage by adding noninteracting fields *only* in the left-moving sector. This breaks the left-moving supersymmetry, and one might therefore hope to obtain an action for (0, 2) models from (2.27), at least *classically*. On the other hand, a possible (0, 2) action is also restricted because this will be the action for the heterotic string. Therefore let us start with the following *naive* steps to find the classical (0, 2) action from a given (2, 2) action (see [7] for more details):

(i) Keep the right-moving sector unchanged; i.e. ψ^p remain as before.

(ii) In the left-moving sector, replace $\psi^{\dot q}$ by eight fermions Ψ^a, $a = 1,\ldots 8$. Also add 24 additional

[8] We thank Juan Maldacena and Edward Witten for discussions on the above issues.

noninteracting fermions Ψ^b, $b = 9, \ldots 32$. In other words,

$$\Psi^A = \begin{pmatrix} \psi^{\dot q} \\ \Psi^9 \\ \ldots \\ \Psi^{32} \end{pmatrix}. \tag{2.29}$$

(iii) Replace ω_+ by gauge fields A, i.e. embed the *torsional* spin connection into the gauge connection.

The above set of transformations will convert the classical (2, 2) action given in (2.27) to a classical (0, 2) one. One might, however, wonder about the Bianchi identity in the heterotic theory in light of the discussions that we had in the previous subsection. The type IIB three-form fields are closed, whereas heterotic three-form fields satisfy the Bianchi identity. One immediate reconciliation would be that because of the *embedding* $\omega_+ = A$, the heterotic three-form should be closed. This may seem like an admissible solution to the problem, but because of subtleties mentioned earlier[9] this cannot be the story here. Therefore an embedding of the form

$$A_i^{AB} = \begin{pmatrix} \omega_{i+}^{ab} & 0 \\ 0 & \mathcal{O}(\alpha') \end{pmatrix} \tag{2.30}$$

(where for simplicity we have left the off-diagonal part vanishing) cannot quite be the solution for our case as we require

$$d\mathcal{H}_{\text{small}} = \alpha'[d\omega_+ \wedge d\omega_+ + \mathcal{O}(\omega_+^4)]. \tag{2.31}$$

Thus one possibility will be to make the gauge field vanishing and replace all connections by the torsional connection ω_+. Using this the new action with (0, 2) supersymmetry becomes

$$S = \frac{1}{8\pi\alpha'} \int d^2\sigma[(g_{ij} + B_{ij})\partial_+ X^i \partial_- X^j + i\psi^p(\Delta_+\psi)^p + i\Psi^A(\Delta_-\Psi)^A + \mathcal{O}(\alpha')],$$

where due to the Bianchi identity (2.31) and our choice, there is no F_{ij}^a Yang-Mills field strength. The fermion indices are $A = 1, \ldots, 32$, which means there are 32 fermions, and T^a form tensors of rank 16. The Laplacians are given as follows:

$$\begin{aligned} \Delta_-\Psi^A &= \partial_-\Psi^A, \\ \Delta_+\psi^p &= \partial_+\psi^p + \tfrac{1}{2}(\omega_+)^{ab}\sigma_{ab}^{pq}\psi^q H_{ijk} \\ &= \tfrac{1}{2}(B_{ij,k} + B_{jk,i} + B_{kj,i})_{(\text{MN})} + \mathcal{O}(\alpha'). \end{aligned} \tag{2.32}$$

As expected, this set of actions determines the (0, 1) supersymmetric heterotic sigma model. This is similar to the (1, 1) action for the type II case. The full (0, 2) SUSY will be determined by additional actions on the fields [exactly as for the (1, 1) case before].

The above discussion is a simple way to see how certain IIB backgrounds can be dragged directly to the heterotic side by making small modifications to the field contents. The above discussion was solely for the heterotic background after the transition where the heterotic gauge group is completely broken. The situation then is completely different for the case before the transition.

To study the heterotic background before geometric transition using our sigma model identification we take the type IIB background given as Eq. (4.13) in [9] and "deboost" the system so that we can have

$$F_3 = F_5 = 0. \tag{2.33}$$

The deboosting procedure follows the *reverse* chain of dualities depicted in Fig. 1 of [9]. Once we have this, then it is obvious that the background before GT is precisely (2.15). In the type I language, which is the S dual of (2.15), the type I D5-branes are the small instantons of the nine-branes gauge theory. For N D5-branes, or N small instantons in the heterotic theory, the gauge symmetry before the transition can be written succinctly as

$$Sp(2N) \times \mathcal{G}, \tag{2.34}$$

where $\mathcal{G}$ is a *subgroup* of the full $SO(32)$ group in the heterotic theory. The $SO(32)$ is broken by the Wilson lines. These Wilson lines are related to the distances between the type IIB seven-branes in the full F-theory picture.

This means that the sigma model before the transition is exactly given by an ADHM sigma model [29] (much like the one discussed in [30]). One may then understand the geometric transition to be related to an Affleck-Dine-Seiberg (ADS) [31] type of superpotential being added to the usual ADHM sigma model superpotential. This is much like the discussion in Klebanov-Strassler [18], where the addition of an equivalent ADS superpotential to the usual $\mathcal{N} = 1$ quartic superpotential shows how one could go from a conifold to a deformed conifold. More details of this will appear elsewhere.

C. A more direct analysis

The third evidence comes from *dissolving* the NS three-form completely in the metric in the IIB picture. We can do it in the presence of an orientifold action also. First, however, let us try without involving any orientifold action, i.e. keeping only the wrapped D5-branes. The local geometry is well known to have the following form:

$$\begin{aligned} ds^2 = dr^2 &+ \left(dz + \sqrt{\frac{\gamma'}{\gamma}} r_0 \cot\langle\theta_1\rangle dx + \sqrt{\frac{\gamma'}{(\gamma + 4a^2)}} r_0 \cot\langle\theta_2\rangle dy\right)^2 + \left[\frac{\gamma\sqrt{h}}{4} d\theta_1^2 + dx^2\right] \\ &+ \left[\frac{(\gamma + a^2)\sqrt{h}}{4} d\theta_2^2 + dy^2\right] + \cdots, \end{aligned} \tag{2.35}$$

[9] See also [26–28] for additional subtleties that come from the above embedding. In fact even in the usual case this embedding will not allow any compact non-Kähler manifolds to appear in the heterotic theory.

where all the coefficients are described in [7,9]. There is also a B field given by

$$B_{\rm NS} = b_{x\theta_1} dx \wedge d\theta_1 + b_{y\theta_2} dy \wedge d\theta_2, \tag{2.36}$$

plus of course there are F_3 and F_5 fields whose orientations will be discussed soon. Recall also that all the coefficients in the above metric are constants. This is going to be useful soon.

Under two T dualities along (x, θ_2) directions, the B field (2.36) completely dissolves in the metric to give us the following non-Kähler geometry:

$$\begin{aligned} ds^2 = dr^2 &+ [dz + \Delta_1 \cot\langle\theta_1\rangle(dx - b_{x\theta_1} d\theta_1) \\ &+ \Delta_2 \cot\langle\theta_2\rangle dy]^2 + [\alpha_1 d\theta_1^2 + (dx - b_{x\theta_1} d\theta_1)^2] \\ &+ [dy^2 + \alpha_2 (d\theta_2 - b_{y\theta_2} dy)^2], \end{aligned} \tag{2.37}$$

where the coefficients appearing in the metric are defined in terms of the coefficients of (2.35) appearing above. The metric (2.37) is a non-Kähler deformation of (2.35). The complex structures of the base tori change from $\tau_k = i\sqrt{\alpha_k}$ to

$$\begin{aligned} \tau_1 &= -b_{x\theta_1} + i\sqrt{\alpha_1}, \\ \tau_2 &= -\frac{\alpha_2 b_{y\theta_2}}{1 + \alpha_2 b_{y\theta_2}^2} + i\frac{\sqrt{\alpha_2}}{1 + \alpha_2 b_{y\theta_2}^2} \\ dz_1 &\equiv dx + \tau_1 d\theta_1, \\ dz_2 &= dy + \tau_2 d\theta_2, \\ \mathcal{D}z &\equiv dz - \Delta_1 b_{x\theta_1} \cot\langle\theta_1\rangle d\theta_1, \end{aligned} \tag{2.38}$$

so that the metric (2.37) takes the following suggestive format:

$$\begin{aligned} ds^2 = dr^2 &+ (\mathcal{D}z + \Delta_1 \cot\langle\theta_1\rangle dx + \Delta_2 \cot\langle\theta_2\rangle dy)^2 \\ &+ |dz_1|^2 + \frac{|\tau_2|_0^2}{|\tau_2|^2} |dz_2|^2, \end{aligned} \tag{2.39}$$

where $|\tau_2|_0$ is the complex structure of the second torus in the absence of B fields. Note that the coefficients in front of the dz_1 and dz_2 tori are different. This means that the global extension of the local metric (2.37) should have two two-spheres of unequal sizes which, in other words, should be a resolved conifold. Thus the five-branes wrap the two-cycle of a non-Kähler resolved conifold, exactly as we have been considering earlier.

There are still a few loose ends that we need to tie up before we go to the analysis of the geometry after the transition. For example, what happens at the orientifold point? How do the seven-branes behave in the final T-dual setup? What happened to the RR three- and five-forms?

To understand these issues let us analyze the system carefully. As discussed in [9] there are two possible ways to perform the orientifolding operation here. The first O action has already been discussed in the previous subsections. The second O action is given by Eq. (4.6) of [9] i.e. $(x, \theta_1) \to (-x, \pi - \theta_1)$. For this action we can keep the D5-branes *parallel* to the seven-branes once we are away from the orientifold point. Such a configuration breaks supersymmetry. However, we can form a *bound state* of D5- and D7-branes that is supersymmetric. This configuration is then different from the one studied before. In the full global geometry we can assume that the bound state is embedded in a nontrivial way in the non-Kähler resolved conifold space, and the fluxes give rise to a dipole deformation of the bound state (see [17,22] for more details).

This picture can also be understood as an N D5-brane bound states on a single D7-brane with all other seven-branes moved away in the (x, θ_1) direction. To go to the heterotic side we need to go to the orientifold point. At the O point the world-volume gauge fluxes are all projected out because only dipole deformations are allowed. However, note that in the far IR the resolution cycle is very small and therefore the wrapped D5-branes are fractional three-branes in the IIB setup. Thus at the orientifold point we may rotate the three-form fluxes (i.e. the five-brane sources) so that the local RR field is the following (see also [5])[10]:

$$\begin{aligned} B_{\rm RR} = \tilde{b}_{xz} dx \wedge dz &+ \tilde{b}_{xy} dx \wedge dy + \tilde{b}_{x\theta_2} dx \wedge d\theta_2 \\ &+ \tilde{b}_{y\theta_1} dy \wedge d\theta_1 + \tilde{b}_{z\theta_1} dz \wedge d\theta_1 + \tilde{b}_{\theta_1\theta_2} d\theta_1 \wedge d\theta_2, \end{aligned} \tag{2.40}$$

where $\tilde{b}_{\alpha\beta} = \tilde{b}_{\alpha\beta}(r, \theta_1, \theta_2)$, which also means that the D5-branes at the O point form a nontrivial surface that could still be viewed as a bound state with the seven-branes. The $B_{\rm NS}$ in (2.36) will give rise to dipole deformation at the O point. The heterotic metric then is (2.37) whose global extension should be the non-Kähler resolved conifold mentioned before.

The rest of the analysis follows a similar route as outlined earlier. The gravity dual should in general be nongeometric, but if we concentrate only on the geometric portion of the moduli space of solutions, the gravity dual is a small deformation over the Maldacena-Nunez geometry at the far IR. This small deformation cannot be ignored; otherwise, the Bianchi identity will not be satisfied.

III. ANALYSIS OF TORSION CLASSES

Now that we have a few pieces of evidence that suggest that there is a possibility to describe the strongly coupled theory on the heterotic five-branes using a gravitational description, we should seriously check the supersymmetry of the underlying gravitational solutions. In fact as pointed out in [9] and in Sec. I A, the issue of supersymmetry is subtle: There is no immediate guarantee that the backgrounds in the IIA, IIB, and M theories are SUSY after performing all the transformations. In this section we will

[10]Note that the rotation keeps one component of the three-form fluxes along the orbifold direction. As is well known, this requirement is enough to survive the O action.

therefore analyze the torsion classes in all the intermediate theories and ask under what conditions SUSY could be preserved. Such an analysis will also help us fix many of the free parameters in the intermediate theories.

A manifold with $SU(3)$ structure has all the group-theoretical features of a Calabi-Yau, namely, invariant two- and three forms, J and Ω, respectively. On a manifold of $SU(3)$ holonomy, not only are J and Ω well defined, but they are also closed: $dJ = 0 = d\Omega$. If they are not closed, dJ and $d\Omega$ give a good measure of how far the manifold is from having $SU(3)$ holonomy [32]:

$$\begin{aligned} dJ &= -\tfrac{3}{2}\mathrm{Im}(W_1\bar{\Omega}) + W_4 \wedge J + W_3 \\ d\Omega &= W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega. \end{aligned} \tag{3.1}$$

The W's are the $(3 \oplus \bar{3} \oplus 1) \otimes (3 \oplus \bar{3})$ components of the intrinsic torsion: W_1 is a complex zero-form in $1 \oplus 1$, W_2 is a complex primitive two-form, so it lies in $8 \oplus 8$, W_3 is a real primitive $(2, 1) \oplus (1, 2)$-form and it lies in $6 \oplus \bar{6}$, W_4 is a real one-form in $3 \oplus \bar{3}$, and finally W_5 is a complex $(1, 0)$-form, so its degrees of freedom are again $3 \oplus \bar{3}$.

It is sometime convenient to express the torsion classes using another definition. These have appeared in the literature in the following guise (see also [33] for more details):

$$\begin{aligned} d\Omega_{\pm} \wedge J &= \bar{W}_1^{\mp} J \wedge J \wedge J, \\ d\Omega_{\pm}^{(2,2)} &= \bar{W}_1^{\mp} J \wedge J + \bar{W}_2^{\mp} \wedge J, \\ dJ^{(2,1)} &= [J \wedge \bar{W}_4]^{(2,1)} + \bar{W}_3, \\ \bar{W}_4 &= \tfrac{1}{2} J \lrcorner dJ, \\ \bar{W}_5 &= \tfrac{1}{2}\Omega_+ \lrcorner d\Omega_+, \end{aligned} \tag{3.2}$$

where $\bar{W}_1 = \bar{W}_1^+ + \bar{W}_1^-$, $\bar{W}_2 = \bar{W}_2^+ + \bar{W}_2^-$, and the contraction operator $\lrcorner$ is defined as

$$\lrcorner : \bigwedge^k T^* \otimes \bigwedge^n T^* \to \bigwedge^{n-k} T^*, \tag{3.3}$$

$$(L_k, M_n) \mapsto \frac{1}{n!} c_n^k L^{a_1 \ldots a_k} M_{a_1 \ldots a_n} e^{a_{k+1} \ldots e^{a_n}}. \tag{3.4}$$

The two definitions are related as

$$\begin{aligned} &W_1 = \bar{W}_1^+ + i\bar{W}_1^-, \qquad W_2 = \bar{W}_2^+ + i\bar{W}_2^-, \qquad W_3^{(2,1)} = \bar{W}_3, \\ &W_4 = \bar{W}_4, \qquad \mathrm{Re}(W_5) = -\bar{W}_5. \end{aligned} \tag{3.5}$$

A. Torsion classes in the heterotic theory

In the following we will study the torsion classes for the heterotic string theories before and after geometric transition. Before geometric transition we have the heterotic string theory as in [9]:

$$ds^2 = k^{-2} e^{2\phi} F_0 ds^2_{0123} + k^2 ds^2_6, \tag{3.6}$$

where $k^2(r, \theta_1, \theta_2) = h^{-1/2} e^{\phi}$ and

$$\begin{aligned} ds_6^2 = {}& F_1 dr^2 + F_2(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &+ F_3(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + F_4(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \end{aligned} \tag{3.7}$$

where F_i are functions[11] of r. The metric is of course related to the type IIB metric studied in [9] and discussed further in Sec. I A. The NS three-form flux is

$$\begin{aligned} H_3 = {}& e^{2\phi} *_6 d(e^{-2\phi} J) \\ = {}& \frac{F_4\sqrt{F_2}\sin\theta_2}{F_3\sqrt{F_1}\sin\theta_1}[k^2(\sqrt{F_1F_2}\sin\theta_1 + 2\phi_r F_3\sin\theta_1 - F_{3r}\sin\theta_1) - 2kF_3k_r\sin\theta_1]d\theta_2 \wedge d\phi_2 \\ &\wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) + \frac{F_3\sqrt{F_2}\sin\theta_1}{F_4\sqrt{F_1}\sin\theta_2}[k^2(\sqrt{F_1F_2}\sin\theta_2 + 2\phi_r F_4\sin\theta_2 - F_{4r}\sin\theta_2) \\ &- 2kF_4k_r\sin\theta_2]d\theta_1 \wedge d\phi_1 \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) + 2k\sqrt{F_1F_2}\sin\theta_2(k\phi_{\theta_2} - k_{\theta_2})d\phi_2 \\ &\wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \wedge dr + 2k\sqrt{F_1F_2}\sin\theta_1(k\phi_{\theta_1} - k_{\theta_1})d\phi_1 \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \wedge dr \\ &+ 2kF_4\sin\theta_1\sin\theta_2(k\phi_{\theta_1} - k_{\theta_1})d\phi_1 \wedge d\theta_2 \wedge d\phi_2 + 2kF_3\sin\theta_1\sin\theta_2(k\phi_{\theta_2} - k_{\theta_2})d\theta_1 \wedge d\phi_1 \wedge d\phi_2, \end{aligned} \tag{3.8}$$

which in turn will soon be related to W_3. Here we have defined $k_\alpha \equiv \partial_\alpha k$ and $F_{n\alpha} \equiv \partial_\alpha F_n$, with α being any of the internal coordinates. Now to see the precise connection of H_3 with W_3 we first need to write the vierbeins for the internal space. They are given by

[11] In this paper we will not investigate the case where F_i are more generic functions of $(r, \theta_i, \phi_i, \psi)$. The analysis with generic F_i will definitely be technically challenging and will give us a bigger moduli space of solutions, but the physics will remain unchanged.

$$
\begin{aligned}
e^1 &= k\sqrt{F_3}(\cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1),\\
e^2 &= k\sqrt{F_3}(-\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1),\\
e^3 &= k\sqrt{F_4}(\cos\psi_2 d\theta_2 + \sin\psi_2 \sin\theta_2 d\phi_2),\\
e^4 &= k\sqrt{F_4}(-\sin\psi_2, d\theta_2 + \cos\psi_2 \sin\theta_2 d\phi_2),\\
e^5 &= k\sqrt{F_2}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2),\\
e^6 &= k\sqrt{F_1}dr.
\end{aligned} \tag{3.9}
$$

The fundamental two-form J and holomorphic three-form Ω are defined in terms of these vierbeins as

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6, \tag{3.10}$$

$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6), \tag{3.11}$$

which will immediately give us the following values for W_1 and W_2:

$$W_1 = W_2 = 0, \tag{3.12}$$

implying that the internal manifold is a complex manifold. The three-form NS flux would be directly related to W_3 if we demand $d(*_6 H_3) = 0$. In that case $W_3 = *_6 H_3$. Thus for our case W_3 is

$$W_3 = \frac{k^2}{2}(F_4F_{3r} - F_3F_{4r} + F_3\sqrt{F_1F_2} - F_4\sqrt{F_1F_2}), \qquad \left(\frac{\sin\theta_1}{F_4} dr \wedge d\theta_1 \wedge d\phi_1 - \frac{\sin\theta_2}{F_3} dr \wedge d\theta_2 \wedge d\phi_2\right). \tag{3.13}$$

The above three torsion classes told us how to put three-form flux on a complex manifold. However, supersymmetry is still not guaranteed. To demand supersymmetry we first require to compute W_4 and ReW_5. For our case they are given by

$$
\begin{aligned}
W_4 &= \left(\frac{F_{3r} - \sqrt{F_1F_2}}{2F_3\sqrt{F_1}} + \frac{F_{4r} - \sqrt{F_1F_2}}{2F_4\sqrt{F_1}} + \frac{2k_r}{k}\right)dr + \frac{2k_{\theta_1}}{k} d\theta_1 + \frac{2k_{\theta_2}}{k} d\theta_2,\\
\mathrm{Re}W_5 &= \left(\frac{3k_r}{2k} + \frac{F_{2r}}{4F_2} + \frac{F_{3r}}{4F_3} + \frac{F_{4r}}{4F_4} - \frac{1}{2}\sqrt{\frac{F_1}{F_2}}\right)dr + \frac{3k_{\theta_1}}{k} d\theta_1 + \frac{3k_{\theta_2}}{k} d\theta_2.
\end{aligned} \tag{3.14}
$$

The SUSY condition requires $2W_4 = \mathrm{Re}W_5$ so k is a function of only r. The SUSY requirement from the torsion classes gives rise to the following constraint equation for the warp factors F_i:

$$\left(\frac{1}{\sqrt{F_1}} - \frac{1}{4}\right)\partial_r \log F_3 + \left(\frac{1}{\sqrt{F_1}} - \frac{1}{4}\right)\partial_r \log F_4 - \frac{1}{4}\partial_r \log F_2 + \frac{5}{2}\partial_r \log k = \left(\frac{1}{F_3} + \frac{1}{F_4}\right)\sqrt{F_2} - \frac{1}{2}\sqrt{\frac{F_1}{F_2}}. \tag{3.15}$$

This constraint equation would also be the one that we will need to impose on the type IIB side. From [26] we expect $e^\phi = h^{-1/2}F_0^{-1}$ to also come out of the torsional constraint (3.15). However, observe that (3.15) is independent of F_0, so F_0 is a *free* parameter for the background. Therefore without loss of generality one may choose

$$e^{-\phi} = h^{1/2}F_0 = \frac{F_0^{3/2}\cosh\beta}{\sqrt{1 + F_0^2\sinh^2\beta}} \tag{3.16}$$

as in (1.2), which will keep the spacetime part of the metric independent of any warp factor. This would then be consistent with the solutions of [26].

After the geometric transition, as we discussed in detail earlier, we expect the generic solution to look like (2.18). The torsion classes for this case is a special case of (3.17) with all the fibration vanishing, i.e. $b_{ij} = 0$, where $i, j = r, \psi, \theta_i, \phi_i$ (see also Appendix B for more details).

B. Torsion classes in the type IIA theory

We want to make sure that during our duality chain discussed in [9] the solutions we take are all supersymmetric, as the final heterotic metric after the transition will come from dualizing the type IIB case after geometric transition. The starting point of the IIB solution is obviously supersymmetric [provided the warp factors satisfy the constraint Eq. (3.15)]. However, after the coordinate transformation and shift in the metric it is not obvious that the IIA solutions before and after flop are still supersymmetric. Therefore in this section we will calculate the torsion classes for the IIA solutions and impose constraints to make them supersymmetric.

Before the flop the IIA metric takes the similar form as the deformed conifold as discussed in [9]:

$$\begin{aligned} ds_{10}^2 = {} & F_0 ds_{0123}^2 + F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}[d\psi - b_{\psi r}dr - b_{\psi\theta_2}d\theta_2 + \Delta_1\cos\theta_1(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr) \\ & + \Delta_2\cos\theta_2\cos\psi_0(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)]^2 + \alpha j_{\phi_2\phi_2}[d\theta_1^2 + (d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr)^2] \\ & + \alpha j_{\phi_1\phi_1}[d\theta_2^2 + (d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)^2] \\ & + 2\alpha j_{\phi_1\phi_2}\cos\psi[d\theta_1 d\theta_2 - (d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr)(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)] \\ & + 2\alpha j_{\phi_1\phi_2}\sin\psi[(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr)d\theta_2 + (d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)d\theta_1]. \end{aligned} \tag{3.17}$$

As before, to compute the torsion classes we need the vierbeins. For our case, they are

$$\begin{aligned} & e^1 = \sqrt{F_1}dr, \qquad e^3 = Kd\theta_1, \qquad e^4 = -K(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr), \\ & e^2 = G(d\psi - b_{\psi r}dr + \Delta_1\cos\theta_1(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr) + \Delta_2\cos\theta_2(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)), \\ & e^5 = L[\sin\psi(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr) + \cos\psi d\theta_2 - ad\theta_1], \\ & e^6 = L[\cos\psi(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr) - \sin\psi d\theta_2 - a(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr)], \end{aligned} \tag{3.18}$$

where G, L, K, and a are defined as

$$G = \sqrt{\frac{\alpha F_2}{\Delta_1\Delta_2}}, \qquad L = \sqrt{\alpha j_{\phi_1\phi_1}}, \qquad K = \sqrt{\alpha\left(j_{\phi_2\phi_2} - \frac{j_{\phi_1\phi_2}^2}{j_{\phi_1\phi_1}}\right)}, \qquad a = \frac{j_{\phi_1\phi_2}}{j_{\phi_1\phi_1}}. \tag{3.19}$$

From the above vierbeins we can write the complex vierbeins as

$$E_1 = e^1 + ie^2, \qquad E_2 = e^3 + i(Ae^4 + Be^6), \qquad E_3 = e^5 + i(Be^4 - Ae^6) \tag{3.20}$$

with A and B as functions of the radial direction r which in turn are determined by the $SU(3)$ structure of the underlying manifold satisfying $A^2 + B^2 = 1$.

With all these preparations, we are now ready to write the torsion classes. They are given by

$$\begin{aligned} W_1 = {} & -\frac{1}{6L^2K^2\sqrt{F_1}GA}(-iL^2GK^2b_{\phi_2\theta_2,r}AB - 2iL^2AGb_{\psi r}K^2B + 2L^2A\sqrt{F1}K^2B + 2iGL^3K\cos\psi^2 b_{\phi_2\theta_2,r}aB^2 \\ & - iL^3GKb_{\phi_2\theta_2,r}aB^2 + iL^3Gb_{\phi_1\theta_1,r}Ka - iL^3Gb_{\phi_1\theta_1,r}KaB^2 - iL^2Gb_{\phi_1\theta_1,r}K^2AB + G^2B\sqrt{F1}AL^2\Delta_1\sin\theta_1 a^2 \\ & + G^2B\sqrt{F_1}B\Delta_2\sin\theta_2 L^2 + G^2B\sqrt{F_1}A\Delta_2\sin\theta_2 K^2 + 2GL^3K\sin\psi b_{\phi_2\theta_2,r}\cos\psi Aa + 2L^2GBK^2 \\ & - 2iGL^3K\cos\psi^2 b_{\phi_2\theta_2,r}a + iL^3GKb_{\phi_2\theta_2,r}aL^2K^2\sqrt{F_1}GB) \\ W_4 = {} & w_{4r}dr + w_{4\theta_1}d\theta_1 + w_{4\theta_2}d\theta_2 + w_{4\phi_1}d\phi_1 + w_{4\phi_2}d\phi_2 \\ \mathrm{Re}W_5 = {} & w_{5e^1}e^1 + w_{5e^2}e^2 + w_{5e^3}e^3 + w_{5e^4}e^4 + w_{5e^5}e^5 + w_{5e^6}e^6, \end{aligned} \tag{3.21}$$

where w_i are given in Appendix B. Once we know W_1, W_4, and W_5, it is easy to calculate W_2 and W_3 from (3.1). We will not give the explicit expressions here. Note also that since W_1 and W_2 are not zero, the type IIA manifold is not complex. This is of course consistent with our earlier works [5,7,9,22]. The supersymmetry condition imposes the following constraints:

$$\begin{aligned} 2w_{4\theta_1} & = Kw_{5e^3} + Kb_{\phi_1\theta_1}w_{5e^4} - aLw_{5e^5} + aLb_{\phi_1\theta_1}w_{5e^6}, \\ 2w_{4r} & = \sqrt{F_1}w_{5e^1} + Kb_{\phi_1 r}w_{5e^4} - L\sin\psi b_{\phi_2 r}w_{5e^5} - L(ab_{\phi_1 r} - \cos\psi b_{\phi_2 r})w_{5e^6}, \\ 2w_{4\theta_2} & = L(-\sin\psi b_{\phi_2\theta_2} + \cos\psi)w_{5e^5} - L(\cos\psi b_{\phi_2\theta_2} + \sin\psi)w_{5e^6}, \\ 2w_{4\phi_2} & = L(\sin\psi w_{5e^5} + \cos\psi w_{5e^6}), \qquad 2w_{4\phi_1} = -aLw_{5e^6} - Kw_{5e^4} \end{aligned} \tag{3.22}$$

with $w_{5e^2} = 0$. These conditions are in addition to the condition (3.15), and therefore would constrain the warp factors further.

After the flop the IIA metric takes the similar form as the resolved conifold [9]:

$$\begin{aligned} ds_{10}^2 = & F_0 ds_{0123}^2 + F_1 dr^2 + e^{2\phi}[d\psi - b_{\psi\mu}dx^\mu + \Delta_1\cos\theta_1(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr) \\ & + \tilde{\Delta}_2\cos\theta_2(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)]^2 + e^{(2\phi)/3}a^2(k^2G_2 + kG_3 + G_1)[d\theta_1^2 + (d\phi_1^2 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr)^2] \\ & + e^{(2\phi)/3}b^2(\mu^2G_2 + \mu G_3 + G_1)[d\theta_2^2 + (d\phi_2^2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)^2], \end{aligned} \tag{3.23}$$

and the definitions of coefficients are the same as in [9]. We define

$$\mathcal{F}_1^2 = e^{(2\phi)/3}a^2(k^2G_2 + kG_3 + G_1), \qquad \mathcal{F}_2^2 = e^{(2\phi)/3}b^2(\mu^2G_2 + \mu G_3 + G_1). \tag{3.24}$$

To determine the supersymmetry condition now, we follow the same procedure, namely, compute the torsion classes. In the following we give the general torsion classes assuming that the fields and metric can depend on angular coordinate θ_1 and θ_2 also. But first we need the vierbeins for the metric (3.23). They are

$$\begin{aligned} & e^1 = \mathcal{F}_1 d\theta_1, \qquad e^3 = \mathcal{F}_2 d\theta_2, \qquad e^6 = \sqrt{F_1}dr, \\ & e^2 = \mathcal{F}_1(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr), \qquad e^4 = \mathcal{F}_2(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr), \\ & e^5 = e^\phi[d\psi - b_{\psi r}dr + \Delta_1\cos\theta_1(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr) + \Delta_2\cos\theta_2(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)]. \end{aligned} \tag{3.25}$$

Using these vierbeins it is a straightforward (but nevertheless tedious) exercise to determine the torsion classes. This time we find the following values for the torsion classes:

$$\begin{aligned} W_1 = & \frac{1}{6\mathcal{F}_1\mathcal{F}_2\sqrt{F_1}}(\mathcal{F}_1^2 b_{\phi_1 r,\theta_2} + \mathcal{F}_2^2 b_{\phi_2 r,\theta_1} - e^{2\phi}\Delta_2\cos\theta_2\sqrt{F_1}b_{\phi_2\theta_2,\theta_1} + e^{2\phi}\Delta_1\cos\theta_1\sqrt{F_1}b_{\phi_1\theta_1,\theta_2} \\ & + ie^{2\phi}\sqrt{F_1}(\Delta_{2,\theta_1}\cos\theta_2 - \Delta_{1,\theta_2}\cos\theta_1)), \\ W_4 = & \frac{\mathcal{F}_{2,\theta_1} + \mathcal{F}_2\phi_{\theta_1}}{\mathcal{F}_2}d\theta_1 + \frac{\mathcal{F}_{1,\theta_2} + \mathcal{F}_1\phi_{\theta_2}}{\mathcal{F}_1}d\theta_2 - \frac{1}{2\mathcal{F}_1\mathcal{F}_2}(\mathcal{F}_1^2 e^{2\phi}\Delta_2\sin\theta_2\sqrt{F_1} - \mathcal{F}_2^2 e^{2\phi}\Delta_1\sin\theta_1\sqrt{F_1} \\ & - \mathcal{F}_1^2 e^{2\phi}\Delta_{2,\theta_2}\cos\theta_2\sqrt{F_1} - \mathcal{F}_2^2 e^{2\phi}\Delta_{1,\theta_1}\cos\theta_1\sqrt{F_1} - 2\mathcal{F}_1^2\mathcal{F}_2\mathcal{F}_{2,r} - 2\mathcal{F}_2^2\mathcal{F}_1\mathcal{F}_{1,r})dr, \ \mathrm{Re}W_5 \\ = & \frac{e_1}{\sqrt{F_1}\mathcal{F}_1^2\mathcal{F}_2}(e^\phi\mathcal{F}_1\mathcal{F}_2\Delta_{1,r}\cos\theta_1 - 2\mathcal{F}_1\mathcal{F}_{2,\theta_1}\sqrt{F_1} - 2\mathcal{F}_{1,\theta_1}\mathcal{F}_2\sqrt{F_1} - \mathcal{F}_1\mathcal{F}_2\phi_{\theta_1}) \\ & + \frac{-e^2}{\mathcal{F}_1^2\mathcal{F}_2\sqrt{F_1}}(\Delta_2\cos\theta_2 e^\phi\mathcal{F}_1\mathcal{F}_2 b_{\phi_2 r\theta_1} - \Delta_1\cos\theta_1 e^\phi\mathcal{F}_1 b_{\phi_1\theta_1 r}\mathcal{F}_2 + \Delta_1\cos\theta_1 e^\phi\mathcal{F}_1 b_{\phi_1 r\theta_1}\mathcal{F}_2 + e^\phi\mathcal{F}_1\mathcal{F}_2 b_{\psi r\theta_1} \\ & - \sqrt{F_1}\mathcal{F}_1\mathcal{F}_2 b_{\phi_2\theta_2\theta_1}) + \frac{e^3}{\sqrt{F_1}\mathcal{F}_2^2\mathcal{F}_1}(-2\mathcal{F}_{1,\theta_2}\mathcal{F}_2\sqrt{F_1} - 2\mathcal{F}_1\mathcal{F}_{2,\theta_2}\sqrt{F_1} - \mathcal{F}_1\mathcal{F}_2\phi_{\theta_2}\sqrt{F_1} + e^\phi\mathcal{F}_1\mathcal{F}_2\Delta_{2r}\cos\theta_2) \\ & + \frac{-e^4}{\sqrt{F_1}\mathcal{F}_2^2\mathcal{F}_1}(-\Delta_2\cos\theta_2 e^\phi\mathcal{F}_1\mathcal{F}_2 b_{\phi_2\theta_2 r} + \Delta_2\cos\theta_2 e^\phi\mathcal{F}_1\mathcal{F}_2 b_{\phi_2 r\theta_2} + \Delta_1\cos\theta_1 e^\phi\mathcal{F}_1 b_{\phi_1 r\theta_2}\mathcal{F}_2 \\ & + e^\phi\mathcal{F}_1\mathcal{F}_2 b_{\psi r\theta_2} - \sqrt{F_1}\mathcal{F}_1 b_{\phi_1\theta_1\theta_2}\mathcal{F}_2) + \frac{e^5}{\sqrt{F_1}\mathcal{F}_1\mathcal{F}_2}(-kb_{\phi_1\theta_1 r}\mathcal{F}_2 + kb_{\phi_1 r\theta_1}\mathcal{F}_2 - \mathcal{F}_1\mathcal{F}_2 b_{\phi_2\theta_2 r} \\ & + \mathcal{F}_1\mathcal{F}_2 b_{\phi_2 r,\theta_2}) + \frac{-e^6}{\sqrt{F_1}\mathcal{F}_1\mathcal{F}_2}(2\mathcal{F}_{1,r}\mathcal{F}_2 + 2\mathcal{F}_1\mathcal{F}_2\phi_r + 2\mathcal{F}_1\mathcal{F}_{2,r}), \end{aligned} \tag{3.26}$$

where W_2 and W_3 can be easily determined from the above information. Note that the type IIA manifold after geometric transition is again a noncomplex non-Kähler manifold as we would have expected. The supersymmetry condition

$$2W_4 = \mathrm{Re}W_5$$

will put further constraints on the parameters of the background. Combining the other two set of constraints (3.15) and (3.22), we can fix most of the parameters of our background. The remaining parameters, which are not fixed by our constraint equations, will give rise to a class of backgrounds corresponding to various gauge theory deformations, as shown in Fig. 3.

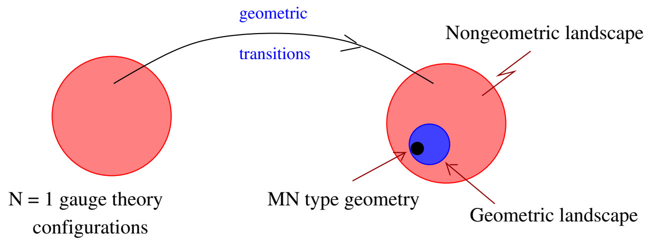

FIG. 3 (color online). The heterotic transitions corresponding to various gauge theory deformations. We see that the dual configurations are generically nongeometric. The geometric (but non-Kähler) regions are shown in blue. The black dot is the warped-deformed conifold geometry, or the Maldacena-Nunez type of geometry.

Combining all the ideas together, we see that a careful analysis of the torsion classes for various intermediate configurations allows us to present explicit supersymmetric solutions for the geometric transitions. This would then justify not only the supersymmetric cases studied in [9], but also the new heterotic configurations that we present in this paper. Therefore combining the above set of arguments, stemming from torsion classes and explicit backgrounds analysis, we believe, should strongly justify the new heterotic duality that we conjecture in this paper.

IV. VECTOR BUNDLES THROUGH CONIFOLD TRANSITION

To study the vector bundles we will start from the anomaly condition for the heterotic theory with torsion. As emphasized in [26] a more useful way to express the anomaly condition is to *complexify* the heterotic three-form H to G and write the anomaly condition as

$$G = dB + \alpha'[\Omega_3(\omega_+) - \Omega_3(A)], \tag{4.1}$$

where ω_+ is the modified spin connection, now described using the one-form $\tilde{G}$, defined in the following way:

$$\omega_+ = \omega - \tfrac{1}{2}\tilde{G}, \quad \text{with} \quad \tilde{G} \equiv Ge^{-2} \equiv G_{ijk}e^{aj}e^{bk}. \tag{4.2}$$

The complexified three-form is very useful in many analyses, for example, writing the superpotential [23] or the action, and can be expressed in terms of the real three-form $\mathcal{H}$ of the heterotic theory complexified with the geometrical data [26]. One may easily show that the Chern-Simons term related to the torsional-spin connection ω_+ is given by

$$\Omega_3(\omega_+) = \Omega_3(\omega) + \tfrac{1}{4}\Omega_3(\tilde{G}) - \tfrac{1}{2}(\omega \wedge R_{\tilde{G}} + \tilde{G} \wedge R_\omega), \tag{4.3}$$

where we define $\Omega_3(\tilde{G})$ in a somewhat similar way as $\Omega_3(A)$ or $\Omega_3(\omega)$:

$$\Omega_3(\tilde{G}) = \tilde{G} \wedge d\tilde{G} - \tfrac{1}{3}\tilde{G} \wedge \tilde{G} \wedge \tilde{G}. \tag{4.4}$$

The quantity $R_{\tilde{G}}$ is the curvature polynomial due to the torsion and is defined as

$$R_{\tilde{G}} = d\tilde{G} - \tfrac{1}{3}\tilde{G} \wedge \tilde{G}, \tag{4.5}$$

whereas R_ω differs from the usual curvature polynomial by $-\frac{1}{3}\omega \wedge \omega$. In fact, we can write in a more compact form as

$$\Omega_3(\omega_+) = (\omega - \tfrac{1}{2}\tilde{G})(R_\omega - \tfrac{1}{2}R_{\tilde{G}}) \tag{4.6}$$

with the curvature polynomials defined above.[12]

From the above analysis it is easy to infer what the background torsion is. If we concentrate only to the lowest order in α' and linear order in G, the three-form background is given by

$$G = dB\left(1 - \frac{\alpha'}{2}R_\omega e^{-2}\right) + \alpha'\Omega_3(\omega) + \mathcal{O}(\alpha'^2), \tag{4.7}$$

where we have imposed $\Omega_3(A) = 0$ because the gauge group is completely broken. To all orders in G and α' the equation that we need to solve is

$$G + \frac{\alpha'}{2}\left[\omega \wedge R_{\tilde{G}} + \tilde{G} \wedge \mathcal{R}_\omega - \frac{1}{2}\tilde{G} \wedge R_{\tilde{G}}\right] = dB + \alpha'\Omega_3(\omega) \equiv f. \tag{4.8}$$

Thus f will have a term linear in α'. Using this, the solution for G is written in terms of powers of α' in the following way:

$$G = \sum \alpha'^n H_n + \frac{i}{\sqrt{\alpha'}}\sum \alpha'^n h_n, \tag{4.9}$$

where n goes from zero onwards. As discussed in [26], the various terms in G can be presented as

[12] In this form it is instructive to compare with the other choice of torsional-spin connection ω_-:

$$\Omega_3(\omega_-) = (\omega + \tfrac{1}{2}\tilde{\mathcal{H}})(\mathcal{R}_\omega + \tfrac{1}{2}\mathcal{R}_{\tilde{\mathcal{H}}} + \tfrac{1}{3}\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}}),$$

which differs in relative signs and an additional term.

$$
\begin{aligned}
&h_0 - \tfrac{1}{12}\tilde{h}_0 \wedge \tilde{h}_0 \wedge \tilde{h}_0 = 0, \qquad H_0 = -\frac{f}{2} + \frac{1}{4}\,\mathrm{Tr}(\tilde{h}_0 \wedge d\tilde{h}_0) + \frac{1}{6}\,\mathrm{Tr}(\omega_0 \wedge \tilde{h}_0 \wedge \tilde{h}_0),\\
&h_1 - \tfrac{1}{4}\mathrm{Tr}(\tilde{h}_0 \wedge \tilde{h}_0 \wedge \tilde{h}_1) = -\tfrac{1}{2}\mathrm{Tr}(\omega_0 \wedge d\tilde{h}_0) + \tfrac{1}{3}\mathrm{Tr}(\omega_0 \wedge \tilde{H}_0 \wedge \tilde{h}_0) - \tfrac{1}{2}\mathrm{Tr}(\tilde{h}_0 \wedge R_{\omega_0})\\
&\quad + \tfrac{1}{4}\mathrm{Tr}(\tilde{H}_0 \wedge d\tilde{h}_0 + \tilde{h}_0 \wedge d\tilde{H}_0) - \tfrac{1}{4}\mathrm{Tr}(\tilde{H}_0 \wedge \tilde{H}_0 \wedge \tilde{h}_0),
\end{aligned} \tag{4.10}
$$

where the tilde terms are one-forms constructed out of three-forms using vierbeins as in (4.2) and the subscript 0 denotes zeroth order[13] in α'. Solving the above set of equations gives us the following:

$$G = -\tfrac{1}{2}d(B + iJ_0) + \alpha'\Omega_3(\omega_0) + \text{corrections}, \tag{4.11}$$

where, as one would have expected, the complexified Kähler form appears naturally from our analysis. The corrections are both to J_0 as well as well as to higher orders in α'. The $-\frac{1}{2}$ coefficient can be absorbed by redefining G. Once we do that, we could rewrite the real part of G, i.e. $\mathcal{H}$, in the following way:

$$
\begin{aligned}
\mathcal{H} = f + \frac{\alpha'}{2}\bigg(&\omega_0 \wedge \tilde{f} \wedge \tilde{f} + \tilde{f} \wedge R_{\omega_0} + \frac{1}{2}\tilde{f} \wedge d\tilde{f}\\
&- \frac{1}{6}\tilde{f} \wedge \tilde{f} \wedge \tilde{f}\bigg),
\end{aligned} \tag{4.12}
$$

where $f = dB + \alpha'\Omega_3(\omega)$ as defined in (4.8) above. Since we know $\mathcal{H}$ from Appendix A, we can determine f or $\tilde{f}$ to lowest order in α' by solving the cubic equation (4.12).

Therefore the story after the transition is simple: The torsion and the metric are the only information needed to specify the dual geometry. On the other hand, before the transition the situation is more involved. There is a nontrivial vector bundle:

$$Sp(2N) \times \mathcal{G}, \tag{4.13}$$

where, as mentioned earlier, the gauge group $\mathcal{G}$ comes from the type IIB seven-branes. Various distributions of the F-theory seven-branes *á la* [34] will give various $\mathcal{G}$. If $\tilde{\mathcal{H}}$ denotes the torsion before the transition, we expect $d\tilde{\mathcal{H}}$ to have contributions from $\mathrm{Tr}F_{\mathcal{G}} \times F_{\mathcal{G}}$ as well as from $\mathrm{Tr}F_{Sp(2N)} \times F_{Sp(2N)}$. As before, the torsion $\tilde{\mathcal{H}}$ will have two parts: one proportional to N and the other independent of N. The part independent of N could be balanced by the torsional curvature and the $\mathcal{G}$ bundle. It will be interesting to work out the full picture as both sides, before and after the transition, require careful analysis of the Bianchi identity. A more detailed analysis of how to pull the bundle (4.13) through the conifold point will be discussed elsewhere.

[13]We write ω and J as $\omega = \sum \alpha'^n \omega_n$ and $J = \sum \alpha'^n J_n$, respectively, to compare terms order by order in α'. This is discussed in more detail in [26].

V. CONCLUSION AND DISCUSSIONS

In this paper we gave some evidence for the gravity dual of large N heterotic small instantons. We pointed out that geometric transition in the heterotic setup is related to a small instanton transition under which the small instantons smoothen out. This way the $Sp(2N)$ gauge symmetry before the transition is completely broken, and therefore in the dual side we no longer have branes or vector bundles but only torsion. For certain cases the gravity duals are *deformations* of the corresponding type II cases because of the underlying Bianchi identity.

We left many questions unanswered. For example,

(i) Can this way of thinking be extended to the type IIB case also? Recall that the IIB D3-branes are small instantons on the seven-branes in the full F-theory setup. Therefore before the transition we can move the seven-branes along the Coulomb branch so that the SUSY remains unbroken at low scales. Then presumably the large N limit of D3-branes could be studied via this mechanism. It would be miraculous to recover ADS target space from the ADHM sigma model.

(ii) In the heterotic side the vector bundle is completely broken. So to satisfy the Bianchi identity we cannot allow a *closed* three-form. However, in IIB there might be a situation where all the D3-brane small instantons smoothen out on a *subset* of the allowed seven-branes. The gauge fields on these seven-branes become all massive, but we can still have nonzero $\mathrm{Tr}F \wedge F$ from the other seven-branes. Therefore we might be able to allow for a closed three-form and still satisfy the Bianchi identity.

(iii) One of the issues that we skimmed over is the ADHM sigma model and possible contributions to the world-sheet superpotential. The precise question is as follows. Could there be an ADS-like contribution to the ADHM sigma model that can tell us how the target space changes from a non-Kähler resolved conifold to a non-Kähler deformed conifold (or even to a nongeometric one)?

(iv) In the type IIB case the effect of a world-volume quartic potential plus the ADS contribution can also be seen from the Gukov-Vafa-Witten type of *bulk* superpotential [35]. Now that we know the heterotic superpotential [23,24], we should be able to see the connection between this superpotential and the total ADHM superpotential.

(v) Is it possible to understand the full cascading dualities from this perspective? This may be more tricky because in type I we do not have D3-brane degrees of freedom. But maybe they all can be understood directly from the F-theory viewpoint where the D3-branes are small instantons on the seven-branes and the D5-branes (that are not parallel to the seven-branes) are in fact T-dual to type I small instantons.[14]

(vi) We discussed how the Maldacena-Nunez solution [19] should be deformed slightly to satisfy the Bianchi identity. However, we did not compute the actual deviations of the components of the metric or the three-form in this paper. Although this is technically challenging as the components of $\mathrm{tr} R_+ \wedge R_+$ from the metric (2.18) are rather unwieldy (see also Appendix A for the form for $\mathcal{H}$), it will nevertheless be an interesting exercise to get the background precisely. This will then provide another confirmation of the heterotic duality.

(vii) Last but not least, we have not said anything about the $E_8 \times E_8$ case. As discussed in the introduction, here the story may follow a similar line of thought as in [14]. We will discuss about this case and hopefully some of the above mentioned points in what follows.

Note: As we were writing this draft two interesting papers appeared in the archive that studied some aspect of the story in a slightly different guise [37]. The second paper in [37] studied some aspect of heterotic-conformal field theory $(0, 2)$ sigma model. It would be interesting to relate them to our results. Some other papers with some indirect relations to our work can be found in [38].

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APPENDIX A: THE TORSION FOR THE DUAL GRAVITATIONAL BACKGROUND

The modified $\mathcal{H}$ for the heterotic background (2.18) is rather involved because of the nontrivial fibration structure in the definition of $\mathcal{D}\phi_i$. However, if we consider the simpler case where $\mathcal{D}\phi_i \approx d\phi_i$, the three-form, or the torsion, can be easily found. Under this simplification $\mathcal{H}$ is given by

$$\begin{aligned}\mathcal{H} = \frac{1}{\sqrt{A_5}K^2L\sqrt{1-B^2}}\big(&-BKL^2a_r + B^3KL^2a_r - L^2a(1-B^2)BK_r - 2L^2a^3\sqrt{1-B^2}\sin(\psi)L_rB^2\\
&- L^2a^2(1-B^2)\sin(\psi)B_rK - (\cos(\psi))^2L^3a\sqrt{1-B^2}a_r + L^3a^2\sqrt{1-B^2}\sin(\psi)a_r - aBK^2\sin(\psi)LB_r\sqrt{1-B^2}\\
&- L^2a^2(1-B^2)\sin(\psi)BK_r + (\cos(\psi))^2La(1-B^2)L_rBK + (\cos(\psi))^2L^3a\sqrt{1-B^2}a_rB^2\\
&- 2(\cos(\psi))^2L^2a^2\sqrt{1-B^2}L_r + 2aB^2K^2\sin(\psi)\phi_rL\sqrt{1-B^2} + 2L^3a^2\sqrt{1-B^2}\phi_rB^2 - 2L^3a^3\sqrt{1-B^2}\sin(\psi)\phi_r\\
&+ 2(\cos(\psi))^2L^3a^2\sqrt{1-B^2}\phi_r - La^2(1-B^2)\sin(\psi)L_rBK + (\cos(\psi))^2L^3a^2\sqrt{1-B^2}BB_r\\
&- La^2(1-B^2)\sqrt{A_5}G\Delta_2\sin(\theta_2) - 2(\cos(\psi))^2L^2a(1-B^2)\phi_rBK - aB^2K^2\sin(\psi)L_r\sqrt{1-B^2}\\
&- (1-B^2)L\sqrt{A_5}G\Delta_1\sin(\theta_1) + (\cos(\psi))^2L^2a(1-B^2)B_rK - La(1-B^2)L_rBK + (\cos(\psi))^2L^2a(1-B^2)BK_r\\
&- aB^3K\sin(\psi)L^2a_r - a^2B^2K\sin(\psi)L^2B_r + 2L^2a^2(1-B^2)\sin(\psi)\phi_rBK + aBK\sin(\psi)L^2a_r - \sqrt{1-B^2}L^3aa_rB^2\end{aligned}$$

[14] In fact there is already a hint that such deformation of the instantons that we see in the heterotic side should have an equivalent story in the T-dual of the IIB geometric transition. This construction has appeared in the second reference of [4] some time back, and here we will elaborate the story very briefly. The IIA brane construction after the last cascade will be M D4-branes in the interval *between* the two orthogonal NS5-branes, and no D4-branes on the other side. Once we shrink the $\mathbf{P}^1$ to zero size, the two NS5-branes in the T-dual picture come together. To see the subsequent behavior, we lift this to M theory. There the SUSY condition is preserved only when the shrunk D4-branes (or M5-branes now) deform to form a *diamond* structure between the two M5-branes (see [36] for more details about this construction). Therefore the final configuration is like two intersecting M5-branes with the intersection "point" blown up and the M5-branes between the two orthogonal M5-branes virtually dissolved. The T-dual type IIB configuration will give us a deformed conifold and no D3-branes. Note that the deformation of the shrunken M5-branes exactly creates the extra metric components required to convert the resolved conifold geometry to a deformed one in the T-dual IIB side. This is almost like the small instanton story: The small instantons deform and become geometry. The only difference is that in the IIA and M theories the curved M5-branes become M planar M5-branes and consequently lose the $U(M)$ gauge symmetry to end up with M $U(1)$'s. In the heterotic theory the k instantons blow up to lose the $Sp(2k)$ gauge symmetry, but now due to the background $\mathcal{G}$ subgroup of $SO(32)$ all the gauge symmetries are completely broken.

$$
\begin{aligned}
&- L^3a^3\sqrt{1-B^2}\sin(\psi)BB_r - L^2a(1-B^2)B_rK - 2L^3a^2\sqrt{1-B^2}\phi_r + 2\sqrt{1-B^2}LKK_r + 2L^2a^2\sqrt{1-B^2}L_r\\
&- 2\sqrt{1-B^2}L\phi_rK^2 - B^2K^2L_r\sqrt{1-B^2} + \sqrt{1-B^2}L^3aa_r - 2\sqrt{1-B^2}LBB_rK^2 - 2(\cos(\psi))^2L^3a^2\sqrt{1-B^2}\phi_rB^2\\
&- aBK\sqrt{A_5}G\Delta_2\sin(\theta_2)\sqrt{1-B^2} - L^3a^2\sqrt{1-B^2}\sin(\psi)a_rB^2 + 2L^2a^3\sqrt{1-B^2}\sin(\psi)L_r + 2L^2a(1-B^2)\phi_rBK\\
&+ 2(\cos(\psi))^2L^2a^2\sqrt{1-B^2}L_rB^2 + 2a^2B^3K\sin(\psi)\phi_rL^2 - 2a^2BK\sin(\psi)\phi_rL^2 - 2a^2B^3K\sin(\psi)L_rL\\
&+ 2a^2BK\sin(\psi)L_rL - L^3a^2\sqrt{1-B^2}BB_r + 4\sqrt{1-B^2}L\phi_rK^2B^2 + 2L^3a^3\sqrt{1-B^2}\sin(\psi)\phi_rB^2\\
&- aB^2K\sin(\psi)LK_r\sqrt{1-B^2} - 3\sqrt{1-B^2}LKK_rB^2 - 2L^2a^2\sqrt{1-B^2}L_rB^2)E_1\wedge E_3\wedge E_4\\
&+ \frac{1}{\sqrt{A_5}K^2L\sqrt{1-B^2}}(-2(\cos(\psi))^2B^3L^3a^2\phi_r + 2(\cos(\psi))^2BL^3a^2\phi_r + (\cos(\psi))^2B^2L^3a^2B_r\\
&+ 2(\cos(\psi))^2B^3L^2a^2L_r - 2(\cos(\psi))^2BL^2a^2L_r + a(1-B^2)K\sin(\psi)LBK_r + a(1-B^2)K^2\sin(\psi)LB_r\\
&- B^2LB_rK^2 - a\sqrt{1-B^2}KL^2BB_r - 2BL\phi_rK^2 + 2BLKK_r + 2B^3L\phi_rK^2 - 2B^3LKK_r\\
&+ a(1-B^2)K\sqrt{A_5}G\Delta_2\sin(\theta_2) - BL\sqrt{A_5}G\Delta_1\sin(\theta_1)\sqrt{1-B^2} + 2B^3L^3a^2\phi_r\\
&- 2(1-B^2)K^2\phi_rLB - 2BL^3a^2\phi_r + (1-B^2)K^2LB_r + 2BL^2a^2L_r + 2a\sqrt{1-B^2}K\phi_rL^2B^2\\
&+ (\cos(\psi))^2B^3L^3aa_r + \sqrt{1-B^2}KL^2a_r + (\cos(\psi))^2B^2LaL_r\sqrt{1-B^2}K - B^2L^2aK_r\sqrt{1-B^2} - 2B^3L^2a^2L_r\\
&- a\sqrt{1-B^2}K\sin(\psi)L^2a_r - 2a^2\sqrt{1-B^2}K\sin(\psi)L_rL + (1-B^2)K^2L_rB - B^3L^3a^2\sin(\psi)a_r\\
&- BLa^2\sqrt{A_5}G\Delta_2\sin(\theta_2)\sqrt{1-B^2} + a\sqrt{1-B^2}K\sin(\psi)L^2a_rB^2 - 2a(1-B^2)K^2\sin(\psi)\phi_rLB\\
&+ (\cos(\psi))^2BL^2aB_rK\sqrt{1-B^2} + a^2\sqrt{1-B^2}K\sin(\psi)L_rLB^2 - \sqrt{1-B^2}KL^2a_rB^2 + 2B^3L^3a^3\sin(\psi)\phi_r\\
&+ a(1-B^2)K^2\sin(\psi)L_rB - B^2L^3a^3\sin(\psi)B_r + (1-B^2)KLBK_r + 2a^2\sqrt{1-B^2}K\sin(\psi)\phi_rL^2 + BL^3aa_r\\
&- B^3L^3aa_r - a\sqrt{1-B^2}KLL_rB^2 - 2B^3L^2a^3\sin(\psi)L_r + 2BL^2a^3\sin(\psi)L_r + BL^3a^2\sin(\psi)a_r - B^2L^3a^2B_r\\
&- 2(\cos(\psi))^2B^2L^2a\phi_r\sqrt{1-B^2}K - B^2L^2a^2\sin(\psi)K_r\sqrt{1-B^2} + (\cos(\psi))^2B^2L^2aK_r\sqrt{1-B^2}\\
&- (\cos(\psi))^2BL^3aa_r - 2BL^3a^3\sin(\psi)\phi_r)E_1\wedge E_3\wedge E_6\\
&+ \frac{1}{\sqrt{A_5}L^2K\sqrt{1-B^2}}(-2B^3K\phi_rL^2 + B^2KL^2B_r + 2BK\sin(\psi)\phi_rL^2a - 2B^3K\sin(\psi)\phi_rL^2a\\
&- 2L^2a(1-B^2)\sin(\psi)\phi_rBK + 2B^3KLL_r - 2BK\sin(\psi)L_rLa + La(1-B^2)\sin(\psi)L_rBK + 2B^3K\sin(\psi)L_rLa\\
&- BK\sin(\psi)L^2a_r + B^2K^2\sin(\psi)L_r\sqrt{1-B^2} - 2L^3a^2\sqrt{1-B^2}\sin(\psi)\phi_rB^2 - (\cos(\psi))^2\sqrt{1-B^2}L^3a_rB^2\\
&+ 2L^2a^2\sqrt{1-B^2}\sin(\psi)L_rB^2 + 2(\cos(\psi))^2\sqrt{1-B^2}L^2L_ra - (\cos(\psi))^2(1-B^2)L^2B_rK + B^2K\sin(\psi)L^2aB_r\\
&+ B^2K\sin(\psi)LK_r\sqrt{1-B^2} + 2(\cos(\psi))^2(1-B^2)L^2\phi_rBK - (\cos(\psi))^2(1-B^2)LL_rBK\\
&+ L^3a^2\sqrt{1-B^2}\sin(\psi)BB_r + L^2a(1-B^2)\sin(\psi)B_rK - (\cos(\psi))^2\sqrt{1-B^2}L^3aBB_r + 2L^2a\sqrt{1-B^2}L_rB^2\\
&+ 2(\cos(\psi))^2\sqrt{1-B^2}L^3\phi_raB^2 + L^3a\sqrt{1-B^2}\sin(\psi)a_rB^2 - L^3a\sqrt{1-B^2}\sin(\psi)a_r\\
&+ La(1-B^2)\sqrt{A_5}G\Delta_2\sin(\theta_2) - 2(\cos(\psi))^2\sqrt{1-B^2}L^2L_raB^2 + (\cos(\psi))^2\sqrt{1-B^2}L^3a_r\\
&+ 2L^3a\sqrt{1-B^2}\phi_r - 2L^2a\sqrt{1-B^2}L_r + L^3a\sqrt{1-B^2}BB_r + B^3K\sin(\psi)L^2a_r
\end{aligned}
$$

$$
\begin{aligned}
&+ 2BK\phi_r L^2 - 2BKLL_r + BK^2\sin(\psi)LB_r\sqrt{1-B^2} + L^2a(1-B^2)\sin(\psi)BK_r - 2L^3a\sqrt{1-B^2}\phi_r B^2\\
&- (\cos(\psi))^2(1-B^2)L^2BK_r - 2L^2a^2\sqrt{1-B^2}\sin(\psi)L_r + 2L^3a^2\sqrt{1-B^2}\sin(\psi)\phi_r\\
&- 2B^2K^2\sin(\psi)\phi_r L\sqrt{1-B^2} + BK\sqrt{A_5}G\Delta_2\sin(\theta_2)\sqrt{1-B^2} - 2(\cos(\psi))^2\sqrt{1-B^2}L^3\phi_r a)E_1\wedge E_4\wedge E_5\\
&- \frac{\sin(\psi)\cos(\psi)}{\sqrt{A_5}KL\sqrt{1-B^2}}(-2\phi_r L^2a + 2\phi_r L^2aB^2 + 2\phi_r L\sqrt{1-B^2}BK + 2L_rLa - 2L_rLaB^2 - L_r\sqrt{1-B^2}BK\\
&- L^2aBB_r + L^2a_r - L^2a_rB^2 - LB_rK\sqrt{1-B^2} - LBK_r\sqrt{1-B^2})E_1\wedge E_4\wedge E_6\\
&- \frac{1}{\sqrt{A_5}L^2K\sqrt{1-B^2}}(-(1-B^2)K^2\sin(\psi)L_rB - 2BL^2aL_r - 2\sqrt{1-B^2}KLL_rB^2\\
&- B^2La\sin(\psi)L_r\sqrt{1-B^2}K - (\cos(\psi))^2B^3L^3a_r - \sqrt{1-B^2}K\sin(\psi)L^2a_rB^2 + BLa\sqrt{A_5}G\Delta_2\sin(\theta_2)\sqrt{1-B^2}\\
&+ 2\sqrt{1-B^2}KLL_r - 2\sqrt{1-B^2}K\phi_r L^2 + B^2L^3aB_r + 2B^3L^2aL_r + B^2L^2a\sin(\psi)K_r\sqrt{1-B^2} - 2B^3L^3a\phi_r\\
&- (1-B^2)K^2\sin(\psi)LB_r + (\cos(\psi))^2BL^3a_r - (\cos(\psi))^2B^2L^2K_r\sqrt{1-B^2} + \sqrt{1-B^2}K\sin(\psi)L^2a_r\\
&+ 2\sqrt{1-B^2}K\sin(\psi)L_rLa + 2(\cos(\psi))^2B^3L^3\phi_r a + B^2L^3a^2\sin(\psi)B_r + 2(\cos(\psi))^2BL^2L_ra\\
&+ 2B^3L^2a^2\sin(\psi)L_r - (\cos(\psi))^2BL^2B_rK\sqrt{1-B^2} + 2(\cos(\psi))^2B^2L^2\phi_r\sqrt{1-B^2}K\\
&+ 2BL^3a^2\sin(\psi)\phi_r - 2(\cos(\psi))^2BL^3\phi_r a - (\cos(\psi))^2B^2LL_r\sqrt{1-B^2}K - 2\sqrt{1-B^2}K\sin(\psi)\phi_r L^2a\\
&+ B^3L^3a\sin(\psi)a_r - (1-B^2)K\sin(\psi)LBK_r - 2(\cos(\psi))^2B^3L^2L_ra - 2B^3L^3a^2\sin(\psi)\phi_r\\
&- \sqrt{1-B^2}KL^2BB_r + 2(1-B^2)K^2\sin(\psi)\phi_r LB - (1-B^2)K\sqrt{A_5}G\Delta_2\sin(\theta_2) + 2BL^3a\phi_r\\
&- 2BL^2a^2\sin(\psi)L_r + 2\sqrt{1-B^2}K\phi_r L^2B^2 - (\cos(\psi))^2B^2L^3aB_r - BL^3a\sin(\psi)a_r)E_1\wedge E_5\wedge E_6,\\
&- \frac{\sqrt{1-B^2}La + BK}{GK}E_2\wedge E_3\wedge E_5,\\
&- \frac{-\sqrt{1-B^2}La + BK}{GK}E_2\wedge E_4\wedge E_6,\\
&+ \frac{1}{LK^2}(\sqrt{1-B^2}La + BK)(\cos(\psi)\Delta_2\cos(\theta_2)La\sqrt{1-B^2} + \cos(\psi)\Delta_2\cos(\theta_2)BK\\
&+ \Delta_1\cos(\theta_1)\sqrt{1-B^2}L)E_3\wedge E_4\wedge E_5 + \frac{a\Delta_2\cos(\theta_2)(-\sqrt{1-B^2}La + BK)\sin(\psi)}{K^2}E_3\wedge E_4\wedge E_6\\
&+ \frac{1}{LK^2}(\sqrt{1-B^2}La + BK)(-\cos(\psi)\Delta_2\cos(\theta_2)BLa + \cos(\psi)\Delta_2\cos(\theta_2)\sqrt{1-B^2}K\\
&- B\Delta_1\cos(\theta_1)L)E_3\wedge E_5\wedge E_6 - \frac{(-\sqrt{1-B^2}La + BK)\Delta_2\cos(\theta_2)\sin(\psi)}{LK}E_4\wedge E_5\wedge E_6,
\end{aligned}
\tag{A1}
$$

where the function B is a function of radial direction r and is determined by the $SU(3)$ structure of the manifold.

$$
G = \sqrt{A_1},\qquad L = \sqrt{A_2},\qquad K = \sqrt{A_3 - A_2 b_{y\theta_1}^2},\qquad a = b_{y\theta_1},\qquad \Delta_1 = a_1,\qquad \Delta_2 = b_1,
$$

and E_i are defined in the following way:

$$
\begin{aligned}
&E_1 = \sqrt{A_5}dr,\qquad E_3 = Kd\theta_1,\qquad E_4 = -Kd\phi_1,\qquad E_2 = G(d\psi + \Delta_1\cos\theta_1 d\phi_1 + \Delta_2\cos\theta_2 d\phi_2),\\
&E_5 = L(\sin\psi d\phi_2 + \cos\psi d\theta_2 - ad\theta_1),\qquad E_6 = L(\cos\psi d\phi_2 - \sin\psi d\theta_2 - ad\phi_1).
\end{aligned}
\tag{A2}
$$

APPENDIX B: GENERAL TORSION CLASSES FOR TYPE IIA

The type IIA torsion classes before geometric transition are given by the following expressions:

$$\begin{aligned} W_1 &= \frac{-1}{6L^2K^2\sqrt{F_1}GA}(-iL^2GK^2b_{\phi_2\theta_2,r}AB - 2iL^2AGb_{\psi r}K^2B + 2L^2A\sqrt{F1}K^2B + 2iGL^3K\cos\psi^2 b_{\phi_2\theta_2,r}aB^2 \\ &\quad - iL^3GKb_{\phi_2\theta_2,r}aB^2 + iL^3Gb_{\phi_1\theta_1,r}Ka - iL^3Gb_{\phi_1\theta_1,r}KaB^2 - iL^2Gb_{\phi_1\theta_1,r}K^2AB + G^2B\sqrt{F1}AL^2\Delta_1\sin\theta_1 a^2 \\ &\quad + G^2B\sqrt{F_1}B\Delta_2\sin\theta_2 L^2 + G^2B\sqrt{F_1}A\Delta_2\sin\theta_2 K^2 + 2GL^3K\sin\psi b_{\phi_2\theta_2,r}\cos\psi Aa + 2L^2GBK^2 \\ &\quad - 2iGL^3K\cos\psi^2 b_{\phi_2\theta_2,r}a + iL^3GKb_{\phi_2\theta_2,r}aL^2K^2\sqrt{F_1}GB), \\ W_4 &= w_{4r}dr + w_{4\theta_1}d\theta_1 + w_{4\theta_2}d\theta_2 + w_{4\phi_1}d\phi_1 + w_{4\phi_2}d\phi_2, \\ \mathrm{Re}W_5 &= w_{5e^1}e^1 + w_{5e^2}e^2 + w_{5e^3}e^3 + w_{5e^4}e^4 + w_{5e^5}e^5 + w_{5e^6}e^6, \end{aligned} \tag{B1}$$

where w_i's appearing above are defined in the following way:

$$\begin{aligned} w_{4r} &= -\frac{1}{2L^2K^2}[-\sin\psi\Delta_2\cos\theta_2 L^4 b_{\phi_1 r}a^3B^2 - \cos\theta_1\Delta_1 L^4\sin\psi b_{\phi_2 r}aB^2 - \sin\psi\Delta_2\cos\theta_2 L^2K^2b_{\phi_1 r}aB^2 \\ &\quad - \sin\psi\Delta_2\cos\theta_2 L^3Kb_{\phi_1 r}\sqrt{1-B^2}Ba^2 + \cos\theta_1\Delta_1 L^3\sin\psi b_{\phi_2 r}K\sqrt{1-B^2}B - \sin\psi\Delta_2\cos\theta_2 LK^3b_{\phi_1 r}\sqrt{1-B^2}B \\ &\quad + \sin\psi\Delta_2\cos\theta_2 L^2K^2b_{\phi_1 r}a - 2LL_rK^2 + \cos\theta_1\Delta_1 L^4\sin\psi b_{\phi_2 r}a + \sin\psi\Delta_2\cos\theta_2 L^4b_{\phi_1 r}a^3 \\ &\quad + \sqrt{F_1}G\Delta_2\sin\theta_2\sqrt{1-B^2}L^2a^2 + L^2\sqrt{F_1}G\Delta_1\sin\theta_1\sqrt{1-B^2} + \sqrt{F_1}G\Delta_2\sin\theta_2 K^2\sqrt{1-B^2} - 2L^2KK_r], \end{aligned}$$

$$\begin{aligned} w_{4\theta_1} &= \frac{1}{2K^2\sqrt{1-B^2}L}[\Delta_2\cos\theta_2\cos\psi B^3KL^2a^2 - \Delta_2\cos\theta_2\sin\psi B^3Kb_{\phi_1\theta_1}L^2a^2 - \Delta_2\cos\theta_2\sin\psi B^3K^3b_{\phi_1\theta_1} \\ &\quad + \Delta_2\cos\theta_2\cos\psi B^3K^3 + 2L^2aB^3K\Delta_1\cos\theta_1 + \Delta_2\cos\theta_2 L\sin\psi\sqrt{1-B^2}ab_{\phi_1\theta_1}K^2B^2x \\ &\quad + \Delta_2\cos\theta_2 L\cos\psi\sqrt{1-B^2}aK^2B^2 - \Delta_2\cos\theta_2\cos\psi BKL^2a^2 + \Delta_2\cos\theta_2 L^3\sin\psi\sqrt{1-B^2}a^3b_{\phi_1\theta_1}B^2 \\ &\quad + L^3\sqrt{1-B^2}a^2\Delta_1\cos\theta_1 B^2 + \Delta_2\cos\theta_2 L\cos\psi\sqrt{1-B^2}aK^2B^2 - \Delta_2\cos\theta_2\cos\psi BKL^2a^2 \\ &\quad + \Delta_2\cos\theta_2\sin\psi BKb_{\phi_1\theta_1}rL^2a^2 + \Delta_2\cos\theta_2\sin\psi BK^3b_{\phi_1\theta_1} - \Delta_2\cos\theta_2\cos\psi BK^3 - 2L^2aBK\Delta_1\cos\theta_1 \\ &\quad - L^3\sqrt{1-B^2}a^2\Delta_1\cos\theta_1 - \Delta_2\cos\theta_2 L^3\cos\psi\sqrt{1-B^2}a^3 - \Delta_2\cos\theta_2 L^3\sin\psi\sqrt{1-B^2}a^3b_{\phi_1\theta_1} \\ &\quad - \Delta_2\cos\theta_2 L\cos\psi\sqrt{1-B^2}aK^2 - \Delta_2\cos\theta_2 L\sin\psi\sqrt{1-B^2}ab_{\phi_1\theta_1}K^2], \end{aligned}$$

$$\begin{aligned} w_{4\theta_2} &= \frac{1}{2\sqrt{1-B^2}K^2}[-L^2ar^2\sqrt{1-B^2}\Delta_2\cos\theta_2 + \Delta_2\cos\theta_2 B^2K^2\sqrt{1-B^2} + L\cos\psi B^3K\Delta_1\cos\theta_1 \\ &\quad - L\Delta_1\cos\theta_1\sin\psi b_{\phi_2\theta_2}B^3K - L\cos\psi BK\Delta_1\cos\theta_1 - \cos\psi\sqrt{1-B^2}L^2a\Delta_1\cos\theta_1 \\ &\quad + L^2\Delta_1\cos\theta_1\sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2}a + L\Delta_1\cos\theta_1\sin\psi b_{\phi_2\theta_2}BK - L^2\Delta_1\cos\theta_1\sin\psi b_{\phi_2\theta_2}aB^2\sqrt{1-B^2} \\ &\quad + \cos\psi L^2a\Delta_1\cos\theta_1 B^2\sqrt{1-B^2} + \Delta_2\cos\theta_2 L^2a^2B^2\sqrt{1-B^2}], \\ w_{4\phi_1} &= \frac{1}{2K^2\sqrt{1-B^2}L}[(B^3K^3 + B^3KL^2a^2 - L\sqrt{1-B^2}aK^2B^2 - L^3\sqrt{1-B^2}a^3B^2 - BK^3 - BKL^2a^2 \\ &\quad + L\sqrt{1-B^2}aK^2 + L^3\sqrt{1-B^2}a^3)\cos\theta_2\sin\psi\Delta_2], \\ w_{r\phi_2} &= \frac{1}{2}[\sqrt{1-B^2}(\sqrt{1-B^2}La + BK)\Delta_1 L\cos\theta_1\sin\psi K^2], \end{aligned} \tag{B2}$$

$$
\begin{aligned}
w_{5e^1} = \frac{1}{\sqrt{F_1}GK^2L^2}[&\Omega_{r\psi\theta_2\phi_2}BK^2 - \Omega_{r\psi\theta_1\phi_2}L\sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2}K - \Omega_{r\psi\theta_1\theta_2}\sin\psi L\sqrt{1-B^2}K \\
&- \Omega_{\psi\theta_2\phi_1\phi_2}K\cos\psi\sqrt{1-B^2}b_{\phi_2 r}L - \Omega_{r\psi\theta_1\phi_1}BL^2 + \Omega_{r\psi\phi_1\phi_2}L^2 b_{\phi_1\theta_1}\sin\psi b_{\phi_2\theta_2}Ba \\
&+ \Omega_{r\psi\theta_1\phi_2}L\cos\psi\sqrt{1-B^2}K + \Omega_{r\psi\phi_1\phi_2}Lb_{\phi_1\theta_1}\cos\psi\sqrt{1-B^2}K - \Omega_{r\psi\theta_2\phi_1}L^2\cos\psi Ba \\
&- \Omega_{r\psi\phi_1\phi_2}L\sqrt{1-B^2}Kb_{\phi_2\theta_2}\cos\psi - \Omega_{r\psi\phi_1\phi_2}L\sqrt{1-B^2}K\sin\psi - \Omega_{\psi\theta_1\theta_2\phi_2}\sin\psi b_{\phi_2 r}L^2Ba \\
&+ \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_2 r}\sin\psi b_{\phi_1\theta_1}BL^2a - \Omega_{\psi\theta_2\phi_1\phi_2}Lb_{\phi_2 r}\sin\psi b_{\phi_1\theta_1}\sqrt{1-B^2}K \\
&+ \Omega_{r\psi\theta_2\phi_1}L\sin\psi b_{\phi_1\theta_1}\sqrt{1-B^2}K + 2\Omega_{r\psi\theta_2\phi_2}aL\sqrt{1-B^2}K + \Omega_{r\psi\phi_1\phi_2}L^2Bab_{\phi_2\theta_2}\cos\psi \\
&- \Omega_{\psi\theta_1\phi_1\phi_2}Lb_{\phi_1 r}\sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2}K - \Omega_{r\psi\phi_1\phi_2}Lb_{\phi_1\theta_1}\sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2}K \\
&- \Omega_{r\psi\phi_1\phi_2}L^2b_{\phi_1\theta_1}\cos\psi Ba + \Omega_{\psi\theta_1\phi_1\phi_2}L^2b_{\phi_1 r}\sin\psi b_{\phi_2\theta_2}Ba + \Omega_{\psi\theta_1\theta_2\phi_2}\sin\psi b_{\phi_2 r}L\sqrt{1-B^2}K \\
&+ \Omega_{\psi\theta_1\phi_1\phi_2}Lb_{\phi_1 r}\cos\psi\sqrt{1-B^2}K - \Omega_{r\psi\theta_2\phi_1}L^2\sin\psi b_{\phi_1\theta_1}Ba + \Omega_{r\psi\theta_2\phi_1}L\cos\psi\sqrt{1-B^2}K \\
&+ 2\Omega_{\psi\theta_2\phi_1\phi_2}Lb_{\phi_1 r}a\sqrt{1-B^2}K + \Omega_{\psi\theta_1\phi_1\phi_2}L^2Bb_{\phi_2 r} - \Omega_{r\psi\theta_2\phi_2}a^2L^2B + \Omega_{r\psi\theta_1\phi_2}L^2\sin\psi b_{\phi_2\theta_2}Ba \\
&- \Omega_{\psi\theta_1\theta_2\phi_1}\sin\psi b_{\phi_1 r}L^2Ba + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1 r}BK^2 + \Omega_{r\psi\theta_1\theta_2}\sin\psi L^2Ba - \Omega_{\psi\theta_1\phi_1\phi_2}L^2b_{\phi_1 r}\cos\psi Ba \\
&+ \Omega_{r\psi\phi_1\phi_2}L^2Ba\sin\psi - \Omega_{r\psi\theta_1\phi_2}L^2\cos\psi Ba + \Omega_{\psi\theta_1\theta_2\phi_1}\sin\psi b_{\phi_1 r}L\sqrt{1-B^2}K \\
&- \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1 r}a^2BL^2 + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_2 r}\cos\psi BL^2a],
\end{aligned}
$$

$$
\begin{aligned}
w_{5e^2} = \frac{1}{\sqrt{F_1}GKL}[&\Omega_{r\psi\theta_1\theta_2}\cos\psi + \Omega_{r\psi\theta_1\phi_2}b_{\phi_2\theta_2}\cos\psi + \Omega_{r\psi\theta_1\phi_2}\sin\psi - \Omega_{r\psi\theta_2\phi_1}\cos\psi b_{\phi_1\theta_1} + \Omega_{r\psi\theta_2\phi_1}\sin\psi \\
&+ \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}b_{\phi_2\theta_2}\cos\psi + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}\sin\psi + \Omega_{r\psi\phi_1\phi_2}\cos\psi - \Omega_{r\psi\phi_1\phi_2}\sin\psi b_{\phi_2\theta_2} \\
&- \Omega_{\psi\theta_1\theta_2\phi_1}\cos\psi b_{\phi_1 r} - \Omega_{\psi\theta_1\theta_2\phi_2}\cos\psi b_{\phi_2 r} + \Omega_{\psi\theta_1\phi_1\phi_2}b_{\phi_1 r}b_{\phi_2\theta_2}\cos\psi + \Omega_{\psi\theta_1\phi_1\phi_2}b_{\phi_1 r}\sin\psi \\
&+ \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_2 r}\cos\psi b_{\phi_1\theta_1} - \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_2 r}\sin\psi],
\end{aligned}
\tag{B3}
$$

$$
\begin{aligned}
w_{5e^3} = {} & \frac{1}{K^2\sqrt{F_1}L^2G}[-\Omega_{r\psi\theta_1\theta_2}\sin\psi\Delta_1\cos\theta_1 LG - \Omega_{r\psi\theta_1\phi_1}\Delta_2\cos\theta_2\cos\psi LG + \Omega_{r\psi\theta_2\phi_2}a\Delta_1\cos\theta_1 LG \\
& - \Omega_{\theta_1\theta_2\phi_1\phi_2}\sin\psi b_{\phi_2 r}LG - \Omega_{r\theta_1\phi_1\phi_2}LG\sin\psi b_{\phi_2\theta_2} + \Omega_{r\psi\theta_1\phi_2}\Delta_1\cos\theta_1 LG\cos\psi \\
& - \Omega_{r\psi\theta_1\phi_2}\Delta_1\cos\theta_1 LG\sin\psi b_{\phi_2\theta_2} + \Omega_{r\psi\theta_2\phi_1}LG\sin\psi\Delta_1\cos\theta_1 b_{\phi_1\theta_1} + \Omega_{\psi\theta_1\theta_2\phi_1}L\sin\psi Gb_{\psi r} \\
& - \Omega_{r\psi\theta_2\phi_1}LGa\Delta_2\cos\theta_2 + \Omega_{r\psi\phi_1\phi_2}LGb_{\phi_1\theta_1}\Delta_1\cos\theta_1\cos\psi \\
& - \Omega_{r\psi\phi_1\phi_2}LGb_{\phi_1\theta_1}\Delta_1\cos\theta_1\sin\psi b_{\phi_2\theta_2} + \Omega_{r\psi\phi_1\phi_2}LG\Delta_2\cos\theta_2 b_{\phi_2\theta_2}a + \Omega_{\psi\theta_1\theta_2\phi_1}L\sin\psi G\Delta_1\cos\theta_1 b_{\phi_1 r} \\
& - \Omega_{\psi\theta_1\theta_2\phi_1}L\cos\psi\sqrt{-B^2}\sqrt{F_1} - \Omega_{\psi\theta_1\theta_2\phi_2}\sqrt{F_1}La\sqrt{1-B^2} + \Omega_{\psi\theta_1\theta_2\phi_2}\sin\psi\Delta_1\cos\theta_1 b_{\phi_2 r}LG \\
& + \Omega_{\psi\theta_1\phi_1\phi_2}LG\Delta_2\cos\theta_2\cos\psi b_{\phi_2 r} + \Omega_{\psi\theta_1\phi_1\phi_2}LGb_{\psi r}\cos\psi + \Omega_{\psi\theta_1\phi_1\phi_2}LGb_{\phi_1 r}\Delta_1\cos\theta_1\cos\psi \\
& - \Omega_{\psi\theta_1\phi_1\phi_2}LGb_{\phi_1 r}\Delta_1\cos\theta_1\sin\psi b_{\phi_2\theta_2} + \Omega_{\psi\theta_1\phi_1\phi_2}L\sqrt{1-B^2}\sqrt{F_1}\sin\psi \\
& - \Omega_{\psi\theta_1\phi_1\phi_2}LGb_{\psi r}\sin\psi b_{\phi_2\theta_2} + \Omega_{\psi\theta_1\phi_1\phi_2}L\sqrt{1-B^2}\sqrt{F_1}b_{\phi_2\theta_2}\cos\psi - \Omega_{\psi\theta_2\phi_1\phi_2}LGb_{\phi_2 r}\sin\psi\Delta_1\cos\theta_1 b_{\phi_1\theta_1} \\
& + \Omega_{\psi\theta_2\phi_1\phi_2}LGb_{\phi_2 r}a\Delta_2\cos\theta_2 + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}\sqrt{F_1}La\sqrt{1-B^2} + \Omega_{\psi\theta_2\phi_1\phi_2}LGa\Delta_1\cos\theta_1 b_{\phi_1 r} \\
& + \Omega_{\psi\theta_2\phi_1\phi_2}LGab_{\psi r} + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}\sqrt{F_1}BK + \Omega_{r\theta_1\theta_2\phi_1}\sin\psi LG - \Omega_{\psi\theta_1\theta_2\phi_2}\sqrt{F_1}BK + \Omega_{r\theta_2\phi_1\phi_2}aLG \\
& + \Omega_{r\theta_1\phi_1\phi_2}LG\cos\psi],
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
w_{5e^4} = {} & \frac{1}{K^2\sqrt{F_1}L^2G}[\Omega_{r\psi\theta_1\theta_2}G\cos\psi\Delta_1\cos\theta_1 L\sqrt{1-B^2} + \Omega_{r\psi\theta_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}La\sqrt{1-B^2} + \Omega_{r\psi\theta_1\theta_2}G\Delta_2 \\
& \times\cos\theta_2 La\sqrt{1-B^2} + \Omega_{r\psi\theta_1\theta_2}G\Delta_2\cos\theta_2 BK + \Omega_{r\theta_1\phi_1\phi_2}\sqrt{1-B^2}LG\sin\psi - \Omega_{r\psi\theta_1\phi_1}\sin\psi\Delta_2\cos\theta_2\sqrt{1-B^2}LG \\
& - \Omega_{r\theta_1\theta_2\phi_1}\cos\psi\sqrt{1-B^2}LG - \Omega_{r\theta_1\theta_2\phi_2}GLa\sqrt{1-B^2} + \Omega_{r\psi\theta_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BK \\
& + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\phi_2 r}\cos\psi\Delta_1\cos\theta_1 L\sqrt{1-B^2} - \Omega_{r\psi\theta_2\phi_1}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 La\sqrt{1-B^2} \\
& - \Omega_{r\psi\theta_2\phi_1}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 BK + \Omega_{\theta_1\theta_2\phi_1\phi_2}G\cos\psi\sqrt{1-B^2}b_{\phi_2 r}L - \Omega_{\theta_1\theta_2\phi_1\phi_2}Gb_{\phi_1 r}La\sqrt{1-B^2} \\
& + \Omega_{r\theta_1\phi_1\phi_2}\sqrt{1-B^2}LGb_{\phi_2\theta_2}\cos\psi + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\psi r}La\sqrt{1-B^2} + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\psi r}BK \\
& - \Omega_{\theta_1\theta_2\phi_1\phi_2}Gb_{\phi_1 r}BK + \Omega_{r\psi\theta_1\phi_2}G\Delta_1\cos\theta_1 L\sqrt{1-B^2}b_{\phi_2\theta_2}\cos\psi + \Omega_{r\psi\theta_1\phi_2}G\Delta_1\cos\theta_1 L\sqrt{1-B^2}\sin\psi \\
& - \Omega_{r\psi\theta_2\phi_1}b_{\phi_1\theta_1}G\cos\psi\Delta_1\cos\theta_1 L\sqrt{1-B^2} + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}La\sqrt{1-B^2} \\
& + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}G\Delta_1\cos\theta_1 L\sqrt{1-B^2}\sin\psi \\
& + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BK + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}G\Delta_1\cos\theta_1 L\sqrt{1-B^2}b_{\phi_2\theta_2}\cos\psi \\
& + \Omega_{r\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}GLa\sqrt{1-B^2} + \Omega_{r\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}GBK - \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\phi_1 r}\Delta_2\cos\theta_2 BK \\
& - \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\psi r}\cos\psi\sqrt{1-B^2}L - \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\phi_1 r}\cos\psi\Delta_1\cos\theta_1 L\sqrt{1-B^2} \\
& - \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\phi_1 r}\Delta_2\cos\theta_2 La\sqrt{1-B^2} - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\psi r}La\sqrt{1-B^2} - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\phi_2 r}\Delta_2\cos\theta_2 La\sqrt{1-B^2} \\
& + \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\psi r}\sqrt{1-B^2}Lb_{\phi_2\theta_2}\cos\psi + \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\psi r}\sqrt{1-B^2}L\sin\psi + \Omega_{\psi\theta_1\phi_1\phi_2}\sqrt{F_1}L\sin\psi b_{\phi_2\theta_2} \\
& + \Omega_{\psi\theta_1\phi_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_1 r}b_{\phi_2\theta_2}BK + \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\phi_1 r}\Delta_1\cos\theta_1 L\sqrt{1-B^2}b_{\phi_2\theta_2}\cos\psi \\
& + \Omega_{\psi\theta_1\phi_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_1 r}b_{\phi_2\theta_2}La\sqrt{1-B^2} + \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\phi_1 r}\Delta_1\cos\theta_1 L\sqrt{1-B^2}\sin\psi \\
& - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\psi r}BK + \Omega_{\psi\theta_1\phi_1\phi_2}G\sin\psi\Delta_2\cos\theta_2\sqrt{1-B^2}b_{\phi_2 r}L - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\phi_2 r}\Delta_2\cos\theta_2 BK \\
& - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\phi_2 r}\cos\psi\Delta_1\cos\theta_1 L\sqrt{1-B^2} + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\phi_2 r}\Delta_2\cos\theta_2 La\sqrt{1-B^2} \\
& + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\phi_2 r}\Delta_2\cos\theta_2 BK - \Omega_{\psi\theta_1\theta_2\phi_1}\sin\psi\sqrt{F_1}L - \Omega_{\psi\theta_2\phi_1\phi_2}a\sqrt{F_1}L - \Omega_{\psi\theta_1\phi_1\phi_2}\sqrt{F_1}L\cos\psi \\
& - \Omega_{r\theta_1\theta_2\phi_2}GBK],
\end{aligned}
\tag{B5}
$$

$$w_{5e^5} = \frac{1}{K^2\sqrt{F_1}L^2G}[-\Omega_{r\psi\theta_2\phi_1}\Delta_2\cos\theta_2 KG + \Omega_{r\psi\theta_2\phi_2}\Delta_1\cos\theta_1 KG + \Omega_{r\psi\phi_1\phi_2}\Delta_2\cos\theta_2 b_{\phi_2\theta_2}KG + \Omega_{r\theta_2\phi_1\phi_2}KG$$
$$- \Omega_{\psi\theta_1\theta_2\phi_1}\cos\psi B\sqrt{F_1}L - \Omega_{\psi\theta_1\theta_2\phi_2}\sqrt{F_1}BLa + \Omega_{\psi\theta_1\theta_2\phi_2}\sqrt{F_1}\sqrt{1-B^2}K + \Omega_{\psi\theta_1\phi_1\phi_2}B\sqrt{F_1}Lb_{\phi_2\theta_2}\cos\psi$$
$$+ \Omega_{\psi\theta_1\phi_1\phi_2}B\sqrt{F_1}L\sin\psi + \Omega_{\psi\theta_2\phi_1\phi_2}KG\Delta_2\cos\theta_2 b_{\phi_2 r} + \Omega_{\psi\theta_2\phi_1\phi_2}KGb_{\psi r} + \Omega_{\psi\theta_2\phi_1\phi_2}KG\Delta_1\cos\theta_1 b_{\phi_1 r}$$
$$+ \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}\sqrt{F_1}BLa - \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}\sqrt{F_1}\sqrt{1-B^2}K], \qquad \text{(B6)}$$

$$w_{5e^6} = \frac{1}{K^2\sqrt{F_1}L^2G}[\Omega_{r\theta_1\theta_2\phi_2}G\sqrt{1-B^2}K - \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\phi_2 r}\Delta_2\cos\theta_2\sqrt{1-B^2}K + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\psi r}BLa$$
$$- \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\psi r}\sqrt{1-B^2}K + \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\phi_2 r}\Delta_2\cos\theta_2 BLa + \Omega_{\theta_1\theta_2\phi_1\phi_2}G\cos\psi Bb_{\phi_2 r}L$$
$$- \Omega_{\theta_1\theta_2\phi_1\phi_2}Gb_{\phi_1 r}BLa + \Omega_{\theta_1\theta_2\phi_1\phi_2}Gb_{\phi_1 r}\sqrt{1-B^2}K - \Omega_{r\psi\theta_1\phi_1}\sin\psi\Delta_2\cos\theta_2 BLG - \Omega_{r\theta_1\theta_2\phi_1}\cos\psi BLG$$
$$+ \Omega_{r\psi\theta_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BLa - \Omega_{r\psi\theta_1\theta_2}G\Delta_2\cos\theta_2\sqrt{1-B^2}K + \Omega_{r\psi\theta_1\theta_2}G\cos\psi B\Delta_1\cos\theta_1 L$$
$$+ \Omega_{r\psi\theta_1\theta_2}G\Delta_2\cos\theta_2 BLa + \Omega_{r\psi\theta_1\phi_2}GB\Delta_1\cos\theta_1 L\sin\psi + \Omega_{r\psi\theta_1\phi_2}GB\Delta_1\cos\theta_1 Lb_{\phi_2\theta_2}\cos\psi$$
$$- \Omega_{r\psi\theta_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}\sqrt{1-B^2}K - \Omega_{r\psi\theta_2\phi_1}b_{\phi_1\theta_1}G\cos\psi B\Delta_1\cos\theta_1 L$$

$$- \Omega_{r\psi\theta_2\phi_1}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 BLa + \Omega_{r\psi\theta_2\phi_1}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2\sqrt{1-B^2}K + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}GB\Delta_1\cos\theta_1 Lb_{\phi_2\theta_2}\cos\psi$$
$$+ \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}GB\Delta_1\cos\theta_1 L\sin\psi + \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BLa$$
$$- \Omega_{r\psi\phi_1\phi_2}b_{\phi_1\theta_1}G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}\sqrt{1-B^2}K - \Omega_{r\theta_1\theta_2\phi_2}GBLa + \Omega_{r\theta_1\phi_1\phi_2}BLGb_{\phi_2\theta_2}\cos\psi$$
$$+ \Omega_{r\theta_1\phi_1\phi_2}BLG\sin\psi + \Omega_{r\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}GBLa - \Omega_{r\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}G\sqrt{1-B^2}K$$
$$- \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\phi_1 r}\cos\psi B\Delta_1\cos\theta_1 L - \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\phi_1 r}\Delta_2\cos\theta_2 BLa$$
$$+ \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\phi_1 r}\Delta_2\cos\theta_2\sqrt{1-B^2}K - \Omega_{\psi\theta_1\theta_2\phi_1}Gb_{\psi r}\cos\psi BL$$

$$- \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\phi_2 r}\cos\psi B\Delta_1\cos\theta_1 L + \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\phi_2 r}\Delta_2\cos\theta_2\sqrt{1-B^2}K - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\psi r}BLa$$
$$+ \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\psi r}\sqrt{1-B^2}K - \Omega_{\psi\theta_1\theta_2\phi_2}Gb_{\phi_2 r}\Delta_2\cos\theta_2 BLa + \Omega_{\psi\theta_1\phi_1\phi_2}G\sin\psi\Delta_2\cos\theta_2 Bb_{\phi_2 r}L$$
$$+ \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\phi_1 r}B\Delta_1\cos\theta_1 Lb_{\phi_2\theta_2}\cos\psi + \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\phi_1 r}B\Delta_1\cos\theta_1 L\sin\psi + \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\psi r}BLb_{\phi_2\theta_2}\cos\psi$$
$$+ \Omega_{\psi\theta_1\phi_1\phi_2}Gb_{\psi r}BL\sin\psi + \Omega_{\psi\theta_1\phi_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_1 r}b_{\phi_2\theta_2}BLa - \Omega_{\psi\theta_1\phi_1\phi_2}G\Delta_2\cos\theta_2 b_{\phi_1 r}b_{\phi_2\theta_2}\sqrt{1-B^2}K$$
$$+ \Omega_{\psi\theta_2\phi_1\phi_2}b_{\phi_1\theta_1}Gb_{\phi_2 r}\cos\psi B\Delta_1\cos\theta_1 L - \Omega_{\psi\theta_2\phi_1\phi_2}\sqrt{F_1}K], \qquad \text{(B7)}$$

where Ω_{ijkl} are now given by the following components:

$$
\begin{aligned}
\Omega_{r\psi\theta_1\theta_2} = \frac{1}{\sqrt{1-B^2}}[& GLK_r \sin\psi B^2 + G_r L b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2} - G_r L b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2} B^2 + GL b_{\phi_1\theta_1} K_r \sin\psi b_{\phi_2\theta_2} \\
& - GL b_{\phi_1\theta_1,r} K \cos\psi - G_r LK \cos\psi b_{\phi_2\theta_2} - GL b_{\phi_1\theta_1} K_r \sin\psi b_{\phi_2\theta_2} B^2 + GL_r b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2} \\
& + GL_r K \sin\psi B^2 - LG\Delta_2 \cos\theta_2 b_{\phi_2 r} b_{\phi_1\theta_1} K \sin\psi B^2 + LG b_{\psi r} K \sin\psi b_{\phi_2\theta_2} + LG b_{\psi r} K \cos\psi B^2 \\
& - LG\Delta_2 \cos\theta_2 b_{\phi_2 r} K \cos\psi + LG\Delta_2 \cos\theta_2 b_{\phi_2 r} K \cos\psi B^2 - LG b_{\psi r} K \sin\psi b_{\phi_2\theta_2} B^2 \\
& - LG\Delta_1 \cos\theta_1 b_{\phi_1 r} K \cos\psi + LG\Delta_1 \cos\theta_1 b_{\phi_1 r} K \cos\psi B^2 - GL b_{\phi_1\theta_1,r} K \sin\psi b_{\phi_2\theta_2} B^2 \\
& + GL b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2,r} + G_r LK \sin\psi B^2 - GL_r b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2} B^2 \\
& + GL b_{\phi_1\theta_1,r} K \sin\psi b_{\phi_2\theta_2} - G_r L b_{\phi_1\theta_1} K \cos\psi - GLK \cos\psi b_{\phi_2\theta_2}, r + GLK \cos\psi b_{\phi_2\theta_2}, r B^2 \\
& + GL b_{\phi_1\theta_1} K_r \cos\psi B^2 - GLK_r \cos\psi b_{\phi_2\theta_2} + GLK_r \cos\psi b_{\phi_2\theta_2} B^2 - GL_r K \cos\psi b_{\phi_2\theta_2} \\
& + GL_r K \cos\psi b_{\phi_2\theta_2} B^2 - GL b_{\phi_1\theta_1} K_r \cos\psi + GL b_{\phi_1\theta_1,r} K \cos\psi B^2 + G_r LK \cos\psi b_{\phi_2\theta_2} B^2 \\
& + GL b_{\phi_1\theta_1,r} K \cos\psi B^2 - GL b_{\phi_1\theta_1} K \sin\psi \frac{d}{dr} b_{\phi_2\theta_2} B^2 + G_r L b_{\phi_1\theta_1} K \cos\psi B^2 \\
& - GL_r b_{\phi_1\theta_1} K \cos\psi + L\sqrt{1-B^2}\sqrt{F1} K \sin\psi - L\sqrt{1-B^2}\sqrt{F1} b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2} \\
& - L^2\sqrt{1-B^2} BG\Delta_1 \cos\theta_1 b_{\phi_1 r} \sin\psi b_{\phi_2\theta_2} a + L\sqrt{1-B^2}\sqrt{F1} b_{\phi_1\theta_1} K \cos\psi \\
& + G_r L^2\sqrt{1-B^2} b_{\phi_1\theta_1} Ba \sin\psi b_{\phi_2\theta_2} + G_r L^2\sqrt{1-B^2} B \cos\psi b_{\phi_2\theta_2} a - G_r L^2\sqrt{1-B^2} b_{\phi_1\theta_1} Ba \cos\psi \\
& - GL b_{\phi_1\theta_1} K \sin\psi b_{\phi_2\theta_2} BB_r - L^2\sqrt{1-B^2} G b_{\psi r} Ba \sin\psi b_{\phi_2\theta_2} + L^2\sqrt{1-B^2} G b_{\psi r} b_{\phi_1\theta_1} Ba \sin\psi \\
& + L^2\sqrt{1-B^2} G b_{\psi r} b_{\phi_1\theta_1} Ba \cos\psi b_{\phi_2\theta_2} + L^2\sqrt{1-B^2} BG\Delta_1 \cos\theta_1 b_{\phi_1 r} \cos\psi a \\
& + L^2\sqrt{1-B^2} G\Delta_2 \cos\theta_2 b_{\phi_2 r} b_{\phi_1\theta_1} Ba \sin\psi + L^2\sqrt{1-B^2} G\Delta_2 \cos\theta_2 b_{\phi_2 r} Ba \cos\psi \\
& + L^2\sqrt{1-B^2} G b_{\psi r} Ba \cos\psi + L\sqrt{1-B^2}\sqrt{F1} K \cos\psi b_{\phi_2\theta_2} - GLK_r \sin\psi + GLK \cos\psi b_{\phi_2\theta_2} BB_r \\
& + GLK \sin\psi BB_r + 2GL_r\sqrt{1-B^2} b_{\phi_1\theta_1} BLa \sin\psi b_{\phi_2\theta_2} + GL b_{\phi_1\theta_1} K \cos\psi BB_r + 2GL_r\sqrt{1-B^2} BLa \sin\psi \\
& + 2GL_r\sqrt{1-B^2} B \cos\psi b_{\phi_2\theta_2} La - 2GL_r\sqrt{1-B^2} b_{\phi_1\theta_1} BLa \cos\psi + GL^2 b_{\phi_1\theta_1} B_r a \sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2} \\
& - GL_r K \sin\psi + GL^2 b_{\phi_1\theta_1} Ba_r \sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2} - GL^2 b_{\phi_1\theta_1,r} Ba \cos\psi\sqrt{1-B^2} \\
& + GL^2 Ba_r \sin\psi\sqrt{1-B^2} + GL^2 b_{\phi_1\theta_1,r} Ba \sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2} + LG\Delta_1 \cos\theta_1 b_{\phi_1 r} K \sin\psi b_{\phi_2\theta_2} \\
& - LG\Delta_1 \cos\theta_1 b_{\phi_1 r} K \sin\psi b_{\phi_2\theta_2} B^2 + LG b_{\psi r} b_{\phi_1\theta_1} K \sin\psi - LG b_{\psi r} b_{\phi_1\theta_1} K \sin\psi B^2 \\
& + GL^2 B_r a \sin\psi\sqrt{1-B^2} + LG b_{\psi r} b_{\phi_1\theta_1} K \cos\psi b_{\phi_2\theta_2} - LG b_{\psi r} b_{\phi_1\theta_1} K \cos\psi b_{\phi_2\theta_2} B^2 \\
& + LG\Delta_2 \cos\theta_2 b_{\phi_2 r} b_{\phi_1\theta_1} K \sin\psi + GL^2 B_r \cos\psi b_{\phi_2\theta_2} a\sqrt{1-B^2} + GL^2 B \cos\psi b_{\phi_2\theta_2,r} a\sqrt{1-B^2} \\
& - GL^2 b_{\phi_1\theta_1} B_r a \cos\psi\sqrt{1-B^2} - GL^2 b_{\phi_1\theta_1} Ba_r \cos\psi\sqrt{1-B^2} + GL^2 B \cos\psi b_{\phi_2\theta_2} a_r\sqrt{1-B^2} \\
& + GL^2 b_{\phi_1\theta_1} Ba \sin\psi b_{\phi_2\theta_2,r}\sqrt{1-B^2} - GL \cos\psi b_{\psi r} K + G_r L^2\sqrt{1-B^2} Ba \sin\psi - G_r LK \sin\psi],
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{r\psi\theta_1\phi_1} = {}& G_r BL^2 a^2 + GB_r L^2 a^2 + 2GBLa^2 L_r + 2GBL^2 aa_r + G_r BK^2 + GB_r K^2 + 2GBKK_r \\
& - G\Delta_1 \cos\theta_1 L^2 b_{\phi_2 r} Ba \sin\psi + G\Delta_1 \cos\theta_1 L b_{\phi_2 r}\sqrt{1-B^2} K \sin\psi,
\end{aligned}
$$

$$\begin{aligned}\Omega_{r\psi\theta_1\phi_2} = \frac{1}{\sqrt{1-B^2}}[&-G_rLK\cos\psi B^2 - GLK_r\cos\psi B^2 + GL_rK\cos\psi - GL_rK\cos\psi B^2 - G_rLb_{\phi_1\theta_1}K\sin\psi\\
&+ G_rLb_{\phi_1\theta_1}K\sin\psi B^2 + GLb_{\phi_1\theta_1,r}K\sin\psi B^2 - LG\Delta_1\cos\theta_1 b_{\phi_1 r}K\sin\psi - GL\sin\psi b_{\phi_1\theta_1,r}K\\
&- LGb_{\psi r}K\sin\psi + LGb_{\psi r}K\sin\psi B^2 + LGb_{\psi r}b_{\phi_1\theta_1}K\cos\psi B^2 + LG\Delta_1\cos\theta_1 b_{\phi_1 r}K\sin\psi B^2 + G_rLK\cos\psi\\
&+ GLK_r\cos\psi - G_rL^2\sqrt{1-B^2}B\cos\psi a - G_rL^2\sqrt{1-B^2}b_{\phi_1\theta_1}Ba\sin\psi - GL^2B_r\cos\psi a\sqrt{1-B^2}\\
&- GL^2B\cos\psi a_r\sqrt{1-B^2} - GL^2b_{\phi_1\theta_1}B_ra\sin\psi\sqrt{1-B^2} - GL^2b_{\phi_1\theta_1}Ba_r\sin\psi\sqrt{1-B^2}\\
&- 2GL_r\sqrt{1-B^2}B\cos\psi La - 2GL_r\sqrt{1-B^2}b_{\phi_1\theta_1}BLa\sin\psi - GL^2b_{\phi_1\theta_1,r}Ba\sin\psi\sqrt{1-B^2}\\
&- \sqrt{F1}K\sqrt{1-B^2}L\cos\psi + GLb_{\phi_1\theta_1}K\sin\psi BB_r - GLK\cos\psi BB_r + L^2\sqrt{1-B^2}Gb_{\psi r}Ba\sin\psi\\
&+ L^2\sqrt{1-B^2}BG\Delta_1\cos\theta_1 b_{\phi_1 r}\sin\psi a + L\sqrt{1-B^2}\sqrt{F1}b_{\phi_1\theta_1}K\sin\psi - L^2\sqrt{1-B^2}Gb_{\psi r}b_{\phi_1\theta_1}Ba\cos\psi\\
&- GL_r\sin\psi b_{\phi_1\theta_1}K + GLb_{\phi_1\theta_1}K_r\sin\psi B^2 + GL_rb_{\phi_1\theta_1}K\sin\psi B^2 - GLb_{\phi_1\theta_1}K_r\sin\psi - GL\cos\psi b_{\psi r}Kb_{\phi_1\theta_1}],\end{aligned}$$

$$\begin{aligned}\Omega_{r\psi\theta_2\phi_1} = \frac{1}{\sqrt{1-B^2}}[&G_rLK\cos\psi B^2 + GLK_r\cos\psi B^2 - GL_rK\cos\psi + GL_rK\cos\psi B^2 + LGb_{\psi r}K\sin\psi\\
&- LGb_{\psi r}K\sin\psi B^2 - G_rLK\cos\psi - GLK_r\cos\psi - G_rL^2\sqrt{1-B^2}B\cos\psi a - GL^2B_r\cos\psi a\sqrt{1-B^2}\\
&- GL^2B\cos\psi a_r\sqrt{1-B^2} - 2G\frac{d}{dr}L\sqrt{1-B^2}B\cos\psi La + \sqrt{F1}K\sqrt{1-B^2}L\cos\psi + GLK\cos\psi BB_r\\
&- GLK\sin\psi b_{\phi_2\theta_2}B\frac{d}{dr}B + 2GL_r\sqrt{1-B^2}BLa\sin\psi b_{\phi_2\theta_2} + G_rL^2\sqrt{1-B^2}Ba\sin\psi b_{\phi_2\theta_2}\\
&- G_rLK\sin\psi b_{\phi_2\theta_2}B^2 + GL_rK\sin\psi b_{\phi_2\theta_2} - GL_rK\sin\psi b_{\phi_2\theta_2}B^2 + GLK_r\sin\psi b_{\phi_2\theta_2} - GLK_r\sin\psi b_{\phi_2\theta_2}B^2\\
&+ GLK\sin\psi b_{\phi_2\theta_2,r} - GLK\sin\psi b_{\phi_2\theta_2,r}B^2 + G_rLK\sin\psi b_{\phi_2\theta_2} + GL^2Ba\sin\psi b_{\phi_2\theta_2,r}\sqrt{1-B^2}\\
&+ GL^2B_ra\sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2} + GL^2Ba_r\sin\psi b_{\phi_2\theta_2}\sqrt{1-B^2} + L^2\sqrt{1-B^2}Gb_{\psi r}Ba\sin\psi\\
&+ LG\Delta_2\cos\theta_2 b_{\phi_2 r}K\sin\psi + LGb_{\psi r}K\cos\psi b_{\phi_2\theta_2} + L^2\sqrt{1-B^2}G\Delta_2\cos\theta_2 b_{\phi_2 r}Ba\sin\psi\\
&+ L^2\sqrt{1-B^2}Gb_{\psi r}Ba\cos\psi b_{\phi_2\theta_2} - L\sqrt{1-B^2}\sqrt{F1}K\sin\psi b_{\phi_2\theta_2} - LG\Delta_2\cos\theta_2 b_{\phi_2 r}K\sin\psi B^2\\
&- LGb_{\psi r}K\cos\psi b_{\phi_2\theta_2}B^2],\end{aligned}$$

$$\Omega_{r\psi\theta_2\phi_2} = -L - G_rBL - GB_rL - 2GBL_r + G\Delta_2\cos\theta_2 b_{\phi_1 r}BLa\sin\psi + G\Delta_2\cos\theta_2 b_{\phi_1 r}\sqrt{1-B^2}K\sin\psi$$

$$\begin{aligned}\Omega_{r\psi\phi_1\phi_2} = \frac{-1}{\sqrt{1-B^2}}[&-G_rL^2\sqrt{1-B^2}Ba\sin\psi - G_rLK\sin\psi + G_rLK\sin\psi B^2 - GL^2B_ra\sin\psi\sqrt{1-B^2}\\
&- 2GL_r\sqrt{1-B^2}BLa\sin\psi - GL^2Ba_r\sin\psi\sqrt{1-B^2} + GLK\sin\psi BB_r - GLK_r\sin\psi + GLK_r\sin\psi B^2\\
&- GL_rK\sin\psi + GL_rK\sin\psi B^2 - L^2\sqrt{1-B^2}Gb_{\psi r}Ba\cos\psi - GL\cos\psi b_{\psi r}K + LGb_{\psi r}K\cos\psi B^2\\
&+ L\sqrt{1-B^2}\sqrt{F1}K\sin\psi],\end{aligned}$$

$$\Omega_{r\psi\theta_2\phi_2} = -L - G_rBL - GB_rL - 2GBL_r + G\Delta_2\cos\theta_2 b_{\phi_1 r}BLa\sin\psi + G\Delta_2\cos\theta_2 b_{\phi_1 r}\sqrt{1-B^2}K\sin\psi$$

$$\begin{aligned}\Omega_{r\psi\phi_1\phi_2} = \frac{-1}{\sqrt{1-B^2}}[&-G_rL^2\sqrt{1-B^2}Ba\sin\psi - G_rLK\sin\psi + G_rLK\sin\psi B^2 - GL^2B_ra\sin\psi\sqrt{1-B^2}\\
&- 2GL_r\sqrt{1-B^2}BLa\sin\psi - GL^2Ba_r\sin\psi\sqrt{1-B^2} + GLK\sin\psi BB_r - GLK_r\sin\psi + GLK_r\sin\psi B^2\\
&- GL_rK\sin\psi + GL_rK\sin\psi B^2 - L^2\sqrt{1-B^2}Gb_{\psi r}Ba\cos\psi - GL\cos\psi b_{\psi r}K + LGb_{\psi r}K\cos\psi B^2\\
&+ L\sqrt{1-B^2}\sqrt{F1}K\sin\psi],\end{aligned}$$

$$\Omega_{r\theta_1\theta_2\phi_1} = \frac{1}{\sqrt{1-B^2}}[GBb_{\phi_2 r}\sqrt{1-B^2}\Delta_2\sin\theta_2 L^2a^2 + GBb_{\phi_2 r}\sqrt{1-B^2}\Delta_2\sin\theta_2 K^2 + G\Delta_{2,r}\cos\theta_2 b_{\phi_2\theta_2}K^2B\sqrt{1-B^2}$$
$$+ G\Delta_2\cos\theta_2 b_{\phi_2\theta_2,r}K^2B\sqrt{1-B^2} + 2G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}KBK_r\sqrt{1-B^2} + G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}K^2B_r\sqrt{1-B^2}$$
$$+ G\Delta_{1,r}\cos\theta_1 BL^2\cos\psi b_{\phi_2\theta_2}a\sqrt{1-B^2} + G\Delta_1\cos\theta_1 B_rL^2\cos\psi b_{\phi_2\theta_2}a\sqrt{1-B^2}$$
$$+ 2G\Delta_1\cos\theta_1 BL\cos\psi b_{\phi_2\theta_2}aL_r\sqrt{1-B^2} + G_r\Delta_1\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2}B^2 + G\Delta_1\cos\theta_1 KL_r\sin\psi B^2$$
$$- G\Delta_1\cos\theta_1 KL_r\cos\psi b_{\phi_2\theta_2} + G\Delta_1\cos\theta_1 KL_r\cos\psi b_{\phi_2\theta_2}B^2 - G\Delta_1\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2,r}$$
$$+ G\Delta_1\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2,r}B^2$$
$$- G\Delta_1\cos\theta_1 K_rL\cos\psi b_{\phi_2\theta_2} + G\Delta_1\cos\theta_1 K_rL\cos\psi b_{\phi_2\theta_2}B^2 - G\Delta_{1,r}\cos\theta_1 KL\sin\psi + G\Delta_{1,r}\cos\theta_1 KL\sin\psi B^2$$
$$- G\Delta_1\cos\theta_1 K_rL\sin\psi + G\Delta_1\cos\theta_1 K_rL\sin\psi B^2 - G\Delta_{1,r}\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2} + G\Delta_{1,r}\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2}B^2$$
$$- G_r\Delta_1\cos\theta_1 KL\sin\psi + \frac{d}{dr}G\Delta_1\cos\theta_1 KL\sin\psi B^2 - G_r\Delta_1\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2}$$
$$+ G\Delta_1\cos\theta_1 BL^2\cos\psi b_{\phi_2\theta_2,r}a\sqrt{1-B^2} + G\Delta_1\cos\theta_1 BL^2\cos\psi b_{\phi_2\theta_2}a_r\sqrt{1-B^2}$$
$$+ 2G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BLa^2L_r\sqrt{1-B^2}$$
$$+ 2G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BL^2aa_r\sqrt{1-B^2} + G\Delta_{1,r}\cos\theta_1 BL^2\sin\psi a\sqrt{1-B^2} + G\Delta_1\cos\theta_1 B_rL^2\sin\psi a\sqrt{1-B^2}$$
$$+ 2G\Delta_1\cos\theta_1 BL\sin\psi aL_r\sqrt{1-B^2} + G\Delta_1\cos\theta_1 BL^2\sin\psi a_r\sqrt{1-B^2} + G\Delta_{2,r}\cos\theta_2 b_{\phi_2\theta_2}BL^2a^2\sqrt{1-B^2}$$
$$+ G\Delta_2\cos\theta_2 b_{\phi_2\theta_2,r}BL^2a^2\sqrt{1-B^2} + G\Delta_2\cos\theta_2 b_{\phi_2\theta_2}B_rL^2a^2\sqrt{1-B^2} + G\Delta_1\cos\theta_1 KL\sin\psi BB_r$$
$$+ G\Delta_1\cos\theta_1 KL\cos\psi b_{\phi_2\theta_2}BB_r + G_r\sqrt{1-B^2}\Delta_1\cos\theta_1 BL^2\cos\psi b_{\phi_2\theta_2}a + G_r\sqrt{1-B^2}\Delta_2\cos\theta_2 b_{\phi_2\theta_2}BL^2a^2$$
$$+ G_r\sqrt{1-B^2}\Delta_1\cos\theta_1 BL^2\sin\psi a + G_r\sqrt{1-B^2}\Delta_2\cos\theta_2 b_{\phi_2\theta_2}K^2B - G\Delta_1\cos\theta_1 KL_r\sin\psi$$
$$+ GBb_{\phi_2 r}\sqrt{1-B^2}L^2\Delta_1\sin\theta_1],$$

$$\Omega_{r\theta_1\theta_2\phi_2} = \frac{1}{\sqrt{1-B^2}}[-G_rL\Delta_2\cos\theta_2 K\sin\psi + G_rL\Delta_2\cos\theta_2 K\sin\psi B^2 - GL_r\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi$$
$$+ GL_r\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi B^2 - GL_r\Delta_2\cos\theta_2 K\sin\psi + GL_r\Delta_2\cos\theta_2 K\sin\psi B^2$$
$$- BGb_{\phi_1 r}\sqrt{1-B^2}L^2\Delta_1\sin\theta_1 - BGb_{\phi_1 r}\sqrt{1-B^2}\Delta_2\sin\theta_2 K^2 - BGb_{\phi_1 r}\sqrt{1-B^2}\Delta_2\sin\theta_2 L^2a^2$$
$$- GL\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K_r\cos\psi + GL\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K_r\cos\psi B^2 - GL\Delta_{2,r}\cos\theta_2 K\sin\psi$$
$$+ GL\Delta_{2,r}\cos\theta_2 K\sin\psi B^2 + G_rL\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi B^2 - GL\Delta_2\cos\theta_2 K_r\sin\psi + GL\Delta_2\cos\theta_2 K_r\sin\psi B^2$$
$$- GL\Delta_2\cos\theta_2 b_{\phi_1\theta_1,r}K\cos\psi + GL\Delta_2\cos\theta_2 b_{\phi_1\theta_1,r}K\cos\psi B^2$$
$$- GL\Delta_{2,r}\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi + GL\Delta_{2,r}\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi B^2 - G_rL\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi$$
$$- \frac{d}{dr}GL^2\sqrt{1-B^2}\Delta_1\cos\theta_1 b_{\phi_1\theta_1}B + G_rL^2\sqrt{1-B^2}\Delta_2\cos\theta_2 B\sin\psi a - G_rL^2\sqrt{1-B^2}\Delta_2\cos\theta_2 b_{\phi_1\theta_1}Ba\cos\psi$$
$$- 2GL_r\sqrt{1-B^2}\Delta_1\cos\theta_1 b_{\phi_1\theta_1}BL + 2GL_r\sqrt{1-B^2}\Delta_2\cos\theta_2 B\sin\psi La - 2GL_r\sqrt{1-B^2}\Delta_2\cos\theta_2 b_{\phi_1\theta_1}BLa\cos\psi$$
$$+ GL\Delta_2\cos\theta_2 K\sin\psi BB_r$$
$$+ GL\Delta_2\cos\theta_2 b_{\phi_1\theta_1}K\cos\psi BB_r - GL^2\Delta_{1,r}\cos\theta_1 b_{\phi_1\theta_1}B\sqrt{1-B^2} - GL^2\Delta_1\cos\theta_1 b_{\phi_1\theta_1,r}B\sqrt{1-B^2}$$
$$- GL^2\Delta_1\cos\theta_1 b_{\phi_1\theta_1}B_r\sqrt{1-B^2} - GL^2\Delta_{2,r}\cos\theta_2 b_{\phi_1\theta_1}Ba\cos\psi\sqrt{1-B^2} - GL^2\Delta_2\cos\theta_2 b_{\phi_1\theta_1,r}Ba\cos\psi\sqrt{1-B^2}$$
$$- GL^2\Delta_2\cos\theta_2 b_{\phi_1\theta_1}B_ra\cos\psi\sqrt{1-B^2} - GL^2\Delta_2\cos\theta_2 b_{\phi_1\theta_1}Ba_r\cos\psi\sqrt{1-B^2} + GL^2\Delta_{2,r}\cos\theta_2 B\sin\psi a\sqrt{1-B^2}$$
$$+ GL^2\Delta_2\cos\theta_2 B_r\sin\psi a\sqrt{1-B^2} + GL^2\Delta_2\cos\theta_2 B\sin\psi\frac{d}{dr}a\sqrt{1-B^2}],$$

$$\begin{aligned}
\Omega_{r\theta_1\phi_1\phi_2} &= \frac{1}{\sqrt{1-B^2}}[G_r\sqrt{1-B^2}\Delta_2\cos\theta_2 BL^2a^2 + G_r\sqrt{1-B^2}\Delta_1\cos\theta_1 BL^2\cos\psi a + G_r\sqrt{1-B^2}\Delta_2\cos\theta_2 K^2B \\
&\quad - G_r\Delta_1\cos\theta_1 KL\cos\psi + G_r\Delta_1\cos\theta_1 KL\cos\psi B^2 + G\Delta_{2,r}\cos\theta_2 BL^2a^2\sqrt{1-B^2} \\
&\quad + G\Delta_2\cos\theta_2 B_rL^2a^2\sqrt{1-B^2} + 2G\Delta_2\cos\theta_2 BLa^2L_r\sqrt{1-B^2} + 2G\Delta_2\cos\theta_2 BL^2aa_r\sqrt{1-B^2} \\
&\quad + G\Delta_{1,r}\cos\theta_1 BL^2\cos\psi a\sqrt{1-B^2} + G\Delta_1\cos\theta_1 B_rL^2\cos\psi a\sqrt{1-B^2} \\
&\quad + 2G\Delta_1\cos\theta_1 BL\cos\psi a\frac{d}{dr}L\sqrt{1-B^2} + G\Delta_1\cos\theta_1 BL^2\cos\psi a_r\sqrt{1-B^2} + G\Delta_{2,r}\cos\theta_2 K^2B\sqrt{1-B^2} \\
&\quad + 2G\Delta_2\cos\theta_2 KBK_r\sqrt{1-B^2} + G\Delta_2\cos\theta_2 K^2B_r\sqrt{1-B^2} - G\Delta_{1,r}\cos\theta_1 KL\cos\psi \\
&\quad + G\Delta_{1,r}\cos\theta_1 KL\cos\psi B^2 - G\Delta_1\cos\theta_1 K_rL\cos\psi + G\Delta_1\cos\theta_1 K_rL\cos\psi B^2 - G\Delta_1\cos\theta_1 KL_r\cos\psi \\
&\quad + G\Delta_1\cos\theta_1 KL_r\cos\psi B^2 + G\Delta_1\cos\theta_1 KL\cos\psi BB_r], \\
\Omega_{r\theta_2\phi_1\phi_2} &= \frac{1}{\sqrt{1-B^2}}[-G_rL^2\sqrt{1-B^2}\Delta_2\cos\theta_2 Ba\cos\psi - G_rL\Delta_2\cos\theta_2 K\cos\psi + G_rL\Delta_2\cos\theta_2 K\cos\psi B^2 \\
&\quad - G_rL^2\sqrt{1-B^2}\Delta_1\cos\theta_1 B - 2GL_r\sqrt{1-B^2}\Delta_2\cos\theta_2 BLa\cos\psi - G\frac{d}{dr}L\Delta_2\cos\theta_2 K\cos\psi \\
&\quad + GL_r\Delta_2\cos\theta_2 K\cos\psi B^2 - 2G\frac{d}{dr}L\sqrt{1-B^2}\Delta_1\cos\theta_1 BL - GL^2\Delta_{2,r}\cos\theta_2 Ba\cos\psi\sqrt{1-B^2} \\
&\quad - GL^2\Delta_2\cos\theta_2 B_ra\cos\psi\sqrt{1-B^2} - GL^2\Delta_2\cos\theta_2 Ba_r\cos\psi\sqrt{1-B^2} - GL\Delta_{2,r}\cos\theta_2 K\cos\psi \\
&\quad + GL\Delta_{2,r}\cos\theta_2 K\cos\psi B^2 + GL\Delta_2\cos\theta_2 K\cos\psi BB_r - GL\Delta_2\cos\theta_2 K_r\cos\psi \\
&\quad + GL\Delta_2\cos\theta_2 K_r\cos\psi B^2 - GL^2\Delta_{1,r}\cos\theta_1 B\sqrt{1-B^2} - GL^2\Delta_1\cos\theta_1 B_r\sqrt{1-B^2}], \\
\Omega_{\psi\theta_1\theta_2\phi_1} &= G\Delta_1\cos\theta_1 LB\cos\psi La - BLa\sin\psi b_{\phi_2\theta_2} - K\sqrt{1-B^2}\cos\psi + \sqrt{1-B^2}K\sin\psi b_{\phi_2\theta_2}, \\
\Omega_{\psi\theta_1\theta_2\phi_2} &= GL\Delta_2\cos\theta_2 B\cos\psi La + b_{\phi_1\theta_1}BLa\sin\psi + b_{\phi_1\theta_1}\sqrt{1-B^2}K\sin\psi - K\sqrt{1-B^2}\cos\psi, \\
\Omega_{\psi\theta_1\phi_1\phi_2} &= -G\Delta_1\cos\theta_1 L\sin\psi BLa - \sqrt{1-B^2}K, \\
\Omega_{\psi\theta_2\phi_1\phi_2} &= GL\Delta_2\cos\theta_2\sin\psi BLa + \sqrt{1-B^2}K, \\
\Omega_{\theta_1\theta_2\phi_1\phi_2} &= GBL^2\Delta_1\sin\theta_1 + \Delta_2\sin\theta_2 L^2a^2 + \Delta_2\sin\theta_2 K^2.
\end{aligned} \tag{B8}$$

Once we have all the components, we can plug this in the SUSY constraint equations (3.22) and get the additional relations between the parameters introduced in [9]. Together with (3.15) we can finally write the precise SUSY backgrounds in the chain of geometric transitions.

[1] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985); Nucl. Phys. **B256**, 253 (1985); **B267**, 75 (1986).

[2] C. Vafa, Nucl. Phys. **B469**, 403 (1996).

[3] C. Vafa, J. Math. Phys. (N.Y.) **42**, 2798 (2001); M. Atiyah, J. M. Maldacena, and C. Vafa, J. Math. Phys. (N.Y.) **42**, 3209 (2001); F. Cachazo, K. A. Intriligator, and C. Vafa, Nucl. Phys. **B603**, 3 (2001).

[4] J. D. Edelstein, K. Oh, and R. Tatar, J. High Energy Phys. 05 (2001) 009; K. Dasgupta, K. Oh, and R. Tatar, Nucl. Phys. B610, 331 (2001); J. High Energy Phys. 08 (2002) 026; K. Dasgupta, K. h. Oh, J. Park, and R. Tatar, J. High Energy Phys. 01 (2002) 031; K. h. Oh and R. Tatar, Adv. Theor. Math. Phys. **6**, 141 (2003).

[5] S. Alexander, K. Becker, M. Becker, K. Dasgupta, A. Knauf, and R. Tatar, Nucl. Phys. **B704**, 231 (2005); M. Becker, K. Dasgupta, S. H. Katz, A. Knauf, and R. Tatar, Nucl. Phys. **B738**, 124 (2006).

[6] L. Carlevaro and D. Israel, J. High Energy Phys. 01 (2010) 083.

[7] M. Becker, K. Dasgupta, A. Knauf, and R. Tatar, Nucl. Phys. **B702**, 207 (2004).
[8] A. M. Uranga, J. High Energy Phys. 01 (1999) 022; K. Dasgupta and S. Mukhi, Nucl. Phys. **B551**, 204 (1999); J. High Energy Phys. 07 (1999) 008.
[9] F. Chen, K. Dasgupta, P. Franche, S. Katz, and R. Tatar, arXiv:1007.5316.
[10] L. A. Pando Zayas and A. A. Tseytlin, J. High Energy Phys. 11 (2000) 028.
[11] M. Cvetic, G. W. Gibbons, H. Lu, and C. N. Pope, Commun. Math. Phys. **232**, 457 (2003).
[12] R. Gwyn and A. Knauf, Rev. Mod. Phys. **80**, 1419 (2008).
[13] E. Witten, Nucl. Phys. **B460**, 541 (1996).
[14] O. J. Ganor and A. Hanany, Nucl. Phys. **B474**, 122 (1996).
[15] B. A. Ovrut, T. Pantev, and J. Park, J. High Energy Phys. 05 (2000) 045.
[16] A. Strominger, S. T. Yau, and E. Zaslow, Nucl. Phys. **B479**, 243 (1996).
[17] A. Bergman and O. J. Ganor, J. High Energy Phys. 10 (2000) 018; K. Dasgupta, O. J. Ganor, and G. Rajesh, J. High Energy Phys. 04 (2001) 034; A. Bergman, K. Dasgupta, O. J. Ganor, J. L. Karczmarek, and G. Rajesh, Phys. Rev. D **65**, 066005 (2002).
[18] I. R. Klebanov and M. J. Strassler, J. High Energy Phys. 08 (2000) 052.
[19] J. M. Maldacena and C. Nunez, Phys. Rev. Lett. **86**, 588 (2001).
[20] A. Butti, M. Grana, R. Minasian, M. Petrini, and A. Zaffaroni, J. High Energy Phys. 03 (2005) 069.
[21] M. Mia, K. Dasgupta, C. Gale, and S. Jeon, Nucl. Phys. **B839**, 187 (2010); Phys. Rev. D **82**, 026004 (2010); Phys. Lett. B **694**, 460 (2011).
[22] K. Dasgupta, M. Grisaru, R. Gwyn, S. H. Katz, A. Knauf, and R. Tatar, Nucl. Phys. **B755**, 21 (2006); K. Dasgupta, J. Guffin, R. Gwyn, and S. H. Katz, Nucl. Phys. **B769**, 1 (2007).
[23] K. Becker, M. Becker, K. Dasgupta, and S. Prokushkin, Nucl. Phys. **B666**, 144 (2003).
[24] G. Lopes Cardoso, G. Curio, G. Dall'Agata, and D. Lust, J. High Energy Phys. 10 (2003) 004; Fortschr. Phys. **52**, 483 (2004).
[25] G. Papadopoulos and A. A. Tseytlin, Classical Quantum Gravity **18**, 1333 (2001).
[26] K. Dasgupta, G. Rajesh, and S. Sethi, J. High Energy Phys. 08 (1999) 023; K. Becker and K. Dasgupta, J. High Energy Phys. 11 (2002) 006; K. Becker, M. Becker, K. Dasgupta, and P. S. Green, J. High Energy Phys. 04 (2003) 007; K. Becker, M. Becker, K. Dasgupta, P. S. Green, and E. Sharpe, Nucl. Phys. **B678**, 19 (2004).
[27] B. de Wit, D. J. Smit, and N. D. Hari Dass, Nucl. Phys. **B283**, 165 (1987).
[28] S. Ivanov and G. Papadopoulos, Phys. Lett. B **497**, 309 (2001).
[29] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, Phys. Lett. **65A**, 185 (1978).
[30] M. R. Douglas, J. Geom. Phys. **28**, 255 (1998).
[31] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B241**, 493 (1984); **B256**, 557 (1985).
[32] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, J. High Energy Phys. 08 (2004) 046.
[33] G. Lopes Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis, and G. Zoupanos, Nucl. Phys. **B652**, 5 (2003).
[34] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, and C. Vafa, Nucl. Phys. **B481**, 215 (1996); A. Sen, *ibid.* **B475**, 562 (1996); K. Dasgupta and S. Mukhi, Phys. Lett. B **385**, 125 (1996).
[35] S. Gukov, C. Vafa, and E. Witten, Nucl. Phys. **B584**, 69 (2000); **B608**, 477(E) (2001).
[36] K. Ohta and T. Yokono, J. High Energy Phys. 02 (2000) 023.
[37] D. Martelli and J. Sparks, arXiv:1010.4031; M. Beccaria, M. Kreuzer, and A. Puhm, arXiv:1010.4564.
[38] J. Maldacena and D. Martelli, J. High Energy Phys. 01 (2010) 104; N. Halmagyi, arXiv:1003.2121; L. Anguelova, F. Larsen, and R. O'Connell, J. High Energy Phys. 11 (2010) 010; L. Anguelova, C. Quigley, and S. Sethi, J. High Energy Phys. 10 (2010) 065; L. Anguelova and C. Quigley, arXiv:1007.5047.