

Higher spins and open strings: Quartic interactions

Dimitri Polyakov*

National Institute for Theoretical Physics (NITHeP) and School of Physics, University of the Witwatersrand, WITS 2050 Johannesburg, South Africa

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We analyze quartic gauge-invariant interactions of massless higher spin fields by using vertex operators constructed in our previous works and computing their 4-point amplitudes in superstring theory. The kinematic part of the quartic interactions of the higher spins is determined by the matter structure of their vertex operators; the nonlocality of the interactions is the consequence of the specific ghost structure of these operators. We compute explicitly the 4-point amplitude describing the complete gauge-invariant $1-1-3-3$ quartic interaction (two massless spin 3 particles interacting with two photons) and comment on more general $1-1-s-s$ cases, particularly pointing out the structure of $1-1-5-5$ coupling.

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I. INTRODUCTION

Constructing consistent gauge-invariant field theories of interacting higher spins is an important and fascinating problem that has attracted deep interest. Despite many efforts by the leading experts in the field and some remarkable results over recent years (for an incomplete list of references, see [1–45]), the entire subject is well known to be difficult to approach. In particular, while there has been some progress in formulating free higher spin field theories as well as those with cubic interactions, our understanding of higher order interactions (such as quartic) is still very limited.

There are many reasons why the field theories of spins greater than 2 are of interest and importance. To mention some of them, while it may not seem plausible that higher spin particles could ever be observed in a 4-dimensional world, objects such as higher spins are likely to be present in higher dimensional physics. Higher spin fields in anti-de Sitter space are known to be important ingredients in anti-de Sitter/conformal field theory correspondence [46]; in addition, constructing gauge invariant interactions of higher spins is by itself an interesting and challenging mathematical problem. String theory appears to be a particularly efficient framework for approaching the problem of higher spins. One reason for this is that the vertex operators describing the emissions of higher spins by a string appear very naturally in the massive sector of string theory (although the mass-to-spin relations for such operators are usually quite rigid, with m^2 roughly proportional to the spin value s). One could then consider the tensionless limit $\alpha' \rightarrow \infty$, in which the higher spin operators formally become massless. One faces several difficulties in this approach. First, the space-time fields coupling to the massive operators usually would lack the gauge symmetries necessary to ensure the consistency of the interactions, and

it is not clear how to recover these symmetries in the tensionless limit. Second, to recover the gauge-invariant interactions of the higher spins from string theory correlators, one generally has to consider the low energy limit of string theory, which of course is different from the tensionless limit. In our previous works [47,48], we have constructed the open string vertex operators that describe the higher spin fields with spin values from 3–9, which are massless at an arbitrary tension due to their nontrivial couplings to the $\beta - \gamma$ ghost system. The explicit expressions for these vertex operators are given by

$$\begin{aligned} V_{s=3}(p) &= H_{a_1 a_2 a_3}(p) c e^{-3\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\tilde{p}\tilde{X}} \\ V_{s=4}(p) &= H_{a_1 \dots a_4}(p) c \eta e^{-4\phi} \partial X^{a_1} \partial X^{a_2} \partial \psi^{a_3} \psi^{a_4} e^{i\tilde{p}\tilde{X}} \\ V_{s=5}(p) &= H_{a_1 \dots a_5}(p) c e^{-4\phi} \partial X^{a_1} \dots \partial X^{a_3} \partial \psi^{a_4} \psi^{a_5} e^{i\tilde{p}\tilde{X}} \\ V_{s=6}(p) &= H_{a_1 \dots a_6}(p) c \eta e^{-5\phi} \partial X^{a_1} \dots \partial X^{a_3} \partial^2 \psi^{a_4} \partial \psi^{a_5} \psi^{a_6} e^{i\tilde{p}\tilde{X}} \\ V_{s=7}(p) &= H_{a_1 \dots a_7}(p) c e^{-5\phi} \partial X^{a_1} \dots \partial X^{a_4} \partial^2 \psi^{a_5} \partial \psi^{a_6} \psi^{a_7} e^{i\tilde{p}\tilde{X}} \\ V_{s=8}(p) &= H_{a_1 \dots a_8}(p) c \eta e^{-5\phi} \partial X^{a_1} \dots \partial X^{a_7} \psi^{a_8} e^{i\tilde{p}\tilde{X}} \\ V_{s=9}(p) &= H_{a_1 \dots a_9}(p) c e^{-5\phi} \partial X^{a_1} \dots \partial X^{a_8} \psi^{a_9} e^{i\tilde{p}\tilde{X}}, \end{aligned} \quad (1)$$

where X^a and ψ^a are the Ramond-Neveu-Schwarz formulation of superstring theory (RNS) world sheet bosons and fermions ($a = 0, \dots, d-1$), and the ghost fields are bosonized according to

$$\begin{aligned} b &= e^{-\sigma}, & c &= e^{\sigma} \\ \gamma &= e^{\phi-\chi} \equiv e^{\phi} \eta & \beta &= e^{\chi-\phi} \partial \chi \equiv \partial \xi e^{-\phi} \end{aligned} \quad (2)$$

The operators (1) are picture inequivalent and are the elements of ghost cohomologies H_{-3} , H_{-4} , and H_{-5} . All the expressions for the operators (1) are given at their minimal negative superconformal ghost pictures (e.g. -3 for $s=3$ and -5 for $s=9$) at which they are annihilated by the direct picture-changing transformation. The symmetric tensors $H_{a_1 \dots a_s}(p)$ describe massless higher spin

*dimitri.polyakov@wits.ac.za

fields in space-time, with the spin values $3 \leq s \leq 9$. The equations of motion and the gauge symmetry transformations follow from the Becchi-Rouet-Stora-Tyutin charge or transformation on strings (BRST) constraints on the operators (1) [47]. Namely, the on-shell Fierz-Pauli constraints

$$\begin{aligned} H_{a_1 a_3 \dots a_s}^{a_1} &= 0 \\ p^{a_1} H_{a_1 \dots a_s} &= 0 \\ p^2 H_{a_1 \dots a_s} &= 0 \end{aligned} \quad (3)$$

follow from the invariance condition $\{Q, V_s\} = 0$, where

$$Q = Q_1 + Q_2 + Q_3 \quad (4)$$

is the BRST operator with

$$\begin{aligned} Q_1 &= \oint \frac{dz}{2i\pi} \{cT - bc\partial c\}, \\ Q_2 &= -\frac{1}{2} \oint \frac{dz}{2i\pi} \gamma \psi_a \partial X^a, \\ Q_3 &= -\frac{1}{4} \oint \frac{dz}{2i\pi} b \gamma^2. \end{aligned} \quad (5)$$

The BRST nontriviality conditions, in turn, entail the gauge symmetry transformations for the higher spins [48]. For the symmetric tensors, the transformations are given by

$$H_{a_1 \dots a_s}(p) \rightarrow H_{a_1 \dots a_s}(p) + p_{(a_1} \Lambda_{a_2 \dots a_s)}(p) \quad (6)$$

(where Λ is also traceless and symmetric) as, under the shift of symmetric H -tensors by symmetrized derivatives of Λ , the vertex operators (1) are shifted by the terms not contributing to correlation functions. Therefore, the gauge invariance of the interaction terms for the higher spins, obtained in the field theory limit of string theory, is ensured by construction, since the structure of these terms is entirely determined by the correlation functions in string theory. For detailed BRST analysis of the operators (1), see [47]. Below, we shall briefly review the relation between BRST constraints, equations of motion, and gauge symmetries on the example of the $s = 3$ operator. (The $s > 3$ cases are treated similarly.) The vertex operator for $s = 3$ is given by:

$$V_{s=3}(p) = H_{abc}(p) c e^{-3\phi} \partial X^a \partial X^b \psi^c e^{i\vec{p} \cdot \vec{X}}. \quad (7)$$

This operator commutes with Q_2 and Q_3 of the BRST charge. To commute with Q_1 , it has to be a dimension 0 primary, i.e. its operator product expansion (OPE) with stress-energy tensor must not contain singularities stronger than a simple pole. This entails constraints on the rank 3 H -tensor. For general H , the OPE contains singularities up to quartic pole, so, to ensure the commutation with Q_1 , the coefficients in front of quartic, triple, and double poles must vanish separately. This leads to tracelessness, transversality, and masslessness conditions, respectively, i.e. to the Fierz-Pauli constraints (3) on H . At the same time, the shift (6) shifts the operator (7) by terms not contributing to

correlation functions. To see this, consider the general (not necessarily symmetric) tensor $H_{a|bc}$. (Note that the form of constraints (3) following from BRST-invariance arguments does not depend on the symmetric properties of H and remains the same.) Under the shift $H_{a|bc}(p) \rightarrow H_{a|bc}(p) + p_c \Lambda_{ab}(p)$, where Λ is symmetric and traceless, the operator (7) is shifted by the BRST-exact part,

$$\begin{aligned} &\sim c e^{-3\phi} (\vec{p} \cdot \vec{\psi}) \Lambda_{ab} \partial X^a \partial X^b e^{i\vec{p} \cdot \vec{X}} \sim \{Q, c e^{\chi-4\phi} \partial \chi (\vec{p} \cdot \vec{\psi}) \\ &\quad \times (\vec{\psi} \cdot \vec{X}) \Lambda_{ab} \partial X^a \partial X^b e^{i\vec{p} \cdot \vec{X}}\}, \end{aligned} \quad (8)$$

which insertion to any correlator is zero. On the other hand, the tensor $p_c \Lambda_{ab}$ can be decomposed as

$$p_c \Lambda_{ab} = \frac{1}{2} (p_{(c} \Lambda_{ab)} + p_{[c} \Lambda_{a]b}), \quad (9)$$

and insertions of operators corresponding to different Young tableau to correlation functions vanish separately. As a matter of fact, vanishing of $p_{[c} \Lambda_{a]b}$ -type insertions to correlators is just a particular example of a general property of S -matrix elements of $s = 3$ vertex operators coupling 3-tensors with hooklike Young diagrams—It can be shown that such operators do not contribute to S -matrices, which is reminiscent of what happens in the framelike approach [11,42,49–51], where contributions with hooklike symmetries are eliminated by algebraic constraints.

Therefore, the correlators are invariant under shifting symmetric tensor H_{abc} by symmetrized derivative of Λ , implying the gauge symmetry (6) in the field theory limit. In order to compute the correlation functions involving the operators (1), one also needs their representations in dual positive ghost pictures. In order to obtain the positive picture presentation for elements of H_{-n-2} (physical operators existing at minimal negative picture $-n-2$ and below; $n = 1, 2, \dots$), one has to replace $e^{-(n+2)\phi}$ with $e^{n\phi}$ (without changing the matter part) and perform the homotopy transformation using the K -operator [52]. Namely, if a higher spin vertex at minimal negative picture $-n-2$ has the structure

$$V_{-n-2} = c e^{-(n+2)\phi} F_{(n^2/2)+n+1}(X, \psi), \quad (10)$$

where $F_{(n^2/2)+n+1}(X, \psi)$ is the matter primary field of conformal dimension $\frac{n^2}{2} + n + 1$, one starts with the operator

$$\oint V_n \equiv \oint dz e^{n\phi} F_{(n^2/2)+n+1}(X, \psi). \quad (11)$$

This charge commutes with Q_1 since it is a world sheet integral of dimension 1 and $b - c$ ghost number zero but it does not commute with Q_2 and Q_3 . To make it BRST-invariant, one has to add the correction terms by using the following procedure [52,53]. We write

$$[Q_{\text{brst}}, V_n(z)] = \partial U(z) + W_1(z) + W_2(z), \quad (12)$$

and therefore

$$[Q_{\text{brst}}, \oint dz V_n] = \oint dz (W_1(z) + W_2(z)), \quad (13)$$

where

$$\begin{aligned} U(z) &\equiv cV_n(z), & [Q_1, V_n] &= \partial U, \\ W_1 &= [Q_2, V_n], & W_2 &= [Q_3, V_n]. \end{aligned} \quad (14)$$

Let us introduce the dimension 0 K -operator,

$$K(z) = -4ce^{2\chi-2\phi}(z) \equiv \xi\Gamma^{-1}(z), \quad (15)$$

satisfying

$$\{Q_{\text{brst}}, K\} = 1. \quad (16)$$

It is easy to check that this operator has a nonsingular operator product with W_1 :

$$K(z_1)W_1(z_2) \sim (z_1 - z_2)^{2n}Y(z_2) + O((z_1 - z_2)^{2n+1}), \quad (17)$$

where Y is some operator of dimension $2n + 1$. Then the complete BRST-invariant operator can be obtained from $\oint dz V_n(z)$ by the following transformation:

$$\begin{aligned} \oint dz V_n(z) &\rightarrow A_n(w) \\ &= \oint dz V_n(z) + \frac{1}{(2n)!} \oint dz (z - w)^{2n} :K \partial^{2n} (W_1 + W_2):(z) \\ &\quad + \frac{1}{(2n)!} \oint dz \partial_z^{2n+1} [(z - w)^{2n} K(z)] K\{Q_{\text{brst}}, U\}, \end{aligned} \quad (18)$$

where w is some arbitrary point on the world sheet. It is then straightforward to check the invariance of A_n by using some partial integration along with the relation (34) as well as the obvious identity:

$$\{Q_{\text{brst}}, W_1(z) + W_2(z)\} = -\partial(\{Q_{\text{brst}}, U(z)\}). \quad (19)$$

Although the invariant operators $A_n(w)$ depend on an arbitrary point w on the world sheet, this dependence is irrelevant in the correlators since all the w derivatives of A_n

are BRST exact—The triviality of the derivatives ensures that there will be no w -dependence in any correlation functions involving A_n . An alternative (yet technically more complicated) method to obtain the positive picture representations for the higher spin operators is to use sequences of Z -transformations combined with picture changing [52]. Namely, introduce the Z -operator, transforming the $b - c$ pictures (in particular, mapping integrated vertices to unintegrated) given by [54],

$$\begin{aligned} Z(w) &= b\delta(T)(w) \\ &= \oint dz (z - w)^3 (bT + 4c\partial\xi\xi e^{-2\phi}T^2)(z), \end{aligned} \quad (20)$$

where T is the full stress-energy tensor in RNS theory. The usual picture-changing operator, transforming the $\beta - \gamma$ ghost pictures, is given by $\Gamma(w) = :\delta(\beta)G:(w) = :e^\phi G:(w)$. Now we introduce the *integrated* picture-changing operators $R_n(w)$ according to

$$R_n(w) = Z(w):\Gamma^n:(w), \quad (21)$$

where $:\Gamma^n:$ is the n th power of the standard picture-changing operator:

$$\begin{aligned} :\Gamma^n:(w) &= :e^{n\phi} \partial^{n-1} G \cdots \partial G G:(w) \\ &\equiv :\partial^{n-1} \delta(\beta) \cdots \partial \delta(\beta) \delta(\beta):. \end{aligned} \quad (22)$$

Then the positive picture representations for the higher spin operators A_n can be obtained from the negative ones, V_{-n-2} (1), by the transformation

$$A_n(w) = (R_2)^{n+1}(w)V_{-n-2}(w). \quad (23)$$

Since both Z and Γ are BRST-invariant and nontrivial, the A_n -operators by construction satisfy the BRST-invariance and nontriviality conditions identical to those satisfied by their negative picture counterparts V_{-2n-2} and therefore lead to the same Pauli-Fierz on-shell conditions (3) and the gauge symmetries (6) for the higher spin fields. For the $s = 3$ operator, the above procedure gives

$$\begin{aligned} V_{s=3} &= ce^{-3\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}} H_{a_1 a_2 a_3}(p) \rightarrow \oint dz V_1 = H_{a_1 a_2 a_3}(p) \oint e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}} \\ [Q_1, V_1] &= \partial U = H_{a_1 a_2 a_3}(p) \partial (ce^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}) \\ [Q_2, V_1] &= W_1 = \frac{1}{2} H_{a_1 a_2 a_3}(p) e^{2\phi-\chi} \{(-\vec{\psi} \partial \vec{X}) + i(\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} + i(\vec{p} \partial \vec{\psi}) \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}} \\ &\quad + \partial X^{a_1} (\partial^2 \psi^{a_2} + 2\partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} - \partial X^{a_1} \partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)})\} e^{i\vec{p}\vec{X}} \\ [Q_3, V_1] &= W_2 = -\frac{1}{4} H_{a_1 a_2 a_3}(p) e^{3\phi-2\chi} P_{2\phi-2\chi-\sigma}^{(1)} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}, \end{aligned} \quad (24)$$

where the conformal weight n polynomials in the derivatives of the ghost fields ϕ, χ, σ are defined according to [52,53]

$$P_{f(\phi, \chi, \sigma)}^{(n)} = e^{-f(\phi(z), \chi(z), \sigma(z))} \frac{\partial^n}{\partial z^n} e^{f(\phi(z), \chi(z), \sigma(z))}, \quad (25)$$

where f is some linear function in ϕ, χ, σ . For example, $P_{\phi-\chi}^{(1)} = \partial\phi - \partial\chi$, etc. Note that the product (43) is defined in the algebraic sense (not as an operator product).

Accordingly,

$$\begin{aligned} :K\partial^2 W_1: &= 4H_{a_1 a_2 a_3}(p) c \xi \{(-(\vec{\psi} \partial \vec{X}) + i(\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} + i(\vec{p} \partial \vec{\psi})) \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}} \\ &\quad + \partial X^{a_1} (\partial^2 \psi^{a_2} + 2\partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} - \partial X^{a_1} \partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)})\} e^{i\vec{p} \vec{X}} \\ :K\partial^2 W_2: &= H_{a_1 a_2 a_3}(p) \{-\partial^2 (e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}) + P_{2\phi-2\chi-\sigma}^{(2)} e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}\} \end{aligned} \quad (26)$$

and

$$:\partial^{2n+1} KK\{Q_{\text{brst}}, U\}: = -24H_{a_1 a_2 a_3}(p) \partial c c \partial \xi \xi e^{-\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}} : \partial^m KK\{Q_{\text{brst}}, U\}: = 0 (m < 2n + 1), \quad (27)$$

and therefore, upon integrating out total derivatives, the complete BRST-invariant expression for the $s = 3$ operator at picture 1 is

$$\begin{aligned} A_{s=3}(w) &= H_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 \left\{ \frac{1}{2} P_{2\phi-2\chi-\sigma}^{(2)} e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} + 2c \xi [(-(\vec{\psi} \partial \vec{X}) + i(\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} \right. \\ &\quad + i(\vec{p} \partial \vec{\psi})) \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}} + \partial X^{a_1} (\partial^2 \psi^{a_2} + 2\partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} - \partial X^{a_1} \partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)})] \\ &\quad \left. - 12\partial c c \partial \xi \xi e^{-\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} \right\} e^{i\vec{p} \vec{X}}. \end{aligned} \quad (28)$$

To abbreviate notations for our calculations of the correlation functions in the following sections, it is convenient to write the vertex operator $A_{s=3}$ (46) as a sum,

$$A_{s=3} = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \quad (29)$$

where

$$A_0(w) = \frac{1}{2} H_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 P_{2\phi-2\chi-\sigma}^{(2)} e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}(z) \quad (30)$$

and

$$A_6(w) = -12H_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 \partial c c \partial \xi \xi e^{-\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}(z) \quad (31)$$

have ghost factors proportional to e^ϕ and $\partial c c \partial \xi \xi e^{-\phi}$, respectively, and the rest of the terms carry ghost factors proportional to $c\xi$:

$$\begin{aligned} A_1(w) &= -2H_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 c \xi (\vec{\psi} \partial \vec{X}) \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}(z) \\ A_2(w) &= 2H_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 c \xi (\partial^2 \psi^{a_2} + 2\partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} e^{i\vec{p} \vec{X}}(z) \\ A_3(w) &= -2H_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 c \xi \partial X^{a_1} \partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)}) e^{i\vec{p} \vec{X}}(z) \\ A_4(w) &= 2iH_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 c \xi (\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}(z) \\ A_5(w) &= 2iH_{a_1 a_2 a_3}(p) \oint dz(z-w)^2 c \xi (\vec{p} \partial \vec{\psi}) \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p} \vec{X}}(z). \end{aligned} \quad (32)$$

We are now prepared to analyze the four-point $1-1-3-3$ amplitude (leading to the gauge-invariant quartic interaction of spin 3 and spin 1 particles), which will be computed in the next sections.

II. $1-1-3-3$ QUARTIC INTERACTIONS—PRELIMINARIES

The goal of the next two sections is to compute the 4-point function of two $s = 3$ vertex operators with 2-photons, describing the gauge-invariant $1-1-3-3$ interactions in the low energy limit of string theory. The

photon vertex operators are the standard ones, and it is convenient to take them unintegrated at superconformal pictures -1 and -2 :

$$\begin{aligned} V_{s=1}^{(-1)}(p) &= c e^{-\phi} \psi^m e^{i\vec{p} \vec{X}} A_m(p) \\ V_{s=1}^{(-2)}(p) &= c e^{-2\phi} \partial X^m e^{i\vec{p} \vec{X}} A_m(p). \end{aligned} \quad (33)$$

To cancel the background charges, the operators in the 4-point $3-3-1-1$ amplitude must be chosen to have total $b-c$ ghost number $+3$, ϕ ghost number -2 , and χ ghost number $+1$. Therefore, with the picture choice (33)

for the photons, it is clear that both of the $s = 3$ operators have to be taken at their positive picture $+1$ representation (32). It is furthermore clear that the amplitude $A(1 - 1 - 3 - 3)(p_1, \dots, p_4)$ will only be contributed by the terms

$$\begin{aligned}
 & A(1 - 1 - 3 - 3)(p_1, \dots, p_4) \\
 &= S(1 - 1 - 3 - 3)(p_1, \dots, p_4) \\
 &\quad + (p_3 \leftrightarrow p_4)S(1 - 1 - 3 - 3) \\
 &\equiv \langle V_{s=1}(p_1)V_{s=1}(p_2)V_{s=3}(p_3)V_{s=3}(p_4) \rangle \\
 &= \sum_{j=1}^5 \langle V_{s=1}(p_1)V_{s=1}(p_2)A_j(p_3)A_0(p_4) \rangle + (p_3 \leftrightarrow p_4),
 \end{aligned} \tag{34}$$

with A_0, A_j given in (32). Note that, with the picture choice (33) for the $s = 1$ operators, the A_6 part of $V_{s=3}$ at positive picture does not contribute to the correlator at all due to the ghost balance constraint. The structure of the amplitude (34) is remarkably different from the standard Veneziano form. Recall that the standard Veneziano expression for 4-point amplitude in string theory arises as a result of 3 out of 4 operators taken unintegrated (multiplied by the c ghosts) and one integrated (with the $b - c$ ghost number 0), in order to ensure the $b - c$ ghost anomaly cancellation. (This choice is related to fixing the $SL(2, R)$ global symmetry with the ghost part of the correlator producing the standard Koba-Nielsen's determinant.) The single integration then leads to the Veneziano structures $\sim \frac{\Gamma}{\Gamma}$ in the open string case or $\sim \frac{\Gamma}{\Gamma}$ for closed strings where Γ are the gamma-functions of Mandelstam variables. With the $s = 3$ vertex operators, the structure of amplitudes is different, as their ghost couplings (both $b - c$ and $\beta - \gamma$) are non-standard. For example, the $s = 3$ operators at positive pictures exist in the integrated form only (unlike the standard operators that can be taken integrated or unintegrated); at the same time, their integrands contain terms with $b - c$ ghost numbers 1 and 2 (as opposed to the standard integrated vertices for which the integrands have ghost number zero). As is clear from (32)–(34), the ghost number balance of the $1 - 1 - 3 - 3$ 4-point function requires both of the $s = 3$ operators to be taken integrated at positive pictures. Therefore, the 4-point amplitude involves the double world sheet integration, and its form is quite different from Veneziano type. In particular, it leads to nonlocalities appearing in the quartic interactions involving the higher spins. Our goal now is to analyze the $\langle VVA_jA_0 \rangle$ correlators contributing to the 4-point amplitude (34) one by one. The first step is to fix the points u_1, u_2, w_1, w_2 in the amplitude $\langle V_{s=1}(u_1)V_{s=1}(u_2)A_{s=3}(w_1)A_{s=3}(w_2) \rangle$ by using the remnant gauge symmetry on the world sheet. Note that, while u_1, u_2 are the actual points of the unintegrated $s = 1$ vertices, w_1 and w_2 are the points defining the contours for the integrated $s = 3$ vertices at positive

pictures (corresponding to the w -points in the expression (32) for the $A_{s=3}$ vertex). In the standard N -point amplitude case (involving 3 unintegrated vertices and $N - 3$ integrated), the remnant gauge symmetry is well known to be given by $SL(2, R)$ subgroup of conformal symmetry, allowing us to fix the locations of the unintegrated operators at 3 particular points (with the standard choice 0, 1 and ∞). In the operator language, the $SL(2, R)$ symmetry simply reflects the fact that translating an unintegrated vertex operator of the form $\sim cV(z_1)$ to some new point z_2 changes it by BRST-exact terms not contributing to correlation functions (since all the z -derivatives of the unintegrated vertices are BRST exact, e.g. $\partial(cV)(z) = [Q, V(z)]$ etc. In our case, because of the nonstandard ghost structure of the spin 3 operators, the situation is different, and the actual remnant gauge symmetry is bigger than $SL(2, R)$. Namely, all w -derivatives of the $A_{s=3}(w)$ operators are BRST exact, so the w -points can be chosen arbitrarily. So, in case of the 4-point amplitude (34), the remnant gauge symmetry on the world sheet allows us to fix 4 rather than 3 points, i.e. it contains an extra generator in addition to the standard $SL(2, R)$ part. As has been pointed out in [52], the extra gauge symmetries on the world sheet are closely related to the *global* space-time α -symmetries that are realized nonlinearly and stem from hidden space-time dimensions in string theory. Just like the higher spin vertices, the α -symmetry generators are essentially mixed with the ghosts, being the elements of nontrivial ghost cohomologies $H_{-3} \sim H_1$, $H_{-4} \sim H_2$, and $H_{-5} \sim H_3$, with each cohomology essentially contributing an extra space-time dimension. In this context, as the higher spin vertex operators and the α -symmetries have similar ghost cohomology structures, the appearance of extra gauge symmetries on the world sheet is not surprising. Therefore, using the $SL(2, R)$ symmetries plus the extra symmetry, it is convenient to set

$$z_1 = 0, \quad z_2 = \infty \quad w_1 = w_2 = 0. \tag{35}$$

Such a choice may appear somewhat unusual; indeed, in the standard case, the unintegrated vertices are set at three different points (e.g. such as 0, 1, ∞), since, if one formally fixes two operators at coincident (or infinitely close) points, one faces the normal ordering issue (although the $SL(2, R)$ symmetry in principle allows us to fix the operators at 3 infinitely close points). In the case of the higher spin operators, however, fixing the w -points is merely related to the choice of their integration contours; thus the gauge choice (35) is appropriate.

III. $1 - 1 - 3 - 3$ AMPLITUDE: THE CALCULATION

It is convenient to start with evaluating the ghost part of the 4-point function, common for all the terms in (34). We get

$$F_{\text{gh}}(u, z_1, z_2) = \lim_{u \rightarrow \infty} \langle c e^{-2\phi}(0) c e^{-\phi}(u) c e^{\chi}(z_1) P_{2\phi-2\chi-\sigma}^{(2)} e^{\phi}(z_2) \rangle = \frac{6u z_1 (z_1^2 + z_2^2)}{(z_1 - z_2)^2}. \quad (36)$$

The first contribution is given by

$$\begin{aligned} S_1(1-1-3-3) &= \langle V_{s=1}(p_1; 0) V_{s=1}(p_2; \infty) A_1(p_3; 0) A_0(p_4, 0) \rangle = A_m(p_1) A_n(p_2) H_{a_2 a_3 a_4}(p_3) H_{b_1 b_2 b_3}(p_4) \\ &\times \lim_{u \rightarrow \infty} \int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 F_{\text{gh}}(u, z_1, z_2) \times \langle \partial X^m e^{i\tilde{p}_1 \tilde{X}}(0) \psi^n e^{i\tilde{p}_2 \tilde{X}}(u) (\psi_{a_1} \partial X^{a_1}) \psi^{a_4} \partial X^{a_2} \partial X^{a_3} \\ &\times e^{i\tilde{p}_3 \tilde{X}} \partial X^{b_1} \partial X^{b_2} \psi^{b_3} e^{i\tilde{p}_4 \tilde{X}} \rangle. \end{aligned} \quad (37)$$

The ψ -correlator gives

$$\begin{aligned} &\lim_{u \rightarrow \infty} \langle \psi^n(u) \psi^{a_1} \psi^{a_4}(z_1) \psi^{b_3}(z_2) \rangle \\ &= \frac{\eta^{na_1} \eta^{a_4 b_3} - \eta^{na_4} \eta^{a_1 b_3}}{u(z_1 - z_2)}, \end{aligned} \quad (38)$$

so the ψ -correlator multiplied by $F_{\text{gh}}(u, z_1, z_2)$ gives $[6(\eta^{na_1} \eta^{a_4 b_3} - \eta^{na_4} \eta^{a_1 b_3}) z_1 (z_1 + z_2)^2] / [(z_1 - z_2)^3]$ with the u -factor canceled. Because of conformal invariance, it is clear that the remaining X -correlator will contribute terms of the order of u^0 to the overall correlator, with all other terms vanishing on-shell—In other words, no pairings of ∂X 's with $e^{i\tilde{p}_2 \tilde{X}}$ contribute to the overall 4-point amplitude. For this reason, the relevant contributions from the X -correlator are reduced to the 3-point function

$$\begin{aligned} S_X &= \langle \partial X^m e^{i\tilde{p}_1 \tilde{X}}(0) \partial X^{a_1} \partial X^{a_2} \partial X^{a_3} e^{i\tilde{p}_3 \tilde{X}}(z_1) \\ &\times \partial X^{b_1} \partial X^{b_2} \partial X^{b_3} e^{i\tilde{p}_4 \tilde{X}}(z_2) \rangle. \end{aligned} \quad (39)$$

This function is not difficult to evaluate. To keep our expressions as compact as possible for the subsequent integrations in z_1, z_2 , it is convenient to use the following notations for computing the X -correlators.

Namely, each term contributing to the correlator (39) can be classified in terms of numbers of pairings between

∂X 's with the exponents and between each other. That is, let M_1, M_2 be pairing numbers between $\partial X_m(0)$ and $e^{i\tilde{p}_3 \tilde{X}}(z_1), e^{i\tilde{p}_4 \tilde{X}}(z_2)$, respectively, with the obvious constraint $0 \leq M_1, M_2 \leq 1$ (since there is only one ∂X in the expression for the photon). Next, let N_1, N_2 be pairing numbers of ∂X 's in the $s = 3$ operator at z_1 , with $e^{i\tilde{p}_1 \tilde{X}}(0)$ and $e^{i\tilde{p}_4 \tilde{X}}(z_2)$ with $0 \leq N_1, N_2 \leq 3$. Finally, let P_1, P_2 satisfying $0 \leq P_1, P_2 \leq 2$ stand for the pairings between ∂X 's of the second $s = 3$ vertex at z_2 with $e^{i\tilde{p}_1 \tilde{X}}(0)$ and $e^{i\tilde{p}_3 \tilde{X}}(z_1)$. It is then straightforward to show that the correlator is contributed by two types of terms. The first type includes the kinematic factors sextic in momentum (accordingly, leading to six derivative interactions in the low energy limit). These terms appear when all ∂X 's in the correlator (39) (total number 6) are contracted with the exponents. The second type involves the kinematic factors quartic in momentum, appearing when 4 out of 6 ∂X 's are contracted with the exponents, while the remaining 2 are contracted with each other. These terms lead to four derivative quartic interactions in space-time. Given the Pauli-Fierz conditions (3) on the $s = 3$ fields, there are no terms quadratic in momentum or momentum-independent.

Computing the X -correlator (39) and multiplying by the ψ -ghost factor (38), we obtain the six-derivative part of the correlator $S_1^{6\text{-der}}(1-1-3-3)$ (37), given by

$$\begin{aligned} S_1^{6\text{-der}}(1-1-3-3) &= 72 A_m(p_1) A_n(p_2) H_{a_2 a_3 a_4}(p_3) H_{b_1 b_2 b_3}(p_4) \times (\eta^{na_1} \eta^{a_4 b_3} - \eta^{na_4} \eta^{a_1 b_3}) \\ &\times \sum_{M_1=0}^1 \sum_{N_1=0}^3 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{N_1! (3-N_1)! P_1! (2-P_1)!} \times \prod_{\alpha=1}^{n_1} \prod_{\beta=N_1+1}^3 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^3 (ip_1)^{a_\alpha} (ip_4)^{a_\beta} (ip_1)^{b_\gamma} (ip_3)^{b_\lambda} \\ &\times (ip_3^m)^{M_1} (ip_4^m)^{1-M_1} \times \int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 (z_1^2 + z_2^2) z_1^{1+\tilde{p}_1 \tilde{p}_3 - M_1 - N_1} z_2^{\tilde{p}_1 \tilde{p}_4 - 1 + M_1 - P_1} \\ &\times (z_1 - z_2)^{\tilde{p}_3 \tilde{p}_4 - 8 + N_1 + P_1}. \end{aligned} \quad (40)$$

A few comments should be made to explain our notations here and below. First, regarding the products appearing in (40): For example, $\prod_{\alpha=1}^{N_1} (ip_1)^{a_\alpha}$ stands for the usual product $(ip_1^{a_1}) \dots (ip_1^{a_{N_1}})$ for $1 \leq N_1 \leq 3$, but it is set to 1 if $N_1 = 0$. Similarly, $\prod_{\beta=N_1+1}^3 (ip_4^{a_\beta})$ stands for the product $ip_4^{a_{N_1+1}} \dots ip_4^{a_3}$ if $N_1 = 0, 1, 2$, but it is set to 1 if $N_1 = 3$, and similarly for all other products of that type. The product $(ip_3^m)^{M_1} (ip_4^m)^{1-M_1}$ obviously stands for ip_3^m for $M_1 = 0$ and ip_4^m for $M_1 = 1$ and similarly for all other products of that type. The next step is to perform the integration of (37) in z_1 and z_2 . This can be done by using

$$\int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^a z_2^b (z_1 - z_2)^c = \frac{\Gamma(a+1)\Gamma(c+1)}{(a+b+c+2)\Gamma(a+c+2)}. \quad (41)$$

Integrating (37) then gives the following answer for $S_1^{6\text{-der}}(1-1-3-3)$:

$$\begin{aligned} S_1^{6\text{-der}}(1-1-3-3) &= 72A_m(p_1)A_n(p_2)H_{a_2a_3a_4}(p_3)H_{b_1b_2b_3}(p_4) \times (\eta^{na_1}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_1b_3}) \\ &\times \sum_{M_1=0}^1 \sum_{N_1=0}^3 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{N_1!(3-N_1)!P_1!(2-P_1)!} \times \prod_{\alpha=1}^{n_1} \prod_{\beta=N_1+1}^3 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^3 (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1)^{b_\gamma} \\ &\times (ip_3)^{b_\lambda}(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(\vec{p}_3\vec{p}_4 - \vec{p}_1\vec{p}_2)^{-1} \times \Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \\ &\times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 6)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 3)} \right]. \end{aligned} \quad (42)$$

We find that the expression (42) contains the factor $G(p_1, p_2, p_3, p_4) = (\vec{p}_3\vec{p}_4 - \vec{p}_1\vec{p}_2)^{-1}$. [This factor will actually appear in all the terms in the 4-point amplitude (34).] If we are on-shell, the denominator in this expression is zero and the correlator (42) diverges. It must be stressed, however, that terms in the low energy effective action, appearing in the field theory limit of string theory, are determined by appropriate terms in conformal beta-functions on the world sheet, rather than by the on-shell correlators. The conformal beta-function, in turn, is determined by the structure constants that are essentially taken off-shell. (The on-shell limit then corresponds to the constraint $\beta = 0$.) For example, if Φ is a scalar massless space-time field, to obtain the linear term in its β -function proportional to $\sim \Delta\Phi = -p^2\Phi$ (corresponding to the free field part of its low energy effective action), one has to take the dilaton's vertex operator initially off-shell (so that $p^2 \neq 0$) and perform the internal normal ordering in this vertex operator, leading to the flow $\sim p^2\Phi \log\Lambda$,

where Λ is the world sheet cutoff. Similarly, the denominator of $G(p_1, \dots, p_4)$ is nonzero in the off-shell case relevant to the β -function computations, so the corresponding quartic terms in the low energy effective action include the factor

$$G(p_1, p_2, p_3, p_4) = (p_1^2 + p_2^2 - p_3^2 - p_4^2)^{-1}, \quad (43)$$

where we used $(p_1 + p_2)^2 = (p_3 + p_4)^2$. This is the factor reflecting the nonlocality of the quartic couplings of the higher spin fields in the position space. We find that, from the string theory point of view, this nonlocality is the consequence of the specific ghost structure of the higher spin vertex operators, as we already noted above. The calculation of the 4-*derivative* part of the correlator (37) (quartic in momentum) is similar. The result is given by

$$S_1^{4\text{-der}}(1-1-3-3) = D_1 + D_2 + D_3, \quad (44)$$

where

$$\begin{aligned} D_1 &= -72A_m(p_1)A_n(p_2)H_{a_1a_2a_4}(p_3)H_{b_1b_2b_3}(p_4)(\eta^{na_3}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_3b_3})\eta^{ma_3} \times \sum_{N_1=0}^2 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{N_1!(2-N_1)!P_1!(2-P_1)!} \\ &\times \prod_{\alpha=1}^{n_1} \prod_{\beta=N_1+1}^3 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^3 (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1)^{b_\gamma}(ip_3)^{b_\lambda} \times G(p_1, p_2, p_3, p_4)\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \\ &\times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 4)} \right], \end{aligned} \quad (45)$$

$$\begin{aligned} D_2 &= -72A_m(p_1)A_n(p_2)H_{a_1a_2a_4}(p_3)H_{b_1b_2b_3}(p_4)(\eta^{na_3}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_3b_3})\eta^{mb_2} \sum_{N_1=0}^3 \sum_{P_1=0}^1 \frac{(-1)^{P_1}}{N_1!(3-N_1)!} \\ &\times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^3 (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1} \times G(p_1, p_2, p_3, p_4)\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \\ &\times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 6)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 2)} \right], \end{aligned} \quad (46)$$

$$\begin{aligned}
D_3 = & -72A_m(p_1)A_n(p_2)H_{a_1a_2a_4}(p_3)H_{b_1b_2b_3}(p_4) \times (\eta^{na_3}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_3b_3})\eta^{mb_2} \sum_{M_1=0}^1 \sum_{N_1=0}^2 \sum_{P_1=0}^1 \frac{(-1)^{P_1}}{N_1!(2-N_1)!} \\
& \times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^3 (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1} \times G(p_1, p_2, p_3, p_4)\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \\
& \times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 6)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 3)} \right]. \tag{47}
\end{aligned}$$

This concludes the computation of the $S_1(1-1-3-3)$ contribution to the quartic interaction of $1-1-3-3$.

The next contribution, $S_2(1-1-3-3)$, is given by

$$\begin{aligned}
S_2(1-1-3-3) = & \langle V_{s=1}(p_1;0)V_{s=1}(p_2;\infty)A_2(p_3;0)A_0(p_4,0) \rangle \\
= & A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \lim_{u \rightarrow \infty} \int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 \times \langle ce^{-2\phi} \partial X^m e^{i\vec{p}_1\vec{X}}(0) ce^{-\phi} \psi^n e^{i\vec{p}_2\vec{X}}(u) \\
& \times ce^{\chi} \partial X^{a_1} \psi^{a_2} (\partial^2 \psi^{a_3} + 2\partial \psi^{a_3} P_{\phi-\chi}^{(1)}) e^{i\vec{p}_3\vec{X}}(z_3) P_{2\phi-2\chi-\sigma}^{(2)} e^{\phi} \partial X^{b_1} \partial X^{b_2} \psi^{b_3} e^{i\vec{p}_4\vec{X}}(z_2) \rangle. \tag{48}
\end{aligned}$$

The $\langle \psi \times \text{ghost} \rangle$ factor of this contribution is

$$\lim_{u \rightarrow \infty} \langle ce^{-2\phi}(0) ce^{-\phi} \psi^n(u) ce^{\chi} \psi^{a_2} (\partial^2 \psi^{a_3} + 2\partial \psi^{a_3} P_{\phi-\chi}^{(1)})(z_1) e^{\phi} P_{2\phi-2\chi-\sigma}^{(2)} \psi^{b_3}(z_2) \rangle = \frac{24\eta^{na_2}\eta^{a_3b_3}z_1z_2^3(z_1^2+z_2^2)}{(z_1-z_2)^3} + O(u^{-1}). \tag{49}$$

Computing the $\langle X \rangle$ part using the same conventions as above and integrating the overall correlator over z_1 and z_2 , we obtain

$$\begin{aligned}
S_2(1-1-3-3) = & 48A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{na_2}\eta^{a_3b_3} \sum_{M_1=0}^1 \sum_{N_1=0}^1 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{P_1!(2-P_1)!} \times \prod_{\alpha=1}^{P_1} \prod_{\beta=P_1+1}^2 (ip_1^{b_\alpha}) \\
& \times (ip_3^{b_\beta})G(p_1, p_2, p_3, p_4)(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1} \times \Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \\
& \times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 5)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 4)} \right]. \tag{50}
\end{aligned}$$

This contribution is quartic in momentum. The next contribution is given by

$$\begin{aligned}
S_3(1-1-3-3) = & \langle V_{s=1}(p_1;0)V_{s=1}(p_2;\infty)A_2(p_3;0)A_0(p_4,0) \rangle = A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \\
& \times \lim_{u \rightarrow \infty} \int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 \times \langle ce^{-2\phi} \partial X^m e^{i\vec{p}_1\vec{X}}(0) ce^{-\phi} \psi^n e^{i\vec{p}_2\vec{X}}(u) ce^{\chi} \partial X^{a_1} \\
& \times \partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)}) e^{i\vec{p}_3\vec{X}}(z_1) P_{2\phi-2\chi-\sigma}^{(2)} e^{\phi} \partial X^{b_1} \partial X^{b_2} \psi^{b_3} e^{i\vec{p}_4\vec{X}}(z_2) \rangle. \tag{51}
\end{aligned}$$

The computation gives

$$S_3(1-1-3-3) = S_3^{(1)} + S_3^{(2)} + S_3^{(3)}, \tag{52}$$

where $S_3^{(1)}$ and $S_3^{(2)}$ are the contributions quartic in momentum while $S_3^{(3)} = S_3^{(3)4\text{-der}} + S_3^{(3)6\text{-der}}$ contains both 4 and 6 derivative terms. These contributions are given by, accordingly,

$$\begin{aligned}
S_3^{(1)} = & 24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \times \eta^{nb_3}\eta^{ma_3} \sum_{N_1=0}^2 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{P_1!(2-P_1)!N_1!(2-N_1)!} \\
& \times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^2 (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_\gamma})(ip_3^{b_\lambda})G(p_1, p_2, p_3, p_4) \times \Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \\
& \times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 4)} \right]. \tag{53}
\end{aligned}$$

$$\begin{aligned}
S_3^{(2)} = & 24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{ma_3} \sum_{M_1=0}^1 \sum_{N_1=0}^2 \sum_{P_1=0}^1 \frac{(-1)^{P_1}}{N_1!(2-N_1)!} \times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^2 (ip_1^{a_\alpha}) \\
& \times (ip_4^{a_\beta})(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1, p_2, p_3, p_4) \times \Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \\
& \times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 6)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 3)} + \frac{2\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 5)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 2)} \right. \\
& \left. + \frac{2\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 4)} \right]. \tag{54}
\end{aligned}$$

The 6-derivative part of $S_3^{(3)}$ is given by

$$\begin{aligned}
S_3^{(3)6\text{-der}} = & 24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \times \eta^{nb_3} \sum_{M_1=0}^1 \sum_{N_1=0}^2 \sum_{P_1=0}^2 \left\{ \frac{(-1)^{P_1}}{P_1!(2-P_1)!N_1!(2-N_1)!} \right. \\
& \times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^2 (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_\gamma})(ip_3^{b_\lambda})(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}G(p_1, p_2, p_3, p_4) \\
& \times \left\{ \times (ip_1)^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 4)} \right] \right. \\
& \left. \left. + (2ip_4^{a_3})\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 5)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 4)} \right] \right\} \right\}. \tag{55}
\end{aligned}$$

The 4-derivative part of $S_3^{(3)}$ is given by

$$\begin{aligned}
S_3^{(3)4\text{-der}} = & -24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{ma_3} \sum_{N_1=0}^1 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{P_1!(2-P_1)!} \\
& \times \prod_{\alpha=1}^{P_1} \prod_{\beta=P_1+1}^2 (ip_1^{b_\alpha})(ip_3^{b_\beta})(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}G(p_1, p_2, p_3, p_4) \\
& \times \left\{ ip_1^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 5) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 3)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 5)} \right] \right. \\
& \left. + ip_4^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 3)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 1)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 5)} \right] \right\} \\
& - 24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{mb_2} \sum_{N_1=0}^2 \sum_{P_1=0}^1 \frac{(-1)^{P_1}}{P_1!(2-P_1)!} \\
& \times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1, p_2, p_3, p_4) \\
& \times \left\{ ip_1^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 5) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 3)} \right] \right. \\
& \left. + ip_4^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 5)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - 3)} \right] \right\} \\
& - 24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{a_2b_2} \sum_{M_1=0}^1 \sum_{N_1=0}^1 \sum_{P_1=0}^1 \\
& \times (ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1, p_2, p_3, p_4) \\
& \times \left\{ ip_1^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 - M_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 2)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 4)} \right] \right. \\
& \left. + ip_4^{a_3}\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 - M_1 + 5)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 2)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 + P_1 - M_1 - 4)} \right] \right\} \tag{56}
\end{aligned}$$

This concludes the computation of $S_3^{(3)}$ and of $S_3(1-1-3-3)$. The final contribution to the amplitude, $S_4(1-1-3-3)$, is given by

$$\begin{aligned} S_4(1-1-3-3) &= \langle V_{s=1}(p_1; 0) V_{s=1}(p_2; \infty) (A_4 + A_5)(p_3; 0) A_0(p_4, 0) \rangle = iA_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) \\ &\times \lim_{u \rightarrow \infty} \int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 \times \langle c e^{-2\phi} \partial X^m e^{i\vec{p}_1 \vec{X}}(0) c e^{-\phi} (\psi^n e^{i\vec{p}_2 \vec{X}}(u) c \xi(\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} + \vec{p} \partial \vec{\psi}) \\ &\times \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}_3 \vec{X}}(z_1) \times P_{2\phi-2\chi-\sigma}^{(2)} e^{\phi} \partial X^{b_1} \partial X^{b_2} \psi^{b_3} e^{i\vec{p}_4 \vec{X}}(z_2) \rangle. \end{aligned} \quad (57)$$

As previously, it is convenient to split this contribution into 6 and 4 derivative parts:

$$S_4(1-1-3-3) = S_4^{6\text{-der}} + S_4^{4\text{-der}}. \quad (58)$$

The 6-derivative part is computed to give

$$\begin{aligned} S_4^{6\text{-der}} &= -24iA_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) \times \sum_{M_1=0}^1 \sum_{N_1=0}^2 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{N_1! (2-N_1)! P_1! (2-P_1)!} \\ &\times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^2 (ip_1^{a_\alpha}) (ip_4^{a_\beta}) (ip_1^{b_\gamma}) (ip_3^{b_\lambda}) (ip_3^m)^{M_1} (ip_3^m)^{1-M_1} G(p_1, p_2, p_3, p_4) \\ &\times \left\{ 2(\eta^{na_3} p_3^{b_3} + \eta^{a_3 b_3} p_3^n) \Gamma(\vec{p}_3 \vec{p}_4 + N_1 + P_1 - 7) \left[\frac{\Gamma(\vec{p}_1 \vec{p}_3 - M_1 - N_1 + 6)}{\Gamma(-\vec{p}_2 \vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1 \vec{p}_3 - M_1 - N_1 + 4)}{\Gamma(-\vec{p}_2 \vec{p}_3 - M_1 + P_1 - 3)} \right] \right. \\ &\left. + (2\eta^{na_3} p_3^{b_3} - \eta^{a_3 b_3} p_3^n) \Gamma(\vec{p}_3 \vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1 \vec{p}_3 - M_1 - N_1 + 5)}{\Gamma(-\vec{p}_2 \vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1 \vec{p}_3 - M_1 - N_1 + 3)}{\Gamma(-\vec{p}_2 \vec{p}_3 - M_1 + P_1 - 3)} \right] \right\}. \end{aligned} \quad (59)$$

Finally, the 4-derivative part of $S_4^{4\text{-der}}$ contributes

$$S_4^{4\text{-der}} = S_4^{(1)4\text{-der}} + S_4^{(2)4\text{-der}} + S_4^{(3)4\text{-der}}, \quad (60)$$

where

$$\begin{aligned} S_4^{(1)4\text{-der}} &= 24iA_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) \sum_{N_1=0}^1 \sum_{P_1=0}^2 \frac{(-1)^{P_1}}{P_1! (2-P_1)!} \\ &\times \prod_{\alpha=1}^{P_1} \prod_{\beta=P_1+1}^2 (ip_1^{b_\alpha}) (ip_3^{b_\beta}) (ip_1^{a_1})^{N_1} (ip_4^{a_1})^{1-N_1} G(p_1, p_2, p_3, p_4) \\ &\times \left\{ 2\eta^{a_2 m} (\eta^{na_3} p_3^{b_3} + \eta^{a_3 b_3} p_3^n) \Gamma(\vec{p}_3 \vec{p}_4 + N_1 + P_1 - 5) \left[\frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 3)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1 - 2)} + \frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 1)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1 - 4)} \right] \right. \\ &\left. + \eta^{a_2 m} (2\eta^{na_3} p_3^{b_3} - \eta^{a_3 b_3} p_3^n) \Gamma(\vec{p}_3 \vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 4)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1 - 2)} + \frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 2)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1 - 4)} \right] \right\}, \end{aligned} \quad (61)$$

$$\begin{aligned} S_4^{(2)4\text{-der}} &= 24iA_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) \sum_{N_1=0}^2 \sum_{P_1=0}^1 \frac{(-1)^{P_1}}{N_1! (2-N_1)!} \\ &\times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 (ip_1^{a_\alpha}) (ip_4^{a_\beta}) (ip_1^{b_1})^{P_1} (ip_3^{b_1})^{1-P_1} G(p_1, p_2, p_3, p_4) \\ &\times \left\{ 2\eta^{b_2 m} (\eta^{na_3} p_3^{b_3} + \eta^{a_3 b_3} p_3^n) \Gamma(\vec{p}_3 \vec{p}_4 + N_1 + P_1 - 5) \left[\frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 5)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1)} + \frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 3)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1 - 2)} \right] \right. \\ &\left. + \eta^{b_2 m} (2\eta^{na_3} p_3^{b_3} - \eta^{a_3 b_3} p_3^n) \Gamma(\vec{p}_3 \vec{p}_4 + N_1 + P_1 - 6) \left[\frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 6)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1)} + \frac{\Gamma(\vec{p}_1 \vec{p}_3 - N_1 + 4)}{\Gamma(-\vec{p}_2 \vec{p}_3 + P_1 - 2)} \right] \right\}, \end{aligned} \quad (62)$$

and

$$\begin{aligned}
S_4^{(3)4\text{-der}} &= 24iA_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \sum_{M_1=0}^1 \sum_{N_1=0}^1 \sum_{P_1=0}^1 (-1)^{P_1} \times (ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}(ip_1^{b_1})^{P_1} \\
&\times (ip_3^{b_1})^{1-P_1} G(p_1, p_2, p_3, p_4) \times \left\{ 2\eta^{a_2b_2}(\eta^{na_3}p_3^{b_3} + \eta^{a_3b_3}p_3^n)\Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 6) \right. \\
&\times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 5)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 3)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 3)} \right] + \eta^{a_2b_2}(2\eta^{na_3}p_3^{b_3} - \eta^{a_3b_3}p_3^n) \\
&\times \Gamma(\vec{p}_3\vec{p}_4 + N_1 + P_1 - 7) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 6)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 - M_1 - N_1 + 4)}{\Gamma(-\vec{p}_2\vec{p}_3 - M_1 + P_1 - 3)} \right] \Big\}. \quad (63)
\end{aligned}$$

This concludes the computation of $S_4(1-1-3-3)$ and of $S(1-1-3-3)(p_1, \dots, p_4)$ in general. The overall 4-point amplitude $A(1-1-3-3)(p_1, \dots, p_4)$ describing the $1-1-3-3$ quartic interaction is obtained from $S(1-1-3-3)(p_1, \dots, p_4)$ by adding $A(1-1-3-3) \times (p_1, \dots, p_4) = S(1-1-3-3)(p_1, \dots, p_4) + (p_3 \leftrightarrow p_4)$, according to (34).

IV. 4-POINT AMPLITUDE AND $1-1-3-3$ QUARTIC INTERACTION

Now that our computation of the $1-1-3-3$ point amplitude is complete, the concluding step is to deduce the related quartic interaction from the structure of $A(1-1-3-3)$. The momentum factors of $ip_J (J=1, \dots, 4)$ translate into derivatives of the space-time fields $A^m, A^n, H^{a_1a_2a_3}$ and $H^{b_1b_2b_3}$ in the position space, while the common $G(p_1, p_2, p_3, p_4)$ factor reflects the nonlocality of the interaction. In addition, all the terms in the amplitude (34) contain the Γ -function factors with the structure

$$\begin{aligned}
\Xi(M_1, N_1, P_1) &\sim \Gamma(\vec{p}_3\vec{p}_4 - a(M_1, N_1, P_1)) \\
&\times \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 + b(M_1, N_1, P_1))}{\Gamma(-\vec{p}_2\vec{p}_3 - c(M_1, N_1, P_1))} \right. \\
&\left. + \frac{\Gamma(\vec{p}_1\vec{p}_3 + b(M_1, N_1, P_1) - 2)}{\Gamma(-\vec{p}_2\vec{p}_3 - c(M_1, N_1, P_1) - 2)} \right], \quad (64)
\end{aligned}$$

where a, b , and c are the numbers appearing in summations over M_1, N_1 , and P_1 . The γ function factor is of some subtlety. While the numbers a, b , and c differ from term to term, it is easy to see that, in general, $a > 0$, $b > 0$, and $b \geq 2$ for each term in the amplitude. For this reason, in the field theory limit $\vec{p}_I\vec{p}_J \rightarrow 0$ that we are interested in,

$\Xi(M_1, N_1, P_1)$ generally includes the singular part, with simple poles in $\vec{p}_3\vec{p}_4$ and $\vec{p}_1\vec{p}_3$ (the latter only appear in terms with $b=2$), as well as the part regular in $\vec{p}_I\vec{p}_J$. The singular part is actually related to the flow of the cubic part of the effective action, rather than to the genuine quartic interaction we are looking for. That is, the singular terms in the γ -function factors are related to two types of exchanges: The first is the $s=1$ field exchange between two $s=3$ vertices, while the second is the $s=3$ field exchange between $s=3$ and $s=1$ operators. These are the exchanges that induce the renormalization group flows on the world sheet for $s=1$ and $s=3$ fields, resulting in the leading (cubic) order terms in the low energy effective action. Schematically, the β -function of the $s=1$ field in the $s=3$ background is given by $\beta_A \sim -\Delta A + CH^2$, where C are the structure constants appearing in the $1-3-3$ 3-point amplitude (expressed in the position space). This particularly leads to cubic terms of the type $\sim CAH^2$ in the low energy effective action. At the same time, the low energy effective equations of motion for the $s=3$ gauge field in the presence of $s=1$ background are given by, in the leading order, $\beta_H \sim \Delta H - CAH = 0$ which, if substituted into cubic terms, lead to “nonlocal” quartic terms of the type $\sim \frac{C^2 A^2 H^2}{\Delta}$ which structurally coincide with the contribution of the singular part of the Γ -function factor to the 4-point amplitude. To obtain the genuine quartic $1-1-3-3$ interaction from the 4-point amplitude (34), one has to subtract the singularities from each of the Γ -function factors appearing in the expressions (42)–(63), similar to the procedure explained in [55]. The Γ -function factors with $b(M_1, N_1, P_1) > 2$ can be expanded in $\vec{p}_I\vec{p}_J$ with the leading order terms given by

$$\begin{aligned}
&\Xi(a(M_1, N_1, P_1), b(M_1, N_1, P_1), c(M_1, N_1, P_1)) \\
&= \Gamma(\vec{p}_3\vec{p}_4 - a) \left[\frac{\Gamma(\vec{p}_1\vec{p}_3 + b)}{\Gamma(-\vec{p}_2\vec{p}_3 - c)} + \frac{\Gamma(\vec{p}_1\vec{p}_3 + b - 2)}{\Gamma(-\vec{p}_2\vec{p}_3 - c - 2)} \right] \\
&\approx \frac{(-1)^{a+c}(b-3)!(c-2)!\vec{p}_2\vec{p}_3}{a!\vec{p}_3\vec{p}_4} \{ 1 + (b-2)(b-1)(c-1)c + (\vec{p}_1\vec{p}_3)[(b-2)(b-1)(c-1)cL(b-1) + L(b-3)] \\
&\quad + (\vec{p}_2\vec{p}_3)[(b-2)(b-1)(c-1)cL(c) + L(c-2)] - (\vec{p}_3\vec{p}_4)[(b-2)(b-1)(c-1)c + 1] + \dots \} \quad (65)
\end{aligned}$$

Then the related factors in the quartic terms in the low energy effective action are given by replacing each of the $\Xi(a, b, c)$ factors in the amplitude according to

$$\begin{aligned} \Xi(a, b, c) &\rightarrow \tilde{\Xi}(a, b, c) \\ &= \Xi(a(M_1, N_1, P_1), b(M_1, N_1, P_1), c(M_1, N_1, P_1)) \\ &\quad - \frac{(-1)^{a+c}(b-3)!(c-2)!\tilde{p}_2\tilde{p}_3}{a!\tilde{p}_3\tilde{p}_4} \{1 + (b-2)(b-1)(c-1)c + \tilde{p}_1\tilde{p}_3[(b-2)(b-1)(c-1)cL(b-1) + L(b-3)] \\ &\quad + \tilde{p}_2\tilde{p}_3[(b-2)(b-1)(c-1)cL(c) + L(c-2)] - \tilde{p}_3\tilde{p}_4[(b-2)(b-1)(c-1)c + 1]\}, \end{aligned} \quad (66)$$

where

$$L(n) = \sum_{m=1}^n \frac{1}{m}. \quad (67)$$

Each of $\tilde{\Xi}$ contributes, in the leading order in $\tilde{p}_I\tilde{p}_J$, the factor given by

$$\begin{aligned} \tilde{\Xi}(a, b, c) &\approx (-1)^{a+c+1}\tilde{p}_2\tilde{p}_3 \frac{(b-3)!(c-2)!}{a!} \\ &\quad \times L(a)[1 + (b-2)(b-1)(c-1)c]. \end{aligned} \quad (68)$$

Similarly, in the special case of $b = 2$, the $\Xi \rightarrow \tilde{\Xi}$ replacement for the quartic term in the low energy effective action is given by

$$\begin{aligned} \Xi(a, 2, c) &\rightarrow \tilde{\Xi}(a, 2, c) \\ &= \Xi(a, 2, c) + \frac{(-1)^{a+c+1}a!c!(\tilde{p}_2\tilde{p}_3)}{(\tilde{p}_3\tilde{p}_4)} \\ &\quad \times [1 + \tilde{p}_1\tilde{p}_3 + L(c)\tilde{p}_2\tilde{p}_3] \\ &\quad + (-1)^{a+c}a!(c-2)!\left(\frac{1}{\tilde{p}_3\tilde{p}_4} + \frac{1}{\tilde{p}_1\tilde{p}_3}\right) \\ &\quad \times (1 + L(c-2)\tilde{p}_2\tilde{p}_3) \end{aligned} \quad (69)$$

with the leading order contribution

$$\begin{aligned} \tilde{\Xi}(a, 2, c) &\approx (-1)^{a+c+1}a!c!L(a)\tilde{p}_2\tilde{p}_3 \\ &\quad + (-1)^{a+c+1}a!(c-2)!(L(a) - 2L(c-2)) \end{aligned} \quad (70)$$

to the amplitude. This concludes the evaluation of the gauge-invariant $1-1-3-3$ quartic interaction in the low-energy effective action.

V. 1-1-5-5 STRUCTURE: CONCLUSION AND DISCUSSION

In this paper we have obtained the gauge-invariant quartic interaction of massless higher spin fields in string theory approach. Although we have concentrated on the $1-1-3-3$ case, with the structure of higher spin vertex operators, basic properties of amplitudes discussed in this paper (such as nonlocality and derivative structure of the

kinematic part of the amplitude) will also hold for more general $1-1-s-s$ cases. The nonlocality structure of the 4-point amplitude calculated in this paper is the consequence of the specific ghost structure of the vertex operators for the massless $s = 3$ fields. In particular, non-standard ghost coupling of $s = 3$ vertices leads to two integrated vertices appearing in the 4-point amplitude (contrary to 1 out of 4 integrated vertex in the standard Veneziano case) producing the factor that diverges on-shell but leads to nonlocalities in the β -function equations (which essentially are the off-shell equations). Thus the emergence of the nonlocalities is closely related to the OPE structures in nontrivial $\beta - \gamma$ and $b - c$ ghost cohomologies that the higher spin vertex operators belong to. In our future works, we hope to perform this OPE analysis in more details. Below, we shall just briefly comment on the relation of the nonlocalities with the OPE structure of the higher spin operators. The nonlocalities stem from the fact that the $s = 3$ operators at the positive pictures exist only at integrated $b - c$ -pictures, namely, they are the elements of the R_2 $b - c$ ghost cohomology (see [52] for discussion). Their OPE with each other produces a massless vertex proportional to the $s = 1$ operator at a “double-integrated” $b - c$ -picture (that is $\sim A(p):Z^2V_{\text{ph}}(p):$), where V_{ph} is the standard operator of a photon and the Z -transformation operator is given in (20). It is straightforward to check that the proportionality coefficient is $A(p) \sim p^{-2}$. (Note that the expression for the double-integrated photon $\sim :Z^2V_{\text{ph}}(p):$ is by itself quadratic in momentum that also can be seen from direct computation.) Thus the inverse Z -transform, applied to the OPE of two $s = 3$ operators, produces the $s = 1$ operator times the factor of $A(p) \sim p^{-2}$ which ultimately leads to the nonlocality in the space-time effective action. A separate question is whether these nonlocalities can be cured by introducing auxiliary compensator fields. To answer this question from the string theory point of view, one has to construct vertex operators that could be interpreted as the sources of the compensator. Construction of such objects in open string theory may not be easy, as they would probably carry higher ghost numbers and their BRST analysis would be quite cumbersome. An important step in this direction would be to relate vertex operator formalism in string theory to the framelike formulation of higher spin field

theories (rather than to the metric one consider in this paper), as it could be more natural to describe the objects like compensators in the framelike approach. Developing a string-theoretic description of higher spin theories in the framelike approach (e.g. finding suitable operators to describe frame fields and connections) is a particularly interesting problem and is currently work in progress.

As for the local part of the $3-3-1-1$ interaction terms, it is structurally reminiscent of the 3-point $3-3-2$ amplitude on the disc describing *cubic* gravitational couplings of massless spin 3 field, that can be expressed in terms of linearized Weyl tensor [41]. In particular, the minimal number of derivatives in the kinematic part of $1-1-3-3$ is equal to $2 \times 3 - 2 = 4$, similar to the $3-3-2$ case. While it is known that cubic $s-s-2$ gauge-invariant couplings always contain a minimum of $2s-2$ space-time derivatives [41], it looks plausible that a similar derivative rule applies to the kinematic part quartic $1-1-s-s$ couplings as well, as the disc $2-s-s$ amplitude (with two spin s operators on the boundary and the spin 2 operator in the bulk) is structurally similar to the special limit ($p_3 = p_4$) of the kinematic part of the $1-1-s-s$ point amplitude in open string theory (where p_3 and p_4 are the momenta of the spin s particles). So at least string theory appears to predict that the minimal derivative rule for the quartic $1-1-s-s$ couplings should be similar to the one established for the $2-s-s$ case. This certainly is the case for $s=3$, and it would be interesting to check if this rule also works for spins higher than 3. Conceptually, the calculation performed in this paper for the $1-1-3-3$ case should be quite similar for $1-1-s-s$ amplitudes with higher values of s as well. In any case, the derivative/momentum structure of the amplitudes is tightly controlled by the ghost structure of the vertex operators and by the overall ghost number balance. For example, the $1-1-5-5$ amplitude $A(1-1-5-5)(p_1, \dots, p_4)$ is structurally

$$\begin{aligned} & A(1-1-5-5)(p_1, \dots, p_4) \\ &= S(1-1-5-5)(p_1, \dots, p_4) \\ &+ (p_3 \leftrightarrow p_4) S(1-1-5-5) \\ &= \langle V_{s=1}^{(-2)} V_{s=1}^{(-2)}(0); \Gamma^{-1} V_{s=1}^{(-2)}; (\infty) A_{s=3}^{(0)}(z_1) A_{s=3}^{(0)}(z_2) \rangle, \quad (71) \end{aligned}$$

where, as previously, $V_{s=1}^{(-2)} = c e^{-2\phi} \partial X^n e^{ipX} A_n(p)$ are un-integrated photon operators at picture -2 , and $\Gamma^{-1} = -4c e^{\chi-2\phi} \partial \chi$ is the inverse picture changing, so the photon at picture -3 has the overall ghost structure $\sim e^{\chi-4\phi}$, while

$$\begin{aligned} A_{s=3}^{(0)}(z) &\sim H_{a_1 \dots a_5} \oint dw (z-w)^4 e^{2\phi} P_{2\phi-2\chi-\sigma}^{(4)} \\ &\times \partial X^{a_1} \partial X^{a_2} \partial X^{a_3} \partial \psi^{a_4} \psi^{a_5} e^{ipX}(w). \quad (72) \end{aligned}$$

Note that, although the full BRST-invariant expression for spin 5 operators contains, apart from $A^{(0)}$, terms with ghost structures $\sim c e^{\chi+\phi}$ and $\sim \partial c c e^{2\chi}$, the ghost balance condition only allows the contributions from $A^{(0)}$ -part with the ghost structure $\sim e^{2\phi}$ (provided, of course, that the photons are chosen at pictures -2 and -3). Again, we see that, first of all, the amplitude contains a double world sheet integration (as in the $1-1-3-3$ case), leading to the non-locality of the interaction. While the computation of the matter/kinematic part of this amplitude is relatively straightforward and similar to the $1-1-3-3$ case described above, the evaluation of the ghost part of this amplitude is quite tedious due to lengthy operator products of the ghost polynomials $P_{2\phi-2\chi-\sigma}^{(4)}$ with the ghost exponents and between themselves. Below we present the expression for the $1-1-5-5$ amplitude up to numerical coefficients which can be fixed by explicit evaluation of these operator products. Evaluating the correlator (71) and integrating in z_1, z_2 , we get the answer

$$\begin{aligned} S_{1-1-5-5} &= A_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) \eta^{a_4 b_4} \eta^{a_5 b_5} \eta^{a_3 n} \\ &\times \sum_{L=0}^4 \sum_{Q=0}^{2L} \sum_{Q_1=0}^Q \sum_{Q_2=0}^{2L-Q} \sum_{R_1=0}^{4-Q_2} \sum_{M_1=0}^1 \sum_{N_1=0}^2 \sum_{P_1=0}^3 \frac{(-1)^{P_1}}{N_1! (2-N_1)! P_1! (3-P_1)!} \alpha_{L,Q,Q_1,Q_2,R_1} \\ &\times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^2 \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^2 (i p_1^{a_\alpha}) (i p_4^{a_\beta}) (i p_1^{b_\gamma}) (i p_3^{b_\lambda}) (i p_3^m)^{M_1} (i p_3^m)^{1-M_1} \times G(p_1, p_2, p_3, p_4) \\ &\times \frac{\Gamma(9+Q_1-Q+R_1-M_1-N_1+\vec{p}_1 \vec{p}_3) \Gamma(2L-20+N_1+P_1+\vec{p}_3 \vec{p}_4)}{\Gamma(P_1-M_1+R_1+Q_1-Q-11-\vec{p}_2 \vec{p}_3)} \\ &+ (m \leftrightarrow n, N_1 \leftrightarrow P_1) \times (-1)^{N_1+P_1}, \quad (73) \end{aligned}$$

where α_{L,Q,Q_1,Q_2,R_1} are the numerical coefficients to be extracted from the ghost OPEs. The overall amplitude $A(1-1-5-5)(p_1, \dots, p_4)$ is again obtained from $S(1-1-5-5)(p_1, \dots, p_4)$ by adding $A(1-1-5-5) = S(1-1-5-5) + (p_3 \leftrightarrow p_4)$ according to (71). The

kinematic part of this amplitude contains a minimum number of 6 space-time derivatives. At the same time, all the Γ -functions in the denominator of (73) are proportional to $\sim (\vec{p}_2 \vec{p}_3)^{-1}$ in the field theory limit, for all the values of M_1, P_1, Q_1, R_1 , and Q . For this reason, the local part of the

amplitude (73) is of at least 8 powers in momentum, again according to the $2s - 2$ -rule conjectured above. One has to check explicitly, however, whether this rule holds in each separate case for different values of s . As the massless higher spins are described in terms of physical vertex operators in open string theory, it would be interesting to generalize the calculations presented in this paper (as well as in our previous works [47,48]) by coupling these operators to Chan-Paton matrices and computing their correlators. This could lead to new interesting non-Abelian extensions of theories with massless higher spin fields.

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