First-order quantum correction to the Larmor radiation from a moving charge in a spatially homogeneous time-dependent electric field

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First-order quantum correction to the Larmor radiation is investigated on the basis of the scalar QED on a homogeneous background of a time-dependent electric field, which is a generalization of a recent work by Higuchi and Walker so as to be extended for an accelerated charged particle in a relativistic motion. We obtain a simple approximate formula for the quantum correction in the limit of the relativistic motion when the direction of the particle motion is parallel to that of the electric field.

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I. INTRODUCTION

The Larmor radiation is the classical radiation from a charged particle in an accelerated motion [1]. In a recent paper by Higuchi and Walker [2], the quantum correction to the Larmor radiation is investigated on the basis of the scalar quantum electrodynamics (QED). In their approach, the mode function for the complex scalar field is constructed with the Wentzel-Kramers-Brillouin (WKB) approximation, in a form expanded with respect to $\hbar$. In a series of Higuchi and Martin's work [3–5] (see also references therein), it has been well understood that the mode function reproduces the classical Larmor formula when the radiation energy is evaluated at the order of $\hbar^0$. The first-order quantum correction to the classical Larmor radiation is evaluated at the order of $\hbar$ in Ref. [2], though the investigation is limited to the nonrelativistic motion of the charged particle.

In the present paper, we consider a simple generalization of Higuchi and Walker's work [2], in order to investigate the case of a relativistic motion of an accelerated charge. Assuming a homogeneous but time-varying background of an electric field, we derive a formula for radiation energy of the order of $\hbar$, the first-order correction due to the quantum effect. This generalized formula is applicable to an accelerated charge in a relativistic motion, and we focus our investigation on the first-order quantum correction to the Larmor radiation in the limit of the relativistic motion. This paper is organized as follows: In Sec. II, we present the general formula for the first-order quantum correction to the Larmor radiation. In Sec. III, we show that the formula reproduces the same result obtained in Ref. [2], in the limit of the nonrelativistic motion of an accelerated charge. Then an approximate formula in the limit of the relativistic motion is presented. Section IV is devoted to summary and conclusions. In Appendix A, a brief derivation of the approximate formulas is summarized. In Appendix B, we consider the validity of the WKB approximation. Throughout this paper, we use units in which the velocity of light equals 1, unless stated otherwise.

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II. FORMULATION

We consider the scalar QED with the action

$$S = \int dt d^3\mathbf{x}\Big[(D_\mu \phi)^\dagger D^\mu \phi - \frac{m^2}{\hbar^2}\phi^\dagger \phi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\Big], \tag{1}$$

where $D_\mu = (\partial/\partial x^\mu + ieA_\mu/\hbar)$, e and m are the charge and mass of the massive scalar field, respectively, and μ_0 is the magnetic permeability of vacuum. We work in the Minkowski spacetime, but consider the homogeneous electric background field $\mathbf{E}(t)$, which is related to the vector potential by $\bar{A}_\mu = (0, \mathbf{A}(t))$ and $\dot{\mathbf{A}}(t) = -\mathbf{E}(t)$, where the dot denotes the differentiation with respect to the time. The equation of motion of the free scalar field yields

$$\bigg(\frac{\partial^2}{\partial t^2} + \frac{(\mathbf{p} - e\mathbf{A}(t))^2 + m^2}{\hbar^2}\bigg)\varphi_{\mathbf{p}}(t) = 0, \tag{2}$$

where $\varphi_{\mathbf{p}}(t)$ is the coefficient of the Fourier expansion of the field, i.e., the mode function. Using the mode function, which is normalized so as to be $\dot{\varphi}^*_{\mathbf{p}}\varphi_{\mathbf{p}} - \varphi^*_{\mathbf{p}}\dot{\varphi}_{\mathbf{p}} = i$, the quantized field is constructed as

$$\phi(x) = \sqrt{\frac{\hbar}{L^3}}\sum_{\mathbf{p}}(\varphi_{\mathbf{p}}(t)b_{\mathbf{p}} + \varphi^*_{-\mathbf{p}}(t)c^\dagger_{-\mathbf{p}})e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{3}$$

where L^3 is the volume of the space, the creation and annihilation operators satisfy the commutation relations,

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = \delta_{\mathbf{p},\mathbf{p}'}, \qquad [b_{\mathbf{p}}, b_{\mathbf{p}'}] = [b^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = 0, \tag{4}$$

and the same relations hold for $c_{\mathbf{p}}$ and $c^\dagger_{\mathbf{p}}$. We also quantize the free electromagnetic field as

$$A_\mu = \sqrt{\frac{\mu_0\hbar}{L^3}}\sum_{\lambda=1,2}\sum_{\mathbf{k}}\epsilon^\lambda_\mu\Big(\frac{e^{-ikt}}{\sqrt{2k}}a^\lambda_{\mathbf{k}} + \text{H.c.}\Big)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5}$$

where ϵ_μ^λ denotes the polarization vector, and $a_{\mathbf{k}}^{\lambda\dagger}$ and $a_{\mathbf{k}}^\lambda$ are the creation and annihilation operators which satisfy the following commutation relation,

$$[a_{\mathbf{k}}^\lambda, a_{\mathbf{k}'}^{\lambda'\dagger}] = \delta^{\lambda\lambda'}\delta_{\mathbf{k},\mathbf{k}'}. \tag{6}$$

We consider the process in which one photon is emitted from a charged particle, as shown in Fig. 1. Note that this process is prohibited without the background electric field because of the Lorentz invariance of the Minkowski spacetime, which ensures existence of the frame that the charged particle is at rest. However, on an electric field background, we have the radiation energy from the process, which can be evaluated, as follows. Using the in-in formalism [6,7], we may compute the radiation energy at the lowest order of the coupling constant,

$$\begin{aligned} E &= \sum_\lambda \int d^3\mathbf{k}\hbar k \langle a_{\mathbf{k}}^{\lambda\dagger} a_{\mathbf{k}}^\lambda\rangle \\ &= \hbar^{-2}\sum_\lambda \int d^3\mathbf{k}\hbar k \,\mathrm{Re}\int_{-\infty}^{\infty} dt_2 \\ &\quad\times \int_{-\infty}^{\infty} dt_1 \langle \mathrm{in}|H_I(t_1) a_{\mathbf{k}}^{\lambda\dagger} a_{\mathbf{k}}^\lambda H_I(t_2)|\mathrm{in}\rangle, \end{aligned} \tag{7}$$

where we adopted the range of the integration from the infinite past to the infinite future, and $|\mathrm{in}\rangle$ denotes the initial state, which we choose as one charged particle state with the momentum $\mathbf{p}_i$, i.e., $|\mathrm{in}\rangle = b_{\mathbf{p}_i}^\dagger|0\rangle$, and

$$\begin{aligned} H_I(t) = -\frac{ie}{\hbar}\int d^3\mathbf{x} A^\mu \bigg\{\bigg(\partial_\mu - \frac{ie}{\hbar}\bar{A}_\mu\bigg)\phi^\dagger\phi \\ - \phi^\dagger\bigg(\partial_\mu + \frac{ie}{\hbar}\bar{A}_\mu\bigg)\phi\bigg\}. \end{aligned} \tag{8}$$

Expression (7) leads to the lowest contribution corresponding to the process of Fig. 1,

$$\begin{aligned} E = -\frac{e^2}{\epsilon_0}\int \frac{d^3\mathbf{k}}{(2\pi)^3} k \bigg\{ \bigg| \int dt \frac{e^{ikt}}{\sqrt{2k}} \bigg(\frac{\partial}{\partial t}\varphi_{\mathbf{p}_f}(t)^*\varphi_{\mathbf{p}_i}(t) \\ - \varphi_{\mathbf{p}_f}^*(t)\frac{\partial}{\partial t}\varphi_{\mathbf{p}_i}(t)\bigg)\bigg|^2 \\ - \bigg|\int dt \frac{e^{ikt}}{\sqrt{2k}}\bigg(\frac{i(\mathbf{p}_f - e\mathbf{A})}{\hbar}\varphi_{\mathbf{p}_f}(t)^*\varphi_{\mathbf{p}_i}(t) \\ + \varphi_{\mathbf{p}_f}^*(t)\frac{i(\mathbf{p}_i - e\mathbf{A})}{\hbar}\varphi_{\mathbf{p}_i}(t)\bigg)\bigg|^2\bigg\}, \end{aligned} \tag{9}$$

where $\mathbf{p}_f = \mathbf{p}_i - \hbar\mathbf{k}$, and ϵ_0 is the permittivity of vacuum, which is related to μ_0 by $\epsilon_0\mu_0 = 1/c^2 = 1$. Performing the partial integral and using Eq. (2), we have

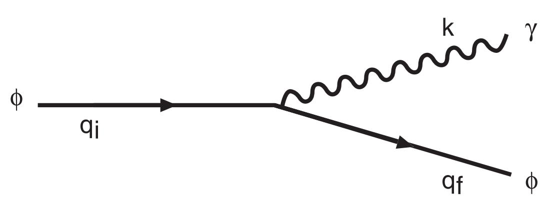

FIG. 1. Feynman diagram for the process.

$$\begin{aligned} E = -\frac{e^2}{\epsilon_0}\int \frac{d^3\mathbf{k}}{(2\pi)^3} k \\ \times \bigg\{ \bigg|\int dt \frac{e^{ikt}}{\sqrt{2k}} \frac{\hat{\mathbf{k}}\cdot(\mathbf{p}_i + \mathbf{p}_f - 2e\mathbf{A})}{\hbar}\varphi_{\mathbf{p}_f}^*(t)\varphi_{\mathbf{p}_i}(t)\bigg|^2 \\ - \bigg|\int dt \frac{e^{ikt}}{\sqrt{2k}} \frac{\mathbf{p}_i + \mathbf{p}_f - 2e\mathbf{A}}{\hbar}\varphi_{\mathbf{p}_f}^*(t)\varphi_{\mathbf{p}_i}(t)\bigg|^2\bigg\}, \end{aligned} \tag{10}$$

where $\hat{\mathbf{k}}$ is the unit vector of $\mathbf{k}$, i.e., $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. We consider the following WKB solution for the mode function

$$\varphi_{\mathbf{p}}(t) = \frac{1}{\sqrt{2\Omega_{\mathbf{p}}(t)}}\exp\bigg[-i\int^t \Omega_{\mathbf{p}}(t')dt'\bigg] \tag{11}$$

with

$$\Omega_{\mathbf{p}}(t) = \sqrt{(\mathbf{p} - e\mathbf{A}(t))^2 + m^2}/\hbar; \tag{12}$$

then Eq. (10) gives

$$\begin{aligned} E = -\frac{e^2}{2\epsilon_0}\frac{1}{2^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\bigg\{\bigg|\int dt \frac{\hat{\mathbf{k}}\cdot(2\mathbf{p}_i - 2e\mathbf{A}(t) - \hbar\mathbf{k})}{\hbar\sqrt{\Omega_{\mathbf{p}_i}(t)}\sqrt{\Omega_{\mathbf{p}_f}(t)}} \\ \times\exp\bigg[ikt + i\int^t(\Omega_{\mathbf{p}_f}(t') - \Omega_{\mathbf{p}_i}(t'))dt'\bigg]\bigg|^2 \\ - \bigg|\int dt\frac{2\mathbf{p}_i - 2e\mathbf{A}(t) - \hbar\mathbf{k}}{\hbar\sqrt{\Omega_{\mathbf{p}_i}(t)}\sqrt{\Omega_{\mathbf{p}_f}(t)}} \\ \times\exp\bigg[ikt + i\int^t(\Omega_{\mathbf{p}_f}(t') - \Omega_{\mathbf{p}_i}(t'))dt'\bigg]\bigg|^2\bigg\} \end{aligned} \tag{13}$$

with

$$\begin{aligned} \Omega_{\mathbf{p}_i}(t) &= \frac{1}{\hbar}\sqrt{(\mathbf{p}_i - e\mathbf{A}(t))^2 + m^2}, \\ \Omega_{\mathbf{p}_f}(t) &= \frac{1}{\hbar}\sqrt{(\mathbf{p}_i - \hbar\mathbf{k} - e\mathbf{A}(t))^2 + m^2}. \end{aligned} \tag{14}$$

In order to evaluate the quantum correction, we consider the expansion in terms of a power series of $\hbar$. Up to the order of $\mathcal{O}(\hbar)$, we have

$$\begin{aligned} \Omega_{\mathbf{p}_f} - \Omega_{\mathbf{p}_i} &\simeq -\frac{\mathbf{k}\cdot(\mathbf{p}_i - e\mathbf{A})}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \\ &\quad + \frac{\hbar}{2}\bigg(\frac{k^2}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \\ &\quad - \frac{(\mathbf{k}\cdot(\mathbf{p}_i - e\mathbf{A}))^2}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}^3}\bigg), \\ \frac{1}{\hbar\sqrt{\Omega_{\mathbf{p}_i}}\sqrt{\Omega_{\mathbf{p}_f}}} &\simeq \frac{1}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \\ &\quad \times\bigg(1 + \frac{\hbar}{2}\frac{\mathbf{k}\cdot(\mathbf{p}_i - e\mathbf{A})}{[(\mathbf{p}_i - e\mathbf{A})^2 + m^2]}\bigg), \end{aligned}$$

then

$$E = -\frac{e^2}{2\epsilon_0}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int d\xi\int d\xi'\bigg\{\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) - \frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg\}e^{ik\xi-ik\xi'}\bigg[1+\frac{\hbar k}{2}\frac{\hat{\mathbf{k}}\cdot(\mathbf{p}_i-e\mathbf{A})}{(\mathbf{p}_i-e\mathbf{A})^2+m^2} + \frac{\hbar k}{2}\frac{\hat{\mathbf{k}}\cdot(\mathbf{p}_i-e\mathbf{A}')}{(\mathbf{p}_i-e\mathbf{A}')^2+m^2} + \frac{i\hbar}{2}\int_{t'}^{t}\bigg(\frac{k^2}{\sqrt{(\mathbf{p}_i-e\mathbf{A}'')^2+m^2}} - \frac{k^2(\hat{\mathbf{k}}\cdot(\mathbf{p}_i-e\mathbf{A}''))^2}{\sqrt{(\mathbf{p}_i-e\mathbf{A}'')^2+m^2}^3}\bigg)dt'' + \mathcal{O}(\hbar^2)\bigg], \tag{15}$$

where we used the notations $\mathbf{x}=\mathbf{x}(t)$, $\mathbf{x}'=\mathbf{x}(t')$, $\mathbf{A}=\mathbf{A}(t)$, $\mathbf{A}'=\mathbf{A}(t')$, $\mathbf{A}''=\mathbf{A}(t'')$, and we introduced the new variable ξ instead of t by

$$\xi = t - \int^t \frac{\hat{\mathbf{k}}\cdot(\mathbf{p}_i - e\mathbf{A}(t''))}{\sqrt{(\mathbf{p}_i-e\mathbf{A}(t''))^2+m^2}}dt'', \tag{16}$$

and ξ' is defined in the same way as ξ but with replacing t by t'. Furthermore, we introduce the quantities parametrized by t (or ξ),

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{p}_i - e\mathbf{A}, \tag{17}$$

$$\frac{dt}{d\tau} = \sqrt{(\mathbf{p}_i - e\mathbf{A})^2+m^2}; \tag{18}$$

then Eq. (15) is rephrased as

$$E = -\frac{e^2}{2\epsilon_0}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int d\xi\int d\xi'\bigg\{\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) - \frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg\}e^{ik\xi-ik\xi'}\bigg[1+\frac{\hbar k}{2}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt} + \hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg) + \frac{i\hbar k^2}{2}\int_{t'}^{t}d\tau''\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\bigg)^2\bigg)\bigg] + \mathcal{O}(\hbar^2). \tag{19}$$

The mathematical technique adopted in Ref. [2] is equivalent to replacing k in Eq. (19) by the partial differentiation with respect to ξ or ξ' which operates to $e^{ik\xi-ik\xi'}$. Partial integrations lead to

$$E = E^{(0)} + E^{(1)} + \mathcal{O}(\hbar^2), \tag{20}$$

where we defined

$$E^{(0)} = -\frac{e^2}{2\epsilon_0(2\pi)^3}\int d\Omega_{\hat{\mathbf{k}}}\int_0^\infty dk\int d\xi \times\int d\xi' e^{ik(\xi-\xi')}\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg) - \frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg), \tag{21}$$

$$E^{(1)} = -\frac{e^2}{2\epsilon_0(2\pi)^3}\int d\Omega_{\hat{\mathbf{k}}}\int_0^\infty dk\int d\xi\int d\xi' e^{ik(\xi-\xi')} \times\bigg\{\frac{i\hbar}{4}\bigg(\frac{d}{d\xi}-\frac{d}{d\xi'}\bigg)\frac{d}{d\xi}\frac{d}{d\xi'}\bigg[\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) - \frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt} + \hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg)\bigg] + \frac{i\hbar}{2}\frac{d^2}{d\xi^2}\frac{d^2}{d\xi'^2}\bigg[\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) - \frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) \times\int_{\xi'(t')}^{\xi(t)}d\xi''\frac{d\tau''}{d\xi''}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\bigg)^2\bigg)\bigg]\bigg\}, \tag{22}$$

where $E^{(0)}$ and $E^{(1)}$ are the terms of the order of $\hbar^0$ and $\hbar^1$, respectively. Here, we assumed the boundary terms can be neglected, as is the case in Ref. [2]. The integration with respect to k yields

$$E^{(0)} = \frac{e^2}{(4\pi)^2\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}\int d\xi\bigg(\bigg(\frac{d^2\mathbf{x}}{d\xi^2}\bigg)^2 - \bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)^2\bigg). \tag{23}$$

The expression (23) yields the classical formula of the Larmor radiation from a charged particle. The first-order quantum correction of the order of $\hbar$ is described by Eq. (22), which yields

$$E^{(1)} = \frac{e^2\hbar}{(4\pi)^3\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}\int d\xi\int d\xi'\frac{1}{\xi-\xi'} \times\bigg\{\bigg(\frac{d}{d\xi}-\frac{d}{d\xi'}\bigg)\frac{d}{d\xi}\frac{d}{d\xi'}\bigg[\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) - \frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt} + \hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg)\bigg] + 2\frac{d^2}{d\xi^2}\frac{d^2}{d\xi'^2}\bigg[\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) - \frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg) \times\int_{\xi'(t')}^{\xi(t)}d\xi''\frac{d\tau''}{d\xi''}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\bigg)^2\bigg)\bigg]\bigg\}. \tag{24}$$

Equation (24) transforms into Eq. (A3). Other useful formulas are summarized in Appendix A.

III. APPROXIMATE FORMULAS

In the nonrelativistic limit, where the velocity $\mathbf{v} = d\mathbf{x}/dt$ is small compared with the velocity of light, $|\mathbf{v}| \ll 1$, Eqs. (23) and (24) reduce to

$$E^{(0)} = \frac{e^2}{6\pi\epsilon_0}\int dt\dot{\mathbf{v}}(t)\cdot\dot{\mathbf{v}}(t), \tag{25}$$

$$E^{(1)} = \frac{e^2\hbar}{6\pi^2\epsilon_0 m}\int dt\int dt'\frac{\ddot{\mathbf{v}}(t)\cdot\dot{\mathbf{v}}(t') - \dot{\mathbf{v}}(t)\cdot\ddot{\mathbf{v}}(t')}{t-t'}, \tag{26}$$

respectively. A brief derivation is summarized in Appendix A. Equation (26) was found for the first time by Higuchi and Walker in Ref. [2]. In the case of the periodic electric field, $|\mathbf{E}| = E_0\sin\omega t$, where E_0 is a constant, we have the periodic acceleration, $|\dot{\mathbf{v}}| = (eE_0/m)\times\sin\omega t$. Then

$$\frac{dE^{(0)}}{dt} = \frac{e^4 E_0^2}{m^2}\frac{\sin^2\omega t}{6\pi\epsilon_0}, \tag{27}$$

$$\frac{dE^{(1)}}{dt} = -\frac{\hbar e^4 E_0^2}{m^2}\frac{\omega}{12\pi\epsilon_0 m}. \tag{28}$$

After taking an average over a long time-duration, we have

$$\frac{E^{(1)}}{E^{(0)}} = -\frac{\hbar\omega}{mc^2}, \tag{29}$$

where c is the light velocity, which is restored here. The quantum effect becomes important when the time scale of the acceleration multiplied by c is comparable to the Compton wavelength, namely, when the wavelike feature of the particle appears.

Let us consider a more general case, when the electric field $|\mathbf{E}| = E_0 f(t/t_0)$, where f is a function of t/t_0 with a constant $t_0(>0)$. In this case, the acceleration is $|\dot{\mathbf{v}}| = (eE_0/m)f(t/t_0)$, and Eqs. (23) and (24) give

$$E^{(0)} = \frac{e^4 E_0^2 t_0}{6\pi\epsilon_0 m^2}\int d\tau f^2(\tau), \tag{30}$$

$$E^{(1)} = -\frac{\hbar e^4 E_0^2}{6\pi^2\epsilon_0 m^3}\iint d\tau d\tau' \frac{f(\tau)f_{,\tau'}(\tau') - f(\tau')f_{,\tau}(\tau)}{\tau - \tau'}, \tag{31}$$

respectively. Then we have

$$\frac{E^{(1)}}{E^{(0)}} = -\frac{\hbar}{\pi mc^2 t_0}D, \tag{32}$$

where we defined

$$D = \left(\int d\tau'' f^2(\tau'')\right)^{-1}\iint d\tau d\tau' \frac{f(\tau)f_{,\tau'}(\tau') - f(\tau')f_{,\tau}(\tau)}{\tau - \tau'}, \tag{33}$$

where $f_{,\tau}(\tau)$ means the differentiation of $f(\tau)$ with respect to τ. Here, let us consider the following four cases: (1) $f(\tau) = 1 - (\tau^2)^n$ for $|\tau| \le 1$ and $f(\tau) = 0$ for $|\tau| > 1$, (2) $f(\tau) = 1/(1+\tau^2)^n$ for $-\infty < \tau < \infty$, (3) $f(\tau) = 1/(\cosh\tau)^n$ for $-\infty < \tau < \infty$, and (4) $f(\tau) = (1+\tau)^n$ for $-1 \le \tau \le 0$, $f(\tau) = (1-\tau)^m$ for $0 < \tau \le 1$, and $f(\tau) = 0$ for $|\tau| > 1$. Figure 2 shows D as a function of n for cases (1)–(3), in which one can see that D is positive. Figure 3 shows D as a function of n and m for case (4), in which too we find that D is positive.

Thus, in all of the above cases, the first-order quantum correction $E^{(1)}$ is negative. Also, the quantum effect is very small as long as the motion of the particle is nonrelativistic. This result is consistent with that found in Refs. [2,8]. The quantum effect might become important when the emitted photon energy becomes of order of mc^2 [2]. Note that this speculation is based on the result with the nonrelativistic approximation.

Next, let us consider the relativistic limit, $|\mathbf{p}_i| \gg |e\mathbf{A}|$, m. For simplicity, we consider the case when the direction of the particle motion is always parallel to that of the background electric field, i.e., $\mathbf{v} \propto \mathbf{A}$. Namely, we consider the case when the directions of the particle's motion and the background electric field are parallel at any moment, and adopt this direction as the z axis. Then we may write $\mathbf{A} = (0, 0, A(t))$, $\dot{\mathbf{A}} = (0, 0, -E(t))$, $\mathbf{v} = (0, 0, v)$, and $\mathbf{p}_i = (0, 0, p_i)$. In this case, we have

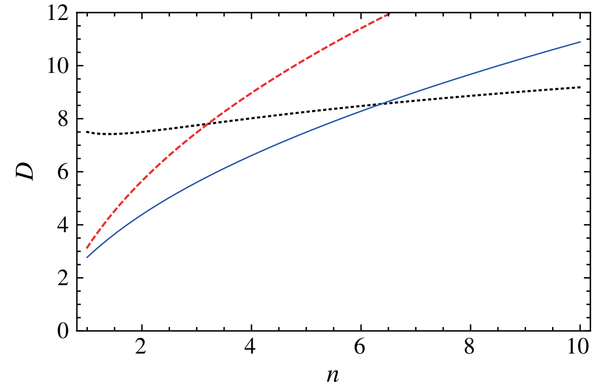

FIG. 2 (color online). D as a function of n for cases (1)–(3). The dotted curve is case (1) $f(\tau) = 1 - (\tau^2)^n$ for $-1 \le \tau \le 1$ and $f(\tau) = 0$ for $|\tau| > 1$, the (red) dashed curve is case (2) $f(\tau) = 1/(1+\tau^2)^n$, and the (blue) solid curve is case (3) $f(\tau) = 1/(\cosh\tau)^n$.

$$E^{(0)} = \frac{e^2}{(4\pi)^2\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}(1-\cos^2\theta)\int dt \frac{m^4}{p_i^6}\frac{e^2\dot{A}^2(t)}{(1-v\cos\theta)^5}. \tag{34}$$

The integration with respect to $\hat{\mathbf{k}}$ yields

$$E^{(0)} = \frac{1}{6\pi\epsilon_0}\frac{m^4 e^4}{p_i^6}\int dt \frac{\dot{A}^2(t)}{(1-v^2)^3}. \tag{35}$$

We consider the case $p_i \gg |eA|$, m. We also assume $|A| \sim |\dot{A}/\omega| \sim |\ddot{A}/\omega^2|$, where $1/\omega$ is a time scale of a time-varying background electric field. In this relativistic limit, we have the leading order expression for the quantum correction (see also Appendix A),

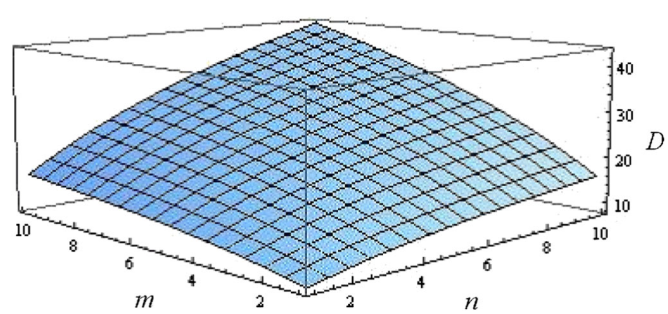

FIG. 3 (color online). D as a function of m and n of case (4), $f(\tau) = (1+\tau)^n$ for $-1 \le \tau \le 0$, $f(\tau) = (1-\tau)^m$ for $0 < \tau \le 1$, and $f(\tau) = 0$ for $|\tau| > 1$.

$$E^{(1)} \simeq -\frac{e^2\hbar}{(4\pi)^3\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}(1-\cos^2\theta)\int d\xi\int d\xi' \times \frac{1}{\xi-\xi'}\frac{m^2}{p_i^5}\frac{1}{(1-v\cos\theta)^2(1-v'\cos\theta)^2}e^2 \times\bigg\{\ddot{A}(t)\dot{A}(t')\bigg(\frac{-v^2\cos\theta}{(1-v\cos\theta)(1-v'\cos\theta)} + \frac{(2+v'\cos\theta)v'}{(1-v\cos\theta)^2}\bigg) - \ddot{A}(t')\dot{A}(t) \times\bigg(\frac{-v'^2\cos\theta}{(1-v\cos\theta)(1-v'\cos\theta)}+\frac{(2+v\cos\theta)v}{(1-v'\cos\theta)^2}\bigg)\bigg\}. \tag{36}$$

Adopting the approximation, $v = v' = \bar{v} \simeq 1$, and $\xi - \xi' \simeq (t-t')(1-\bar{v}\cos\theta)$, we have

$$E^{(1)} \simeq -\frac{e^2\hbar}{4(2\pi)^3\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}(1-\cos^2\theta) \times\int dt\int dt'\frac{1}{(1-\bar{v}\cos\theta)^5}\frac{m^2}{p_i^5} \times\frac{e^2(\ddot{A}(t)\dot{A}(t')-\dot{A}(t)\ddot{A}(t'))}{t-t'}. \tag{37}$$

The integration with respect to $\hat{\mathbf{k}}$ yields

$$E^{(1)} \simeq -\frac{e^4\hbar}{3(2\pi)^2\epsilon_0}\frac{m^2}{p_i^5}\int dt\int dt'\frac{1}{(1-\bar{v}^2)^3} \times\frac{\ddot{A}(t)\dot{A}(t')-\dot{A}(t)\ddot{A}(t')}{t-t'}. \tag{38}$$

In the case of the periodic background of the electric field, $\dot{A}(t) = -E_0\sin\omega t$, where E_0 is a constant, we have

$$\frac{dE^{(0)}}{dt} = \frac{e^4m^4}{6\pi\epsilon_0 p_i^6}\frac{E_0^2\cos^2\omega t}{(1-v^2)^3}, \tag{39}$$

$$\frac{dE^{(1)}}{dt} = \frac{\hbar e^4 m^2}{12\pi\epsilon_0 p_i^5}\frac{E_0^2\omega}{(1-v^2)^3}. \tag{40}$$

After averaging over sufficiently long time-duration, we have

$$\frac{E^{(1)}}{E^{(0)}} = \frac{p_i}{mc}\frac{\hbar\omega}{mc^2}. \tag{41}$$

Note that the quantum correction $E^{(1)}$ is positive, which is a contrast to the nonrelativistic case.

Similar to the nonrelativistic limit, we next consider the case $\dot{A}(t) = -E_0 f(t/t_0)$, with a general function $f(\tau)$. In this case, we have

$$E^{(0)} = \frac{e^4m^4E_0^2t_0}{6\pi\epsilon_0 p_i^6(1-\bar{v})^3}\int d\tau f^2(\tau), \tag{42}$$

$$E^{(1)} = \frac{\hbar e^4 m^2 E_0^2}{12\pi^2\epsilon_0 p_i^5(1-\bar{v})^3} \times\iint d\tau d\tau'\frac{f(\tau)f_{,\tau'}(\tau')-f(\tau')f_{,\tau}(\tau)}{\tau-\tau'}, \tag{43}$$

and

$$\frac{E^{(1)}}{E^{(0)}} = \frac{p_i}{\pi mc}\frac{\hbar}{mc^2 t_0}D, \tag{44}$$

where D is defined by Eq. (33). When we adopt the four function of $f(\tau)$ considered in the case of the nonrelativistic limit, D is positive. Thus, in contrast to the nonrelativistic case, the quantum correction $E^{(1)}$ is positive again, for all the cases in the present paper.

For the radiation from an electron in a periodic electric field, e.g., by a laser field, Eq. (41) is estimated as

$$\frac{E^{(1)}}{E^{(0)}} \sim 2.6\times 10^{-3}\bigg(\frac{p_i c}{\mathrm{GeV}}\bigg)\bigg(\frac{mc^2}{0.5\ \mathrm{MeV}}\bigg)^{-2}\bigg(\frac{\omega}{10^{15}\ \mathrm{s}^{-1}}\bigg), \tag{45}$$

where $\omega \sim 10^{15}\ \mathrm{s}^{-1}$ corresponds to an x-ray laser. The quantum effect becomes significant when the electron kinetic energy reaches the TeV scale. The above formula is derived under the condition $p_i \gg |eA|, m$. For a periodic electric field of large amplitude, $p_i \sim |eA|$, the condition of the relativistic motion cannot be always guaranteed, because the physical momentum might become $|\mathbf{p}_i - e\mathbf{A}| \sim m$. In this case, it is difficult to express the quantum correction in a simple analytic form. We need a more general treatment including fully numerical calculation, because the nonlocality plays an important role. Potentially, there is a lot of room for discussion about how to detect the quantum effect of the Larmor radiation experimentally, but this is outside of the scope of the present paper.

IV. SUMMARY

In the present paper, we obtained the general formula, Eq. (24) or Eq. (A3), for the first-order quantum correction to the Larmor radiation from a charged particle moving in a spatially homogeneous but time-dependent electric field. This formula reproduces the same result as that in Ref. [2], in the limit of a nonrelativistic motion of the charged particle. Our result is useful to investigate the case of a relativistic motion. When the direction of a particle's motion is parallel to that of the background electric field, a simple formula was derived. In the limit of the relativistic motion of the charged particle, we obtained the formula (36). Similar to the case of the nonrelativistic motion [2], the leading quantum effect is described by a nonlocal difference between $\dot{A}(t)\ddot{A}(t')$ and $\ddot{A}(t)\dot{A}(t')$, as is demonstrated in Eq. (38). This quantum effect disappears when $\dot{A}$ is constant. Note that Eq. (38) is the leading term in the limit of the ultrarelativistic motion, assuming $p_i \gg eA, m$ and $|A| \sim |\dot{A}/\omega| \sim |\ddot{A}/\omega^2|$. We discarded the other subleading terms. For example, the term in proportion to $\dot{A}(t)\dot{A}(t')(\dot{A}(t)-\dot{A}(t'))$ appears in the subleading terms, but also disappears when $\dot{A}$ is constant. Thus, the essence of the quantum effect of the Larmor radiation should be the nonlocality, which reflects the fact that the exact solution of motion cannot be represented with simple classical trajectories in quantum theory [2].

We also note that the expression in the nonrelativistic limit (26) is not simply connected to that in the relativistic limit (38). The leading contributions for these opposite limits come from different sources. In the nonrelativistic limit, the leading contribution comes only from terms in (A2), i.e., the phase of the mode function. On the other hand, in the relativistic limit, the leading contribution comes from both (A1) and (A2), i.e., the amplitude and the phase of the mode function. An interesting question might be how these facts are related to the difference of our final results in the opposite limits.

By applying the formula to the cases of a periodic acceleration and possible function of acceleration, it was demonstrated that the leading quantum effect enhances the radiation in the relativistic limit and that it decreases in the nonrelativistic limit. This quantum effect will become important when the incident kinetic electron energy approaches TeV scale for a periodic electric field background with an x-ray laser. However, this result is obtained assuming that the charged particle is moving in the direction parallel to that of the background electric field. In a practical situation, this assumption is somewhat ideal. Here too there is a lot of room for further investigations of more general cases (cf. Ref. [2]), but this is outside of the scope of the present paper.

Our work, which is based on the QED theoretical framework, will be useful to investigate the feature of the radiation from an electron under a strong electric field. Investigation of the quantum effect in the Larmor radiation could be related to the subject of testing the QED process in the strong field background. For example, Chen and Tajima claimed the possibility of detecting the Unruh effect in the radiation from an electron under an ultraintense laser background [9]. Possible signature of the Unruh effect in the radiation from an electron accelerated by an electric field of strong lasers is under debate (cf. [10–12]). According to Ref. [9], the radiation from the Unruh effect could be of order of $\hbar$. The characteristic signature of the Unruh effect claimed in [9] is in proportion to E_0^3 at order of $\hbar$. As mentioned in the above, in our approach, the term in proportion to $\dot{A}(t)\dot{A}(t')(\dot{A}(t)-\dot{A}(t'))$ appears in the subleading terms in evaluating Eq. (24). This might give a contribution in proportion to E_0^3. However, the angular dependence is different. In our approach, the quantum radiation of the order $\hbar$ emitted in the direction of the motion, $\theta = 0$, is exactly zero from Eq. (19). This is a difference between our result and the prediction in Ref. [9], which might be tested experimentally. However, in our approach, it is difficult to separate the signature of the Unruh effect from other effects, even if they existed. This is a disadvantage of our approach.

In Ref. [8], the quantum radiation from a charged particle moving in an expanding or contracting universe was investigated. It was shown that the radiation can be regarded as the Larmor radiation from a charged particle in an decelerated (accelerated) motion, because the physical momentum of the particle decreases (increases) as the background universe expands (contracts) [8,13]. The approach developed in the present paper is useful to investigate the quantum effect of this process [14].

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APPENDIX A: BRIEF SUMMARY OF DERIVATION OF APPROXIMATE FORMULAS

It is straightforward to derive the following formulas:

$$\begin{aligned}
&\left(\frac{d}{d\xi}-\frac{d}{d\xi'}\right)\frac{d}{d\xi}\frac{d}{d\xi'}\left[\left(\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\right)\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\right)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\right)\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}+\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\right)\right]\\
&\quad=\left(\left(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\right)\left(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\right)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}-\left(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\right)\left(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\right)+\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\right)\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}+\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\right)\\
&\qquad+2\left(\left(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\right)\left(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\right)-\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\right)\left(\frac{d}{d\xi}\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}\right)-\frac{d}{d\xi'}\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\right)\right)\\
&\qquad+\left(\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\right)\left(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\right)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\right)\frac{d^2}{d\xi^2}\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}\right)-\left(\left(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\right)\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\right)-\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d\mathbf{x}'}{d\xi'}\right)\frac{d^2}{d\xi'^2}\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\right)\\
&\qquad+\left(\left(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\right)\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\right)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d\mathbf{x}'}{d\xi'}\right)\frac{d}{d\xi'}\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\right)-\left(\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\right)\left(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\right)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\right)\frac{d}{d\xi}\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}\right)
\end{aligned}\tag{A1}$$

and

$$\frac{d^2}{d\xi^2}\frac{d^2}{d\xi'^2}\bigg[\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\int_{\xi'(t')}^{\xi(t)}d\xi''\frac{d\tau''}{d\xi''}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\bigg)^2\bigg)\bigg]$$
$$=\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\int_{\xi'}^{\xi}d\xi''\frac{d\tau''}{d\xi''}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\bigg)^2\bigg)-2\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)$$
$$\times\frac{d\tau'}{d\xi'}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\bigg)^2\bigg)+2\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\frac{d\tau}{d\xi}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\bigg)^2\bigg)$$
$$-\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\frac{d}{d\xi'}\bigg(\frac{d\tau'}{d\xi'}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\bigg)^2\bigg)\bigg)$$
$$+\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\frac{d}{d\xi}\bigg(\frac{d\tau}{d\xi}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\bigg)^2\bigg)\bigg). \qquad (A2)$$

Then we find

$$E^{(1)}=\frac{e^2\hbar}{(4\pi)^3\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}\int d\xi\int d\xi'\frac{1}{\xi-\xi'}\bigg\{\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}-\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)+\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)$$
$$\times\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}+\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg)+2\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)-\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)\bigg(\frac{d}{d\xi}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}\bigg)-\frac{d}{d\xi'}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg)\bigg)$$
$$+\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)\frac{d^2}{d\xi^2}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}\bigg)-\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)-\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\frac{d^2}{d\xi'^2}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg)$$
$$+\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\frac{d}{d\xi'}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\frac{d\tau'}{dt'}\bigg)-\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\frac{d}{d\xi}\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\frac{d\tau}{dt}\bigg)$$
$$+2\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\int_{\xi'}^{\xi}d\xi''\frac{d\tau''}{d\xi''}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\bigg)^2\bigg)-4\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d^2\mathbf{x}'}{d\xi'^2}\bigg)$$
$$\times\frac{d\tau'}{d\xi'}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\bigg)^2\bigg)+4\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^2\mathbf{x}}{d\xi^2}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d^2\mathbf{x}}{d\xi^2}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\frac{d\tau}{d\xi}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\bigg)^2\bigg)-2\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}}{d\xi^3}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)$$
$$-\frac{d^3\mathbf{x}}{d\xi^3}\cdot\frac{d\mathbf{x}'}{d\xi'}\bigg)\frac{d}{d\xi'}\bigg(\frac{d\tau'}{d\xi'}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}'}{dt'}\bigg)^2\bigg)\bigg)+2\bigg(\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{d\xi}\bigg)\bigg(\hat{\mathbf{k}}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)-\frac{d\mathbf{x}}{d\xi}\cdot\frac{d^3\mathbf{x}'}{d\xi'^3}\bigg)\frac{d}{d\xi}\bigg(\frac{d\tau}{d\xi}\bigg(1-\bigg(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}\bigg)^2\bigg)\bigg)\bigg\}. \qquad (A3)$$

From the definition of ξ by Eq. (16), we have

$$\frac{d\xi}{dt}=1-\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}}{dt}=1-\mathcal{v}, \qquad (A4)$$

where we defined $\mathcal{v}=\hat{\mathbf{k}}\cdot\mathbf{v}$. Then we also have

$$\frac{d\mathbf{x}}{d\xi}=\frac{\mathbf{v}}{1-\mathcal{v}}, \qquad (A5)$$

$$\frac{d^2\mathbf{x}}{d\xi^2}=\frac{\dot{\mathbf{v}}}{(1-\mathcal{v})^2}+\frac{\mathbf{v}\dot{\mathcal{v}}}{(1-\mathcal{v})^3}, \qquad (A6)$$

$$\frac{d^3\mathbf{x}}{d\xi^3}=\frac{\ddot{\mathbf{v}}}{(1-\mathcal{v})^3}+\frac{\mathbf{v}\ddot{\mathcal{v}}}{(1-\mathcal{v})^4}+\frac{3\dot{\mathbf{v}}\dot{\mathcal{v}}}{(1-\mathcal{v})^4}+\frac{3\mathbf{v}\dot{\mathcal{v}}^2}{(1-\mathcal{v})^5}, \qquad (A7)$$

and

$$\frac{d\tau}{dt}=\frac{1}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}}, \qquad (A8)$$

$$\frac{d}{d\xi}\bigg(\mathcal{v}\frac{d\tau}{dt}\bigg)=\frac{1}{1-\mathcal{v}}\bigg(\frac{\dot{\mathcal{v}}}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}}-\frac{\mathcal{v}(\mathbf{p}_i-e\mathbf{A})\cdot(-e\dot{\mathbf{A}})}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}^3}\bigg), \qquad (A9)$$

$$\frac{d^2}{d\xi^2}\bigg(\mathcal{v}\frac{d\tau}{dt}\bigg)=\frac{\dot{\mathcal{v}}}{(1-\mathcal{v})^3}\bigg(\frac{\dot{\mathcal{v}}}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}}-\frac{\mathcal{v}(\mathbf{p}_i-e\mathbf{A})\cdot(-e\dot{\mathbf{A}})}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}^3}\bigg)+\frac{1}{(1-\mathcal{v})^2}\bigg(\frac{\ddot{\mathcal{v}}}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}}$$
$$+\frac{3\mathcal{v}((\mathbf{p}_i-e\mathbf{A})\cdot(-e\dot{\mathbf{A}}))^2}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}^5}-\frac{2\dot{\mathcal{v}}(\mathbf{p}_i-e\mathbf{A})\cdot(-e\dot{\mathbf{A}})+\mathcal{v}(-e\dot{\mathbf{A}})\cdot(-e\dot{\mathbf{A}})+\mathcal{v}(\mathbf{p}_i-e\mathbf{A})\cdot(-e\ddot{\mathbf{A}})}{\sqrt{(\mathbf{p}_i-e\mathbf{A})^2+m^2}^3}\bigg). \qquad (A10)$$

In the limit of a nonrelativistic motion of a charged particle, we use the following approximate formulas,

$$\frac{d\mathbf{x}}{d\xi} \simeq \mathbf{v}, \qquad \frac{d^2\mathbf{x}}{d\xi^2} \simeq \dot{\mathbf{v}}, \qquad \frac{d^3\mathbf{x}}{d\xi^3} \simeq \ddot{\mathbf{v}},$$

$$\frac{d\tau}{dt} \simeq \frac{1}{m}, \qquad \frac{d}{d\xi}\left(\varkappa\frac{d\tau}{dt}\right) \simeq \frac{\dot{\varkappa}}{m}, \qquad \frac{d^2}{d\xi^2}\left(\varkappa\frac{d\tau}{dt}\right) \simeq \frac{\ddot{\varkappa}^2}{m};$$

then we have the following expression by neglecting subleading terms,

$$E^{(1)} = \frac{e^2\hbar}{(4\pi)^3\epsilon_0}\int d\Omega_{\hat{\mathbf{k}}}\int dt\int dt'\frac{1}{t-t'}\frac{2}{m}\{(\dot{\varkappa}\dot{\varkappa}' - \dot{\mathbf{v}}\cdot\dot{\mathbf{v}}') \times(\varkappa-\varkappa') + (\ddot{\varkappa}\ddot{\varkappa}' - \ddot{\mathbf{v}}\cdot\ddot{\mathbf{v}}')(t-t') - 2(\ddot{\varkappa}\dot{\varkappa}' - \ddot{\mathbf{v}}\cdot\dot{\mathbf{v}}') + 2(\dot{\varkappa}\ddot{\varkappa}' - \dot{\mathbf{v}}\cdot\ddot{\mathbf{v}}')\}. \tag{A11}$$

We also neglect the first term of the right-hand side of (A11), which is of order of $\dot{v}^3$. After the integration with respect to $\hat{\mathbf{k}}$, we have

$$E^{(1)} = \frac{e^2\hbar}{(4\pi)^2\epsilon_0}\int dt\int dt'\frac{4}{3m}\left\{-\ddot{\mathbf{v}}\cdot\ddot{\mathbf{v}}' + \frac{2(\ddot{\mathbf{v}}\cdot\dot{\mathbf{v}}' - \dot{\mathbf{v}}\cdot\ddot{\mathbf{v}}')}{t-t'}\right\}. \tag{A12}$$

The first term of the right-hand side of (A12) gives no contribution by assuming that the acceleration is zero at the boundaries of the time. Then Eq. (26) is obtained.

Now let us consider the case when the particle is moving with a relativistic speed in the direction parallel to the electric field. We choose the z axis parallel to this direction. Then we may write $\mathbf{A} = (0, 0, A(t))$, $\mathbf{v} = (0, 0, v)$, and $\mathbf{p}_i = (0, 0, p_i)$. In this case, we have

$$\dot{v} \simeq -\frac{m^2 e\dot{A}}{p_i^3}, \qquad \ddot{v} \simeq -\frac{m^2 e\ddot{A}}{p_i^3}.$$

We also have

$$\frac{d^2 z}{d\xi^2} \simeq \frac{-1}{(1-v\cos\theta)^3}\frac{m^2 e\dot{A}}{p_i^3},$$

$$\frac{d^3 z}{d\xi^3} \simeq \frac{-1}{(1-v\cos\theta)^4}\frac{m^2 e\ddot{A}}{p_i^3},$$

$$\frac{d\tau}{dt} \simeq \frac{1}{p_i}, \qquad \frac{d}{d\xi}\left(\varkappa\frac{d\tau}{dt}\right) \simeq \frac{v\cos\theta}{1-v\cos\theta}\frac{e\dot{A}}{p_i^2},$$

and

$$\frac{d^2}{d\xi^2}\left(\varkappa\frac{d\tau}{dt}\right) \simeq \frac{v\cos\theta}{(1-v\cos\theta)^2}\frac{e\ddot{A}}{p_i^2} - \frac{v\cos\theta}{(1-v\cos\theta)^3}\frac{(e\dot{A})^2 m^2}{p_i^5},$$

in the limit of the relativistic motion. These approximate formulas yield the expression (36) at the leading order. In this derivation, we note that the following term included in Eq. (A3),

$$\int_{\xi'}^{\xi} d\xi''\frac{d\tau''}{d\xi''}\left(1-\left(\hat{\mathbf{k}}\cdot\frac{d\mathbf{x}''}{dt''}\right)^2\right) = \int_{\xi'}^{\xi} d\xi''\frac{d\tau''}{dt''}\frac{dt''}{d\xi''}(1-\varkappa^2(t'')), \tag{A13}$$

gives no contribution at the leading order. Using Eqs. (A4) and (A8), the right-hand side of Eq. (A13) is written as

$$\int_{\xi'}^{\xi} d\xi''\frac{d\tau''}{dt''}\frac{dt''}{d\xi''}(1-\varkappa^2(t'')) = \int_{\xi'}^{\xi} d\xi''\frac{(1+\varkappa(t''))}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}}. \tag{A14}$$

In the limit of the relativistic motion, the leading term is

$$\int_{\xi'}^{\xi} d\xi''\frac{(1+\varkappa(t''))}{\sqrt{(\mathbf{p}_i - e\mathbf{A})^2 + m^2}} \simeq \frac{1}{p_i}\int_{\xi'}^{\xi} d\xi''\left(1+\frac{e\mathbf{A}\cdot\mathbf{p_i}}{|p_i|^2}\right)\times(1+\varkappa(t'')) \simeq \frac{1+\cos\theta}{p_i}(\xi-\xi'). \tag{A15}$$

The contribution of this term to $E^{(1)}$ is zero by assuming that the acceleration is zero at the boundaries of the time.

APPENDIX B: VALIDITY OF WKB APPROXIMATION

We consider the validity of using the WKB approximation, which breaks down when the background field varies rapidly. The following condition is necessary to use the WKB approximation (e.g., [15]),

$$\frac{1}{2\Omega_{\mathbf{p}}^2}\left|\frac{\ddot{\Omega}_{\mathbf{p}}}{\Omega_{\mathbf{p}}} - \frac{3}{2}\frac{\dot{\Omega}_{\mathbf{p}}^2}{\Omega_{\mathbf{p}}^2}\right| \ll 1. \tag{B1}$$

Using the expression (12), this condition yields

$$\frac{\hbar^2}{2((\mathbf{p}_i - e\mathbf{A})^2 + m^2)^3}\left|\frac{5}{2}(e\dot{\mathbf{A}}\cdot(\mathbf{p}_i - e\mathbf{A}))^2 + ((\mathbf{p}_i - e\mathbf{A})^2 + m^2)(e\ddot{\mathbf{A}}\cdot(\mathbf{p}_i - e\mathbf{A}) - (e\dot{\mathbf{A}})^2)\right| \ll 1. \tag{B2}$$

In the relativistic limit, $|\mathbf{p}_i| \gg |e\mathbf{A}|, m$, as is considered in Sec. III, (B2) reduces to

$$\frac{\hbar^2}{2(\mathbf{p}_i^2)^3}\left|\frac{5}{2}(e\dot{\mathbf{A}}\cdot\mathbf{p}_i)^2 + \mathbf{p}_i^2(e\ddot{\mathbf{A}}\cdot\mathbf{p}_i)\right| \ll 1. \tag{B3}$$

In the case of the periodic electric field, $\dot{A} = -E_0\sin\omega t$, (B3) gives

$$\frac{\hbar^2 e^2 E_0^2}{p_i^4} \ll 1, \quad \text{and} \quad \frac{\hbar^2 e E_0\omega}{p_i^3} \ll 1, \tag{B4}$$

which can be rewritten as

$$\left(\frac{\hbar\omega}{p_i}\right)^2\left(\frac{eE_0}{p_i\omega}\right)^2 \ll 1, \quad \text{and} \quad \left(\frac{\hbar\omega}{p_i}\right)^2\left(\frac{eE_0}{p_i\omega}\right) \ll 1. \tag{B5}$$

We impose $eE_0/\omega \ll p_i$, as a condition of the limit of the relativistic motion. Then the first inequality (B5) is satisfied when the second inequality (B5) is satisfied. Then the condition required for the WKB approximation is written as

$$1.3\times 10^{-10}\left(\frac{\omega}{10^{15}\ \mathrm{s}^{-1}}\right)\left(\frac{eE_0}{1\times 10^{15}\ \mathrm{eV/m}}\right)\left(\frac{p_i c}{\mathrm{MeV}}\right)^{-3} \ll 1. \tag{B6}$$

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